POLYNOMIALS WITH A SHARP CAUCHY BOUND AND THEIR ZEROS OF MAXIMAL MODULUS

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Abstract. The moduli of zeros of a complex polynomial are bounded by the positive zero of an associated auxiliary polynomial. The bound is due to Cauchy. This note describes polynomials with a sharp Cauchy bound and the location of peripheral zeros.

1. Introduction

Let
\[ g(z) = z^m - (c_{m-1}z^{m-1} + \cdots + c_1z + c_0) \]  
be a complex polynomial.

Define
\[ g_a(z) = z^m - (|c_{m-1}|z^{m-1} + \cdots + |c_1|z + |c_0|). \]

If \( g(z) \neq z^m \) then (see e.g. [4, p. 122], [7, p. 3], [8, p. 243]) there exists a unique positive zero \( R(g) \) of \( g_a(z) \), and all zeros of \( g(z) \) have modulus less or equal to \( R(g) \). The number \( R(g) \) is known (see [8]) as Cauchy bound of \( g(z) \).

Set
\[ \sigma(g) = \{ \lambda \in \mathbb{C}; g(\lambda) = 0 \} \quad \text{and} \quad \rho(g) = \max\{|\lambda|; \lambda \in \sigma(g)\}. \]

Then \( \rho(g) \leq R(g) \). In general, the numbers \( \rho(g) \) and \( R(g) \) do not coincide, that is, \( \rho(g) < R(g) \). For example, the polynomials
\[ g(z) = z^2 - (z - 1) = (z - e^{2\pi i/6})(z - e^{-2\pi i/6}) \]
and
\[ g_a(z) = z^2 - (z + 1) = (z - \frac{1 + \sqrt{5}}{2})(z + \frac{1 + \sqrt{5}}{2}) \]
satisfy \( 1 = \rho(g) < R(g) = (1 + \sqrt{5})/2 \). We say that the Cauchy bound is sharp, if \( \rho(g) = R(g) \). Clearly, if \( g(z) = g_a(z) \) then \( \rho(g) = R(g) \), and \( R(g) \in \sigma(g) \). But the Cauchy bound may be sharp, even if \( g(z) \neq g_a(z) \). An example is the polynomial
\[ g(z) = z^2 - (-z + 2) = (z - 1)(z + 2) \]


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with \( g_a(z) = z^2 - (z + 2) = (z + 1)(z - 2) \) and \( R(g) = \rho(g) = 2 \).

In this note we are concerned with polynomials \( g(z) \) which have the property that \( R(g) = \rho(g) \) and we describe their zeros of maximal modulus. For the straightforward proof of the following result I am indebted to a referee.

**Theorem 1.1.** Let \( g(z) \in \mathbb{C}[z] \) be given as in (1.1). Then \( \rho(g) = R(g) \) if and only if

\[
  g(z) = \lambda^m g_a(\lambda^{-1}z)
\]

for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \).

**Proof.** Suppose \( g(z) \neq z^m \). Let \( R \) be the Cauchy bound of \( g(z) \), that is,

\[
  R^m = \sum_{j=0}^{m-1} |c_j| R^j.
\]

Then \( g(z) \) has a zero of modulus \( R \) if and only if

\[
  (\lambda R)^m = \sum_{j=0}^{m-1} c_j (\lambda R)^j
\]

for some \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \). Because of (1.3) the equation (1.4) is equivalent to

\[
  \sum_{j=0}^{m-1} |c_j| R^j = \sum_{j=0}^{m-1} c_j \lambda^{-m+j} R^j.
\]

All terms on the left-hand side of (1.5) are nonnegative. Thus it is easy to see that (1.5) holds if and only if

\[
  |c_j| = c_j \lambda^{-m+j}, \quad j = 0, \ldots, m - 1,
\]

which is equivalent to (1.2). \( \square \)

In Section 2 we apply Theorem 1.1 to obtain a result on rotational symmetry of zeros of maximal modulus and we consider polynomials with real coefficients. A different approach to deal with the Cauchy bound and its sharpness is described in Section 3. It is based on companion matrices and the Perron-Frobenius theory of nonnegative matrices.

