Control of Ensembles of Single-Input Continuous-Time Linear Systems

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Abstract: In this paper we consider continuous-time linear time-invariant single-input systems that depend on a single parameter. Given a family of desired states we aim to find open-loop control inputs which are independent of the parameter that simultaneously steer the zero initial state arbitrarily close to a family of terminal states. As an application we show that a family of networks of harmonic oscillators can be robustly controlled by a single input function. Thus our goal is to effectively compute open-loop input functions that are cabable of controlling an ensemble of linear systems.

Keywords: approximate controllability, complex approximation, control of ensembles of systems, harmonic oscillators

1. INTRODUCTION

In recent years the goal of controlling entire ensembles of parameter-dependent linear systems using open-loop controls has received considerable attention. Thus, for a given family of desired terminal states, one attempts to construct parameter-independent input functions that steer the zero-state to the family of target states states, simultaneously for all parameter values. Such open-loop control issues for ensembles of systems arise e.g. in designing compensating pulse-sequences in quantum control; see e.g. Li and Khaneja (2006). They are also of interest towards controlling large-scale networks of systems, spatial-temporal systems, or in understanding biological systems, such as flocks of systems; see e.g. Brockett (2010).

We consider parameter-dependent linear time-invariant single-input systems $(A(\theta), b(\theta))$, of the form

$$\frac{\partial}{\partial t} x(t,\theta) = A(\theta)x(t,\theta) + b(\theta)u(t),$$

where the system matrices $A(\theta) \in \mathbb{R}^{n\times n}$ and $b(\theta) \in \mathbb{R}^n$ are assumed to depend continuously on a single real parameter $\theta \in \mathbf{P} := [\theta^-, \theta^+]$. Our goal is to develop conditions for the existence of parameter independent open-loop controls u that steer the zero initial state arbitrarily close to a given family of terminal states. This is referred to as *uniform ensemble controllability.*

We note that in the case that $\mathbf{P} = \{\theta_1, ..., \theta_N\}$ is discrete the family $(A(\theta), b(\theta))$ embodies a network of N systems under parallel connection. Our main result Theorem 1 gives an easily verifiable sufficient condition for uniform ensemble controllability. Concrete computational results are obtained for a one-parameter family of harmonic oscillators. The weaker version of L^2-ensemble controllability has been first introduced by Li and Khaneja (2006) for ensemble control of bilinear systems arising in quantum control. Subsequently, Li (2011) studied the task of L^2-ensemble controllability for time-varying linear multivariable systems. Moreover, sufficient conditions for L^2-ensemble control were proposed. However, these conditions are stated in terms of the singular vectors of the input-to-state operator and thus are very hard to verify, even for low-dimensional systems. Moreover, the results in Li (2011) do not apply to uniform ensemble controllability. In Helmke and Schönlein (2013), necessary as well as sufficient conditions are derived for uniform ensemble controllability of one-parameter families of time-invariant linear systems, both for discrete-time and continuous-time systems. In comparison with the work by Li (2011), the analysis in Helmke and Schönlein (2013) yields a much more concrete and easily verifiable condition for ensemble controllability. However, the analysis in Helmke and Schönlein (2013) has focused on discrete-time systems and was a bit roundabout in the continuous-time case. In this paper we therefore give an independent and more directly applicable approach to the continuous-time case. We also stress applications to networks of harmonic oscillators and to the control of spatial-temporal systems.

The paper is organized as follows. In Section 2 we formulate the problem under consideration and state out main result. Section 3 draws a comparison between ensemble controllability and approximate controllability. A proof of our main result Theorem 1 based on complex approximation theory is given in Section 4. In Section 5 and Section 6 the problem of effectively constructing input functions is considered for a one-parameter family of harmonic oscillators with varying frequencies. Section 5 contains an approach to construct input functions which is based on the features of the harmonic oscillators and uses Lagrange interpolation. In Section 6 we present a new method for L^2-ensemble controllability of a string of harmonic oscillators and in Section 7 we show that a family of networks of harmonic oscillators can be controlled robustly by a single input function. In the subsequent Section 8 we show that our approach also applies to the controlled heat equation considered by Bamieh et al. (2002). The paper closes by conclusions.

2. PROBLEM FORMULATION AND MAIN RESULT

In this paper we consider parameter-dependent linear time-invariant systems of the form

$$\begin{aligned} \tfrac{\partial}{\partial t}x(t,\theta) &= A(\theta)x(t,\theta) + b(\theta)u(t) \\ x(0,\theta) &= 0, \end{aligned} \tag{1}$$

where $A(\theta) \in \mathbb{R}^{n\times n}$ and $b(\theta) \in \mathbb{R}^n$ are assumed to depend continuously on a single parameter $\theta \in \mathbf{P} := [\theta^-, \theta^+]$. Let $x(T,\theta)$ denote the solution to (1) at time $T > 0$. In the following, let $x^*: \mathbf{P} \to \mathbb{R}^n$ denote a given family of desired terminal states. For uniform ensemble controllability it is natural to assume that $x^*(\theta)$ depends continuously on the parameter. Thus, we assume in the following that x^* is continuous. Our aim is to investigate the existence of an open-loop input function $u\colon [0,T] \to \mathbb{R}$ that steers the initial state $x(0,\theta) = 0$ in time T into an ε-neighborhood of the desired terminal state $x^*(\theta)$, simultaneously for all parameters $\theta \in \mathbf{P}$. This is in principle what we call uniform ensemble controllability. The precise definition is as follows.

Definition 1. System (1) is called *uniformly ensemble controllable* if for any $T > 0$ and for any $x^* \in C(\mathbf{P}, \mathbb{R}^n)$ there exists an input function $u\colon [0,T] \to \mathbb{R}$ such that for any $\varepsilon > 0$ it holds

$$\sup_{\theta\in\mathbf{P}} \|x(T,\theta) - x_*(\theta)\| < \varepsilon. \tag{2}$$

Alternatively one may also consider input functions u that minimize the L^2-norm

$$\left(\int_{\mathbf{P}} \|x(T,\theta) - x_*(\theta)\|^2 d\theta\right)^{\frac{1}{2}} < \varepsilon. \tag{3}$$

In this case, system (1) is called L^2*-ensemble controllable.* Obviously, uniform ensemble controllability is stronger than L^2-ensemble controllability. If the conditions in (2) or (3) hold for $\varepsilon = 0$, then the system is called exactly ensemble controllable. We note that there is a difference between the continuous-time and the discrete-time case. In the continuous-time case it is possible to steer the zero initial state to the terminal states in any time $T > 0$ by choosing piecewise constant input functions taking appropriate many values. The number of values for the input function depends on the approximation error. For discrete-time systems the number of input values is directly determined by the approximation error and therefore T cannot be chosen arbitrarily.

