Diagonal matrix solutions of a discrete-time Lyapunov inequality

Harald K. Wimmer
Mathematisches Institut
Universität Würzburg
D-97074 Würzburg, Germany

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Abstract

Diagonal solutions of a Lyapunov inequality for companion matrices are studied. Such solutions are required if states of a discrete-time linear system are computed with a finite precision arithmetic.
1 Introduction

Let
\[ x(i + 1) = Ax(i), \quad x(0) = x_0, \]  
(1.1)
be a discrete-time linear system with \( x(i) = (x_1(i), \ldots, x_n(i))^T \in \mathbb{C}^n \). It is well known that the system (1.1) is asymptotically stable if and only if there exists a matrix \( P > 0 \) (positive-definite) such that
\[ A^*PA - P = -Q^*Q \]  
(1.2)
and
\[ (A, Q) \text{ is observable.} \]  
(1.3)
According to [1] diagonal solutions \( P \) of (1.2) are required if a finite precision arithmetic is used to calculate the states \( x(i) \) of (1.1). Let \( g(\cdot) \) be a scalar function which satisfies
\[ |g(y)| \leq |y| \quad \text{for all } y \in \mathbb{C}, \]  
(1.4)
and let \( \tilde{g}[x(i)] = (g[x_1(i)], \ldots, g[x_n(i)])^T \) be the vector obtained from \( x(i) \) using \( g(\cdot) \) component-wise as a quantizer operator. It was shown in [1] that the quantized system
\[ x(i + 1) = A\tilde{x}(i), \quad \tilde{x}(i + 1) = \tilde{g}[x(i + 1)], \]  
(1.5)
is asymptotically stable if there exists a diagonal matrix \( P \) such that (1.2) and (1.3) hold.

In the case where \( A \) is a companion matrix diagonal solutions of the Lyapunov equation (1.2) were studied in [2]. In this note we clarify some issues of [2] and prove the following result.

**Theorem 1** Let
\[ A = \begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ 1 & \ddots & & \\ & & 1 & 0 \end{pmatrix} \]  
(1.6)
be a complex companion matrix. Put
\[ s = \sum_{\nu=1}^n |a_\nu|. \]

(i) There exists a diagonal matrix \( P = \text{diag}(p_1, \ldots, p_n) \) with properties
\[ P \geq 0 \text{ (positive-semidefinite), } P \neq 0, \]  
(1.7)
and
\[ L(P) = P - A^*PA \geq 0 \]  
(1.8)
if and only if \( s \leq 1 \).
There exists a diagonal matrix \( P > 0 \) (positive-definite) satisfying (1.8) if and only if

\[
either \ s < 1 \ or \ both \ s = 1 \ and \ a_n \neq 0 \quad (1.9)
\]

hold.

In [2] it was shown that \( s \leq 1 \) is necessary for the existence of a diagonal positive-definite solution \( P \) of (1.8). Discrete-time Lyapunov equations (1.2) with a companion matrix \( A \) have been investigated by several authors, we refer to [3], [4], [5]. Theorem 1 will be proved in Section 2. A matrix of the form (1.6) which is important for the critical exponent of the row-sum norm [6] will be discussed in Section 3. A counterpart of Theorem 1 for the continuous-time Lyapunov inequality \( A^*P + PA \leq 0 \) will be derived in Section 4.

2 Explicit solutions of \( L(P) \geq 0 \)

In the following Lemma the matrix inequality (1.8) will be related to a scalar inequality. It will be convenient to allow denominators to be zero. For \( \alpha \in \mathbb{R} \) and \( \pi = 0 \) we set

\[
\frac{\alpha^2}{\pi} = \begin{cases} 0 & \text{if } \alpha = 0 \\ \infty & \alpha \neq 0 \end{cases}
\]  

(2.1)

**Lemma 2** Let \( a_1, \ldots, a_n \in \mathbb{C} \) and \( p_1, \ldots, p_n \in \mathbb{R} \) be given. The matrix \( P = \text{diag} (p_1, \ldots, p_n) \) satisfies

\[
P \geq 0, \quad P \neq 0 \quad (2.2)
\]

and

\[
L(P) = P - A^*PA \geq 0 \quad (2.3)
\]

if and only if the conditions

\[
p_1 > 0, \quad p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad (2.4)
\]

and

\[
\frac{1}{p_1} \geq \frac{|a_1|^2}{p_1 - p_2} + \cdots + \frac{|a_{n-1}|^2}{p_{n-1} - p_n} + \frac{|a_n|^2}{p_n} \quad (2.5)
\]

hold.