**2. Zeros of maximal modulus**

Throughout this paper \( g(z) \) will be a polynomial of the form (1.1) and we assume \( g(z) \neq z^m \). The following notation will be used. With regard to (1.2) we define

\[
  (\kappa \cdot g)(z) = \kappa^m g(\kappa^{-1}z),
\]

where \( \kappa \in \mathbb{C}, \kappa \neq 0 \). If \( g(z) = \prod_{j=1}^{m}(z - \lambda_j) \) then

\[
  (\kappa \cdot g)(z) = \prod_{j=1}^{m}(z - \kappa \lambda_j),
\]
and therefore $\sigma(\kappa \cdot g) = \kappa \sigma(g)$. Let $\partial \mathbb{D}$ denote the unit circle and let $E_n$ be the group of $n$-th roots of unity,

$$E_n = \sigma(z^n - 1) = \{e^{2k\pi i/n}; k = 0, \ldots, n - 1\}.$$ 

The support $\Sigma(q)$ of a polynomial $q(z) = \sum_{j=0}^k q_j z^j$ is the set of indices $j$ with nonzero coefficient $q_j$. Thus, for the polynomial $g(z)$ in (1.1) we have

$$\Sigma(g) = \{j; 0 \leq j \leq m - 1, c_j \neq 0\} \cup \{m\}.$$ 

Define

$$d(g) = \gcd\{j \in \Sigma(g)\} \quad \text{and} \quad \ell(g) = m/d(g).$$ 

If $d(g) = d$ and $\ell(g) = \ell$ then

$$g(z) = \left(z^d\right)^{\ell} - \left(c(\ell-1)d\right)^{\ell-1}c_dz^d + c_0. \quad (2.1)$$ 

Set $\check{c}_k = c_{kd}$, $k = 0, 1, \ldots, \ell - 1$, and

$$\check{g}(z) = z^\ell - \left(\check{c}_{\ell-1}z^{\ell-1} + \cdots + \check{c}_1z + \check{c}_0\right). \quad (2.2)$$ 

Then $g(z) = \check{g}(z^d)$. Moreover, $\Sigma(g) = d\Sigma(\check{g})$ implies $d(\check{g}) = 1$. In accordance with [1] we denote by $\pi_{+1}^{k-1}$ the set of real polynomials $p(z) = \sum_{i=0}^{k-1} a_i z^i$ satisfying

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{k-1} = 1.$$ 

Define $S(g) = \sum_{j=1}^{m-1} |c_j|$. Then $S(g) = 1$ is equivalent to $1 \in \sigma(g_a)$. On the other hand, $R(g) = 1$ means that $\lambda = 1$ is the (unique) positive zero of $g_a(z)$. Hence we have $R(g) = 1$ if and only if $S(g) = 1$.

In this section we deal with polynomials $g(z)$ with a sharp Cauchy bound and we focus on zeros of $g(z)$ of maximal modulus. For the sake of simplicity we shall assume $0 \notin \sigma(g)$. The following theorem can be traced back to Hurwitz [3]. We include a proof to make the note self-contained. The theorem has an interesting history, which is indicated in [1]. Only the special case with $d(g) = 1$ seems to be widely known [6, p. 92]). [7, p. 3].

**Theorem 2.1.** (Hurwitz) Assume $g(z) = g_a(z)$. Suppose $R(g) = 1$ and $g(0) \neq 0$. Let $d(g) = d$ and $\ell(g) = \ell$. Then $g(z) = (z^d - 1)\check{p}(z^d)$ with $\check{p}(z) \in \pi_{+1}^{\ell-1}$ and $\rho(\check{p}) < 1$. The unimodular zeros of $g(z)$ are simple, and $\sigma(g) \cap \partial \mathbb{D} = E_d$.