Certainly, a necessary condition for uniform ensemble controllability is that for each parameter $\theta \in \mathbf{P}$ the pair $(A(\theta), b(\theta))$ is controllable. However, this condition is not sufficient and needs to be complemented by additional assumptions. The following main result gives sufficient conditions for uniform ensemble controllability. The first and the second condition are also necessary for uniform ensemble controllability.

Theorem 1. (Main Theorem: Controllability). *Let $\boldsymbol{P} = [\theta^-, \theta^+]$ be a compact interval in $\mathbb{R}$. A continuous family $(A(\theta), b(\theta))$ of linear systems is uniformly ensemble controllable in arbitrary time T (more generally, L^2-ensemble controllable) provided the following conditions are satisfied:*

(i) $(A(\theta), b(\theta))$ is controllable for all $\theta \in \boldsymbol{P}$.

(ii) For any pair of distinct parameters $\theta, \theta' \in \boldsymbol{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$

(iii) For each $\theta \in \boldsymbol{P}$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.

In Helmke and Schönlein (2013) this result has been shown for multi-input time-invariant linear systems in discrete-time as well as continuous-time. However, the main step in the proof for continuous-time systems consists in a reduction to the case of single-input discrete-time systems via sampling arguments. Due to the presence of these sampling arguments, the construction of ensemble controls for continuous-time systems becomes a bit roundabout and complicated. Thus in this paper, we present a direct approach to the continuous-time case in order to obtain more efficient algorithms.

3. ENSEMBLE CONTROLLABILITY VERSUS APPROXIMATE CONTROLLABILITY

Throughout this section we assume that $\mathbf{P}$ is a compact interval in $\mathbb{R}$. For $\theta \in \mathbf{P}$ we consider the ensemble-control task for continuous-time systems

$$\frac{\partial}{\partial t}x(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \quad x(0,\theta) = 0. \tag{4}$$

We assume continuity of $\theta \mapsto (A(\theta), B(\theta))$, although weaker assumptions could be made. The input to state-operator of (4) at time T is given as $\mathcal{R}_T : L^2([0,T], \mathbb{R}^m) \to L^2(\mathbf{P}, \mathbb{R}^n)$,

$$\mathcal{R}_T(u) = \int_{\mathbf{P}} e^{(T-s)A(\theta)} B(\theta)u(s)\, \mathrm{d}\theta. \tag{5}$$

Note that $\mathcal{R}_T$ is an integral operator with continuous kernel $K : \mathbf{P} \times [0,T] \to \mathbb{R}^n$ as

$$K(\theta, s) = e^{(T-s)A(\theta)} B(\theta). \tag{6}$$

It is a well-known consequence of the continuity of $A(\cdot), B(\cdot)$ on the compact interval $\mathbf{P}$, that $\mathcal{R}_T$ defines a bounded linear operator. In fact, $\mathcal{R}_T$ is a compact operator. Moreover, by continuity of $A(\cdot), B(\cdot)$, the operator $\mathcal{R}_T$ is even compact as an operator from $L^2([0,T])$ to $C(\mathbf{P}, \mathbb{R}^n)$. Further, compactness of $\mathcal{R}_T$ implies that $\mathcal{R}_T$ has a closed image if and only if $\mathcal{R}_T$ has finite rank. If $K(\theta, s)$ would be of the form $\sum_{j=1}^k \phi_j(s)\psi_j(\theta)$, then $\mathcal{R}_T$ has finite rank. Of course, this holds if (A, B) is parameter independent; however for truly parameter dependent systems this cannot be expected. We conclude

Proposition 2. The system (4) is not exactly uniformly ensemble controllable. Uniform ensemble controllability holds if and only if the image of $\mathcal{R}_T$ is dense in $C(\boldsymbol{P}, \mathbb{R}^n)$. Equivalently, uniform ensemble controllability holds if and only if the kernel of the dual operator $\mathcal{R}_T^*$ is trivial.

Proof. Note that the range of the input-operator B is finite-dimensional. Triggiani (1975) has shown that, under this condition, the input-to-state-operator $\mathcal{R}_T$: $L^1([0,T], \mathbb{R}^m) \to X$ is never surjective, for any separable Banach space X. Thus the first claim follows, since $L^2([0,T], \mathbb{R}^m) \subset L^1([0,T], \mathbb{R}^m)$ and since $X = C(\mathbf{P}, \mathbb{R}^n)$ is a separable Banach space. The second claim is obvious. The third follows from the Hahn-Banach theorem together with the fact, that the annihilator of the image space of $\mathcal{R}_T$ in $C(\mathbf{P}, \mathbb{R}^n)$ coincides with the kernel of $\mathcal{R}_T^*$. □

The above result shows that exact (uniform) ensemble controllability is an unrealistic task for linear parameter-dependent systems. We therefore focus on the notion of L^2-ensemble controllability, i.e. on approximate controllability. In a Hilbert space setting explicit characterizations for approximate controllability are well-known; see e.g. Fuhrmann (1972), Curtain and Zwart (1995). Thus consider the input-to-state operator as a (compact) linear operator $\mathcal{R}_T : L^2([0,T], \mathbb{R}^m) \to L^2(\mathbf{P}, \mathbb{R}^n)$. The adjoint map then is the compact linear operator

$$\mathcal{R}_T^* : L^2(\mathbf{P}, \mathbb{R}^n) \to L^2([0,T], \mathbb{R})$$
$$(\mathcal{R}_T^* x)(s) = \int_{\mathbf{P}} B(\theta)^\top e^{(T-s)A(\theta)^\top} x(\theta)\, \mathrm{d}\theta, \;\; s \in [0,T].$$

The controllability gramian of (4) is the compact operator $\mathcal{L}_T = \mathcal{R}_T \circ \mathcal{R}_T^* \colon L^2(\mathbf{P}, \mathbb{R}^n) \to L^2(\mathbf{P}, \mathbb{R}^n)$ defined by

$$(\mathcal{L}_T x)(\theta') = \int_{\mathbf{P}} \left(\int_0^T e^{sA(\theta')} B(\theta') B(\theta)^\top e^{sA(\theta)^\top} \mathrm{d}s \right) x(\theta)\, \mathrm{d}\theta.$$

