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MINIMUM SENSITIVITY REALIZATIONS OF NETWORKS OF LINEAR SYSTEMS

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Abstract. We investigate networks of linear control systems that are interconnected by a fixed network topology. A new class of sensitivity Gramians is introduced whose singular values measure the sensitivity of the network. We characterize the state space realizations of the interconnected node transfer functions such that the overall network has minimum sensitivity. We also develop an optimization approach to the sum of traces of the sensitivity Gramians that determine minimum sensitivity state space realizations of the network. Our work extends previous work by [6, 10, 11] on L^2-minimum sensitivity design.

1. **Introduction.** Mathematical models for interconnected dynamical systems have for long played an important role in various application areas, including, e.g., biological systems, network control systems, flocking and formation control; see, e.g., [15, 16, 20] and the references therein. Typically, the components of a network are represented by linear dynamical systems, the so-called nodes, while the interconnections among the nodes define the network topology. The description of such networks depends on parameters defining node systems and the couplings between them. In such regard it is important to obtain a quantitative measure for the sensitivity of the interconnected system with respect to variations in the node systems and the interconnection parameters. This sensitivity measure for network of systems quantifies robustness properties of interconnected systems with respect to variations in the parameters. For a general approach to sensitivity in a transfer function context we refer to [1, 2, 25]. In the analysis of such coupled dynamical systems it is of fundamental interest to identify the state space representations of the node systems that contribute most, or least, to the sensitivity of the overall network. This problem is studied in the present paper.

Sensitivity issues of linear systems are important in digital controller and filter design, which are subject to finite precision errors and round-off noise [6, 18, 19].

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For linear systems, there are several notions of sensitivity. A class of mixed L_1/L_2-sensitivity measures was studied by [18, 19, 22, 23], and the associated minimum-sensitivity realizations were shown to coincide with *balanced realizations*, first introduced by B.C. Moore [17]. Thus there exists an interesting connection between optimal sensitivity design and model reduction; see also [11]. A natural (and more complicated) L_2-sensitivity measure has been studied by [6, 11], with subsequent contributions by Helmke and Moore [10] and [13]. The corresponding sensitivity Gramians are however harder to compute and one has to recourse to numerical optimization methods, see [24].

In this paper, we extend this earlier work to interconnected linear systems, which poses two new problems. First, the known network structure has to be reflected in the solutions to the optimization tasks. Following [4], this is best done in the frequency domain, using coprime polynomial matrix factorizations of the node transfer functions. Second, the inherent structure of the parameter variations leeds to a structured optimization task.

We consider networked control systems consisting of N node systems Σ_i, $i = 1,...,N$, which are interconnected by constant coupling matrices. In particular, this framework includes the familiar processes of parallel, series and feedback interconnections. We assume that the interconnection structure of the network is described by fixed, constant matrices (K, L, M), where K reflects the internal couplings of the node systems, while L and M describe the input and output couplings, respectively. The dynamics of each linear node system Σ_i are described by strictly proper rational transfer functions G_i, with minimal state space realizations (A_i, B_i, C_i). We introduce an L^2-network sensitivity function that measures the performance degradation of the overall system with respect to variations, or parameter errors, in the realizations of the node system. To this end, we introduce a new class of sensitivity Gramians corresponding to the sensitivity function. Based on the equality of sensitivity Gramians we provide a complete characterization of sensitivity optimal realizations of the node systems. Computing minimum sensitivity realizations for the node systems can be done using optimization algorithms that minimize the sensitivity function. Gradient flows on positive definite matrices can be designed that converge to the minimum of the sensitivity function, similarly to [11]. As parallel, series and tree-like interconnections are the building blocks for connections of networks of linear systems we consider these structures and achieve explicit characterizations of minimum sensitivity realizations.

In this context it also reasonable to investigate the related question of sensitivity analysis with respect to variations in the interconnection matrices (K, L, M), while keeping the node realizations (A_i, B_i, C_i) of the node systems fixed. This is not done here, and we refer to [9] for a sensitivity analysis of interconnection structures for networks of single-input-single-output systems.

The paper is organized as follows. Section 2 introduces the class of networks we are considering. In Section 3 we introduce a new class of Gramians, called the node sensitivity Gramians and discuss their relation to the classical Gramians. We derive necessary and sufficient conditions for positive definiteness of the Gramians. Section 4 introduces the sensitivity function and expresses it in terms of traces of the sensitivity Gramians. Furthermore, we investigate how the sensitivity Gramians transform under changes of coordinates in the node systems. Theorem 4.3 establishes the existence and uniqueness properties of sensitivity optimal realizations.

Section 5 discusses computational issues. In Section 6 we apply the results to special interconnections, i.e., parallel and series interconnections of linear systems as well as networks with a tree structure. Conclusions appear in Section 7.

This work owes much to the early collaboration with John B. Moore. John introduced the first author to the sensitivity optimization problem of state space realizations of linear systems, a problem which is generalized here. Our joint research started in the late 1980s, when working on the related task of optimizing a trace function on positive definite matrices. That led us into studying optimization tasks on manifolds of matrices and resulted in several joint papers and the monograph [11]. The scientific collaboration and friendship with John, our discussions during daily walks at Lake Griffin, or on weekend excursions, have been special highlights of frequent visits at the Australian National University. I (U.H.) will never forget those moments.

2. **Networks of Linear Systems.** In this section we provide the conceptual framework to study the sensitivity of node transfer functions in a given network of linear control systems. Consider a network of N strictly proper linear discrete-time systems

$$\Sigma_i \ : \quad \begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i v_i(t) \\ w_i(t) &= C_i x_i(t), \end{aligned} \qquad i = 1, \dots, N. \tag{1}$$

We refer to (1) as the node systems Σ_i of the network. In the sequel, we focus on discrete-time systems; the equivalent statements for continuous-time systems

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i v_i(t) \\ w_i(t) &= C_i x_i(t), \ i = 1, \dots, N, \end{aligned}$$

hold true mutatis mutandis. Here $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$ are the associated system matrices. We assume that each node system is reachable and observable and has the transfer function $G_i(z) = C_i(zI - A_i)^{-1}B_i$. Without loss of generality we assume that B_i, C_i are full rank matrices, i.e., they satisfy $\operatorname{rank}(B_i) = m_i$ and $\operatorname{rank}(C_i) = p_i$. To interconnect the node systems we use static coupling laws as

$$v_i(t) = \sum_{j=1}^{N} K_{ij} w_j(t) + L_i u(t) \in \mathbb{R}^{m_i}, \ i = 1, \dots, N.$$

with $K_{ij} \in \mathbb{R}^{m_i \times p_j}$ and $L_i \in \mathbb{R}^{m_i \times m}$, although more complex dynamic interconnection laws are possible, too. Here $u = \operatorname{col}(u_1, ..., u_m) \in \mathbb{R}^m$ denotes the external control input applied to the whole network. The interconnected output is given by

$$y(t) = \sum_{i=1}^{N} M_i w_i(t) \in \mathbb{R}^p,$$

with $M_i \in \mathbb{R}^{p \times p_i}$. We define

$$K := (K_{ij})_{ij} \in \mathbb{R}^{\overline{m} \times \overline{p}}, \quad L := \begin{pmatrix} L_1 \\ \vdots \\ L_N \end{pmatrix} \in \mathbb{R}^{\overline{m} \times m}, \quad M := \begin{pmatrix} M_1 & \cdots & M_N \end{pmatrix} \in \mathbb{R}^{p \times \overline{p}},$$

where $\overline{n} := n_1 + \cdots + n_N$, $\overline{m} := m_1 + \cdots + m_N$ and $\overline{p} := p_1 + \cdots + p_N$. To express the closed loop system in compact matrix form, we use for matrices $X_1, ..., X_n$ the notation

$$\operatorname{diag}(X_1, \cdots, X_n) := \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{pmatrix}.$$

