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ASYMPTOTIC STABILITY AND SMOOTH LYAPUNOV FUNCTIONS FOR A CLASS OF ABSTRACT DYNAMICAL SYSTEMS

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Abstract. This paper deals with a characterization of asymptotic stability for a class of dynamical systems in terms of smooth Lyapunov pairs. We point out that well known converse Lyapunov results for differential inclusions cannot be applied to this class of dynamical systems. Following an abstract approach we put an assumption on the trajectories of the dynamical systems which demands for an estimate of the difference between trajectories. Under this assumption, we prove the existence of a C^∞-smooth Lyapunov pair. We also show that this assumption is satisfied by differential inclusions defined by Lipschitz continuous set-valued maps taking nonempty, compact and convex values.

1. Introduction.

After Lyapunov published his stability results in 1892 starting in the 1950s a lot of effort has been spent on the derivation of converse theorems for Lyapunov's second method. Beginning with ordinary differential equations defined by a continuous function, the results have been extended to differential inclusions

$$\dot{x}(t) \in F(x(t)), \qquad x(t) \in \mathbb{R}^n,\ t \geq 0,\ x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

where $F\colon \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map satisfying $0 \in F(0)$. A comprehensive exploration of the connection between stability of differential equations and differential inclusions and Lyapunov functions can be found in [15]. For general references to differential inclusions and set-valued maps the interested reader is referred to [1] and [2], respectively.

A solution to (1) is an absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ with $x(0) = x_0$ such that (1) is satisfied almost everywhere. Following [6, Proposition 2.2] the equilibrium $x = 0$ of differential inclusion (1) is called *strongly asymptotically stable* if each solution can be extended to $[0, \infty)$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that any solution $x(\cdot)$ with $\|x(0)\| < \delta$ satisfies $\|x(t)\| < \varepsilon$ for all $t \geq 0$, and for each individual solution $x(\cdot)$, one has $\lim_{t\to\infty} x(t) = 0$.

The analysis of robust stability has been an active field in the dynamical systems literature. In the wake of this, the investigation of converse Lyapunov theorems and, in particular, the construction of smooth Lyapunov functions is of vital interest, cf. [6, 20, 23].

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Clarke, Ledyaev and Stern [6] (see also [16, 23]) consider the case that $F(x)$ is nonempty, compact and convex for every $x \in \mathbb{R}^n$ and the set-valued map F is upper semicontinuous, i.e. for any $x \in \mathbb{R}^n$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that $F(y) \subset F(x) + \varepsilon B(0,1)$ for all $y \in x + \delta B(0,1)$, where $B(0,1)$ denotes the unit open ball in $\mathbb{R}^n$. In [6, Theorem 2] it is shown that the differential inclusion (1) is strongly asymptotically stable if and only if there is a C^∞-smooth and positive definite pair of functions (V, W) such that V is proper and (V, W) satisfies the *strong infinitesimal decrease condition*

$$\max_{v \in F(x)} \langle \nabla V(x), v \rangle \leq -W(x) \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2}$$

A different proof of this converse Lyapunov theorem has been obtained by Siconolfi and Terrone, where the authors followed a metric approach using weak KAM theory, cf. [20]. Related results for retarded functional equations and difference inclusions can be found in [13] and [14], respectively.

Originating from stochastic systems, such as multiclass queueing networks and semimartingale reflected Brownian motions [8, 9], there is a class of dynamical systems that does not immediately fall into framework mentioned above. More specifically, the remarkable insights of [8, 9, 18] show that the analysis of recurrence behavior of the stochastic processes corresponding to multiclass queueing networks or semimartingale reflected Brownian motions can be reduced to the stability analysis of a related deterministic system, called fluid network and linear Skorokhod problem, respectively. Both models are obtained by taking limits of scaled versions of the stochastic processes. In [9, 19] it is outlined that a wide class of linear Skorokhod problems and fluid networks can be defined by differential inclusions in a natural way. An essential part in [9] is the description of the linear Skorokhod problem in terms of a differential inclusion and the construction of a C^1-Lyapunov function.

The paper by Dupius and Williams [9] was published a few years before the work of Clarke, Ledyaev and Stern [6], but the construction of the C^1-Lyapunov function is limited to the special type of differential inclusion under consideration while the results in [6] are valid for differential inclusions defined by upper semicontinuous set-valued maps taking nonempty, compact and convex values. The techniques used to construct a smooth Lyapunov function have in common that the set-valued map F is embedded into a local Lipschitz set-valued map F_L, which keeps the property of asymptotic stability. Whereas the procedure in [9] uses explicitly the properties of the set-valued map describing the evolution of the linear Skorokhod problem, the embedding technique in [6] is applicable to any upper semicontinuous set-valued map taking nonempty, compact and convex values. The essential feature of the set-valued map F_L being local Lipschitz continuous is that it provides a Lyapunov function which is locally Lipschitz continuous and this property can be carried over to conclude a locally Lipschitz continuous Lyapunov function for the original differential inclusion. Moreover, the local Lipschitz continuity of the set-valued map F_L facilitates to establish that the convolution of the local Lipschitz continuous Lyapunov function and a C^∞-smooth mollifier satisfies locally the decrease condition (2). The construction is completed by using a locally finite open covering of $\mathbb{R}^n$ and a smooth partition of unity subordinate to it.

Considering the simplest possible fluid network, i.e. a single station fluid network serving one type of fluids, it turns out that in general the set-valued map defining the differential inclusion that describes the evolution of a fluid network is not upper-semicontinuous. Thus, although the zero solution may be strongly asymptotically stable the existence of a C^∞-smooth Lyapunov pair cannot be concluded by applying available results from the literature mentioned above. Moreover, the properties of the set-valued map describing the evolution of the fluid network rest critically on the discipline of the fluid network. In order to obtain an unified approach across disciplines we choose an abstract framework based on common properties of fluid networks under various disciplines.

In this paper we follow an abstract point of view, starting with Zubov [26], understanding dynamical systems as abstract mathematical objects with certain properties. This has been further explored by Hale, Infante, Slemrod and Walker, cf. [11, 12, 21, 24]. In the literature there are several terms used, for instance, generalized dynamical system, C^0-semigroup, (semi)flow, process or abstract dynamical system, cf. [24] and the references therein. The class of abstract dynamical systems considered in this paper is defined by the characteristic properties of fluid networks. The trajectories of fluid networks evolve in the positive orthant. In order to get a C^∞-smooth Lyapunov function on the positive orthant we use an extension of a Lyapunov function candidate to $\mathbb{R}^n$ by taking absolute values component-by-component. As this defines a continuous map the extended Lyapunov function is continuous as well. We note that, as the solutions to linear Skorokhod problems also stay within the positive orthant, Dupuis and Williams [9] solved the boundary problem by shifting the orthant by some positive constant. Further, we note that the class of abstract dynamical systems under consideration may in general not be defined by a differential inclusion. As a consequence the constructions of a local Lipschitz continuous Lyapunov function in [6, 9, 20], which are based on the right-hand side of the differential inclusion, are not applicable in the present setting. It turns out that the essential ingredient to obtain a local Lipschitz continuous Lyapunov function is an estimate on the evolution of the difference of trajectories. For this reason, we have to make an assumption on the trajectories of the abstract dynamical system (see assumption (A) in Theorem 3.1). Considering the assumption from the differential inclusions perspective we show that it is automatically satisfied for every differential inclusion with Lipschitz continuous right-hand side.

