A variational characterization of canonical angles between subspaces

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Abstract
Canonical angles between subspaces of a unitary space are characterized by a min-max property which involves inner products.


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Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonzero subspaces of an $n$-dimensional unitary space $\mathcal{V}$. Angles between $\mathcal{X}$ and $\mathcal{Y}$ can be defined in several equivalent ways. The starting point for our note is a recursive definition (see e.g. [1],[2]) based on the inner product in $\mathcal{V}$. Let $S_X = \{x \in \mathcal{X} : |x| = 1\}$ and $S_Y$ be the unit spheres of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Set $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$. The smallest angle $\phi_1 \in [0, \frac{\pi}{2}]$ between $\mathcal{X}$ and $\mathcal{Y}$ is defined by

$$\cos \phi_1 = \max_{x \in S_X, y \in S_Y} |(x, y)|. \quad (1)$$

Let the maximum in (1) be attained at $x_1 \in S_X$ and $y_1 \in S_Y$. Then $\phi_2 \geq \phi_1$ can be defined as the smallest angle between the orthogonal
complements of \( x_i \) in \( X \) and \( y_i \) in \( Y \). Thus, starting from \( x_0 = y_0 = 0 \) one can construct recursively pairs of vectors \( (x_i, y_i) \in S_X \times S_Y \) and a set of angles \( 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \frac{\pi}{2} \) given by

\[
\cos \phi_k = \max_{x \perp x_i, y \perp y_i, i=0, \ldots, k-1} \| (x, y) \|.
\]

(2)

Theorem 1 below shows that the angles \( \phi_k \) are well defined by (2), independent of the choice of the vectors \( x_i, y_i \).

Let \( P_X \) denote the orthogonal projection of \( V \) on \( X \). If \( A : Y \to X \) is a linear map then we assume that the singular values \( \sigma_i(A) \) are ordered by decreasing magnitude such that \( \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_r(A) \).

**Theorem 1** [1, p. 382], [4, p. 43] Let \( \phi_1, \ldots, \phi_r \) be defined as in (2). Then

\[
\cos \phi_k = \sigma_k(P_X P_Y), \quad k = 1, \ldots, r.
\]

(3)

The numbers \( \phi_1, \ldots, \phi_r \) given by (2) or (3) are called the canonical (or principal) angles between \( X \) and \( Y \). It is the purpose of this note to derive a min-max characterization of canonical angles.

We use the following notation. Let \( \mathcal{R}(A) \) be the range of \( A \). If \( A^\dagger \) is the Moore-Penrose inverse of \( A \) then \( AA^\dagger = P_{\mathcal{R}(A)} \) (see e.g. [4, p. 106]). The matrix \( (AA^\dagger)^{1/2} \) is the positive semidefinite square root of \( AA^\dagger \), so that the first \( r \) eigenvalues of \( (AA^\dagger)^{1/2} \) are the singular values of \( A \). For a subspace \( U \subseteq X \) let \( U^\perp \) denote the orthogonal complement of \( U \) in \( X \).

**Lemma 2** The singular values of a linear map \( A : Y \to X \) are given by

\[
\sigma_k(A) = \min_{U \subseteq X} \max_{\dim U = k-1} \max_{x \in U^\perp \cap S_X} \| x, (A^* y) \|, \quad k = 1, \ldots, r.
\]

(4)

**Proof.** It is known (see e.g. [3, p. 148]) that

\[
\sigma_k(A) = \min_{U \subseteq X} \max_{\dim U = k-1} \max_{x \in U^\perp \cap S_X} (x, (AA^\dagger)^{1/2} x) = \min_{U \subseteq X} \max_{\dim U = k-1} \max_{x \in U^\perp \cap S_X} \| A^* x \|.
\]

(5)
Because of $\mathcal{R}[(AA^*)^{1/2}] = \mathcal{R}(A)$ we have

$$(AA^*)^{1/2} [(AA^*)^{1/2}]^T = P_{\mathcal{R}(A)}.$$  

Hence, if $\tilde{y} = A^* [(AA^*)^{1/2}]^T x$ then $(\tilde{y}, \tilde{y}) = (P_{\mathcal{R}(A)} x, P_{\mathcal{R}(A)} x)$ and therefore $|\tilde{y}| \leq |x|$. Set

$$\tau_k = \min_{U \subseteq \mathcal{X}} \max_{\dim U = k-1} \max_{x \in U \cap S_{\mathcal{X}}} \min_{y \in S_Y} |(x, Ay)|,$$

(6)

In (6) we can replace the unit sphere $S_Y$ by the closed unit ball $\{y \in \mathcal{Y} : |y| \leq 1\}$. Then

$$\tau_k \geq \min_{U \subseteq \mathcal{X}} \max_{\dim U = k-1} \max_{x \in U \cap S_{\mathcal{X}}} \min_{y = A^* [(AA^*)^{1/2}]^T x} |(x, Ay)|$$

$$= \min_{U \subseteq \mathcal{X}} \max_{\dim U = k-1} \max_{x \in U \cap S_{\mathcal{X}}} |(x, (AA^*)^{1/2} x)| = \sigma_k(A).$$

On the other hand, if $|y| = 1$ then $|(x, Ay)| \leq |A^* x|$. Hence (5) yields $\tau_k \geq \sigma_k(A)$, which completes the proof. \hfill \square

The case $k = 1$ in (4), i.e.

$$\sigma_1(A) = \max_{x \in S_{\mathcal{X}}, y \in S_{\mathcal{Y}}} |(x, Ay)|$$

can be found in [3, p.155]. Note that $k = 1$ in (7) below yields $\phi_1$ as in (1).

**Theorem 3** The canonical angles $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_r \leq \frac{\pi}{2}$ between $\mathcal{X}$ and $\mathcal{Y}$ satisfy

$$\cos \phi_k = \min_{U \subseteq \mathcal{X}} \max_{\dim U = k-1} \max_{y \in S_{\mathcal{Y}}} |(x, y)|, \ k = 1, \ldots, r.$$  

(7)

**Proof.** We apply Lemma 2 to $A = P_{\mathcal{X}} P_{\mathcal{Y}}$. For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have $(x, P_{\mathcal{X}} P_{\mathcal{Y}} y) = (P_{\mathcal{X}} x, P_{\mathcal{Y}} y) = (x, y)$. Thus (7) is an immediate consequence of (4) and (3). \hfill \square
References


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