Then, for $x(t) = \operatorname{col}(x_1(t), ..., x_N(t)) \in \mathbb{R}^{\overline{n}}$ and block-diagonal matrices,

$$A = \operatorname{diag}(A_1, \cdots, A_N), \quad B = \operatorname{diag}(B_1, \cdots, B_N), \quad C = \operatorname{diag}(C_1, \cdots, C_N) \tag{2}$$

of sizes $\overline{n} \times \overline{n}$, $\overline{n} \times \overline{m}$ and $\overline{p} \times \overline{n}$, respectively, the global state space representation of the node systems is given by

$$\begin{aligned} x(t+1) &= Ax(t) + Bv(t) \\ w(t) &= Cx(t) \end{aligned}$$

and the interconnection is given as

$$\begin{aligned} v(t) &= Kw(t) + Lu(t) \\ y(t) &= Mw(t). \end{aligned}$$

Throughout this paper we call the matrix triple (K, L, M) the *interconnection matrices*, while (A, B, C) are referred to as the *node systems*. Unless stated otherwise, the matrices (A, B, C) are always assumed to be in block-diagonal form (2). Thus the network dynamics has the state space description as

$$\begin{aligned} x(t+1) &= \mathcal{A}x(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{C}x(t), \end{aligned}$$

with

$$\mathcal{A} := A + BKC \in \mathbb{R}^{\overline{n} \times \overline{n}}, \quad \mathcal{B} := BL \in \mathbb{R}^{\overline{n} \times m}, \quad \mathcal{C} := MC \in \mathbb{R}^{p \times \overline{n}}.$$

Switching to a frequency domain description of the network we note that the i-th node transfer function is defined as the strictly proper transfer function of McMillan degree $\delta(G_i) = n_i$ and is given in state space form

$$G_i(z) = C_i(zI - A_i)^{-1}B_i.$$

We define the *node transfer function* as

$$G(z) := \operatorname{diag}(G_1(z), \cdots, G_N(z)) = C(zI - A)^{-1}B \in \mathbb{R}(z)^{\overline{p} \times \overline{m}}.$$

The overall *network transfer function* then is defined as

$$F(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} \in \mathbb{R}(z)^{p \times m}.$$

We obtain the following representations of the global network transfer function

$$F(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} = MC(zI - A - BKC)^{-1}BL.$$

It follows that the network transfer function $F(z)$ is a function of the similarity orbit $\{(SAS^{-1}, SB, CS^{-1} \mid S \in GL_{\overline{n}}(\mathbb{R})\}$ of a realization (A, B, C) of $G(z)$. In the sequel, we aim to find block diagonal state space coordinate transformations S of the node systems such that the network transfer function F is least sensitive to variations in S. We emphasize that the network parameters K, L, M are just coupling parameters which are regarded as fixed.

3. **Network Sensitivity Gramians.** In real-world implementation of control systems, one is often concerned with the fact that different state-space realizations of the same given transfer function might perform differently. This raises the question of constructing state-space realizations with minimal sensitivity properties. In the literature several well-known sensitivity measures have been used; see e.g. [22, 23]. For more extensive studies on minimum sensitivity design of linear systems we refer to [6, 11]. With the exception of [9], the task of sensitivity minimization of networks has not been studied before. When studying sensitivity properties of networked control systems the earlier techniques have to be adapted appropriately. To obtain a measure to quantify the sensitivity of a networked control system with respect to parameterization errors we consider the global network transfer function $F(z)$.

Node Sensitivity Gramians. To analyze the sensitivity of the network with respect to changes in the node system parameters A, B and C, we consider the Jacobians of the global network transfer function

$$F(z) = MC(zI - A - BKC)^{-1}BL = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}$$

with respect to the block-diagonal matrices A, B, C. To this end, let

$$\begin{aligned} X(z) &:= (zI - A - BKC)^{-1}BL = (zI - \mathcal{A})^{-1}\mathcal{B} \in \mathbb{R}(z)^{\overline{n}\times m} \\ Y(z) &:= MC(zI - A - BKC)^{-1} = \mathcal{C}(zI - \mathcal{A})^{-1} \in \mathbb{R}(z)^{p\times\overline{n}}. \end{aligned}$$

More precisely, the matrices X and Y have the form

$$X(z) = \begin{pmatrix} X_1(z) \\ \vdots \\ X_N(z) \end{pmatrix} \qquad Y(z) = \begin{pmatrix} Y_1(z) & \cdots & Y_N(z) \end{pmatrix},$$

where $X_i(z) \in \mathbb{R}(z)^{n_i\times m}$ and $Y_i(z) \in \mathbb{R}(z)^{p\times n_i}$. Before turning to the Jacobians of the transfer function with respect to variations in (A, B, C) we emphasize that the matrices A, B and C are block diagonal. Recall that the *Kronecker product* of the matrices $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{p\times q}$ is given by

$$U \otimes V := (u_{ij}V) = \begin{pmatrix} u_{11}V & \cdots & u_{1m}V \\ \vdots & \ddots & \vdots \\ u_{n1}V & \cdots & u_{nm}V \end{pmatrix} \in \mathbb{R}^{np\times mq}.$$

Moreover, given block matrices

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$$

with $A_{kl} \in \mathbb{R}^{n_k\times n_l}$ and $B_{kl} \in \mathbb{R}^{m_k\times m_l}$ with $k, l \in \{1, ..., N\}$ the *block Kronecker product* (sometimes also called the *Khatri-Rao product*) [12, 14] is

$$A \circledast B := (A_{kl} \otimes B_{kl}) = \begin{pmatrix} A_{11}\otimes B_{11} & \cdots & A_{1N}\otimes B_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1}\otimes B_{N1} & \cdots & A_{NN}\otimes B_{NN} \end{pmatrix} \in \mathbb{R}^{\sum n_k m_k \times \sum n_k m_k}.$$

Lemma 3.1. *The Jacobians of the global network transfer function F with respect to A, B, C are given by*

$$\begin{aligned}
\frac{\partial F(z)}{\partial A} &= \begin{pmatrix} X_1(z)^\top \otimes Y_1(z) & \cdots & X_N(z)^\top \otimes Y_N(z) \end{pmatrix}, \\
\frac{\partial F(z)}{\partial B} &= \begin{pmatrix} L_1^\top \otimes Y_1(z) & \cdots & L_N^\top \otimes Y_N(z) \end{pmatrix}, \\
\frac{\partial F(z)}{\partial C} &= \begin{pmatrix} X_1(z)^\top \otimes M_1 & \cdots & X_N(z)^\top \otimes M_N \end{pmatrix}.
\end{aligned}$$

Proof. We begin with the Jacobian of the global network transfer function F with respect to B. For matrices X, Y, Z of appropriate size one has

$$\operatorname{vec} ZXY = (Y^\top \otimes Z) \operatorname{vec} X.$$

Using this identity it follows that

$$\mathrm{d}\, F(z)\dot{B} = \mathcal{C}(zI - \mathcal{A})^{-1} \begin{pmatrix} \dot{B}_1 L_1 \\ \vdots \\ \dot{B}_N L_N \end{pmatrix} = \sum_{k=1}^{N} Y_k(z) \dot{B}_k L_k = \sum_{k=1}^{N} L_k^\top \otimes Y_k(z) \operatorname{vec} \dot{B}_k.$$

This shows

$$\frac{\partial F(z)}{\partial B} = \begin{pmatrix} L_1^\top \otimes Y_1(z) & \cdots & L_N^\top \otimes Y_N(z) \end{pmatrix}$$

and, similarily,

$$\frac{\partial F(z)}{\partial C} = \begin{pmatrix} X_1(z)^\top \otimes M_1 & \cdots & X_N(z)^\top \otimes M_N \end{pmatrix}.$$