The paper is organized as follows. In Section 2 we state relevant notation and terminology that is used throughout the paper. Section 3 introduces the class of abstract dynamical systems that is considered and the main result is presented. In Section 4 we outline that the class of abstract dynamical systems is motivated by the analysis of fluid networks. We also show that the classical results on smooth Lyapunov functions do not apply to the class of abstract dynamical systems discussed in this paper. In Section 5 we examine the relation of the assumption posed on the trajectories in the light of differential inclusions. Finally, Section 6 is devoted to the proof of the main result.

2. Notation and terminology.

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *proper* if the sublevel sets $\{x \in \mathbb{R}^n | f(x) \leq c\}$ are bounded for all $c > 0$. For $r > 0$ and $x \in \mathbb{R}^n$ let $B(x,r) := \{y \in \mathbb{R}^n | \|x-y\| \leq r\}$. A function $k \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$ is called a *mollifier* if supp $k = B(0,1)$ and

$$\int_{\mathbb{R}^n} k(x)\ \mathrm{d}x = 1.$$

Furthermore, the support of a mollifier can be scaled in the following way. For $r > 0$ consider $k_r(x) := \frac{1}{r^n}\, k(r^{-1}x)$. Then, it follows $k_r \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$, $\operatorname{supp} k_r = B(0,r)$, and

$$\int_{\mathbb{R}^n} k_r(x)\ \mathrm{d}x = 1.$$

Moreover, to consider the convolution of a function $f \in C(\mathbb{R}^n, \mathbb{R})$ and a mollifier k_r, let U be an open subset of $\mathbb{R}^n$ and $U_r = \{x \in U \,|\, d(x, \partial U) > r\}$. Then, the *convolution*, denoted by $f_r : U_r \to \mathbb{R}$, is defined by

$$x \mapsto f_r(x) := f * k_r\,(x) = \int_{B(0,r)} f(x-y)\, k_r(y)\ \mathrm{d}y.$$

By standard convolution results it follows $f_r \in C^\infty(U_r, \mathbb{R}_+)$, see for instance [10, Theorem 6 Appendix C.4]. Furthermore, if f is continuous in U, it holds $f_r \to f$ uniformly on compact subsets (u.o.c.) of U as $r \to 0$. The *Dini subderivative* of a function $f : U \to \mathbb{R}$ at $x \in U$ in the direction $v \in \mathbb{R}^n$ is defined by

$$Df(x;v) := \liminf_{\varepsilon \to 0, v' \to v} \frac{f(x+\varepsilon v') - f(x)}{\varepsilon}.$$

Let $T(x, \mathbb{R}^n_+)$ denote the *contingent cone* to $\mathbb{R}^n_+$ at x defined by

$$T\big(x, \mathbb{R}^n_+\big) = \left\{ v \in \mathbb{R}^n \mid \liminf_{\varepsilon \to 0}\ \frac{d(x+\varepsilon v, \mathbb{R}^n_+)}{\varepsilon} = 0 \right\},$$

with $d(x,K) = \inf\{\|x-y\| \,|y \in K\}$.

3. **Statement of the main result.** We start this section by recalling an abstract definition of a dynamical system from [24]. A *dynamical system* defined on a metric space (X,d) is a continuous mapping $u\colon \mathbb{R}_+ \times X \to X$ such that $u(0,x) = x$ and

$$u(t, u(s,x)) = u(t+s, x) \qquad \text{for all } t,s \in \mathbb{R}_+,\ x \in X.$$

Recall that $x_* \in X$ is an equilibrium if $u(t,x_*) = x_*$ for all $t \geq 0$ and x_* is said to be *stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x,x_*) < \delta$ implies that $d\,(u(t,x), u(t,x_*)) < \varepsilon$ for all $t \geq 0$. If in addition, there is a $M > 0$ so that $d(x,x_*) < M$ implies that $\lim_{t\to\infty} d\,(u(t,x), u(t,x_*)) = 0$, then φ_* is called *asymptotically stable*.

To define the class of dynamical systems that will be considered in this paper, we first specify the metric space and define the mapping afterwards. Consider the set $\mathcal{P} \subset C(\mathbb{R}_+, \mathbb{R}^n_+)$ defined by the following properties:

(a) There is a $L > 0$ such that

$$\|\varphi(t) - \varphi(s)\| \leq L\,|t-s| \qquad \text{for all } \varphi \in \mathcal{P},\ t,s \in \mathbb{R}_+.$$

(b) Scaling invariance: $\frac{1}{r}\,\varphi(r\,\cdot) \in \mathcal{P}$ for all $\varphi \in \mathcal{P}, r > 0$.

(c) Shift invariance: $\varphi(\cdot + t) \in \mathcal{P}$ for all $\varphi \in \mathcal{P}, t \geq 0$.

(d) If a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{P}$ converges to φ_* uniformly on compact sets, then $\varphi_* \in \mathcal{P}$.

(e) Concatenation property: For all $\varphi_1, \varphi_2 \in \mathcal{P}$ with $\varphi_1(t^*) = \varphi_2(0)$ for some $t^* \geq 0$ it holds $\varphi_1 \diamond_{t^*} \varphi_2 \in \mathcal{P}$, where

$$\varphi_1 \diamond_{t^*} \varphi_2(t) := \begin{cases} \varphi_1(t) & t \leq t^*, \\ \varphi_2(t-t^*) & t \geq t^*. \end{cases}$$

(f) There is a $T > 0$ such that the set-valued map $P : \mathbb{R}^n_+ \rightsquigarrow \mathcal{P}$ defined by

$$P(x) = \{\varphi : [0, T] \to \mathbb{R}^n_+ \,|\, \varphi \in \mathcal{P},\, \varphi(0) = x\}$$

is lower semicontinuous, i.e. for any $\varphi \in P(x)$ and for any sequence of elements $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^n_+$ converging to x, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n \in P(x_n)$ converging to φ uniformly on compact sets.

The set $\mathcal{P}$ is equipped with the metric

$$\mathrm{d}(\varphi_1, \varphi_2) := \max_{N\in\mathbb{N}} 2^{-N} \frac{\|\varphi_1 - \varphi_2\|_N}{1 + \|\varphi_1 - \varphi_2\|_N},$$

where $\|\varphi\|_N := \sup_{t\in[0,N]} \|\varphi(t)\|$ so that convergence of functions is equivalent to uniform convergence of the corresponding restrictions on each compact subset of $\mathbb{R}_+$, cf. [17].

By condition (a) the functions $\varphi \in \mathcal{P}$ are Lipschitz continuous with respect to a global Lipschitz constant. In particular, the functions $\varphi \in \mathcal{P}$ are differentiable for almost all $t \geq 0$. Condition (c) is in one-to-one correspondence to time-invariance of differential equations/inclusions. Condition (d) expresses that the set $\mathcal{P}$ is closed in the topology of uniform convergence on compact sets.

To introduce the continuous mapping defining the dynamical system we consider the shift operator

$$S(t)\colon C(\mathbb{R}_+, \mathbb{R}^n_+) \to C(\mathbb{R}_+, \mathbb{R}^n_+), \quad S(t)\varphi(\cdot) = \varphi(\cdot + t).$$

The class of dynamical systems considered in this paper is

$$u\colon \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}, \quad u(t, \varphi) = S(t)\varphi(\cdot) = \varphi(\cdot + t) \tag{3}$$

defined on the metric space $(\mathcal{P}, \mathrm{d})$. Throughout the paper we call a function $\varphi \in \mathcal{P}$ a *trajectory* of the dynamical system. The zero trajectory $\varphi_* \equiv 0$ is the unique fixed point of the shift operator $S(t)$ defined on $\mathcal{P}$ and thus, $\varphi_* \equiv 0$ is the only equilibrium of the dynamical system defined by (3). The scope of this paper is to characterize asymptotic stability of the dynamical system u defined on $\mathcal{P}$ in terms of the existence of a smooth Lyapunov function.

