Roth's Theorems for Matrix Equations With Symmetry Constraints

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ABSTRACT

This note deals with the consistency of complex matrix equations $AX - YB = C$ and $AX - XB = C$ under the constraints $Y = X^*$ and $X = X^*$. Let $F$ be a field, and let $A, B, C$ be matrices over $F$ of respective sizes $m \times n$, $s \times k$, and $m \times k$. Put

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

and

$$M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$ 

The following theorem was given by W. E. Roth.

THEOREM 1 [2].

(1) The matrix equation

$$AX - YB = C$$ 

(1)
has a solution $X \in F^{n \times k}$, $Y \in F^{m \times s}$ if and only if $M_c$ and $M_o$ are equivalent (i.e., have the same rank).

(2) Assume $n = m$, $k = s$. There exists a solution $X \in F^{n \times k}$ of

$$AX - XB = C$$

if and only if $M_c$ and $M_o$ are similar.

In this note we assume $F = \mathbb{C}$ and consider (1) and (2) together with the respective constraints $Y = X^*$ and $X = X^*$. The approach of [1] will be used to prove the following results.

**Theorem 2.** Assume $n = s$, $k = m$. The following statements are equivalent:

(a) The equation

$$AX - X^*B = C$$

has a solution $X \in \mathbb{C}^{n \times k}$.

(b) There exists a nonsingular matrix $S \in \mathbb{C}^{(n+k) \times (n+k)}$ such that

$$\begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} = S \begin{pmatrix} C & -A \\ B & 0 \end{pmatrix} S^*$.

(c) There exist nonsingular matrices $R, S \in \mathbb{C}^{(n+k) \times (n+k)}$ such that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} R = S \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

and

$$S^* \begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix} R = \begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix}.$$

**Theorem 3.** Assume $n = m = k = s$. The following statements are equivalent:

(a) The equation

$$AX - XB = C$$

has a hermitian solution.
(b) There exists a nonsingular matrix $R \in \mathbb{C}^{2n \times 2n}$ which satisfies

\[
R^{-1} \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix} R = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}
\]

and

\[
R^* \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} R = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

Proof of Theorem 2. (a) ⇒ (c): For $X \in \mathbb{C}^{n \times k}$ and $Y \in \mathbb{C}^{m \times s}$ define

\[
G_X = \begin{pmatrix} I_n & X \\ 0 & I_k \end{pmatrix}
\]

and

\[
G_Y = \begin{pmatrix} I_m & Y \\ 0 & I_s \end{pmatrix}.
\]

As was observed in [2], Equation (1) can be written in an equivalent form

\[
M_0 G_X - G_Y M_C = 0.
\] (7)

Put

\[
J = \begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix}.
\]

For a nonsingular matrix $G$ define $G^{-*} = (G^*)^{-1}$. Obviously $Y = X^*$ is equivalent to

\[
G_Y^* J G_X = J.
\]

Hence (7) yields (5) with $R = G_X$ and $S = G_Y$.

(b) ⇔ (c): Put

\[
P_C = \begin{pmatrix} C & -A \\ B & 0 \end{pmatrix}
\]
such that $P_c J = M_c$. Hence (4) is equivalent to

$$M_0 J^{-1} S^{-*} J = SM_c.$$ 

Note that (6) can be written as $R = J^{-1} S^{-*} J$.

(c) $\Rightarrow$ (a): Let

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{pmatrix}$$

be complex $(n + k) \times (n + k)$ matrices where $U_{12} \in C^{n \times k}$ and $W_{12} \in C^{k \times n}$. Put

$$\Gamma_c = \{(U, W) \mid M_0 U - W M_c = 0\}$$

and

$$\Delta_c = \{(U, W) \mid J U^* J^* M_0 - M_c J^* W^* J = 0\}.$$  

The conditions for $(U, W) \in \Gamma_c$ are

$$AU_1 - W_1 A = 0, \quad AU_{12} - W_1 C - W_{12} B = 0,$$

$$BU_{21} - W_{21} A = 0, \quad BU_2 - W_{21} C - W_2 B = 0,$$

and those for $(U, W) \in \Delta_c$ are given explicitly by

$$U_2^* A - AW_2^* + CW_2^* = 0, \quad -U_{12}^* B + AW_{12}^* - CW_1^* = 0,$$

$$-U_{21}^* A + BW_{21}^* = 0, \quad U_1^* B - BW_1^* = 0.$$  

Put

$$D_c = \Gamma_c \cap \Delta_c.$$  

Clearly $D_c$ is a vector space over $C$. For $C = 0$ let $\Gamma_0$, $\Delta_0$, and $D_0$ be defined by (9), (10), and (13). It is not difficult to verify that $M_c = S^{-1} M_0 R$ together with (6) implies that $(U, W) \in D_c$ is equivalent to $(U R^{-1}, W S^{-1}) \in D_0$. Hence

$$\dim D_c = \dim D_0.$$
Suppose there exists a pair \((U, W) \in D_c\) such that \(W_1 = I\). Then (11) and (12) yield

\[ AU_{12} - W_{12} B = C \quad (15) \]

and

\[ AW_{12}^* - U_{12}^* B = C, \quad (16) \]

and

\[ X = \frac{1}{2}(U_{12} + W_{12}^*) \quad (17) \]

is a solution of (3).

Set \(E = \mathbb{C}^{(n+k) \times (n+k)}\). We introduce a linear map \(\varphi : E \times E \to \mathbb{C}^{(n+k) \times k}\) and define

\[ \varphi(U, W) = \begin{pmatrix} W_1 \\ W_{21} \end{pmatrix}. \]

The aim is to prove that

\[ \begin{pmatrix} I \\ 0 \end{pmatrix} \in \varphi(D_c). \quad (18) \]

It is obvious that in the case \(C = 0\) we have \((U, W) = (I, I) \in D_0\) and therefore

\[ \begin{pmatrix} I \\ 0 \end{pmatrix} \in \varphi(D_0). \quad (19) \]

Put \(\varphi_c = \varphi|_{D_c}\) and \(\varphi_0 = \varphi|_{D_0}\). From (11) and (12) we see that

\[ \text{Ker } \varphi_c = \text{Ker } \varphi_0. \quad (20) \]

Let \(U\) and \(W\) be as in (8), and put

\[ \tilde{U} = \begin{pmatrix} U_1 & 0 \\ U_{21} & 0 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} W_1 & 0 \\ W_{21} & 0 \end{pmatrix}. \]

If \((U, W) \in D_c\) then \((\tilde{U}, \tilde{W}) \in D_0\). Therefore we have

\[ \text{Im } \varphi_c \subseteq \text{Im } \varphi_0. \quad (21) \]
Note that
\[ \dim \ker \varphi_c + \dim \text{Im} \varphi_c = \dim D_c \]
and
\[ \dim \ker \varphi_0 + \dim \text{Im} \varphi_0 = \dim D_0. \]
Then (14) and (20) imply \( \dim \text{Im} \varphi_c = \dim \text{Im} \varphi_0 \), and (21) yields \( \varphi(D_c) = \varphi(D_0) \). From (19) we obtain (18), which completes the proof.

**Proof of Theorem 3.** To show that (b) implies (a) let \( \Gamma_c \) and \( \Delta_c \) be defined as in (9) and (10). Put \( \Lambda = \{(U,W) \mid U = W\} \), and replace \( D_c \) in (13) by \( D_c = \Gamma_c \cap \Delta_c \cap \Lambda \). The arguments of the preceding proof remain unchanged. They lead to (15) and (16), but now with \( U_{12} = W_{12} \). Hence in (17) we have \( X = X^* \).

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**REFERENCES**


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