**Proof.** Suppose first that $d = 1$ such that $g(z) = \check{g}(z)$. The assumption $\gcd\{j \in \Sigma(g)\} = 1$ yields a Bezout identity $\sum_{j \in \Sigma(g)} y_j j = 1$ with $y_j \in \mathbb{Z}$. Let $\lambda \in \sigma(g) \cap \partial \mathbb{D}$. From the proof of Theorem 1.1 we know that $\lambda$ satisfies (1.6). We have $y_j = |c_j|$ for all $j$. Then $g(0) = c_0 \neq 0$ and (1.6) imply $\lambda^n = 1$, and we obtain $j \in \Sigma(g)$ if and only if $\lambda^j = 1$. Hence $\lambda = \prod_{j \in \Sigma(g)} \lambda^{y_j} = 1$, that is, $\sigma(g) \cap \partial \mathbb{D} = \{1\}$. From

$$g'(1) = m - \sum_{j=1}^{m-1} jc_j > m - (m - 1) \sum_{j=1}^{m-1} c_j > 1$$

and therefore $\sigma(\kappa \cdot g) = \kappa \sigma(g)$. Let $\partial \mathbb{D}$ denote the unit circle and let $E_n$ be the group of $n$-th roots of unity,
we see that \( \lambda = 1 \) is a simple zero of \( g(z) \). Hence \( g(z) = (z - 1)p(z) \) for some polynomial \( p(z) = \sum_{k=0}^{m-1} a_k z^k \) with \( a_{k-1} = 1 \) and \( \rho(p) < 1 \). The coefficients of \( g(z) \) and \( p(z) \) satisfy \( a_k = \sum_{j=0}^{k} c_j, \ k = 0, \ldots, m - 1 \). Thus \( p(z) \in \pi^{m-1}_1 \). In the general case, if \( d(g) = d \), it suffices to note that \( g(z) = \tilde{g}(z^d) \) with \( d(\tilde{g}) = 1 \). □

Combining Theorem 2.1 with Theorem 1.1 we obtain the following.

**Corollary 2.2.** Let \( d(g) = d \) and \( g(0) \neq 0 \). Suppose \( \rho(g) = R(g) = R \). If \( |\lambda| = R \) and \( g(\lambda) = 0 \) then \( \sigma(g) \cap R \partial \mathbb{D} = \lambda E_d \). In other words, the zeros of maximal modulus are the vertices of a regular \( d \)-sided polygon in the complex plane.

We now consider polynomials \( g(z) \) with real coefficients.

**Theorem 2.3.** Let \( g(z) \in \mathbb{R}[z] \) and \( d = d(g) \). Suppose \( g(0) \neq 0 \). Then \( \rho(g) = R(g) \) if and only if \( g(z) = g_a(z) \) or

\[ g(z) = \eta \cdot g_a(z) = z^{\ell d} - (1)^\ell \sum_{\nu=0}^{\ell-1} (-1)^\nu |c_{\nu d}| z^{\nu d} \]  

(2.3)

where \( \ell d = m \) and \( \eta = e^{\pi i/d} \).

**Proof.** Suppose \( \rho(g) = R(g) \). Let \( \tilde{g}(z) \) be the polynomial in (2.2). Then \( \rho(\tilde{g}) = R(\tilde{g}) \), and it follows from Theorem 1.1 that \( \tilde{g}(z) = \lambda^\ell \tilde{g}_a(\lambda^{-1} z) \) for some \( \lambda \in \partial \mathbb{D} \). Assuming \( R(g) = 1 \) we apply Theorem 2.1. Because of \( d(\tilde{g}) = 1 \) we obtain \( \sigma(\tilde{g}_a) \cap \partial \mathbb{D} = \{1\} \). Therefore

\[ \tilde{g}(z) = \lambda^\ell (\lambda^{-1} z - 1) \tilde{p}(\lambda^{-1} z) = (z - \lambda) \lambda^{\ell-1} \tilde{p}(\lambda^{-1} z) = (z - \lambda) \lambda \cdot \tilde{p}(z). \]