A pair (V, W) of positive definite functions on $\mathbb{R}^n_+$ is called a *Lyapunov pair* for the dynamical system u defined on $\mathcal{P}$ if $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ is proper and for any $\varphi \in \mathcal{P}$,

$$V(\varphi(t)) - V(\varphi(s)) \leq -\int_s^t W(\varphi(r))\, \mathrm{d}r \qquad \text{for all } 0 \leq s \leq t \in \mathbb{R}_+. \tag{4}$$

The pair (V, W) is called a C^∞*-smooth Lyapunov pair* if the functions V and W are C^∞-smooth. In the case we have a C^∞-smooth Lyapunov pair, by property (a), the decrease condition can also be expressed in the differential form

$$\dot{V}(\varphi(t)) := \tfrac{\mathrm{d}}{\mathrm{d}t} V(\varphi(t)) \leq -W(\varphi(t)) \qquad \text{for almost all } t \geq 0. \tag{5}$$

We note that the definition of a Lyapunov function differs from the one in [24, Chapter IV, Definition 1.1]. There a Lyapunov function for the dynamical system $u\colon \mathbb{R}_+ \times X \to X$ defined on a metric space (X, d) is a lower semicontinuous function

$$V : X \to \mathbb{R} \cup \{\infty\} \quad \text{ such that } \quad \dot{V}(x) \leq 0 \text{ for all } x \in X, \tag{6}$$

where

$$\dot{V}(x) := \liminf_{t\searrow 0} \frac{V(u(t, x)) - V(x)}{t}$$

and $\dot{V}(x) := 0$ if $V(x) = \infty$ and $\dot{V}(x) := 1$ if $V(x) = -\infty$. For Lyapunov's second method in this context we refer to [24, Theorem 3.1, Chapter IV]. A possible choice for a Lyapunov function in the sense of (6) for the class of dynamical systems considered in this paper is

$$\tilde{V} : \mathcal{P} \to \mathbb{R}_+, \quad \tilde{V}(\varphi) := \int_0^\infty \|\varphi(s)\| \, \mathrm{d}s.$$

It is shown in [25, Theorem 2.3], see also [19], that (3) is asymptotically stable if and only if there are strictly increasing continuous functions $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3$ satisfying $\omega_i(0) = 0$ such that

$$\omega_1(\|\varphi(0)\|) \le \tilde{V}(\varphi) \le \omega_2(\|\varphi(0)\|)$$

and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \tilde{V}(u(t, \varphi)) \le -\omega_3(\|\varphi(t)\|).$$

We are not following this approach because the drawback is, however, that in the stability analysis of fluid networks under particular disciplines we aim at Lyapunov functions depending only on the state of the fluid network, i.e. on the fluid level. Thus, Lyapunov functions map the positive orthant to the real numbers. The main result of the paper is the following.

Theorem 3.1. *Suppose the dynamical system* $u : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}, u(t, \varphi) = \varphi(\cdot + t)$ *satisfies:*

(A) *For any* $\varphi \in \mathcal{P}$, $\varepsilon > 0$, *and* $T > 0$ *there is a continuous function* $c : [0, T] \to \mathbb{R}_+$ *such that* $\lim_{t \to 0} \frac{c(t)}{t}$ *exists and is positive and for any* $y \in \mathbb{R}^n$ *with* $\varphi(0) - y \in B(\varphi(0), \varepsilon) \cap \mathbb{R}^n_+$ *there is a trajectory* $\psi \in \mathcal{P}$ *with* $\psi(0) = \varphi(0) - y$ *satisfying*

$$\| \varphi(t) - y - \psi(t) \| \le \|y\| \, c(t) \qquad \text{for all } \; t \in [0, T].$$

Then, the dynamical system u *defined on* $\mathcal{P}$ *is asymptotically stable if and only if there is a* C^∞*-smooth Lyapunov pair* (V, W) *such that for every* $\varphi \in \mathcal{P}$ *it holds*

$$\dot{V}(\varphi(t)) \le -W(\varphi(t)) \qquad \text{for almost all } \; t \ge 0. \tag{7}$$

We note that estimates similar to (A) for trajectories of differential inclusions with state constraints were derived by Bressan and Facchi, cf. [4] and the references therein.

4. **Motivation and application of the main result.** To outline the motivation for the consideration of the class of abstract dynamical systems, we give a very brief introduction to fluid networks. For a comprehensive description of multiclass queueing networks and fluid networks we refer to [3, 5, 8]. A fluid network consists of $J \in \mathbb{N}$ stations serving $n \in \mathbb{N}$ different types of fluids with $J \le n$ and each fluid type is served exclusively at a predefined station. This assignment defines the constituency matrix $C \in \mathbb{R}^{J \times K}$ with $c_{jk} := 1$ if fluid type $k \in \{1, ..., n\}$ is served at station $j \in \{1, ..., J\}$ and $c_{jk} = 0$ otherwise. The exogenous inflow rate of fluid type k is denoted by α_k and $\alpha = (\alpha_1, ..., \alpha_n)^\mathsf{T} \in \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \ge 0, \, i = 1, ..., n\}$ is called the exogenous inflow rate. Likewise, $\mu_k \in \mathbb{R}_+$ denotes the potential outflow rate of type k fluids and $\mu = (\mu_1, ..., \mu_n)^\mathsf{T} \in \mathbb{R}^n_+$. The substochastic matrix $P \in [0, 1]^{n \times n}$ describes the transitions in the network, where it is assumed that the spectral radius of the matrix P is strictly less than one, i.e. $1 > \max\{|\lambda| \,|\, \exists x \ne 0 : Px = \lambda x\}$. The initial fluid level and the fluid level at time t of the network are denoted by x_0 and $x(t)$, respectively. We note that, as $x(t)$ described the

deterministic analog of the queue length of the multiclass queueing network, the fluid level process $x(\cdot)$ evolves only in the positive orthant $\mathbb{R}^n_+$.