Finally, we consider the Jacobian of the global network transfer function F with respect to A. From the computation

$$\mathrm{d}\, F(z)\dot{A} = \mathcal{C}(zI - \mathcal{A})^{-1} \begin{pmatrix} \dot{A}_1 & & \\ & \ddots & \\ & & \dot{A}_N \end{pmatrix} (zI - \mathcal{A})^{-1} \mathcal{B} = \sum_{k=1}^{N} Y_k(z) \dot{A}_k X_k(z)$$

we conclude

$$\frac{\partial F(z)}{\partial A} = \begin{pmatrix} X_1(z)^\top \otimes Y_1(z) & \cdots & X_N(z)^\top \otimes Y_N(z) \end{pmatrix}.$$

This shows the assertion. $\square$

We introduce a new class of Gramians Γ_o, Γ_c and Γ_i of sizes $\sum_{i=1}^N n_i m_i \times \sum_{i=1}^N n_i m_i$, $\sum_{i=1}^N p_i n_i \times \sum_{i=1}^N p_i n_i$ and $\sum_{i=1}^N n_i^2 \times \sum_{i=1}^N n_i^2$, respectively, via

$$\begin{aligned}
\Gamma_o &:= \frac{1}{2\pi i} \int_{|z|=1} \left(\frac{\partial F(\frac{1}{z})}{\partial B} \right)^\top \frac{\partial F(z)}{\partial B} \, \frac{\mathrm{d}z}{z} \\
&= \left(\frac{1}{2\pi i} \int_{|z|=1} L_k L_l^\top \otimes Y_k(\tfrac{1}{z})^\top Y_l(z) \, \frac{\mathrm{d}z}{z} \right)_{k,l}, \qquad (3)
\end{aligned}$$

$$\Gamma_c := \frac{1}{2\pi i} \int\limits_{|z|=1} \left(\frac{\partial F(\frac{1}{z})}{\partial C}\right)^{\mathsf{T}} \frac{\partial F(z)}{\partial C} \, \frac{\mathrm{d}z}{z}$$

$$= \left(\frac{1}{2\pi i} \int\limits_{|z|=1} X_k(\tfrac{1}{z}) X_l(z)^{\mathsf{T}} \otimes M_k^{\mathsf{T}} M_l \, \frac{\mathrm{d}z}{z} \right)_{k,l}, \quad (4)$$

$$\Gamma_i := \frac{1}{2\pi i} \int\limits_{|z|=1} \left(\frac{\partial F(\frac{1}{z})}{\partial A}\right)^{\mathsf{T}} \frac{\partial F(z)}{\partial A} \, \frac{\mathrm{d}z}{z}$$

$$= \left(\frac{1}{2\pi i} \int\limits_{|z|=1} X_k(\tfrac{1}{z}) X_l(z)^{\mathsf{T}} \otimes Y_k(\tfrac{1}{z})^{\mathsf{T}} Y_l(z) \, \frac{\mathrm{d}z}{z} \right)_{k,l}. \quad (5)$$

These are symmetric and positive semidefinite matrices, which are called the *input-*, *output-*, and *state interconnection sensitivity Gramians*, respectively. Recall, that the classical reachability and observability Gramians associated with the network realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are

$$\mathcal{W}_c := \frac{1}{2\pi i} \int\limits_{|z|=1} X(z) X(\tfrac{1}{z})^{\mathsf{T}} \, \frac{\mathrm{d}z}{z} \in \mathbb{R}^{\overline{n} \times \overline{n}}.$$

and

$$\mathcal{W}_o := \frac{1}{2\pi i} \int\limits_{|z|=1} Y(\tfrac{1}{z})^{\mathsf{T}} Y(z) \, \frac{\mathrm{d}z}{z} \in \mathbb{R}^{\overline{n} \times \overline{n}}.$$

The Gramians $\mathcal{W}_c$ and $\mathcal{W}_o$ exist and are positive definite, provided $\mathcal{A} = A + BKC$ is (discrete-time) stable, i.e. all eigenvalues are in the open unit disc, and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is reachable and observable. See [4] for a characterization for reachability and observability of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in terms of associated coprime factorizations; see also [8] for a reachability analysis of homogeneous networks, defined by identical SISO node systems. If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is reachable and observable, the Gramians $\mathcal{W}_c$ and $\mathcal{W}_o$ satisfy the discrete-time Lyapunov equations

$$\mathcal{A}\mathcal{W}_c\mathcal{A}^{\mathsf{T}} - \mathcal{W}_c + \mathcal{B}\mathcal{B}^{\mathsf{T}} = 0\,, \qquad \mathcal{A}^{\mathsf{T}}\mathcal{W}_o\mathcal{A} - \mathcal{W}_o + \mathcal{C}^{\mathsf{T}}\mathcal{C} = 0\,. \quad (6)$$

Define the $n_k \times n_k$-matrices

$$\mathcal{W}_o^k := \frac{1}{2\pi i} \int\limits_{|z|=1} Y_k(\tfrac{1}{z})^{\mathsf{T}} Y_k(z) \, \frac{\mathrm{d}z}{z}, \qquad \mathcal{W}_c^k := \frac{1}{2\pi i} \int\limits_{|z|=1} X_k(\tfrac{1}{z}) X_k(z)^{\mathsf{T}} \, \frac{\mathrm{d}z}{z}.$$

Thus $\mathcal{W}_o^k$ and $\mathcal{W}_c^k$ are the block-diagonal parts of $\mathcal{W}_o$ and $\mathcal{W}_o$, respectively. By Lemma 3.1, the sensitivity Gramians and the classical Gramians are related as

$$\Gamma_o = LL^{\mathsf{T}} \circledast \mathcal{W}_o \qquad \text{and} \qquad \Gamma_c = \mathcal{W}_c \circledast M^{\mathsf{T}} M. \quad (7)$$

Define the local sensitivity Gramians for $k = 1, \dots, N$ as

$$\Gamma_o^k := L_k L_k^{\mathsf{T}} \otimes \mathcal{W}_o^k, \qquad \Gamma_c^k := \mathcal{W}_c^k \otimes M_k^{\mathsf{T}} M_k,$$

and

$$\Gamma_i^k := \frac{1}{2\pi i} \int\limits_{|z|=1} X_k(\tfrac{1}{z}) X_k(z)^\top \otimes Y_k(\tfrac{1}{z})^\top Y_k(z) \, \tfrac{\mathrm{d}z}{z}.$$

Thus the local sensitivity Gramians are the block-diagonal parts of Γ_c, Γ_o and Γ_i, respectively. The Cauchy-Schwarz inequality implies the bound on the state interconnection sensitivity Gramian

$$\mathrm{Tr}\,[\Gamma_i] \leq \sqrt{\frac{1}{2\pi i} \int\limits_{|z|=1} \|X(z)\|^4 \tfrac{\mathrm{d}z}{z}} \sqrt{\frac{1}{2\pi i} \int\limits_{|z|=1} \|Y(z)\|^4 \tfrac{\mathrm{d}z}{z}},$$

where $\mathrm{Tr}[X]$ denotes the trace of the matrix X.

In [12, Theorem 3.1], a sufficient condition for positive definiteness of the block Kronecker product $A \circledast B$ of two symmetric matrices A, B is derived, i.e., $A \circledast B$ is positive definite if B is positive definite and A is positive semidefinite with positive definite block diagonal parts. In particular, the Gramian Γ_o is positive definite if $\mathcal{W}_o$ is positive definite and the main diagonal blocks $L_k L_k^\top$ of

$$LL^\top = \begin{pmatrix} L_1 L_1^\top & \cdots & L_1 L_N^\top \\ \vdots & \ddots & \vdots \\ L_N L_1^\top & \cdots & L_N L_N^\top \end{pmatrix}$$

are positive definite. Note that the matrices L and M are typically tall and wide, respectively. For this reason one cannot expect that Γ_c and Γ_o are positive definite. Next, we derive sufficient conditions such that the sensitivity Gramians exist and are positive definite.