Since $\mathcal{R}_T : L^2([0,T], \mathbb{R}^m) \to L^2(\mathbf{P}, \mathbb{R}^n)$ is a compact operator between Hilbert spaces, it admits a singular value decomposition. That is, there exist orthonormal families of basis vectors $(e_n)_n$ and $(f_n)_n$ in $L^2([0,T], \mathbb{R}^m)$ and $L^2(\mathbf{P}, \mathbb{R}^n)$, respectively, and a decreasing sequence of scalars $s_i \geq 0$ such that for any $u \in L^2([0,T], \mathbb{R}^m)$ we have

$$\mathcal{R}_T u = \sum_{n=0}^{\infty} s_n \langle u, e_n \rangle f_n \tag{7}$$
$$\mathcal{R}_T^* v = \sum_{n=0}^{\infty} s_n \langle v, f_n \rangle e_n. \tag{8}$$

It follows, that the singular values are all positive if and only if $\mathcal{R}_T^* \circ \mathcal{R}_T$ is positive definite.

Li (2011) has studied the problem of L^2-ensemble controllability for one-parameter families of time-varying linear systems. Using the singular value decomposition (7), we derive the following characterization by Li (2011) of the image elements of the reachability operator $\mathcal{R}_T$.

Theorem 3. (Li (2011)). *Let $S := \{i \in \mathbb{N}_0 | s_i > 0\}$ denote the set of non-zero singular values of $\mathcal{R}_T$.*

(1) An element $x = \sum_{i \in S} \alpha_i f_i$ is in the image of $\mathcal{R}_T$ if and only if

$$\sum_{i \in S} |\alpha_i|^2 < \infty \quad \text{and} \sum_{\mathrm{i} \in \mathrm{S}} \frac{|\alpha_\mathrm{i}|^2}{\mathrm{s}_\mathrm{i}^2} < \infty.$$

In that case, a solution of $\mathcal{R}_T u = x$ is

$$u = \sum_{i \in S} \frac{\alpha_i}{s_i} e_i.$$

(2) An element $x \in L^2(\boldsymbol{P}, \mathbb{R}^n)$ is in the closure $\overline{\operatorname{Im} \mathcal{R}_T}$ if and only $x = \sum_{i \in S} \alpha_i f_i$ with

$$\sum_{i \in S} |\alpha_i|^2 < \infty. \tag{9}$$

Proof. Since the discussion in Li (2011) is not very comprehensible, we include the short proof. If $x = \mathcal{R}_T u$ for $u \in L^2([0,T], \mathbb{R}^m)$, then

$$x = \sum_{i \in S} \alpha_i f_i = \sum_{i \in S} s_i \langle u, e_i \rangle f_i = \mathcal{R}_T u.$$

This implies $\alpha_i = s_i \langle u, e_i \rangle$ for all $i \in S$. Since $x \in L^2(\mathbf{P}, \mathbb{R}^n)$ we have $\sum_{i \in S} |\alpha_i|^2 < \infty$ and from the above identity we conclude

$$\sum_{i \in S} \frac{|\alpha_i|^2}{s_i^2} = \sum_{i \in S} |\langle u, e_i \rangle|^2 = \|u\|^2 < \infty.$$

Conversely, if condition (9) holds, then $u := \sum_{i \in S} \frac{\alpha_i}{s_i} e_i$ is in $L^2([0,T], \mathbb{R}^m)$ and satisfies $\mathcal{R}_T u = x$. □

As noted above, the system (4) is never exactly L^2-ensemble controllable. However, the situation is much better for approximate ensemble controllability. In order to characterize approximate L^2-ensemble controllability, we use the following equivalent characterizations for approximate controllability in a Hilbert space; see e.g. Theorem 4.1.7 (b) in Curtain and Zwart (1995).

Theorem 4. (Curtain and Zwart (1995)). *The ensemble $(A(\theta), B(\theta))$ is approximately controllable on $[0,T]$ if and only if one of the following conditions hold:*

(i) $\mathcal{L}_T > 0$.
(ii) $\operatorname{Ker} \mathcal{R}_{\mathrm{T}}^* = \{0\}$.
(iii) $\mathcal{R}_T z = 0$ *on* $[0,T]$ *implies* $z = 0$.

Then we can show

Theorem 5. The following conditions are equivalent:

(i) System (4) is L^2-ensemble controllable.
(ii) $\overline{\operatorname{span}\{f_n | s_n > 0\}} = L^2(\boldsymbol{P}, \mathbb{R}^n)$.
(iii) $\overline{\operatorname{span}\{A(\theta)^k b_j(\theta) | k \in \mathbb{N}_0,\, j = 1, ..., m\}} = L^2(\boldsymbol{P}, \mathbb{R}^n)$.
(iv) $\int_{\boldsymbol{P}} x(\theta)^\top A(\theta)^k B(\theta)\, \mathrm{d}\theta = 0 \quad \forall k \in \mathbb{N}_0 \implies x(\cdot) = 0.$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is implicitly contained in Li (2011). Explicitly, $\mathcal{L}_T > 0$ holds if and only if the singular vectors f_n with $s_n > 0$ form a basis of $L^2(\mathbf{P}, \mathbb{R}^n)$. The equivalence $(iii) \Leftrightarrow (iv)$ is obvious and $(i) \Leftrightarrow (iv)$ follows immediately from the equivalence of (iv) with

$$\int_{\mathbf{P}} x(\theta)^\top e^{sA(\theta)} B(\theta)\, \mathrm{d}\theta = 0 \quad \forall s \in [0,T] \implies x(\cdot) = 0.$$

□

Although the above results are precise, they are not very useful in practice. Of course, one can immediately conclude certain necessary conditions for ensemble controllability, e.g. that each individual system $(A(\theta), B(\theta)), \theta \in \mathbf{P}$, is controllable. In the next section we verify the concrete sufficient conditions for ensemble controllability stated in Theorem 1.

4. PROOF OF THEOREM 1

In order to prove Theorem 1 we use a result from complex approximation, namely Mergelyan's Theorem, cf. Gaier (1987) Theorem 1 in Chapter II 2.

Theorem 6. (Mergelyan). *Suppose K is compact in $\mathbb{C}$ and the complement $\mathbb{C} \setminus K$ is connected; suppose further f is continuous on K and analytic in the interior of K. Then, for any $\varepsilon > 0$ there exists a polynomial p such that*

$$|f(z) - p(z)| < \varepsilon$$

for all $z \in K$.