**Proof.** Put

\[
\Pi = \text{diag} (p_1 - p_2, \ldots, p_{n-1} - p_n, p_n - p_{n+1}) \quad p_{n+1} = 0.
\]
As in [2] we note that

\[ L(P) = \Pi - p_1(a_1, \ldots, a_n)^*(a_1, \ldots, a_n) \]

and that (2.3) is equivalent to

\[ \Pi \geq p_1(a_1, \ldots, a_n)^*(a_1, \ldots, a_n). \] (2.6)

If \( a_j = |a_j|e^{i\varphi_j}, \ j = 1, \ldots, n \), set \( D = \text{diag}(e^{-i\varphi_1}, \ldots, e^{-i\varphi_n}) \) and \( a = (|a_1|, \ldots, |a_n|)^* \). Then \( D^*\Pi D = \Pi \). Hence (2.6) is equivalent to

\[ \Pi \geq p_1aa^*. \] (2.7)

Assume that (2.4) and (2.5) hold. Then (2.1) implies \( a_j = 0 \) if \( p_j = p_{j+1} \). Hence in order to prove (2.7) we can discard the indices \( j \) with \( p_j - p_{j+1} = 0 \) and because of (2.4) assume \( \Pi > 0 \). Then (2.5) is equivalent to

\[ 1 \geq p_1a^*\Pi^{-1}a = p_1 (\Pi^{-1/2}a)^* (\Pi^{-1/2}a). \] (2.8)

Note that for a vector \( b \in \mathbb{C}^n \) the eigenvalues of the dyadic product \( bb^* \) are \( b^*b, 0, \ldots, 0 \). Hence (2.8) is equivalent to

\[ I \geq p_1\Pi^{-1/2}aa^*\Pi^{-1/2}, \]

i.e. to (2.7). Starting from (2.7) the converse part of the lemma can be proved along similar lines.

We focus on the inequalities (2.4) and (2.5) assuming \( s \neq 0 \). Set \( \hat{p}_{n+1} = 0 \) and

\[ \hat{p}_\nu = \frac{1}{s}(|a_\nu| + \cdots + |a_n|), \ \nu = 1, \ldots, n. \] (2.9)

Then \( \hat{p}_1 = 1 \) and

\[ \sum_{\nu=1}^n \frac{|a_\nu|^2}{\hat{p}_\nu} = \sum_{\nu=1}^n |a_\nu|^2 = s^2. \]

Hence if \( 0 < s \leq 1 \) then (2.4) and (2.5) are satisfied by \( p_\nu = \hat{p}_\nu, \ \nu = 1, \ldots, n \).

Now consider the case \( a_n = 0 \) and \( s < 1 \). Let \( k \) be such that

\[ a_n = \cdots = a_{k+1} = 0, \ a_k \neq 0. \] (2.10)

Then \( \hat{p}_{k+1} = \cdots = \hat{p}_n = 0, \hat{p}_k = \frac{1}{s}|a_k| > 0 \). For \( \delta \in \mathbb{R}, \ \delta \neq \hat{p}_k \), define

\[ f(\delta) = \frac{|a_1|^2}{\hat{p}_1 - \hat{p}_2} + \cdots + \frac{|a_{k-1}|^2}{\hat{p}_{k-1} - \hat{p}_k} + \frac{|a_k|^2}{\hat{p}_k - \delta}. \]

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Then $f(0) = s^2$. By continuity of $f(\delta)$ there exists an $R$, $R > 0$, such that
\[
f(\delta) \leq 1 \quad \text{if} \quad 0 \leq \delta \leq R.
\] (2.11)

It can be shown that (2.11) holds for
\[
R = \frac{|a_k|(1 - s^2)}{s - s^2(s - |a_k|)}
\] (2.12)

and that (2.12) is the best possible bound for (2.11). Hence any $\delta \in [0, R]$ yields a solution
\[
(p_1, \ldots, p_n) = (\hat{p}_1, \ldots, \hat{p}_k, \delta, \ldots, \delta)
\] (2.13)
of (2.4) and (2.5).