Proposition 1. *Suppose that the node transfer function G is strictly proper with minimal realization (A, B, C), with $\operatorname{rank} B = \overline{m}$, $\operatorname{rank} L_i = m_i$, $\operatorname{rank} L = m$, $\operatorname{rank} C = \overline{p}$, $\operatorname{rank} M_i = p_i$, and $\operatorname{rank} M = p$. Assume that the network realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are reachable and observable, with $\mathcal{A}$ having no eigenvalues on the unit circle. Then, the Gramians $\mathcal{W}_c, \mathcal{W}_o$ and Γ_c, Γ_o and Γ_i exist and are positive definite symmetric matrices.*

Proof. The assumption that $\mathcal{A}$ has no eigenvalues on the unit circle implies that the contour integrals, defining Γ_c, Γ_o and Γ_i, exist. Since $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are reachable and observable, the reachability and observability Gramians $\mathcal{W}_c, \mathcal{W}_o$ are positive definite symmetric matrices. We begin with Γ_o. By [12, Theorem 3.1] and (7) the assertion follows if the main diagonal blocks $L_k L_k^\top$ of $LL^\top$ are positive definite. Clearly, $L_k L_k^\top$ is positive semidefinite. Thus, as $\operatorname{rank} L_k L_k^\top = \operatorname{rank} L_k = m_k$ it follows that zero is no eigenvalue of $L_k L_k^\top$. Consequently, Γ_o is positive definite. Note that $\operatorname{rank} L_k = m_k$ implies $m_k \leq m$. The same argument shows that Γ_c is positive definite. The interconnection sensitivity Gramian

$$\Gamma_i = \left(\frac{1}{2\pi i} \int\limits_{|z|=1} X_k(\tfrac{1}{z}) X_l(z)^\top \otimes Y_k(\tfrac{1}{z})^\top Y_l(z) \, \tfrac{\mathrm{d}z}{z} \right)_{k,l}$$

is positive semidefinite and symmetric. For every $v_k \in \mathbb{R}^{n_k^2}$, $v = \mathrm{col}(v_1, ..., v_N)$ one has

$$v^\top \Gamma_i v = \frac{1}{2\pi i} \int\limits_{|z|=1} \left\| \sum_{k=1}^{N} X_k(z) \otimes Y_k(z)\, v_k \right\|^2 \tfrac{\mathrm{d}z}{z} \geq 0.$$

Hence, it suffice to show that

$$\sum_{k=1}^{N} X_k(z) \otimes Y_k(z)\, v_k = 0 \quad \Longrightarrow \quad v_k = 0. \tag{8}$$

Let $V_k \in \mathbb{R}^{n_k \times n_k}$ with $\mathrm{vec}(V_k) = v_k$, $k = 1, ..., N$ and $V = \mathrm{diag}(V_1, ..., V_N)$. Then (8) is equivalent to

$$Y(z) V X(z) = 0 \quad \Longrightarrow \quad V = 0$$

which is equivalent to

$$\mathcal{C}(zI - \mathcal{A})^{-1} V (zI - \mathcal{A})^{-1} \mathcal{B} = 0 \quad \Longrightarrow \quad V = 0.$$

Since $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is reachable and observable with $\operatorname{rank} \mathcal{C} = p$ and $\operatorname{rank} \mathcal{B} = m$ there exist left and right coprime polynomial matrix factorizations

$$\mathcal{C}(zI - \mathcal{A})^{-1} = T(z)^{-1} U(z), \quad (zI - \mathcal{A})^{-1} \mathcal{B} = \overline{U}(z) \overline{T}(z)^{-1}$$

with $U(z)$ left prime and $\overline{U}(z)$ right prime. Thus there are polynomial matrices $U_1(z), \overline{U}_1(z)$ with $U_1(z)U(z) = I, \overline{U}(z)\overline{U}_1(z) = I$. Therefore $\mathcal{C}(zI - \mathcal{A})^{-1} V (zI - \mathcal{A})^{-1} \mathcal{B} = 0$ implies that

$$U(z) V \overline{U}(z) = 0.$$

By left and right primeness of $U(z)$ and $\overline{U}(z)$ this implies $V = 0$. This shows the assertion. □

4. **Sensitivity optimal realizations.** The L_2-norm of a rational transfer function $T(z)$ with no poles on the unit circle is defined as

$$\|T(z)\|_2^2 := \frac{1}{2\pi i} \int\limits_{|z|=1} \mathrm{Tr} \left[T(\tfrac{1}{z})^\top T(z) \right] \tfrac{\mathrm{d}z}{z}.$$

Following [11, Section 9.2] we define the following.

Definition 4.1. Let $G(z) = C(zI - A)^{-1} B$ denote the transfer function of the decoupled node systems and let (K, L, M) denote the interconnection matrices. Assume that $\mathcal{A}$ does not have eigenvalues on the unit circle. Let $\alpha > 0, \beta > 0$ and $\gamma \geq 0$. Then

$$\mathcal{S}_G(A, B, C) := \alpha \left\| \frac{\partial F(z)}{\partial B} \right\|_2^2 + \beta \left\| \frac{\partial F(z)}{\partial C} \right\|_2^2 + \gamma \left\| \frac{\partial F(z)}{\partial A} \right\|_2^2$$

is called the *weighted L_2-network sensitivity* of $F(z)$ with respect to the node systems (A, B, C).

Using the definition of the sensitivity Gramians we obtain the following expression of the L^2-network sensitivity. The proof is by a straightforward computation.

Proposition 2. *Let $G(z)$ be a strictly proper transfer function describing the decoupled node systems and let (K, L, M) be the interconnection matrices. Then the weighted L_2-network sensitivity is given in terms of the sensitivity Gramians as*

$$\begin{aligned}\mathcal{S}_G(A,B,C) &= \alpha \operatorname{Tr}\Big[\Gamma_c\Big] + \beta \operatorname{Tr}\Big[\Gamma_o\Big] + \gamma \operatorname{Tr}\Big[\Gamma_i\Big] = \sum_{k=1}^{N} \operatorname{Tr}\Big[\alpha\Gamma_c^k + \beta\Gamma_o^k + \gamma\Gamma_i^k\Big] \\ &= \alpha \sum_{k=1}^{N} \|L_k\|^2 \|Y_k(z)\|_2^2 + \beta \sum_{k=1}^{N} \|M_k\|^2 \|X_k(z)\|_2^2 \\ &\quad + \gamma \sum_{k=1}^{N} \frac{1}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(z)X_k(z^{-1})^\top\right] \operatorname{Tr}\left[Y_k(z^{-1})^\top Y_k(z)\right] \tfrac{\mathrm{d}z}{z}.\end{aligned}$$