The evolution of the fluid level process $x(\cdot)$ is basically described by the balance equation

$$x(t) = x_0 + \alpha\, t - (I - P^T)MT(t) \geq 0, \tag{8}$$

where $M = \operatorname{diag}(\mu)$ and $T(\cdot)$ denotes the allocation process according to the discipline determining the rule under which the individual stations of the fluid network are serving the different fluid types. The complete description of the evolution of the fluid network under particular disciplines embraces additional equations characterizing the allocation process, cf. [3] and the references therein. It is well-known that the fluid level as well as the allocation process are Lipschitz continuous and therefore differentiable almost everywhere. In order to consider fluid networks from the differential inclusions point of view, we define $\dot{T}(t) =: u(t)$ and consider the differential form of the flow balance equation

$$\dot{x}(t) = \alpha - (I - P^T)\, Mu(t) \qquad \text{for almost all } t \geq 0.$$

For the class of general work-conserving fluid networks, given $x \in \mathbb{R}^n_+$, the set of admissible allocation rates $u = (u_1, ..., u_n)^\mathsf{T}$ is

$$U(x) = \big\{ u \in \mathbb{R}^n \,\big|\, u \geq 0,\ e - Cu \geq 0,\ (Cx)^\mathsf{T} \cdot (e - Cu) = 0 \big\}, \tag{9}$$

where $e = (1, ..., 1)^\mathsf{T} \in \mathbb{R}^J$ and the inequalities have to be understood component-by-component. The evolution of a general work-conserving fluid network can then be described by the following differential inclusion

$$\dot{x} \in F\big(x\big) := \Big\{ \alpha - (I - P^\mathsf{T})Mu \,\Big|\, u \in U(x) \Big\} \cap T\big(x, \mathbb{R}^n_+\big), \quad x(0) = x_0, \tag{10}$$

where $T(x, \mathbb{R}^n_+)$ denotes the contingent cone to $\mathbb{R}^n_+$ at x. The intersection with the contingent cone to the positive orthant is to ensure the nonnegativity of the solutions. In [5, Theorem 2.1] and [19, Theorem 4] it is shown that for any $x_0 \in \mathbb{R}^n_+$ there is at least one solution, i.e. an absolutely continuous function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ satisfying $\varphi(0) = x_0$ and $\dot{\varphi}(t) \in F(\varphi(t))$ for almost all $t \geq 0$. Let $\mathcal{S}_F(x)$ denote the set of solutions $\varphi\colon \mathbb{R}_+ \to \mathbb{R}^n$ to (10) with $\varphi(0) = x$ and $\mathcal{S}_F := \{\mathcal{S}_F(x) \,|\, x \in \mathbb{R}^n\}$.

From the abstract point of view the mapping

$$u : \mathbb{R}_+ \times \mathcal{S}_F \to \mathcal{S}_F, \quad u(t, \varphi) := \varphi(t + \cdot) \tag{11}$$

defines a dynamical system on $\mathcal{S}_F$. The zero trajectory $\varphi_* \in \mathcal{S}_F$, $\varphi_*(s) = 0$ for all $s \geq 0$ of the differential inclusion (14) satisfies $u(t, \varphi_*) = \varphi_*(t + \cdot) = \varphi_*$ for all $t \geq 0$, i.e. $\varphi_* \equiv 0$ is the only equilibrium of the dynamical system (11). Next, we show that the notions of stability in the two approaches are equivalent if the set of solutions satisfies the conditions (a)-(d).

Before doing this, based on the properties (a)-(d), we present a useful characterization of asymptotic stability for the class of dynamical systems under consideration.

Proposition 1. *The dynamical system u defined on $\mathcal{P}$ is asymptotically stable if and only if there is a $\tau > 0$ such that $u(\|\varphi(0)\|\tau, \varphi) = \varphi(\|\varphi(0)\|\tau + \cdot) \equiv 0$ for all $\varphi \in \mathcal{P}$.*

Proof. Suppose there is a $\tau < \infty$ such that $u(\|\varphi(0)\|\tau, \varphi) = 0$ for all $\varphi \in \mathcal{P}$. To conclude stability let $\varepsilon > 0$ and $\delta := \frac{\varepsilon}{\lceil L\tau \rceil}$, where $\lceil r \rceil := \min\{k \in \mathbb{Z} \,|\, k \geq r\}$ for

$r \in \mathbb{R}$. Let $\varphi \in \mathcal{P}$ with $\mathrm{d}(0,\varphi) < \delta$. By assumption, using the scaling and shift property it holds

$$\varphi(s+t) = 0 \quad \text{for all } t \geq \tau \, \|\varphi(s)\|. \tag{12}$$

Together with the Lipschitz continuity property (a), for every $t \in [0, \tau\|\varphi(s)\|]$ we have

$$\|\varphi(s+t)\| = \| \, \varphi(s+t) - \varphi\big(s + \tau\|\varphi(s)\|\big) \, \| \leq L \, \big|t - \tau\|\varphi(s)\| \, \big| \leq L\tau\|\varphi(s)\|. \tag{13}$$

By (12) we conclude that (13) holds for all $t \geq 0$. Therefore,

$$\|\varphi(\cdot + t)\|_N = \sup_{s\in[0,N]} \|\varphi(s+t)\| \leq L\tau\|\varphi\|_N \leq \lceil L\tau \rceil \|\varphi\|_N = \|\lceil L\tau \rceil \varphi\|_N.$$

In turn, by the triangular inequality, we have

$$\mathrm{d}(0, u(t,\varphi)) \leq \max_{N\in\mathbb{N}} \frac{1}{2^N} \frac{\|\lceil L\tau\rceil\varphi\|_N}{1 + \|\lceil L\tau\rceil\varphi\|_N} = \mathrm{d}(0, \lceil L\tau\rceil\varphi) \leq \lceil L\tau\rceil \, \mathrm{d}(0,\varphi) < \varepsilon.$$

By assumption, it holds $\varphi\big(\|\varphi(0)\|\tau + t\big) = 0$ for all $t \geq 0$. This in turn implies $\lim_{t\to\infty} \mathrm{d}(0, u(t,\varphi)) = 0$ and we have attractivity.

Conversely, let $\varphi_* \equiv 0$ be asymptotically stable. Due to the scaling property it suffices to consider trajectories φ with $\|\varphi(0)\| = 1$. Then, as

$$\lim_{t\to\infty} \mathrm{d}(0, u(t,\varphi)) = \lim_{t\to\infty} \mathrm{d}(0, \varphi(t + \cdot)) = 0$$

we have

$$\lim_{t\to\infty} \|\varphi(t)\| = 0 \quad \text{for all } \varphi \in \mathcal{P}.$$

Hence, $\inf\{\|\varphi(t)\| \,|\, t \geq 0\} = 0$ for any $\varphi \in \mathcal{P}$ with $\|\varphi(0)\| = 1$. The assertion then follows from [22, Theorem 6.1]. □

We note that the closedness property (d) is required as it is an assumption of Theorem 6.1 in [22]. In combination with [19, Lemma 1] the previous Lemma 1 yields

Corollary 1. *Let $F\colon \mathbb{R}^n_+ \rightsquigarrow \mathbb{R}^n$ be a set-valued map such that $0 \in F(0)$ and the set of solutions $\mathcal{S}_F$ to $\dot{x} \in F(x)$ satisfies (a)-(d). Then, the origin $0 \in \mathbb{R}^n_+$ is strongly asymptotically stable if and only if the zero trajectory $\varphi_* \equiv 0$ is an asymptotically stable equilibrium of the dynamical system $u : \mathbb{R}_+ \times \mathcal{S}_F \to \mathcal{S}_F$, $u(t,\varphi) = \varphi(t + \cdot)$.*

Tackling a simple example we show that the differential inclusion (10) does not satisfy the standard assumption posed in the literature mentioned in the introduction.