It should be noted that this theorem includes the special case where the interior of K is empty. Recall that we consider parameter-dependent linear systems of the form

$$\begin{aligned} \dot{x}(t) := \tfrac{\partial}{\partial t}x(t,\theta) = A(\theta)x(t,\theta) + b(\theta)u(t) \\ x(0,\theta) = 0, \end{aligned} \tag{10}$$

where the parameter space is assumed to be a nonempty compact interval $[\theta^-, \theta^+] =: \mathbf{P}$ and the initial state is equal to zero for all parameter values $\theta \in \mathbf{P}$. The family of terminal states is determined by the continuous function $x^*\colon \mathbf{P} \to \mathbb{R}^n$. Let $T > 0$ be fixed.

Given the control input function u, since $x(0,\theta) = 0$, the solution to (10) at time T is

$$x(T,\theta) = \int_0^T \mathrm{e}^{sA(\theta)}\, b(\theta)u(T-s)\,\mathrm{d}s.$$

As the function u is scalar the solution to (10) can equivalently be written as

$$x(T,\theta) = \int_0^T \mathrm{e}^{sA(\theta)}\, u(T-s)\,\mathrm{d}s\; b(\theta). \tag{11}$$

According to the assumptions on reachability and since $A(\theta)$ has n distinct eigenvalues, the family (10) of system can be transformed to a family of single-input systems of the form

$$\dot{x}(t) = \begin{bmatrix} \lambda_1(\theta) & & \\ & \ddots & \\ & & \lambda_n(\theta) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \tag{12}$$

with continuous and injective functions $\lambda_1, \cdots, \lambda_n \colon \mathbf{P} \to \mathbb{C}$, cf. Chapter II, § 5, 3 in Kato (1995). Then, for

$$A(\theta) = \begin{bmatrix} \lambda_1(\theta) & & \\ & \ddots & \\ & & \lambda_n(\theta) \end{bmatrix} \quad \text{and} \quad b(\theta) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

the solution to (12) with input function u is given by

$$x(T,\theta) = \int_0^T \begin{bmatrix} \mathrm{e}^{s\lambda_1(\theta)}\, u(T-s) \\ \vdots \\ \mathrm{e}^{s\lambda_n(\theta)}\, u(T-s) \end{bmatrix} \mathrm{d}s. \tag{13}$$

Note that, since we applied a coordinate transformation to diagonalize the system matrix $A(\theta)$, the terminal state $x^*(\theta)$ has to be transformed accordingly and might become complex in general. For simplicity, we do not introduce a new notation for the transformed terminal states. Hence, the task is to determine for any given terminal state $x_*(\theta) \in C(\mathbf{P}, \mathbb{C}^n)$ and any $\varepsilon > 0$ an input function u so that for some $T > 0$ it holds that

$$\sup_{\theta\in\mathbf{P}} \|x^*(\theta) - x(T,\theta)\| < \varepsilon.$$

We pick a real $\tau > 0$ and $N \in \mathbb{N}$ so that the mappings $\theta \mapsto \mathrm{e}^{\tau\lambda_j(\theta)}$ are injective for all $j = 1, ..., n$ and $T = \tau N$. Furthermore, we consider input functions that are piecewise constant. Let the interval $[0, T]$ be partitioned equidistantly

$$[0,T] = \bigcup_{k=0}^{N-1} I_k,$$

where $I_k = [k\tau, (k+1)\tau)$ for $k = 0, ..., N-2$ and $I_{N-1} = [(N-1)\tau, N\tau]$. Moreover, let $\mathbb{1}_{I_k}$ denote the indicator function defined by

$$s \mapsto \mathbb{1}_{I_k}(s) = \begin{cases} 1 & \text{if } s \in I_k \\ 0 & \text{else }. \end{cases}$$

We pick a piecewise constant input function u of the form

$$u(T-s) = \sum_{k=0}^{N-1} u_k\, \mathbb{1}_{I_k}(T-s) = \sum_{k=0}^{N-1} u_{N-1-k}\, \mathbb{1}_{I_k}(s)$$

for some real coefficients $u_0, ..., u_{N-1}$. Then, the jth component of the solution (13) for $j = 1, ..., n$ is

$$\begin{aligned} x_j(T,\theta) &= \int_0^T \mathrm{e}^{s\lambda_j(\theta)}\, u(T-s)\,\mathrm{d}s \\ &= \sum_{k=0}^{N-1} \int_{k\tau}^{(k+1)\tau} \mathrm{e}^{s\lambda_j(\theta)}\, u_{N-1-k}\mathbb{1}_{I_k}(s)\,\mathrm{d}s \\ &= \sum_{k=0}^{N-1} \frac{u_{N-1-k}}{\lambda_j(\theta)}\, \mathrm{e}^{k\tau\lambda_j(\theta)}\,(\mathrm{e}^{\tau\lambda_j(\theta)} - 1) \\ &= \sum_{k=0}^{N-1} \tau u_{N-1-k}\, \mathrm{e}^{k\tau\lambda_j(\theta)} \left(\frac{\mathrm{e}^{\tau\lambda_j(\theta)} - 1}{\tau\lambda_j(\theta)}\right) \end{aligned}$$

For τ sufficiently small it holds that

$$\left|\frac{\mathrm{e}^{\tau\lambda_j(\theta)} - 1}{\tau\lambda_j(\theta)} - 1\right| < \tfrac{\varepsilon}{2} \tag{14}$$

for any $\lambda_j(\theta)$. For $z \in \mathbb{C}$ let

$$p(z) := \sum_{k=0}^{N-1} u_{N-1-k}\, z^k. \tag{15}$$

Then, the jth component of the solution to (12) reads as

$$x_j(T,\theta) = \tau \left(\frac{\mathrm{e}^{\tau\lambda_j(\theta)} - 1}{\tau\lambda_j(\theta)}\right) p\left(\mathrm{e}^{\tau\lambda_j(\theta)}\right)$$

By (14), independently of the coefficients u_k, it holds that

$$|\tau p(\mathrm{e}^{\tau\lambda_j(\theta)}) - x_j(T,\theta)| < \tfrac{\varepsilon}{2}. \tag{16}$$