Dependence on Coordinate Transformations. We next discuss how the sensitivity function transforms under changes of coordinates. Obviously, the sensitivity function depends on the choice of realizations (A, B, C) of the node transfer function $G(z)$. A realization (A, B, C) of the node transfer function G is said to have *minimal sensitivity* if each realization (A', B', C') of G satisfies $\mathcal{S}_G(A, B, C) \leq \mathcal{S}_G(A', B', C')$. Let

$$\mathrm{GL}(\mathbf{n}) = \mathrm{GL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{n_N}(\mathbb{R}) \ \subset GL_{\overline{n}}(\mathbb{R})$$

denote the direct product of the N groups of invertible real matrices of sizes $n_1, \ldots, n_N$. We regard the elements of $\mathrm{GL}(\mathbf{n})$ as block diagonal matrices of the form

$$S = \operatorname{diag}\,(S_1, \cdots, S_N),$$

where $S_i \in \mathrm{GL}_{n_i}(\mathbb{R})$. Thus $\mathrm{GL}(\mathbf{n})$ is a closed Lie group of $GL_{\overline{n}}(\mathbb{R})$ of dimension $n_1^2 + \cdots + n_N^2$. We now address the existence of sensitivity optimal realizations as well as their characterization. Since reachable and observable realizations of a strictly proper transfer function are similar, we restrict the sensitivity function to the orbits of the state space similarity action

$$(A, B, C) \mapsto (SAS^{-1}, SB, CS^{-1})$$

with $S \in \mathrm{GL}(\mathbf{n})$. Fix the node transfer function G as well as the interconnection parameters K, L, M. Choose a diagonal, reachable and observable realization (A, B, C) of $G(z)$ with associated $\mathrm{GL}(\mathbf{n})$-similarity orbit

$$\mathcal{R}_G := \left\{(A', B', C') = (SAS^{-1}, SB, CS^{-1}) \mid S \in \mathrm{GL}(\mathbf{n})\right\}.$$

Then $\mathcal{R}_G$ is a smooth manifold. Let Γ_c', Γ_o' and Γ_i' denote the sensitivity Gramians of (SAS^{-1}, SB, CS^{-1}). The L^2-sensitivity function then defines the smooth function

$$\mathcal{S}_G\colon \mathcal{R}_G \to \mathbb{R}, \quad \mathcal{S}_G(A', B', C') = \alpha \operatorname{Tr}\left[\Gamma'_c\right] + \beta \operatorname{Tr}\left[\Gamma'_o\right] + \gamma \operatorname{Tr}\left[\Gamma'_i\right],$$

or, equivalently, the smooth function $S \mapsto \alpha \operatorname{Tr}\left[\Gamma'_c\right] + \beta \operatorname{Tr}\left[\Gamma'_o\right] + \gamma \operatorname{Tr}\left[\Gamma'_i\right]$ on the space of block-diagonal invertible coordinate transformations.

Lemma 4.2. *Let (A, B, C) be a block-diagonal realization of the node transfer function G and let $S \in \mathrm{GL}(\mathbf{n})$.*

(a) The sensitivity Gramians transform as

$$(\Gamma_c, \Gamma_o, \Gamma_i) \mapsto \Big((S \otimes I)\Gamma_c(S^\top \otimes I), (I \otimes S^{-\top})\Gamma_o(I \otimes S^{-1}), (S \otimes S^{-\top})\ \Gamma_i\ (S^\top \otimes S^{-1})\Big).$$

(b) The L_2-network sensitivity of (SAS^{-1}, SB, CS^{-1}) is

$$\mathcal{S}_G(SAS^{-1}, SB, CS^{-1}) = \alpha \operatorname{Tr}\left[\Gamma_c(S^\mathsf{T} S \otimes I)\right] + \beta \operatorname{Tr}\left[\Gamma_o\big(I \otimes (S^\mathsf{T} S)^{-1}\big)\right] \\ + \gamma \operatorname{Tr}\left[\Gamma_i(S^\mathsf{T} S \otimes (S^\mathsf{T} S)^{-1}\right].$$

(c) The L^2-network sensitivity function $\mathcal{S}_G$ is unitarily invariant on $\mathcal{R}_G$.

Proof. To show (a), one verifies that

$$\big(zI - SAS^{-1} - SBKCS^{-1}\big)^{-1} SBL = S(zI - \mathcal{A})^{-1}\mathcal{B}$$

and

$$MCS^{-1}\big(zI - SAS^{-1} - SBKCS^{-1}\big)^{-1} = \mathcal{C}(zI - \mathcal{A})^{-1}S^{-1}.$$

Thus, we have

$$\frac{1}{2\pi i}\int\limits_{|z|=1} S_k X_k(z^{-1}) X_l(z)^\mathsf{T} S_l^\mathsf{T} \otimes S_k^{-\mathsf{T}} Y_k(z^{-1})^\mathsf{T} Y_l(z) S_l^{-1} \tfrac{\mathrm{d}z}{z}$$
$$= \frac{1}{2\pi i}\int\limits_{|z|=1} (S_k \otimes S_l^{-\mathsf{T}})\ \big(X_k(z^{-1})X_l(z)^\mathsf{T} \otimes Y_k(z^{-1})^\mathsf{T} Y_l(z)\big)\ (S_k^\mathsf{T} \otimes S_l^{-1}) \tfrac{\mathrm{d}z}{z}$$

and therefore Γ_i transforms via

$$\Gamma_i \mapsto (S \otimes S^{-\mathsf{T}})\ \Gamma_i\ (S^\mathsf{T} \otimes S^{-1}).$$

Similarly, one shows that $\Gamma_c \mapsto (S\otimes I)\Gamma_c(S^\mathsf{T} \otimes I)$ and $\Gamma_o \mapsto (I\otimes S^{-\mathsf{T}})\Gamma_o(I\otimes S^{-1})$.

To conclude (b), observe that $\operatorname{Tr}\left[XYZ\right] = \operatorname{Tr}\left[YZX\right]$ and $(S^\mathsf{T}\otimes S^{-1})(S\otimes S^{-\mathsf{T}}) = (S^\mathsf{T} S \otimes (S^\mathsf{T} S)^{-1})$, and the claim follows. The assertion (c) is immediately implied by (b). □

The preceding lemma yields a convenient expression for the sensitivity function in terms of the positive definite matrix $P = S^\mathsf{T} S$ as

$$\overline{\mathcal{S}}_G(P) := \alpha \operatorname{Tr}\left[\Gamma_c(P\otimes I)\right] + \beta \operatorname{Tr}\left[\Gamma_o\big(I\otimes P^{-1}\big)\right] + \gamma \operatorname{Tr}\left[\Gamma_i(P\otimes P^{-1})\right].$$

Characterization of Sensitivity Optimal Realizations. We next formulate our main result on the existence and uniqueness of L^2-sensitivity optimal realizations.

Theorem 4.3. *Assume that (A, B, C) and (K, L, M) are such that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is reachable and observable, with $\mathcal{A}$ having no eigenvalues on the unit circle. Assume further that* $\operatorname{rank} B = \overline{m}$, $\operatorname{rank} L_i = m_i$, $\operatorname{rank} L = m$, $\operatorname{rank} C = \overline{p}$, $\operatorname{rank} M_i = p_i$, *and* $\operatorname{rank} M = p$. *Then, for $\alpha > 0, \beta > 0, \gamma \geq 0$:*

(a) The L^2-network sensitivity function $\mathcal{S}_G\colon \mathcal{R}_G \to \mathbb{R}$,

$$\mathcal{S}_G\big(A, B, C\big) = \alpha \operatorname{Tr}\left[\Gamma_c\right] + \beta \operatorname{Tr}\left[\Gamma_o\right] + \gamma \operatorname{Tr}\left[\Gamma_i\right],$$

has compact sublevel sets. In particular, the L^2-network sensitivity function $\mathcal{S}_G\colon \mathcal{R}_G \to \mathbb{R}$ attains its global minimum, i.e. there exists a realization $(A', B', C') \in \mathcal{R}_G$ satisfying

$$\mathcal{S}_G\big(A', B', C'\big) = \inf_{(A,B,C)\in\mathcal{R}_G} \mathcal{S}_G\big(A, B, C\big).$$