Lemma 4.1. *The set-valued map F describing the evolution of a general work-conserving fluid network defined in (10) is not upper semicontinuous in general.*

Proof. To show the claim we consider a single station fluid network serving one type of fluid. That is, for $\alpha > 0$, $\mu = \alpha + 1$, and $P = 0$ the differential inclusion (10) is defined by the set-valued map

$$\begin{aligned} G(x) &= \big\{ \alpha - (\alpha+1)\,u \quad \big| \quad 0 \leq u \leq 1, \quad x(1-u) = 0 \big\} \cap T(x, \mathbb{R}_+) \\ &= \Big\{ \alpha - (\alpha+1)\,u \quad \Big| \quad 0 \leq u \leq 1, u = 1 \text{ if } x > 0 \text{ and } u = \tfrac{\alpha}{\alpha+\delta} \text{ else } \Big\}. \end{aligned} \tag{14}$$

To conclude that G is not upper semicontinuous let $x = 0$ and consider the sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k = \frac{1}{k}$. Then, for each $k \in \mathbb{N}$ we have $G(x_k) = -1$. Let $\mathrm{graph}(G) := \{\, (x,y) \in \mathbb{R}_+ \times \mathbb{R} \,\big|\, y \in G(x) \,\}$ denote the graph of G and consider the sequence

$\big(x_k, -1\big)_{k\in\mathbb{N}}$ on the graph of G which converges to $\big(0, -1\big) \notin \operatorname{graph}(G)$. Hence, the graph is not closed and by Proposition 2 in [1, Section 1.1] the set-valued map G is not upper semicontinuous. □

We note that the origin is the only equilibrium of the differential inclusion (14). Moreover, it is strongly asymptotically stable. However, as a consequence of Lemma 4.1, the existence of a C^∞-smooth Lyapunov pair (V, W) satisfying the strong infinitesimal decrease condition (2) cannot be concluded by the results on differential inclusions mentioned above.

We use the main result Theorem 3.1 to conclude that the differential inclusion (14) admits a C^∞-smooth Lyapunov pair. It is well known that a single station general work-conserving fluid network serving only one fluid type satisfies the properties (a)-(f); cf. [5, 19, 22]. Thus, in order to conclude the existence of a C^∞-smooth Lyapunov function it is sufficient to verify assumption (A) and apply the main result (Theorem 3.1) of the paper.

Theorem 4.2. *The differential inclusion* (14) *admits a C^∞-smooth Lyapunov pair, i.e. there is a C^∞-smooth and positive definite pair (V, W) such that V is proper and*

$$\max_{v\in G(x)} \langle \nabla V(x), v\rangle \leq -W(x) \qquad \textit{for all } x \in (0,\infty).$$

Proof. To show the existence of a C^∞-smooth Lyapunov pair we verify that the set of solutions $\mathcal{S}_G$ to the differential inclusion (14) satisfies the assumption (A).

Let $\varphi \in \mathcal{S}_G$, $\varepsilon > 0$ and $T > 0$ be fixed. In a first step, we treat the case that $\varphi(0) > 0$. Then, we have

$$\varphi(t) = \begin{cases} \varphi(0) - t & \text{if } t \leq \varphi(0) \\ 0 & \text{else.} \end{cases}$$

In the case $y = \varphi(0)$ the only solution ψ to the differential inclusion starting in $\psi(0) = \varphi(0) - y = 0$ is the zero solution and we obtain $|\varphi(t) - y - \psi(t)| = |t|$ for all $t \leq \varphi(0)$ and $|\varphi(t) - y - \psi(t)| = |y|$ otherwise. Hence, one has

$$|\varphi(t) - y - \psi(t)| \leq \frac{1}{\varphi(0)}\,|y|\,t \qquad \text{for all } t \geq 0.$$

If $y \neq \varphi(0)$ we consider the solution ψ to the differential inclusion starting in $\psi(0) = \varphi(0) - y$ given by

$$\psi(t) = \begin{cases} \varphi(0) - y - t & \text{if } t \leq \varphi(0) - y \\ 0 & \text{else.} \end{cases}$$

On one hand, if $0 < \varphi(0) - y < \varphi(0)$ it follows $|\varphi(t) - y - \psi(t)| = 0$ for all $t \in [0, \varphi(0) - y]$. Also, for all $t \in [\varphi(0) - y, \varphi(0)]$ we have $|\varphi(t) - y - \psi(t)| = |\varphi(0) - y - t|$ and for all $t \geq \varphi(0)$ one has $|\varphi(t) - y - \psi(t)| = |y|$. On other hand, if $0 < \varphi(0) \leq \varphi(0) - y$ we have $|\varphi(t) - y - \psi(t)| = 0$ for all $t \in [0, \varphi(0)]$. Further, for all $t \in [\varphi(0), \varphi(0) - y]$ one has $|\varphi(t) - y - \psi(t)| = |\varphi(0) - y - t|$ and for all $t \geq \varphi(0) - y$ it follows $|\varphi(t) - y - \psi(t)| = |y|$. Consequently, in either case one obtains

$$|\varphi(t) - y - \psi(t)| \leq \frac{1}{\varphi(0)}\,|y|\,t \qquad \text{for all } t \geq 0.$$

Finally, we consider the case $\varphi(0) = 0$. Then, we have $\varphi \equiv 0$. For $y \in (-\varepsilon, 0]$ a solution ψ of the differential inclusion with $\psi(0) = -y$ is

$$\psi(t) = \begin{cases} -y - t & \text{if } t \leq -y \\ 0 & \text{else.} \end{cases}$$

Therefore, $|\varphi(t) - y - \psi(t)| = t$ for all $t \in [0, -y]$ and $|\varphi(t) - y - \psi(t)| = |y|$ for all $t \geq -y$ and one has

$$|\varphi(t) - y - \psi(t)| \leq \log(t + \mathrm{e})\, |y| \qquad \text{for all } t \geq 0.$$

Thus, assumption (A) is fulfilled and by Theorem 3.1 there is a C^∞-smooth Lyapunov pair (V, W) such that for every solution $\varphi \in \mathcal{S}_G$ one has

$$\dot{V}(\varphi(t)) \leq -W(\varphi(t)) \qquad \text{for almost all } t \geq 0.$$

Consequently, the pair (V, W) satisfies

$$\max_{v \in G(x)} \langle \nabla V(x), v \rangle \leq -W(x) \qquad \text{for all } x \in (0, \infty).$$

This shows the assertion. □

5. **Relating assumption (A) to differential inclusions.** In this section we investigate assumption (A) from the differential inclusions perspective. Due to the fact that Clarke, Ledyaev and Stern [6] as well as Dupius and Williams [9] embed the set-valued map defining the differential inclusion into a Lipschitz one, we consider the differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{15}$$

where $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is Lipschitz continuous, i.e. there is a constant $L > 0$ such that

$$F(x) \subset F(y) + L\, \|x - y\|\, B(0, 1) \qquad \text{for all } \ x, y \in \mathbb{R}^n,$$

and $F(x)$ is nonempty, compact and convex for every $x \in \mathbb{R}^n$. Let $\mathcal{S}_F(x)$ denote the set of solutions $\varphi\colon \mathbb{R}_+ \to \mathbb{R}^n$ to (15) with $\varphi(0) = x$. Let $\mathcal{S}_F := \{\mathcal{S}_F(x) \,|\, x \in \mathbb{R}^n\}$. Next we show that condition (A) is a natural assumption. More precisely, we show that the set of solutions to a differential inclusion defined by a Lipschitz continuous set-valued map automatically has property (A).