**Proof of Theorem 1:** (i) Assume $s \leq 1$. If $s = 0$ then $A^*A = \text{diag}(1, \ldots, 1, 0)$. Hence $P = I$ is a solution of (1.8). If $s \neq 0$ we use the numbers $\hat{p}_\nu$ given by (2.9) and set...
\[
\hat{P} = \text{diag}(\hat{p}_1, \ldots, \hat{p}_n).
\] (2.14)

Then Lemma 2 implies that $P = \hat{P}$ has the properties (1.7) and (1.8). Note that $\hat{P} > 0$ if $a_n \neq 0$. To show that $s \leq 1$ is necessary for (1.7) and (1.8) consider (1.8) in the equivalent form (2.7). Put $e = (1, \ldots, 1)^T$ then (2.7) yields
\[
p_1 = e^*\Pi e \geq p_1(e^*a)^2 = p_1s^2.
\]
Because of (2.4) we have $p_1 > 0$ and therefore $1 \geq s^2$.

(ii) To show that (1.8) has a positive-definite solution if (1.9) holds consider the three cases $s = 0$ and $0 < s \leq 1$ with $a_n \neq 0$, and
\[
0 < s < 1, \quad a_n = 0.
\] (2.15)

In the first two cases we know that a solution $P > 0$ of (1.8) is given by $P = I$ and $P = \hat{P}$ respectively. In the third case (2.15) we assume (2.10) and choose $0 < \delta \leq R$ with $R$ as in (2.12). Using the $n$-tuple (2.13) we obtain the solution
\[
P = \text{diag}(\hat{p}_1, \ldots, \hat{p}_n, \delta, \ldots, \delta) > 0.
\] (2.16)

To complete the proof we now assume that (1.8) holds for some $P > 0$. Then $s \leq 1$ and we have to exclude the case
\[
s = 1, \quad a_n = 0.
\] (2.17)

As before the assumption $a_\nu = |a_\nu|$, $\nu = 1, \ldots, n$, is no loss of generality. Put $g = (1, \ldots, 1, 0)^T$. Then (2.17) implies $Ag = e$ and $g^TL(P)g = -p_n$. From $L(P) \geq 0$ follows
Let $\rho(A)$ denote the spectral radius of $A$. Clearly $P > 0$ and $L(P) \geq 0$ imply $\rho(A) \leq 1$, and as it was mentioned in Section 1 an observability condition (1.3) ensures $\rho(A) < 1$. In the case of a companion matrix $A$ and a diagonal $P$ we note the following result.

**Corollary 3** There exists a diagonal matrix $P = \text{diag}(p_1, \ldots, p_n)$ such that

$$P > 0 \text{ and } L(P) = P - A^*PA \geq 0$$

(2.18)

and the pair

$$(A, L(P)) \text{ is observable}$$

(2.19)

if and only if

either $s < 1$ or both $s = 1$ and $0 < |a_n| < 1$

(2.20)

hold. From (2.20) follows $\rho(A) < 1$.

**Proof.** If $A$ is of the form (1.6) and $H = (h^{(1)}, \ldots, h^{(n)}) \in \mathbb{C}^{m \times n}$, then the pair $(A, H)$ is observable if and only if $h^{(n)} \neq 0$.