(b) *Every critical point of the L^2-network sensitivity function $\mathcal{S}_G\colon \mathcal{R}_G \to \mathbb{R}$,*

$$\mathcal{S}_G\big(A,B,C\big) = \alpha \operatorname{Tr}\left[\Gamma_c\right] + \beta \operatorname{Tr}\left[\Gamma_o\right] + \gamma \operatorname{Tr}\left[\Gamma_i\right],$$

is a point where $\mathcal{S}_G$ assumes a global minimum. The set of global minima of $\mathcal{S}_G$ is given by a single unitary orbit

$$\left\{\big(SAS^{-1}, SB, CS^{-1}\big) \mid S \in \mathrm{GL}(\mathbf{n}), SS^\mathsf{T} = I\right\}.$$

(c) *The sensitivity optimal realizations (A,B,C) of $G(z)$ are characterized as*

$$\alpha\|M_k\|^2\mathcal{W}_c^k + \gamma \int\limits_{|z|=1} \operatorname{Tr}\left[Y_k(\tfrac{1}{z})^\mathsf{T} Y_k(z)\right] X_k(\tfrac{1}{z}) X_k(z)^\mathsf{T} \tfrac{\mathrm{d}z}{z}$$
$$= \beta\|L_k\|^2\mathcal{W}_o^k + \gamma \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(\tfrac{1}{z}) X_k(z)^\mathsf{T}\right] Y_k(\tfrac{1}{z})^\mathsf{T} Y_k(z) \tfrac{\mathrm{d}z}{z}, \quad k = 1,\dots,N.$$

Proof. By Proposition 1, the Gramians $\Gamma_c, \Gamma_o, \Gamma_i$ are positive definite. Obviously, we have $\mathcal{S}_G(A,B,C) \geq 0$ for any $(A,B,C) \in \mathcal{R}_G$. For any $S \in \mathrm{GL}(\mathbf{n})$ it holds

$$\mathcal{S}_G\big(SAS^{-1}, SB, CS^{-1}\big) \geq \alpha \operatorname{Tr}\Big[\Gamma_c(S^\mathsf{T} S \otimes I)\Big] + \beta \operatorname{Tr}\Big[\Gamma_o\big(I \otimes (S^\mathsf{T} S)^{-1}\big)\Big].$$

Note that $\alpha > 0$ and $\beta > 0$. Moreover, by Lemma 6.4.1 in [11], for all $c \in \mathbb{R}$, the set

$$\{P = P^\mathsf{T} > 0 \,|\, \alpha \operatorname{Tr}\left[\Gamma_o(P \otimes I)\right] + \beta \operatorname{Tr}\left[\Gamma_c(I \otimes P^{-1})\right] \leq c\}$$

is compact. Thus, since

$$\{S \in \mathrm{GL}(\mathbf{n}) \,|\, \mathcal{S}_G\big(SAS^{-1}, SB, CS^{-1}\big) \leq c\}$$
$$\subset \{S \in \mathrm{GL}(\mathbf{n}) \,|\, \alpha \operatorname{Tr}\left[\Gamma_o(SS^\mathsf{T} \otimes I)\right] + \beta \operatorname{Tr}\left[\Gamma_c(I \otimes (SS^\mathsf{T})^{-1})\right] \leq c\}$$

is a closed subset, the first claim follows. The second assertion follows immediately from the continuity of the L^2-network sensitivity function.

The set $\mathcal{R}_G$ of real similarity orbits is a closed submanifold of the complex manifold $\mathcal{R}_G^\mathbb{C}$, which is defined by all similarity orbits (SAS^{-1}, SB, CS^{-1}) of complex invertible matrices $S = \operatorname{diag}(S_1, \cdots, S_N)$. By inspection, the L^2-sensitivity extends to a plurisubharmonic function on $\mathcal{R}_G^\mathbb{C}$. Thus the proof of the claim follows as in [11], using the Azad-Loeb Theorem.

We next turn to analyze the critical points of the L^2-sensitivity function. Let $P := S^\mathsf{T} S$ and let $\mathcal{P}(\mathbf{n})$ denote the convex space of all real $\overline{n} \times \overline{n}$ block-diagonal, symmetric matrices $P = \operatorname{diag}(P_1, \cdots, P_N)$, where P_i is $n_i \times n_i$ and positive definite. Similarly, let $\operatorname{Sym}(\mathbf{n})$ denote the vector space of all block-diagonal real symmetric matrices $\xi = \operatorname{diag}(\xi_1, \cdots, \xi_N)$, where ξ_i has size $n_i \times n_i$. Consider the function $\overline{\mathcal{S}}_G\colon \mathcal{P}(\mathbf{n}) \to \mathbb{R}$ defined by

$$\overline{\mathcal{S}}_G(P) := \alpha \operatorname{Tr}\left[\Gamma_c(P \otimes I)\right] + \beta \operatorname{Tr}\left[\Gamma_o\big(I \otimes P^{-1}\big)\right] + \gamma \operatorname{Tr}\left[\Gamma_i(P \otimes P^{-1})\right].$$

Note that the tangent space of $\mathcal{P}(\mathbf{n})$ at any element P is given by the vector space of real symmetric $\mathbf{n} \times \mathbf{n}$-matrices $\operatorname{Sym}(\mathbf{n})$.

The differential $D\overline{\mathcal{S}}_G\colon \operatorname{Sym}(\mathbf{n}) \to \mathbb{R}$ is

$$D\overline{\mathcal{S}}_G(P)(\xi) = \alpha \operatorname{Tr}\Big[\Gamma_c(\xi \otimes I)\Big] - \beta \operatorname{Tr}\Big[\Gamma_o(I \otimes P^{-1}\xi P^{-1})\Big]$$
$$+ \gamma \operatorname{Tr}\Big[\Gamma_i\left(\xi \otimes P^{-1} - P \otimes P^{-1}\xi P^{-1}\right)\Big].$$

To compute the critical points of $\overline{\mathcal{S}}_G$ one has to find all $P \in \mathcal{P}(\mathbf{n})$ with $D\overline{\mathcal{S}}_G(P) = 0$. A computation shows

$$\begin{aligned}
&\operatorname{Tr}\left[\Gamma_i \left(\xi \otimes P^{-1} - P \otimes P^{-1}\xi P^{-1}\right)\right] = \\
&\qquad \sum_{k=1}^{N} \frac{1}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1}\right] \operatorname{Tr}\left[X_k(\tfrac{1}{z}) X_k(z)^\top \xi_k\right] \tfrac{\mathrm{d}z}{z} \\
&\qquad - \sum_{k=1}^{N} \frac{1}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(\tfrac{1}{z})^\top X_k(z) P_k\right] \operatorname{Tr}\left[P_k^{-1} Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1}\xi_k\right] \tfrac{\mathrm{d}z}{z} \\
&\qquad = \sum_{k=1}^{N} \frac{1}{2\pi i} \operatorname{Tr}\Bigg[\Big(\int\limits_{|z|=1} \operatorname{Tr}\left[Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1}\right] X_k(\tfrac{1}{z}) X_k(z)^\top \\
&\qquad\qquad - \operatorname{Tr}\left[X_k(\tfrac{1}{z}) X_k(z)^\top P_k\right] P_k^{-1} Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1} \tfrac{\mathrm{d}z}{z}\Big)\xi_k\Bigg],
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tr}\left[\Gamma_c(\xi \otimes I)\right] &= \sum_{k=1}^{N} \frac{1}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(\tfrac{1}{z})^\top X_k(z)\xi_k \otimes M_k^\top M_k\right] \tfrac{\mathrm{d}z}{z} \\
&= \sum_{k=1}^{N} \frac{\|M_k\|^2}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(\tfrac{1}{z})^\top X_k(z)\xi_k\right] \tfrac{\mathrm{d}z}{z} = \sum_{k=1}^{N} \|M_k\|^2 \operatorname{Tr}\left[\mathcal{W}_c^k \xi_k\right]
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Tr}\left[\Gamma_o(I \otimes P^{-1}\xi P^{-1})\right] &= \sum_{k=1}^{N} \frac{1}{2\pi i} \int\limits_{|z|=1} \operatorname{Tr}\left[L_k L_k^\top \otimes Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1}\xi_k P_k^{-1}\right] \tfrac{\mathrm{d}z}{z} \\
&= \sum_{k=1}^{N} \|L_k\|^2 \operatorname{Tr}\left[P_k^{-1}\mathcal{W}_o^k P_k^{-1}\xi_k\right].
\end{aligned}$$