Theorem 5.1. *Let F be a Lipschitz continuous set-valued map taking nonempty, compact and convex values with $0 \in F(0)$. Then, the set of solutions $\mathcal{S}_F$ satisfies assumption* (A)*.*

Proof. Let $\varphi \in \mathcal{S}_F$, $\varepsilon > 0$ and $T > 0$. We define $c(t) := \mathrm{e}^{Lt} - 1$. Then, for $y \in \mathbb{R}^n$ the function $\varphi_y(\cdot) := \varphi(\cdot) - y$ is absolutely continuous with $\varphi_y(0) = \varphi(0) - y$. Further, as F is Lipschitz it holds $F(\varphi_y(t)) \subset F(\varphi(t)) + L\, \|y\|\, B(0, 1)$ and we have

$$d(\dot{\varphi}_y(t), F(\varphi_y(t)) = d(\dot{\varphi}(t), F(\varphi_y(t)) \leq L\, \|y\|.$$

Thus, by Filippov's Theorem [1, Theorem 1 in Chapter 2, Section 4] there is a solution $\psi(\cdot)$ to (15) defined on the interval $[0, T]$ with $\psi(0) = \varphi_y(0) = \varphi(0) - y$ satisfying

$$\|\varphi_y(t) - \psi(t)\| = \|\varphi(t) - y - \psi(t)\| \leq \|y\| \left(\mathrm{e}^{Lt} - 1\right) \qquad \text{for all } t \in [0, T].$$

This shows the assertion. □

6. **Proof of Theorem 3.1.** In [19, Theorem 2] it is shown that for the dynamical system under consideration there is a continuous Lyapunov pair if and only if the dynamical system is asymptotically stable. Thus, the non-converse implication is already shown.

Conversely, let the dynamical system u defined on $\mathcal{P}$ be asymptotically stable. Then, by Theorem 2 in [19] there is a continuous Lyapunov pair (V, W) such that

$$V(\varphi(t)) - V(\varphi(s)) \leq -\int_s^t W(\varphi(r))\, \mathrm{d}r \quad \text{for all } \varphi \in \mathcal{P},\ 0 \leq s \leq t. \tag{16}$$

Thus, the construction of a C^∞-smooth Lyapunov-pair remains.

To get differentiability on the boundary of the positive orthant, we first extend the pair (V, W) to $\mathbb{R}^n$. To this end, let $|\cdot|_{\text{vec}}$ denote the map that takes componentwise absolute values defined by $|x|_{\text{vec}} := (|x_1|, ..., |x_n|)^\mathsf{T} \in \mathbb{R}^n_+$. The extension of the pair (V, W) to $\mathbb{R}^n$ is defined by

$$V^e(x) := V(|x|_{\text{vec}}), \qquad W^e(x) := W(|x|_{\text{vec}}).$$

Note that, as a composition of continuous functions, the pair (V^e, W^e) is also continuous. As a first consequence of assumption (A) we conclude that V^e is locally Lipschitz.

Lemma 6.1. *Suppose the dynamical system u defined on $\mathcal{P}$ satisfies* (A) *and is asymptotically stable. Then, V^e is locally Lipschitz on $\mathbb{R}^n$.*

Proof. Let $U \subset \mathbb{R}^n$ be open, convex, and bounded and let $x \in U$. Following Corollary 3.7 in [7], since $-V^e$ is lower semicontinuous, it suffices to show that there is a $M > 0$ such that for any $v \in \mathbb{R}^n$ it holds

$$D(-V^e)(x; v) \leq M\|v\|.$$

Let $v' \in \mathbb{R}^n$ and $\xi > 0$. Let $\varphi \in \mathcal{P}$ be a trajectory of the dynamical system satisfying $\varphi(0) = |x|_{\text{vec}}$ and

$$V^e(x) = \int_0^\infty \|\varphi(s)\|\, \mathrm{d}s = \int_0^{\|x\|\tau} \|\varphi(s)\|\, \mathrm{d}s,$$

where in the last equality used the stability of $\mathcal{P}$ and Lemma 1. By continuity of $|\cdot|_{\text{vec}}$ we have

$$\lim_{\xi \to 0} |x + \xi v'|_{\text{vec}} = |x|_{\text{vec}}.$$

So, for every $\varepsilon > 0$ and ξ sufficiently small we have $|x + \xi v'|_{\text{vec}} \in B(|x|_{\text{vec}}, \varepsilon) \cap \mathbb{R}^n_+$. Moreover, there is a continuous mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\|g(v')\| = \|v'\|$ and

$$|x + \xi v'|_{\text{vec}} = |x|_{\text{vec}} + \xi g(v').$$

For $T := \max\{\|x\|\,\tau, \|x+\xi v'\|\,\tau\} < \tau(\|x\|+\varepsilon)$, by assumption (A) and the triangular inequality, there are $c > 0$ and $\psi \in \mathcal{P}$ with $\psi(0) = |x|_{\text{vec}} + \xi g(v')$ such that

$$\|\,\varphi(t)\,\| - \|\,\psi(t)\,\| \leq \|\,\varphi(t) \,-\, \psi(t)\,\| \leq \xi\,\|g(v')\|\,(1 + c(t)) = \xi\,\|v'\|\,(1 + c(t)) \tag{17}$$

for all $t \in [0, T]$. The definition of V^e, the stability of $\mathcal{P}$ together with Lemma 1, and $\|\,|x|_{\text{vec}} + \xi g(v')\| = \|x + \xi v'\|$ yield

$$V^e(x + \xi v') \geq \int_0^\infty \|\,\psi(s)\,\|\, \mathrm{d}s = \int_0^{\|x+\xi v'\|\tau} \|\,\psi(s)\,\|\, \mathrm{d}s.$$

On the one hand, if $\|x\| \le \|x+\xi v'\|$ by using (17) it follows

$$\begin{aligned} V^e(x) - V^e(x+\xi v') &\le \int_0^{\|x\|\tau} \|\,\varphi(s)\,\|\,\mathrm{d}s - \int_0^{\|x+\xi v'\|\tau} \|\,\psi(s)\,\|\,\mathrm{d}s \\ &\le \int_0^{\|x\|\tau} \|\,\varphi(s)\| - \|\psi(s)\,\|\,\mathrm{d}s \\ &\le \int_0^{\|x\|\tau} \xi\,\|v'\|\,(1+c(s))\,\mathrm{d}s \le \xi\,\|v'\|\,\|x\|\,\tau\,C, \end{aligned}$$

where $C := \max_{0\le s\le \|x\|\tau}(1+c(s))$. On the other hand, to consider the case $\|x\| > \|x+\xi v'\|$ we note that the triangle inequality together with the Lipschitz condition imply

$$\|\varphi(t)\| \le \|\varphi(0)\| + Lt \le \|x\|\,(1+L\tau) \qquad \text{for all } t\in[0,\|x\|\tau]. \tag{18}$$

Using (17), (18), and $0 \le \|x\| - \|x+\xi v'\| \le \xi\|v'\|$ we obtain

$$\begin{aligned} V^e(x) - V^e(x+\xi v') &\le \int_0^{\|x\|\tau} \|\,\varphi(s)\,\|\,\mathrm{d}s - \int_0^{\|x+\xi v'\|\tau} \|\,\psi(s)\,\|\,\mathrm{d}s \\ &\le \int_0^{\|x+\xi v'\|\tau} \|\,\varphi(s)\| - \|\psi(s)\,\|\,\mathrm{d}s + \int_{\|x+\xi v'\|\tau}^{\|x\|\tau} \|\,\varphi(s)\,\|\,\mathrm{d}s \\ &\le \int_0^{\|x+\xi v'\|\tau} \xi\,\|v'\|\,(1+c(s))\,\mathrm{d}s + \tau(\,\|x\| - \|x+\xi v'\|\,)\cdot \sup_{s\in[\|x+\xi v'\|\tau,\|x\|\tau]} \|\,\varphi(s)\,\| \\ &\le \xi\,\|v'\|\,\|x+\xi v'\|\,\tau\,C \;+\; \tau\,\xi\,\|v'\|\,\|x\|\,(1+L\tau). \end{aligned}$$