For $j \in \{1, ..., n\}$ let $\Lambda_j := \{\mathrm{e}^{\tau\lambda_j(\theta)} : \theta \in \mathbf{P}\} \subset \mathbb{C}$ and $\Lambda = \bigcup_{j=1}^n \Lambda_j$. Note that, by the assumption of Theorem 1 and by the choice of τ, it holds that Λ is compact with empty interior and $\mathbb{C} \setminus \Lambda$ is connected. Moreover, we consider the continuous function $f\colon \Lambda \to \mathbb{C}$ defined by

$$f|_{\Lambda_j}(z) = x_j^*\left(\lambda_j^{-1}\left(\tfrac{\ln z}{\tau}\right)\right).$$

By Mergelyan's theorem there exists a polynomial p so that

$$|f(z) - p(z)| < \tfrac{\varepsilon}{2} \tag{17}$$

for all $z \in \Lambda$. Note that p is a polynomial with complex coefficients in general. Let the degree of the polynomial p be denoted by $N_p - 1 \in \mathbb{N}$. In order to obtain a polynomial which has real coefficients, we consider the polynomial

$$q(z) := \tfrac{1}{2}\left(p(z) + \overline{p}(z)\right).$$

Then, q defines a real polynomial of degree $N_p - 1 \in \mathbb{N}$. Let $\tau = T/N_p$. So choosing the coefficients in (15) as in q, then the conditions (16) and (17) imply that for any $\theta \in \mathbf{P}$ and for any $j \in \{1, ..., n\}$ it holds that

$$\begin{aligned} &|x_j^*(\theta) - x_j(T,\theta)| \\ &\quad \le |x_j^*(\theta) - \tau q(\mathrm{e}^{\tau\lambda_j(\theta)})| + |\tau q(\mathrm{e}^{\tau\lambda_j(\theta)}) - x_j(T,\theta)| < \varepsilon. \end{aligned}$$

Consequently, given $x^*(\theta)$, $T > 0$ and $\varepsilon > 0$ there exist real coefficients $u_0, ..., u_{N_p-1}$ such that

$$\sup_{\theta\in\mathbf{P}} \|x^*(\theta) - x(T,\theta)\| < \varepsilon.$$

□

Remark 1. Note that the proof of Theorem 1 is not constructive. Hence, the problem of finding the coefficients $u_0, ..., u_{N-1}$ is not covered by our approach. The derivation of effective algorithms for the calculation of the approximating polynomial is the content of future research and requires tools from constructive function theory.

For certain cases there are simple approaches to determine the polynomial. For instance, Helmke and Schönlein (2013) consider an ensemble of discrete-time harmonic oscillators. In this example it is possible to use Bernstein polynomials to approximate any continuous family of terminal states uniformly. Moreover, dependent on the regularity of the family of terminal states with respect to the parameter there are error bounds available in the literature on approximation theory, cf. DeVore and Lorentz (1993).

5. UNIFORM ENSEMBLE CONTROL OF HARMONIC OSCILLATORS

In this section we present a construction procedure for parameter-independent controls that steer an ensemble of harmonic oscillators to a family of target states. We consider a family of harmonic oscillators with frequencies θ varying over the compact interval $\mathbf{P} = [\theta^-, \theta^+] \subset (0, \infty)$. That is, the ensemble of systems under consideration is

$$\tfrac{\partial}{\partial t} x(t,\theta) = A(\theta)x(t,\theta) + b(\theta)u(t) \tag{18}$$

with

$$A(\theta) := \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}, \quad b(\theta) := \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{19}$$

The family of terminal states is defined by the continuous function $x^* \colon \mathbf{P} \to \mathbb{R}^2$. Using the state space transformation

$$S := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \tag{20}$$

the system can be transformed to

$$\dot{x}(t) = \begin{bmatrix} -i\theta & 0 \\ 0 & i\theta \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t). \tag{21}$$

To describe the transformation of the terminal states let $z^*(\theta) := x_1^*(\theta) + i\, x_2^*(\theta) \in \mathbb{C}$. Then, we have

$$Sx^*(\theta) = \begin{bmatrix} x_1^*(\theta) - i\, x_2^*(\theta) \\ x_1^*(\theta) + i\, x_2^*(\theta) \end{bmatrix} = \begin{bmatrix} \overline{z^*(\theta)} \\ z^*(\theta) \end{bmatrix}.$$

Given the input function u the solution to (21) at time $T > 0$ is given by

$$x(T,\theta) = \int_0^T \begin{bmatrix} \mathrm{e}^{-si\theta}\, u(T-s) \\ \mathrm{e}^{si\theta}\, u(T-s) \end{bmatrix} \mathrm{d}s. \tag{22}$$

Similar to Section 2 we consider piecewise constant input functions. So we pick some $\tau < \frac{\pi}{2}$ as then the mappings $\theta \mapsto \mathrm{e}^{i\theta}$ and $\theta \mapsto \mathrm{e}^{-i\theta}$ are injective. Further, given $N \in \mathbb{N}$ define $T := \tau N$ and let the interval $[0, T]$ be partitioned as in Section 4. Then, we consider input functions of the form

$$\begin{aligned} u(T-s) &= \sum_{k=0}^{N-1} u_k\, \mathbb{1}_{I_k}(T-s) \\ &= \sum_{k=0}^{N-1} u_{N-1-k}\, \mathbb{1}_{I_k}(s) \end{aligned}$$

and following the proof of Theorem 1 the task is to determine the coefficients $u_0, ..., u_{N-1}$ such that for a given $\varepsilon > 0$ it holds that

$$x_j(T,\theta) = \left(\frac{\mathrm{e}^{(-1)^j i\tau\theta} - 1}{(-1)^j i\theta} \right) p\left(\mathrm{e}^{(-1)^j i\tau\theta} \right)$$

for $j = 1, 2$ satisfy

$$\sup_{\theta \in \mathbf{P}} \| Sx^*(\theta) - x(T,\theta) \| < \varepsilon.$$

Based on the specific structure of the spectrum of $A(\theta)$ we pick sampling points $\theta^- \le \theta_1 < \cdots < \theta_K \le \theta^+$ in the interval $\mathbf{P}$ and determine the polynomial

$$p(z) = \sum_{k=0}^{N-1} u_{N-1-k} z^k$$

satisfying the following $N = 2K$ conditions

$$\begin{aligned} p(\mathrm{e}^{-i\tau\theta_l}) &= \left(\frac{-i\theta_l}{\mathrm{e}^{-i\tau\theta_l} - 1} \right) \overline{z^*(\theta_l)} \\ p(\mathrm{e}^{i\tau\theta_l}) &= \left(\frac{i\theta_l}{\mathrm{e}^{i\tau\theta_l} - 1} \right) z^*(\theta_l) \end{aligned} \tag{23}$$