The critical points of $\overline{\mathcal{S}}_G$ are thus characterized as the positive definite matrices $P \in \mathcal{P}(\mathbf{n})$ such that the block-diagonal parts P_k satisfy for $k = 1, \ldots, N$

$$\begin{aligned}
&\alpha\|M_k\|^2\mathcal{W}_c^k + \gamma \int\limits_{|z|=1} \operatorname{Tr}\left[Y_k(\tfrac{1}{z})^\top Y_k(z) P_k^{-1}\right] X_k(\tfrac{1}{z}) X_k(z)^\top \tfrac{\mathrm{d}z}{z} \\
&= P_k^{-1}\left(\beta\|L_k\|^2\mathcal{W}_o^k + \gamma \int\limits_{|z|=1} \operatorname{Tr}\left[X_k(\tfrac{1}{z}) X_k(z)^\top P_k\right] Y_k(\tfrac{1}{z})^\top Y_k(z) \tfrac{\mathrm{d}z}{z}\right) P_k^{-1}.
\end{aligned}$$

This implies the last claim. □

One can improve Theorem 4.3 for special classes of networks. To obtain a simplified representation of the sensitivity Gramians, we focus on the case of a *homogeneous network*, where all node systems are identical SISO systems, represented as (A, b, c). Let $g(z) = c(zI_n - A)^{-1}b$ denote the transfer function with inverse $h(z) = g(z)^{-1}$. The state space representation of the network is then given as

$$\mathcal{A} = I_N \otimes A + K \otimes bc, \quad \mathcal{B} = L \otimes b, \quad \mathcal{C} = M \otimes c.$$

As is shown in [18] the following equalities hold

$$(zI - \mathcal{A})^{-1}\mathcal{B} = (I - gK)^{-1} L \otimes (zI - A)^{-1}b$$

and

$$\mathcal{C}(zI - \mathcal{A})^{-1} = M (I - gK)^{-1} \otimes c(zI - A)^{-1}.$$

The associated network transfer function is

$$F(z) = \mathcal{C}(zI_{nN} - \mathcal{A})^{-1}\mathcal{B} = M(h(z)I_N - K)^{-1}L.$$

Recall that a scalar transfer function $g(z)$ of a discrete-time system (A, b, c) is *bounded real*, if A is stable and $|g(z)| \leq 1$ for all z with $|z| = 1$. A transfer function $g(z)$ with all poles inside the open unit disc is called lossless if $|g(z)| = 1$ for all z with $|z| = 1$. By the bounded real lemma, $g(z)$ is bounded real if and only if A is stable and there exists a positive definite solution of the algebraic Riccati equation

$$A^\top P_* A - P_* + \frac{A^\top P_* bb^\top P_* A}{1 - b^\top P_* b} + c^\top c = 0. \tag{9}$$

For symmetric matrices A, B, let $A \leq B$ denote that $B - A$ is positive semidefinite. In [13, Lemma 5] it is shown that, if the transfer function g is bounded real, then for the controllability and observability Gramians for $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (K, L, M) the following inequalities hold

$$W_c(\mathcal{A}, \mathcal{B}) \leq W_c(K, L) \otimes P^{-1}, \qquad W_o(\mathcal{C}, \mathcal{A}) \leq W_o(M, K) \otimes P, \tag{10}$$

where P is the unique positive definite solution to the Riccati equation (9). Moreover, if $g(z)$ is lossless, then

$$W_c(\mathcal{A}, \mathcal{B}) = W_c(K, L) \otimes P^{-1}, \qquad W_o(\mathcal{C}, \mathcal{A}) = W_o(M, K) \otimes P.$$

Note that, if (A, b, c) is replaced by (SAS^{-1}, Sb, cS^{-1}), the unique solution of (9) is given by $S^{-\top} P_* S^{-1}$. Thus, for g bounded real and P_* defined as above, one has for $k = 1, \ldots, N$

$$W_c^k \leq e_k^\top W_c(K, L) e_k P_*^{-1}, \qquad W_o^k \leq e_k^\top W_o(M, K) e_k P_*. \tag{11}$$

If $g(z)$ is lossless, then equalities in (11) hold. Next assume that $\alpha = \beta = 1, \gamma = 0$. Then the input/output sensitivity function is given as

$$\mathcal{S}_{co}(A, b, c) = \operatorname{Tr}\left[\Gamma_c\right] + \operatorname{Tr}\left[\Gamma_o\right] = \sum_{k=1}^{N} \operatorname{Tr}\left[\|L_k\|^2 W_c^k + \|M_k\|^2 W_o^k\right].$$

Theorem 4.4. *Let $g(z) = c(zI - A)^{-1}b$ be bounded real and let P_* be a solution of the Riccati equation (9). Define*

$$\lambda := \sum_{k=1}^{N} \|L_k\|^2 e_k^\top W_c(K, L) e_k, \qquad \mu := \sum_{k=1}^{N} \|M_k\|^2 e_k^\top W_o(M, K) e_k.$$

Then the minimum value $\min \mathcal{S}_{co}$ over all realizations of $g(z)$ satisfies the bound

$$\min \mathcal{S}_{co} \leq 2n\sqrt{\lambda\mu}.$$

If $g(z)$ is lossless, then

$$\min \mathcal{S}_{co} = 2n\sqrt{\lambda\mu},$$

with an optimal state space coordinate transformation $S_* = \sqrt{\mu\lambda^{-1}} P_*^{\frac{1}{2}}$.

Proof. By (11), for all $P = SS^\top$

$$\begin{aligned}\mathcal{S}_{co}(SAS^{-1}, Sb, cS^{-1}) &\le \sum_{k=1}^{N} \|L_k\|^2 e_k^\top W_c^k e_k \operatorname{Tr}\left[SP_*^{-1}S^\top\right] \\ &+ \sum_{k=1}^{N} \|M_k\|^2 e_k^\top W_o^k e_k \operatorname{Tr}\left[S^{-\top}P_*S^{-1}\right] \\ &= \lambda \operatorname{Tr}\left[P_*^{-1}P\right] + \mu \operatorname{Tr}\left[P^{-1}P_*\right]\end{aligned}$$

It is easily seen that the smooth function $Q \mapsto \lambda \operatorname{Tr}\left[Q\right] + \mu \operatorname{Tr}\left[Q^{-1}\right]$ has a unique local and global minimum at $Q_{\min} = \sqrt{\mu\lambda^{-1}} I_n$. Thus the optimal transformation $P_{\min}$ minimizing $\lambda \operatorname{Tr}\left[Q\right] + \mu \operatorname{Tr}\left[Q^{-1}\right]$ is equal to $\sqrt{\mu\lambda^{-1}} P_*$. This implies the result. □

The upper bound in Theorem 4.4 depends on the fact that we work with homogeneous networks of identical SISO systems. This allows one to prove the important inequalities (10). We leave it as an open problem to find extensions for heterogeneous networks of SISO node systems.

We briefly discuss how the preceding computations simplify for transfer functions with symmetries. Recall that a realization (A, B, C) is called signature symmetric provided there exists a diagonal matrix J with diagonal entries ± 1 such that $(AJ)^\top = AJ$ and $CJ = B^\top$ holds. J is called the signature matrix and satisfies $J = J^{-1}$. By a well known result from linear systems theory, any strictly proper rational transfer function $G(z)$ which satisfies $G(z) = G(z)^\top$ has a signature symmetric realization. Moreover, the difference in the numbers of positive and negative diagonal entries in J coincides with the matrix Cauchy-index of $G(z)$.