Consequently, taking limits and using that U is bounded there is a $M>0$ such that

$$\begin{aligned} D(-V^e)(x;v) &= \liminf_{\xi\to 0, v'\to v} \frac{V^e(x) - V^e(x+\xi v')}{\xi} \le \tau\,(C+1+L\tau)\,\|x\|\,\|v\| \\ &\le M\,\|v\|. \end{aligned}$$

The shows Lemma 6.1. □

Proceeding with the construction of a smooth Lyapunov pair, let U be an open subset of $\mathbb{R}^n$ and consider the convolution of V^e and k_r defined by

$$V_r^e(x) := V^e * k_r\,(x) = \int_{\mathbb{R}^n} V^e(x-y)\,k_r(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} V(|x-y|_{\mathrm{vec}})\,k_r(y)\,\mathrm{d}y.$$

Also, we consider the convolution of W^e and k_r given by

$$W_r^e(x) := W^e * k_r\,(x) = \int_{\mathbb{R}^n} W^e(x-y)\,k_r(y)\,\mathrm{d}y.$$

By standard convolution results it follows $V_r^e \in C^\infty(U,\mathbb{R}_+)$ and $W_r^e \in C^\infty(U,\mathbb{R}_+)$. Furthermore, since V^e is continuous on U it holds $V_r^e \to V^e$ uniform on compact subsets of U as $r\to 0$. Consequently, for every $\varepsilon>0$ there is an r_0 such that for all $r\in(0,r_0)$ we have V_r^e and W_r^e are smooth on U and

$$|V_r^e(x) - V^e(x)| \le \varepsilon, \qquad |W_r^e(x) - W^e(x)| \le \tfrac{\varepsilon}{2} \qquad \text{for all } x\in U. \tag{19}$$

The subsequent statement addresses the decrease condition of the convolution along trajectories $\varphi \in \mathcal{P}$.

Lemma 6.2. *Let $U \subset \mathbb{R}^n$ be compact such that $U \cap \mathbb{R}^n_+ \neq \emptyset$ and suppose (V, W) satisfy (16) and assumption* (A) *is satisfied. Then, for every $\varepsilon > 0$ there exists a $r_0 > 0$ such that for all $r \in (0, r_0)$ we have*

$$\dot{V}^e_r(\varphi(t)) \leq -W^e(\varphi(t)) + \varepsilon \tag{20}$$

for all $\varphi \in \mathcal{P}$ and almost all $t \in [0, T]$ with $\varphi(\cdot)|_{[0,T]} \subset U \cap \mathbb{R}^n_+$.

Proof. Let $\varphi \in \mathcal{P}$ be a trajectory satisfying $\varphi(0) = x \in U \cap \mathbb{R}^n_+$. Let $t \in [0, T]$ be such that φ is differentiable at t. Then, for $h > 0$ we have

$$V^e_r\big(\varphi(t+h)\big) - V^e_r\big(\varphi(t)\big) = \int_{\mathbb{R}^n} \Big(V^e\big(\varphi(t+h) - y\big) - V^e\big(\varphi(t) - y\big) \Big)\, k_r(y)\, \mathrm{d}y.$$

There is a continuous mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\|g(y)\| = \|y\|$ and

$$|\varphi(t) - y|_{\text{vec}} = \varphi(t) - g(y).$$

Further, by assumption (A) and for h sufficiently small there is a continuous function $c\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \frac{c(t)}{t} =: c_0 > 0$ and a trajectory $\psi(t + \cdot) \in \mathcal{P}$ with $\psi(t) = \varphi(t) - g(y)$ such that

$$\|\varphi(t+h) - y - \psi(t+h)\| \leq \|y\| c(h). \tag{21}$$

Using this, as $V^e(\psi(t+h)) = V(\psi(t+h))$ we obtain

$$\begin{aligned} V^e_r(\varphi(t+h)) - V^e_r(\varphi(t)) \leq & \int_{\mathbb{R}^n} \Big| V^e(\varphi(t+h) - y)) \;-\; V^e(\psi(t+h)) \Big|\, k_r(y)\, \mathrm{d}y \\ & + \int_{\mathbb{R}^n} \Big(V(\psi(t+h)) - V(\varphi(t) - g(y)) \Big)\, k_r(y)\, \mathrm{d}y. \end{aligned} \tag{22}$$

By the local Lipschitz continuity of V with constant L and (21), the first term on the right hand side in the above inequality can be estimated as follows

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big| V^e(\varphi(t+h) - y) \;-\; V^e(\psi(t+h)) \Big|\, k_r(y)\, \mathrm{d}y \\ &\qquad = \int_{\mathbb{R}^n} \Big| V(\varphi(t+h) - g(y)) \;-\; V(\psi(t+h)) \Big|\, k_r(y)\, \mathrm{d}y \\ &\qquad \leq \int_{\mathbb{R}^n} L \big\|\, \varphi(t+h) - g(y) \;-\; \psi(t+h) \,\big\|\, k_r(y)\, \mathrm{d}y \\ &\qquad\qquad \leq c(h)\, L \int_{\mathbb{R}^n} \|g(y)\|\, k_r(y)\, \mathrm{d}y = c(h)\, L \int_{\mathbb{R}^n} \|y\| k_r(y)\, \mathrm{d}y. \end{aligned}$$

Furthermore, it holds $\displaystyle\int_{B(0,r)} \|y\|\, k_r(y)\, \mathrm{d}y \leq \int_{B(0,r)} r\, k_r(y)\, \mathrm{d}y = r$ and choosing $r_0 := \frac{\varepsilon}{2\, c_0\, L}$ it follows

$$\int_{\mathbb{R}^n} \big| V^e(\varphi(t+h) - y) \;-\; V(\psi(t+h)) \big| k_r(y)\, \mathrm{d}y \leq \tfrac{c(h)}{c_0}\, \tfrac{\varepsilon}{2}.$$

Asymptotic stability of the dynamical system implies that the very last term in (22) can be estimated by means of the function W and its mollification,

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(V\big(\psi(t+h)\big) - V\big(\varphi(t) - g(y)\big)\Big)\, k_r(y)\ \mathrm{d}y \\ \le \int_{\mathbb{R}^n} \left(- \int_t^{t+h} W\big(\psi(s)\big)\ \mathrm{d}s \right) k_r(y)\ \mathrm{d}y \\ = - \int_0^h \Big(\int_{\mathbb{R}^n} W\big(\psi(t+s)\big) k_r(y)\ \mathrm{d}y \Big)\ \mathrm{d}s,\end{aligned}$$

where the last identity is obtained by integration by substitution. Next, we show that the function

$$s \mapsto \int_{\mathbb{R}^n} W\big(\psi(t+s)\big)\, k_r(y)\ \mathrm{d}y$$

is continuous in $[0,h]$. To see this, consider the *modulus of continuity* of the function

$$s \mapsto W\big(\psi(t+s)\big),$$

defined for $\delta \in [0,h]$ by

$$\mathrm{m}\Big(\delta, W\big(\psi(t+\cdot)\big)\Big) := \sup_{|s-s'|\le\delta} \Big| W\big(\psi(t+s)\big) - W\big(\psi(t+s')\,\big) \Big|.$$