Recall

$$L(P) = \text{diag}(p_1 - p_2, \ldots, p_{n-1} - p_n, p_n) - p_1(a_1, \ldots, a_n)^*(a_1, \ldots, a_n).$$

If $L(P) \geq 0$ then the last column of $L(P)$ is nonzero if and only if

$$p_n - p_1|a_n|^2 > 0.$$  

(2.21)

Hence (2.18) and (2.19) implies (1.9) and (2.21). From (2.4) and (2.21) we deduce $|a_n| < 1$ which leads to (2.20). Now assume (2.20) to establish the existence of a diagonal $P$ satisfying (2.18) and (2.19) three cases will be considered, namely $s = 0$, and

$$0 < s < 1, \quad a_n = 0,$$

(2.22)

and

$$0 < s \leq 1, \quad 0 < |a_n| < 1.$$

(2.23)

In the case $s = 0$ again take $P = I$. Then $L(P) = \text{diag}(0, \ldots, 0, 1)$, and (2.18) and (2.19) are satisfied. In the case (2.22) take $P$ as in (2.16). Then $p_n = \delta > p_1|a_n|^2 = 0$ and (2.21) and equivalently (2.19) hold. In the case (2.23) take the matrix $P = \hat{P}$ of (2.14). Then $p_n = \frac{1}{s}|a_n|$, $p_1 = 1$, and

$$p_n - p_1|a_n|^2 = \frac{1}{s}|a_n|(1 - s|a_n|) > 0.$$
We call a matrix $A \in \mathbb{C}^{n \times n}$ diagonally $d$-stable if there exists a diagonal matrix $P$ such that $P > 0$ and $L(P) = P - A^*PA > 0$. For diagonal stability of discrete- and of continuous-time Lyapunov equations we refer to [7]. A slight modification of the proof of Theorem 1 yields the following result.

**Corollary 4** The companion matrix $A$ in (1.6) is diagonally $d$-stable if and only if $s < 1$.

### 3 The critical exponent of the row-sum norm

For a matrix $B = (b_{ij}) \in \mathbb{C}^{n \times n}$ put

$$||B|| = \max_i \left\{ \sum_{j=1}^{n} |b_{ij}| \right\}.$$ 

Mařík and Pták [6] define the critical exponent of the row-sum norm $|| \cdot ||$ on $\mathbb{C}^{n \times n}$ as the positive integer $\kappa$ which has the property that

$$\rho(B) = 1 \text{ if } ||B|| = ||B^2|| = \cdots ||B^\kappa|| = 1,$$

and for which there exists a matrix $C$ with

$$\rho(C) < 1 \text{ and } ||C|| = \cdots = ||C^{n-1}|| = 1;$$

i.e. $\kappa$ is the smallest number with property (3.1). To prove that $\kappa = n^2 - n + 1$ Mařík and Pták use the following matrix $C$ which (up to a permutation of rows and columns) is of the form (1.6) and satisfies (2.20). Let $\tau$ be real, $0 < \tau < 1$, and put

$$C = \begin{pmatrix}
0 & 1 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
-\tau & \tau - 1 & 0 & \cdots & 0 \\
\end{pmatrix}_{n \times n}.$$ 

Then Corollary 3 with $a_n = -\tau$ and $a_{n-1} = \tau - 1$ implies $\rho(C) < 1$. Note that $C^n = -[(1 - \tau)C + \tau I]$ and

$$C^{n^2-n} = (-1)^{n-1} [(1 - \tau)C + \tau I]^{n-1}.$$ 

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Again set \( e = (1, \ldots, 1)^T \). Then
\[
Ce = (1, \ldots, 1, -1), \ldots, C^{n-1}e = (1, -1, \ldots, -1),
\]
and (3.3) implies
\[
C^{n^2-n}e = (-1)^{n-1} \left( \sum_{j=0}^{n-1} \binom{n-1}{j} \tau^{n-1-j}(1-\tau)^j, \ldots \right) = (-1)^{n-1}(1, \ldots).
\]
Hence the first row of \( C^{n^2-n} \) has norm 1 and from \( 1 = \|C\| \leq \|C^2\| \leq \ldots \) follows
\[
\|C\| = \ldots = \|C^{n^2-n}\| = 1,
\]
which shows that \( \kappa \geq n^2 - n + 1 \). A different proof of (4.4) using a graph theoretical approach is contained in [8].