The real polynomial \( \tilde{g}(z) = \lambda \cdot \tilde{p}(z) \) satisfies \( \rho(\tilde{g}) < 1 \). Thus \( \tilde{g}(z) \in \mathbb{R}[z] \) implies \( \lambda \in \{1, -1\} \). If \( \lambda = 1 \) then \( \tilde{g}(z) = \tilde{g}_a(z) \), and therefore

\[ g(z) = g_a(z). \]

If \( \lambda = -1 \) then

\[ \tilde{g}(z) = (-1)^\ell \tilde{g}_a(-z) = z^\ell - (1)^\ell \sum_{\nu=0}^{\ell-1} (-1)^\nu \tilde{c}_\nu |z^\nu. \]

Hence

\[ g(z) = \tilde{g}(z^d) \]

and \( \eta \cdot g = \eta^d \cdot \tilde{g} \) imply (2.3). □

The real polynomial \( g(z) = z^2 + 1 \) is an example with a sharp Cauchy bound and \( g(z) \neq (\pm 1) \cdot g_a(z) \). Here we have \( d = 2, \ \ell = 1, \ \eta = i, \ g_a(z) = z^2 - 1 \), and

\[ g(z) = i \cdot g_a(z). \]
3. Companion matrices

A different approach to study zeros of polynomials uses companion matrices and takes advantage of the theory of Perron-Frobenius-Wielandt (see e.g. [9], [1], [8]). We indicate how results of this note can be viewed in that context. Let

$$F = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ c_0 & c_1 & c_2 & \ldots & c_{m-1} \end{pmatrix} \quad \text{and} \quad F_a = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ |c_0| & |c_1| & |c_2| & \ldots & |c_{m-1}| \end{pmatrix}$$

be companion matrices associated with the polynomials $g(z)$ and $g_a(z)$, respectively. Thus, $g(z) = \det(zI - F)$ and $g_a(z) = \det(zI - F_a)$. If $\sigma(F)$ and $\rho(F)$ denote the spectrum and the spectral radius of $F$ then $\sigma(F) = \sigma(g)$, $\rho(F) = \rho(g)$ and $\rho(F_a) = R(g)$. The matrix $F_a$ is a nonnegative matrix, and $F_a$ is irreducible if and only if $c_0 \neq 0$.

Let $A = (a_{ij}) \in \mathbb{R}^{m \times m}$ be a nonnegative matrix and let $B = (b_{ij}) \in \mathbb{C}^{m \times m}$. We write $|B| \leq A$ if $|b_{ij}| \leq a_{ij}$ for all $i, j$. The following theorem is due to Wielandt (see [2, Theorem 8.4.5] or [5, Chapter 8]).

**Theorem 3.1.** Let $A \in \mathbb{R}^{m \times m}$ be nonnegative and irreducible. Suppose $|B| \leq A$. Then

$$\rho(B) \leq \rho(A).$$

We have $\rho(B) = \rho(A)$ if and only if

$$B = e^{i\phi} D A D^{-1}$$

for some $D = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_m})$.

If $B = F$ and $A = F_a$ are given by (3.1) then (3.2) yields $\rho(g) \leq R(g)$. Moreover, if $\rho(g) = R(g)$ then (3.3) implies

$$F = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ e^{i(\theta_1 - \theta_2)} & e^{i(\theta_2 - \theta_3)} & \ldots & 0 \\ 0 & 0 & \ldots & \ldots \\ e^{i(\theta_{m-1} - \theta)} & e^{i(\theta_{m-2} - \theta)} & \ldots & 0 \\ |c_0| & |c_1| & |c_2| & \ldots & |c_{m-1}| \end{pmatrix}.$$
The matrix $F_\lambda$ is the companion matrix of
\[
z^m - (|c_0|\lambda^m + |c_1|\lambda^{m-1}z + \cdots + |c_{m-1}|\lambda) = \lambda^m g_a\left(\frac{z}{\lambda}\right) = \lambda \cdot g_a(z).
\]
Hence the polynomial $g(z)$ satisfies (1.2), in accordance with Theorem 1.1.

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References


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