for $l = 1, ..., K$. Note that (23) defines a Lagrange interpolation problem, which has a unique solution p. To see that p has real coefficients, observe that

$$\begin{aligned} \overline{p(\mathrm{e}^{-i\tau\theta})} &= \overline{\left(\frac{-i\theta}{\mathrm{e}^{-i\tau\theta} - 1} \right)}\, \overline{\overline{z^*(\theta)}} \\ &= \left(\frac{i\theta}{\mathrm{e}^{i\tau\theta} - 1} \right) z^*(\theta) = p(\mathrm{e}^{i\tau\theta}), \end{aligned}$$

which implies that

$$\begin{aligned} \overline{\sum_{k=0}^{N-1} u_{N-1-k} (\mathrm{e}^{-i\tau\theta})^k} &= \sum_{k=0}^{N-1} \overline{u_{N-1-k}} (\mathrm{e}^{i\tau\theta})^k \\ &= \sum_{k=0}^{N-1} u_{N-1-k} (\mathrm{e}^{i\tau\theta})^k. \end{aligned}$$

To illustrate the procedure we consider the parameter interval $\mathbf{P} = [1, 2]$. The family of target states is supposed to be

$$x^*(\theta) := \begin{bmatrix} x_1^*(\theta) \\ x_2^*(\theta) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+(2\theta-3)^2} \\ 0 \end{bmatrix}. \tag{24}$$

The sampling points in the parameter space are chosen as Chebyshev points, i.e.

$$\theta_l = 1 + \tfrac{1}{2}\Big(\cos\big(\tfrac{2l-1}{2K}\pi \big) + 1 \Big)$$

for $l = 1, ..., K$. The Figures 1 and 2 illustrate the approach for $K = 10$ and $\tau = 1$.

6. L^2-ENSEMBLE CONTROL OF HARMONIC OSCILLATORS

In this section we consider the problem of L^2-ensemble controllability for an ensemble of harmonic oscillators. We follow the notation given in Section 5. The family of terminal states is defined by the continuous function $x^* \colon \mathbf{P} \to \mathbb{R}^2$. Let $T > 0$ be fixed and $\tau > 0$ and $N \in \mathbb{N}$ be such that $T = \tau N$. The input function u is chosen piecewise constant, i.e.

$$u(T-s) = \sum_{k=0}^{N-1} u_{N-1-k}\, \mathbb{1}_{I_k}(s)$$

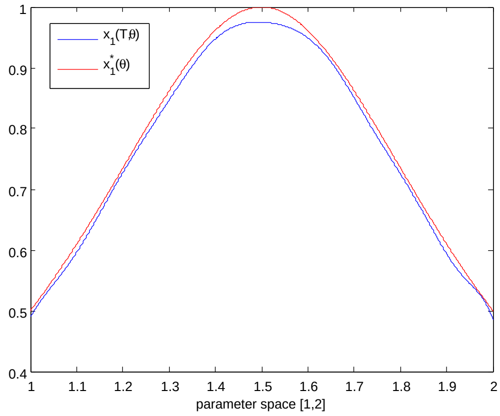

Fig. 1. This figure shows the approximation of the first component of the terminal state $x_1^*(\theta) = \frac{1}{1+(2\theta-3)^2}$, where the input u is defined by the coefficients of the polynomial given by Lagrange interpolation (23).

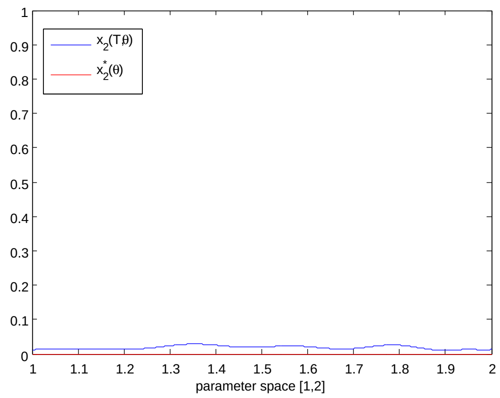

Fig. 2. This figure shows the approximation of the second component of the terminal state $x_1^*(\theta) = 0$, where the input u is defined by the coefficients of the polynomial given by Lagrange interpolation (23).

with real coefficients $u_0, ..., u_{N-1}$. The solution to (18) at time T is the then

$$x(T,\theta) = \int_0^T \mathrm{e}^{sA(\theta)}\, b(\theta) u(T-s)\,\mathrm{d}s = \sum_{k=0}^{N-1} \tau u_{N-1-k}\, \mathrm{e}^{k\tau A(\theta)} (\tau A(\theta))^{-1} (\mathrm{e}^{\tau A(\theta)} - I) b(\theta). \tag{25}$$

For brevity we define

$$p(z) := \sum_{k=0}^{N-1} \tau u_{N-1-k} z^k$$

and

$$\bar{b}(\theta) := (\tau A(\theta))^{-1} (\mathrm{e}^{\tau A(\theta)} - I) b(\theta).$$

The problem of L^2-ensemble controllability is to find coefficients so that the piecewise constant input function u minimizes

$$\|x^* - x(T)\|^2_{L^2(\mathbf{P})} = \int_{\mathbf{P}} \|x^*(\theta) - x(T,\theta)\|^2\,\mathrm{d}\theta.$$

In the following we determine the coefficients $u_0, ..., u_{N-1}$ using least squares approximation, cf. Davis (1963). That is, we consider the problem

$$\min_{u_0,...,u_{N-1}} \int_{\mathbf{P}} \|x^*(\theta) - \sum_{k=0}^{N-1} \tau u_{N-1-k}\, \mathrm{e}^{k\tau A(\theta)}\, \bar{b}(\theta)\|^2\,\mathrm{d}\theta. \tag{26}$$