4 The continuous-time Lyapunov inequality

For a continuous-time \( n \)-dimensional linear system \( \dot{x}(t) = Ax(t) \) the Lyapunov inequality corresponding to (1.8) is
\[
S(P) = A^*P + PA \leq 0. \tag{4.1}
\]
In contrast to the discrete-time inequality (1.8) a matrix \( A \) in companion form is hardly an advantage in (4.1) if diagonal solutions \( P \) are required.

**Theorem 5** Let \( A \in \mathbb{C}^{n \times n}, n \geq 2 \), be a companion matrix of the form (1.6). (i) Then there exists a diagonal matrix \( P \) such that \( P \neq 0, P \geq 0 \) and \( S(P) \leq 0 \) hold if and only if
\[
\det(\lambda I - \lambda) = \lambda^{n-2}(\lambda^2 - a_1\lambda - a_2) \tag{4.2}
\]
and
\[
\bar{a}_1 + a_1 \leq 0 \text{ and } a_2 \in \mathbb{R}, a_2 \leq 0. \tag{4.3}
\]
(ii) If (4.2) and (4.3) hold then a diagonal matrix \( P \) satisfies \( P \geq 0 \) and \( S(P) \leq 0 \) if and only if
\[
P = \text{diag}(p_1, -a_2p_1, 0, \ldots, 0), \quad p_1 \geq 0.
\]
**Proof.** (i) If \( P = \text{diag}(p_1, \ldots, p_n) \) has real entries then
\[
S(P) = \begin{pmatrix}
(\bar{a}_1 + a_1)p_1 & a_2p_1 + p_2 & a_3p_1 & \cdots & a_np_1 \\
\bar{a}_2p_1 + p_2 & 0 & p_3 & \cdots & 0 \\
\bar{a}_3p_1 & p_3 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & p_n \\
\bar{a}_np_1 & 0 & \cdots & \cdots & p_n
\end{pmatrix}. \tag{4.4}
\]
Assume $S(P) \leq 0$ and $P \neq 0$. Then $p_3 = \cdots = p_n = 0$ and $p_1 \neq 0$, and we obtain $a_3 = \cdots = a_n = 0$ which is (4.2). Clearly

$$
\begin{pmatrix}
(\bar{a}_1 + a_1)p_1 & a_2p_1 + p_2 \\
\bar{a}_2p_1 + p_2 & 0
\end{pmatrix} \leq 0, \quad p_1 > 0, \quad p_2 \geq 0
$$

implies (4.3). The converse part of (i) and also of (ii) is obvious.

In accordance with the discrete time concept we call a matrix $A \in \mathbb{C}^{n \times n}$ diagonally $c$-stable if $S(P) = A^*P + PA < 0$ for some positive-definite diagonal matrix $P$. In mathematical biology [9, p. 199] this property is known as Volterra-Lotka stability. From (4.4) it is clear that a companion matrix can never be diagonally $c$-stable if $n \geq 2$.

**Correction of Corollary 3**

Let $u(z) = z^n - (a_1z^{n-1} + \cdots + a_{n-1}z + a_n)$ be the characteristic polynomial of $A$. Set

$$
\hat{u}(z) = z^n - (|a_1|z^{n-1} + \cdots + |a_{n-1}|z + |a_n|).
$$

Suppose $s = 1$. Then $\hat{u}(1) = 1$. Hence $c = 1$ is the Cauchy bound of $u(z)$. Thus, if $\lambda$ is a zero of $u(z)$ then $|\lambda| \leq 1$. It is known that $u(z)$ has a zero of modulus 1 if and only if $u(z) = \omega^n\hat{u}(\omega^{-1})$, that is,

$$a_j = |a_j|\omega^j, \quad j = 1, \ldots, n, \quad \text{for some } \omega \text{ with } |\omega| = 1.
$$

Hence, if $s = 1$ then $\rho(A) = 1$ if and only if (*) holds. Therefore the condition (2.20) in Corollary 3 should be replaced by

(either $s < 1$ or $s = 1$ and (*) is not satisfied) \hspace{1cm} (2.20b)

**References**