Then, for $s, s' \in [0,h]$ it holds

$$W\big(\psi(t+s))\,\big) - W\big(\psi(t+s')\,\big) \le \mathrm{m}\Big(h, W\big(\psi(t+\cdot)\,\big)\Big).$$

By asymptotic stability of the dynamical system $\|\psi(t+s)\|$ is bounded and, hence, $W(\psi(t+\cdot)\,)$ is uniformly continuous. Thus, we have

$$\lim_{h\to 0} \mathrm{m}\left(h, W\big(\psi(t+\cdot)\big)\right) = 0.$$

That is, for every $\varepsilon' > 0$ there is a $\delta' > 0$ such that $\mathrm{m}\Big(h, W\big(\,\psi(t+\cdot)\,\big)\Big) \le \varepsilon'$ for all $h \le \delta'$. For $\varepsilon' > 0$ choose $\delta > 0$ such that $|s-s'| < \delta < \delta'$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(W\big(\psi(t+s)\big) - W\big(\psi(t+s')\big)\Big)\, k_r(y)\ \mathrm{d}y \\ \le \int_{\mathbb{R}^n} \mathrm{m}\Big(\delta, W\big(\psi(t+\cdot)\big)\Big)\, k_r(y)\ \mathrm{d}y \le \int_{\mathbb{R}^n} \varepsilon'\, k_r(y)\ \mathrm{d}y = \varepsilon'.\end{aligned}$$

Moreover, by conditions (19) we have $-W^e_r(x)+\frac{\varepsilon}{2}\leq -W^e(x)+\varepsilon$. Finally, the collection of the above relations yields

$$\begin{aligned}
\dot V^e_r(\varphi(t)) &= \lim_{h\to 0}\frac{V^e_r\big(\varphi(t+h)\big)-V^e_r\big(\varphi(t)\big)}{h}\\
&\leq \frac{\varepsilon}{2}-\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^n}\left(\int_0^h W\big(\psi(t+s)\big)\,k_r(y)\,dy\right)\mathrm{d}s\\
&\leq -\lim_{h\to 0}\frac{1}{h}\int_0^h\left(\int_{\mathbb{R}^n} W\big(\psi(t+s)\big)\,k_r(y)\,dy\right)\mathrm{d}s+\frac{\varepsilon}{2}\\
&= -\int_{\mathbb{R}^n} W\big(\psi(t)\big)\,k_r(y)\;\mathrm{d}y+\frac{\varepsilon}{2} = -\int_{\mathbb{R}^n} W\big(\varphi(t)-g(y)\big)\,k_r(y)\;\mathrm{d}y+\frac{\varepsilon}{2}\\
&= -\int_{\mathbb{R}^n} W^e(\varphi(t)-y)\,k_r(y)\;\mathrm{d}y+\frac{\varepsilon}{2}\\
&= -W^e_r(\varphi(t))+\frac{\varepsilon}{2}\leq -W^e(\varphi(t))+\varepsilon.
\end{aligned}$$

This shows Lemma 6.2. □

Now, let $\mathcal{U}=\{U_i\}_{i=1}^{\infty}$ be a locally finite open cover of $\mathbb{R}^n$ such that for every i the closure $\overline{U_i}$ is compact. Further, let $\{\psi_i\}_{i=1}^{\infty}$ be a smooth partition of unity that is subordinate to $\mathcal{U}$. Define

$$\varepsilon_i=\tfrac{1}{4}\min\{\min_{x\in\bar U_i}V^e(x),\min_{x\in\bar U_i}w^e(x)\}\qquad\text{and}\qquad q_i=\max_{x\in\bar U_i}\|\nabla\psi_i(x)\|. \tag{23}$$

Then, by Lemma 6.2 for every i there is a C^∞-pair (V^e_i,W^e_i) such that for every $x\in U_i$,

$$|V^e(x)-V^e_i(x)|<\frac{\varepsilon_i}{2^{i+1}(1+q_i)}\qquad\text{and}\qquad |W^e(x)-W^e_i(x)|<\varepsilon_i. \tag{24}$$

Moreover, by the conditions (20) and (23) we have that

$$\dot V^e_i\big(\varphi(t)\big)\leq -W^e\big(\varphi(t)\big)+2\,\varepsilon_i\leq -\tfrac{1}{2}W^e\big(\varphi(t)\big). \tag{25}$$

Next, we define

$$V^e_s(x):=\sum_{i=1}^{\infty}\psi_i(x)\,V^e_i(x).$$

The following estimate holds true

$$|V^e_s(x)-V^e(x)|\leq\sum_{i=1}^{\infty}\psi_i(x)\,\big|V^e_i(x)-V^e(x)\big|\leq\frac{V^e(x)}{4}\sum_{i=1}^{\infty}\frac{\psi_i(x)}{2^{i+1}(1+q_i)}\leq\tfrac{1}{8}V^e(x).$$

Using the triangular inequality, the latter estimate shows that V^e_s is proper and positive definite. The next step is to derive that V^e_s is decaying along trajectories of $\mathcal{P}$. To this end, we consider

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}[V_s^e(\varphi(t))] &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[V^e(\varphi(t)) + V_s^e(\varphi(t)) - V^e(\varphi(t))\right] \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}[V^e(\varphi(t))] + \tfrac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(V_i^e(\varphi(t)) - V^e(\varphi(t))\right)\right] \\ &= \dot V^e(\varphi(t)) + \sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(\dot V_i^e(\varphi(t)) - \dot V^e(\varphi(t))\right) \\ &\qquad\qquad + \sum_{i=1}^{\infty}\dot\psi_i(\varphi(t))\left(V_i^e(\varphi(t)) - V^e(\varphi(t))\right) \\ &\le \sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(\dot V_i^e(\varphi(t)) + \sum_{j=1}^{\infty}\dot\psi_j(\varphi(t))\left|V_j^e(\varphi(t)) - V^e(\varphi(t))\right|\right).\end{aligned}$$

Using the conditions (24) and (25) we get the following estimate

$$\dot V_s^e(\varphi(t)) \le \sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(-\tfrac{1}{2}\,W^e(\varphi(t)) + \sum_{j=1}^{\infty}\frac{q_j\varepsilon_j}{2^{j+1}(1+q_j)}\right).$$

Defining $\widetilde\varepsilon_i := \max\{\varepsilon_j \,:\, x \in U_i \cap U_j \neq \emptyset\}$ we have that

$$\begin{aligned}\dot V_s^e(\varphi(t)) &\le \sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(-\tfrac{1}{2}\,W^e(\varphi(t)) + \widetilde\varepsilon_i\sum_{j=1}^{\infty}\tfrac{1}{2^{j+1}}\right) \\ &= \sum_{i=1}^{\infty}\psi_i(\varphi(t))\left(-\tfrac{1}{2}W^e(\varphi(t)) + \widetilde\varepsilon_i\right).\end{aligned}$$

Using (23) and the triangular inequality applied to the second inequality in (24), it holds that

$$-\tfrac{1}{2}W^e(\varphi(t)) + \widetilde\varepsilon_i \le -\tfrac{1}{4}W^e(\varphi(t)) \le -\tfrac{1}{5}W_i^e(\varphi(t)).$$

Finally, we have that

$$\dot V_s^e(\varphi(t)) \le -\tfrac{1}{5}\sum_{i=1}^{\infty}\psi_i(\varphi(t))\,W_i^e(\varphi(t)) =: -W_s^e(\varphi(t)).$$

Consequently, the pair (V_s^e, W_s^e) defines a C^∞-smooth Lyapunov pair, which shows the assertion. □

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