A necessary condition is that for $l = 0, ..., N-1$ it holds that

$$0 = \frac{\partial}{\partial u_l} \int_{\mathbf{P}} \|x^*(\theta) - \sum_{k=0}^{N-1} \tau u_{N-1-k}\, \mathrm{e}^{k\tau A(\theta)}\, \bar{b}(\theta)\|^2\,\mathrm{d}\theta \tag{27}$$

for $l \in \{0, ..., N-1\}$. Furthermore, using

$$\bar{c}_l := \int_{\mathbf{P}} x^*(\theta)^{\mathsf{T}}\, \mathrm{e}^{l\tau A(\theta)}\, \bar{b}(\theta)\,\mathrm{d}\theta$$

and

$$\bar{M}_{kl} := \int_{\mathbf{P}} \bar{b}(\theta)^{\mathsf{T}} {\mathrm{e}^{k\tau A(\theta)}}^{\mathsf{T}}\, \mathrm{e}^{l\tau A(\theta)}\, \bar{b}(\theta)\,\mathrm{d}\theta = \int_{\mathbf{P}} \bar{b}(\theta)^{\mathsf{T}}\, \mathrm{e}^{(l-k)\tau A(\theta)}\, \bar{b}(\theta)\,\mathrm{d}\theta$$

the condition (27) yields the following system of N linear equations

$$\bar{M} u = \bar{c}. \tag{28}$$

Then, the coefficients $u = (u_{N-1} \cdots u_0)^{\mathsf{T}}$ satisfying (26) are given by the solution to (28). To compute the variables recall that the matrix exponential for $l\tau A(\theta)$ is given by

$$\mathrm{e}^{l\tau A(\theta)} = \begin{bmatrix} \cos(l\tau\theta) & -\sin(l\tau\theta) \\ \sin(l\tau\theta) & \cos(\tau\theta) \end{bmatrix}.$$

Using (19) we have

$$\bar{b}(\theta) = \tfrac{1}{\tau\theta} \begin{bmatrix} \sin(\tau\theta) \\ \cos(\tau\theta) \end{bmatrix}.$$

Note that $\bar{b}(\theta) \to b(\theta)$ as $\tau \to 0$. Thus, by choosing τ sufficiently small one might consider approximately $b(\theta)$ instead of $\bar{b}(\theta)$. For the terminal states given in (24) the variables defining the system of linear equations are

$$\bar{c}_l = \int_{\mathbf{P}} \frac{2\sin(\frac{\tau\theta}{2})\cos(\frac{(2l+1)\tau\theta}{2})}{\tau\theta\,(1+(2\theta-3)^2)}\,\mathrm{d}\theta$$

and

$$\bar{M}_{kl} = \int_{\mathbf{P}} \left(\frac{1}{\frac{\tau\theta}{2}} \sin\left(\tfrac{\tau\theta}{2}\right) \right)^2 \cos\left((l-k)\tau\theta\right)\,\mathrm{d}\theta.$$

Moreover, by using $b(\theta)$ approximately instead of $\bar{b}(\theta)$ the approximate variables c_l and M_{lk} are

$$c_l = \int_{\mathbf{P}} \frac{\cos(l\tau\theta)}{(1+(2\theta-3)^2)}\,\mathrm{d}\theta$$

and

$$M_{kl} = \begin{cases} \frac{1}{l-k}\left(\sin\left((l-k)\theta^+\right) - \sin\left((l-k)\theta^-\right)\right) & \text{if } k \neq l \\ \theta^+ - \theta^- & \text{if } k = l, \end{cases}$$

respectively.

7. APPLICATION TO NETWORKS OF HARMONIC OSCILLATORS

In this section we consider networks of N identical single-input-single-output (SISO) harmonic oscillators (A, b, c) defined by

$$A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{29}$$

The interconnection of the harmonic oscillators is described by an undirected weighted connected graph and an input interconnection vector $g \in \mathbb{R}^N$. To describe the graph structure we consider a feasible set of Laplacians. To this end, let $0 < l^- \leq l^+$ be some real numbers and

$$\mathcal{L}_g = \{L \in \mathbb{R}^{N\times N} : L = (l_{ij}) \text{ Laplacian}, \\ (L, g) \text{ controllable }, l^- \leq l_{ij} \leq l^+ \;\forall i \neq j\}.$$

$$\lambda^- := \min\{\lambda \neq 0 \mid \exists L' \in \mathcal{L}_g \;:\; \lambda \in \sigma(L')\}$$

and

$$\lambda^+ := \max_{\lambda \in \mathcal{L}_g} \lambda_{\max}(L)$$

we have $0 < \lambda^- \leq \lambda^+$.

We consider networks of interconnected linear systems, where Laplacians of the underlying graph vary over a compact set

$$\mathcal{L} = \{L = -\mu I + L' \in \mathbb{R}^{N\times N} : L' \in \mathcal{L}_g\}$$

for any real number μ satisfying $\mu > \lambda^+$. Thus, we consider the family of networks given by

$$\begin{aligned} \dot{x}(t) &= \big(I \otimes A + L \otimes bc\big)x(t) + \big(g \otimes b\big)u(t) \\ x(0) &= 0, \end{aligned} \tag{30}$$

where $L \in \mathcal{L}$. Note that (30) describes the general form of networks of identical oscillators with state-interconnection matrix L and input-to-state interconnection matrix g. Thus, the structure of (30) is fairly general. Moreover, by $x(T, L)$ we denote the solution to (30) at time T. Let e_k denote the kth standard basis vector on $\mathbb{R}^N$ and let $\mathbf{e} = \sum_{k=1}^N e_k$.

Theorem 7. Consider the network defined in (30)*. For any* $\xi \in \mathbb{R}^2$, $\varepsilon > 0$ *and* $T > 0$ *there exists an input function* $u\colon [0, T] \to \mathbb{R}$ *such that*

$$\sup_{L\in\mathcal{L}} \|x(T, L) - (\mathbf{e} \otimes \xi)\| < \varepsilon.$$

Thus, u serves as an universal input for a class of networks that steers the zero state to a desired synchronized state $e \otimes \xi$ in finite time T uniformly for all interconnection matrices $L \in \mathcal{L}$. It seems surprising that a single input function exists that robustly synchronizes states for the whole family of networks $\mathcal{L}$. The proof of the theorem rests critically on our main result about uniform ensemble controllability.

Proof. The assumptions on $\mathcal{L}$ imply that

$$\sigma(\mathcal{L}) := \bigcup_{L\in\mathcal{L}} \sigma(L) \subset [-\mu, -\mu + \lambda^+] \subset (-\infty, 0).$$

Notice that $0 \notin \sigma(\mathcal{L})$. We pick some $L \in \mathcal{L}$. Then, there is a change of coordinates $S \in \mathrm{GL}(N)$ so that $Sg = \mathbf{e}$ and using the transformed state $\overline{x} := (S \otimes I)x$, the network can be written as

$$\dot{\overline{x}}(t) = \begin{bmatrix} \begin{bmatrix} 0 & \lambda_1 \\ 1 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & \lambda_2 \\ 1 & 0 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} 0 & \lambda_N \\ 1 & 0 \end{bmatrix} \end{bmatrix} \overline{x}(t) + \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} u(t),$$

with $-(\mu + 1) \leq \lambda_1 < \lambda_2 < \cdots < \lambda_N \leq -(\mu + 1) + \lambda^+$ Since $(S \otimes I)(\mathbf{e} \otimes \xi) = e_1 \otimes \xi$ the task is to determine an input u so that

$$\|\overline{x}(T, L) - (e_1 \otimes \xi)\| < \varepsilon$$

for all $L \in \mathcal{L}$. To this end, as the spectra of the Laplacians in $\mathcal{L}$ are contained in the compact set $[-\mu, -\mu + \lambda^+]$, we regard the following one-parameter dependent family of linear systems

$$\begin{aligned} \tfrac{\partial}{\partial t} z(t, \theta) &= \begin{bmatrix} 0 & -\theta \\ 1 & 0 \end{bmatrix} z(t, \theta) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ z(0, \theta) &= 0, \end{aligned} \tag{31}$$

where $\theta \in \mathbf{P} := [(\mu + 1) - \lambda^+, (\mu + 1)]$. Further, we define $z^*\colon \mathbf{P} \to \mathbb{R}^2$ by

$$z^*(\theta) = \begin{cases} \xi & \text{if } \theta = \mu + 1 \\ \xi \cdot \varphi(\theta) & \text{else}, \end{cases}$$

where $\varphi\colon \mathbf{P} \to \mathbb{R}$ is any continuous function satisfying $\varphi(\mu + 1) = 1$ and $\varphi(\theta) = 0$ for all $\theta \in [(\mu + 1) - \lambda^+, (\mu + 1) - \lambda^-]$. The assertion then follows from Theorem 1. □

8. CONTROLLED HEAT EQUATION

As a second illustration of the method we consider the controllability problem for a class of spatially-invariant systems, described by a system of partial-differential equations on $[0, \infty) \times \mathbb{R}$. Specifically, we consider the example of a controlled heat equation. We refer to the work by Bamieh et al. (2002) for a general study of spatially-invariant systems; our results are however not covered by the results in Bamieh et al. (2002). Let $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}$ be controllable and observable and $\psi(t, z) \in \mathbb{R}^n$ denote a vector-valued valued function in $(t, z) \in \mathbb{R} \times \mathbb{R}$. Recall, that the Fourier-transform on $\mathbb{R}$ defines the linear isometry $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$(\mathcal{F}\psi)(\theta) := \int_{-\infty}^{\infty} e^{i\theta z}\psi(z)\,\mathrm{d}z \tag{32}$$

with inverse transform

$$(\mathcal{F}^{-1}f)(z) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta z} f(\theta)\,\mathrm{d}\theta. \tag{33}$$

Note, that for any $R > 0$, the Fourier-transform of $\frac{\sin(Rz)}{\pi z}$ is equal to $H(R - |z|)$, where $H(x)$ denotes the Heaviside function (i.e. $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise). We then consider the PDE

$$\tfrac{\partial\psi}{\partial t}(t, z) = (A + \tfrac{\partial^2}{\partial z^2} bc)\psi(t, z) + bu(t)\frac{\sin(Rz)}{\pi z}, \quad \psi(0, z) = 0. \tag{34}$$

By Fourier-transforming equation (34), we obtain for $x(t, \theta) := (\mathcal{F}\psi)(t, \theta)$ the parameter-dependent control system

$$\dot{x} = (A - \theta^2 bc)x(t, \theta) + bH(R - |\theta|)u(t), \quad x(0, \theta) = 0. \tag{35}$$

Thus, for the compact parameter domain $\mathbf{P} = [0, R]$ we obtain

$$\dot{x} = (A - \theta^2 bc)x(t, \theta) + bu(t), \quad x(0, \theta) = 0. \tag{36}$$

Our main result implies the following controllability result for the PDE (34).

Theorem 8. Assume, that $A - \theta^2 bc$ *has distinct eigenvalues for* $\theta \in \boldsymbol{P}$*. Let* $x_*(\theta)$ *be any continuous function on* $\boldsymbol{P}$*. Then there exists* $T > 0$ *and a piecewise constant input function* $u : [0, T] \to \mathbb{R}$ *such that the Fourier-transform of the unique solution* $\psi(t, z)$ *of (34) satisfies*

$$\max_{\theta\in\boldsymbol{P}} \|(\mathcal{F}\psi)(T, \theta) - x_*(\theta)\| < \mathrm{e}\,. \tag{37}$$

Proof. By minimality of A, b, c it follows that $(A-\theta^2 bc, b)$ is controllable for all $\theta \in \mathbf{P}$. Moreover, by the same reason, the eigenvalues of $A - \theta^2 bc$ are distinct for distinct pairs of θ, θ'. Thus, by our main result, we conclude that there exists $T > 0$ and an input function u on $[0, T]$ such that $\|x(T, \theta) - x_*(\theta)\| < \mathrm{e}$ for all $\theta \in \mathbf{P}$. Define

$$\psi(t, z) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta z} x(t, \theta) H(R - |\theta|)(\theta) \, \mathrm{d}\theta. \qquad (38)$$

Then the Fourier-transform of $\psi(T, \cdot)$ coincides with $x(T, \theta) H(R - |\theta|) = x(T, \theta)$ for $\theta \in \mathbf{P}$. This completes the proof. □

9. CONCLUSIONS

In this paper we considered the problem of ensemble controllability of linear continuous-time single-input systems. We stated necessary and sufficient conditions for ensemble controllability. Also, we provided a comparison to approximate controllability. The verification of the proposed sufficient conditions stated in Theorem 1 is nonconstructive. To obtain a constructive procedure to determine the input functions we considered an ensemble of harmonic oscillators as an illustrative example. In this setting we examined uniform and L^2-ensemble controllability. In both cases we provide a scheme to determine the values of a piecewise constant input function that steers the initial zero state to a family of terminal states. Moreover, we depict that the concept of uniform ensemble controllability can be used to control a whole family of networks of harmonic oscillators robustly by applying a single input function. In addition, to illustrate that our approach might also be applied to partial differential equations we considered a controlled heat equation.

10. ACKNOWLEDGMENTS

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