Corollary 1. *Assume that* (A_k, B_k, C_k) , $k = 1, \dots, N$, *are signature symmetric, reachable and observable realizations with signature matrices* $J_1, \dots, J_N$. *Then the reachable and observable realization* (A, B, C) *is signature symmetric with signature matrix* $J = \operatorname{diag}(J_1, \dots, J_N)$. *Assume further that* (K, L, M) *is a symmetric minimal realization, i.e.* $K = K^\top, M^\top = L$, *such that* $A + BKC$ *has no eigenvalues on the unit circle. Choose* $\alpha = \beta = \gamma = 1$. *Then* (A, B, C) *is sensitivity minimal if and only if it satisfies for* $k = 1, \dots, N$

$$\begin{aligned}&\int_{|z|=1} \left(\|M_k\|^2 + \operatorname{Tr}\left[X_k(z^{-1})X_k(z)^\top\right]\right) X_k(z^{-1})X_k(z)^\top \tfrac{\mathrm{d}z}{z} \\ &\qquad = J_k \left(\int_{|z|=1} \left(\|M_k\|^2 + \operatorname{Tr}\left[X_k(z^{-1})X_k(z)^\top\right]\right) X_k(z^{-1})X_k(z)^\top \tfrac{\mathrm{d}z}{z}\right) J_k.\end{aligned}$$

In particular, a signature symmetric realization (A, B, C) *with signature matrix* $J = I$ *has minimal* L^2*-sensitivity.*

Proof. We have $Y(z) = JX(z)^\top$ and thus the result follows from part (c) of Theorem 4.3. □

5. **Computational Issues.** We briefly address some computational issues, focusing on input and output sensitivities (i.e., we set $\gamma = 0$).

Computation of Gramians. By inspection of the contour integrals that define the sensitivity Gramians (3), (4) and (5) one might wonder how to effectively compute them. One approach is to use numerical methods for computing contour integrals. Let $\omega = \exp(\frac{2\pi i \tau}{s})$ denote a primitive s-th root of unity we consider the k, l-entries of the approximate Gramians, being defined as

$$(\Gamma_o^{(s)})_{k,l} := \frac{1}{s}\sum_{\tau=0}^{s-1} L_k L_l^\mathsf{T} \otimes Y_k(\omega^{-\tau})^\mathsf{T} Y_l(\omega^\tau),$$

$$(\Gamma_c^{(s)})_{k,l} := \frac{1}{s}\sum_{\tau=0}^{s-1} X_k(\omega^{-\tau}) X_l(\omega^\tau)^\mathsf{T} \otimes M_k^\mathsf{T} M_l,$$

$$(\Gamma_i^{(s)})_{k,l} := \frac{1}{s}\sum_{\tau=0}^{s-1} X_k(\omega^{-\tau}) X_l(\omega^\tau)^\mathsf{T} \otimes Y_k(\omega^{-\tau})^\mathsf{T} Y_l(\omega^\tau) \ .$$

Choose any $0 < r < 1 < \frac{1}{r}$ such that $A + BKC$ has all its eigenvalues outside the open annulus $A(r) = \{z \in \mathbb{C} : 0 < r < |z| < \frac{1}{r}\}$ and not on $|z| = \frac{1}{r}$. Then, using a result by [3], there is a $\kappa > 0$ the such that the approximation error $\Gamma_o - \Gamma_o^s$ satisfies the following bound

$$\|\Gamma_o - \Gamma_o^s\| \le \kappa \|LL^\mathsf{T}\| \frac{r^s}{1 - r^s}.$$

Similar estimates hold for the other two Gramians. Note that r can be chosen arbitrarily small, provided $A + BKC$ is nilpotent.

A second approach utilizes the block-Hadamard product representation (7) and computes the Gramians $\mathcal{W}_c$ and $\mathcal{W}_o$, by solving the discrete-time Lyapunov equations (6). A difficulty with this approach, for N large, lies in the large-scale nature of the Lyapunov equations

$$\mathcal{A}\mathcal{W}_c\mathcal{A}^\mathsf{T} - \mathcal{W}_c + \mathcal{B}\mathcal{B}^\mathsf{T} = 0, \quad \mathcal{A}^\mathsf{T}\mathcal{W}_o\mathcal{A} - \mathcal{W}_o + \mathcal{C}^\mathsf{T}\mathcal{C} = 0.$$

The network structure of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ cries for a distributed algorithm to solve (6). This will be done elsewhere.

Frequency Gramians. If all node transfer functions are SISO and identical, then frequency domain representations of the sensitivity Gramians can be obtained, similarly to [9]. Thus assume that all node systems are identical SISO systems, and are given by (A, b, c). Let $g(z) = c(zI - A)^{-1}b$ be the transfer function, with reciprocal $h(z) := 1/g(z)$. Let

$$g(z) = \frac{p(z)}{q(z)}$$

be a coprime polynomial factorization and $G(z) = g(z)I_N$. Thus $G(z)$ is invertible and $G(z)^{-1} = H(z) = h(z)I_N$. The transfer function of the network then can be expressed as

$$F(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} = M(h(z)I_N - K)^{-1}L.$$

Let

$$\Phi(z) = -M(I - g(z)K)^{-2}L$$

and define

$$\|\Phi(z)\|^2 := \operatorname{Tr}\left[\Phi(z)\Phi^\mathsf{T}(\tfrac{1}{z})\right].$$

The sensitivity function $\mathcal{S}_G(A,b,c)$ then takes the form

$$\mathcal{S}_G(A,b,c) = \alpha \operatorname{Tr}\left[\Gamma_c^g\right] + \beta \operatorname{Tr}\left[\Gamma_0^g\right] + \gamma \operatorname{Tr}\left[\Gamma_i^g\right]$$

with *frequency weighted sensitivity Gramians*

$$\Gamma_o^g := \frac{1}{2\pi i}\int\limits_{|z|=1} \|\Phi(z)\|^2 (\tfrac{1}{z}I - A^\top)^{-1}c^\top c(zI-A)^{-1}\,\tfrac{\mathrm{d}z}{z},$$

$$\Gamma_c^G := \frac{1}{2\pi i}\int\limits_{|z|=1} \|\Phi(z)\|^2 (zI-A)^{-1}bb^\top(\tfrac{1}{z} - A^\top)^{-1}\,\tfrac{\mathrm{d}z}{z},$$

$$\Gamma_i^G := \frac{1}{2\pi i}\int\limits_{|z|=1} \|\Phi(z)\|^2 (zI-A)^{-1}bb^\top(\tfrac{1}{z} - A^\top)^{-1}\otimes(\tfrac{1}{z}I - A^\top)^{-1}c^\top c(zI-A)^{-1}\,\tfrac{\mathrm{d}z}{z}.$$

We emphasize that these frequency weighted Gramians have reduced sizes $n\times n$, $n\times n$, and $n^2\times n^2$, respectively. In contrast to the sizes of sensitivity Gramians, which are $nN\times nN$, $nN\times nN$, and $n^2N\times n^2N$, respectively. Thus the frequency weighted Gramians can be computed with less effort than the sensitivity Gramians. One can also define frequency weighted Gramians for arbitrary heterogeneous networks of SISO node systems, and even for MIMO node systems with invertible square transfer functions. However, the case of node systems with non-square transfer functions is more complicated and cannot be easily handled along such lines.

Computing Sensitivity Optimal Realizations. Numerical algorithms for computing sensitivity optimal realizations have been proposed by [11], using either gradient steepest descent schemes or discrete-time Riccati equations. Here we propose a simple Newton algorithm that is fast (locally quadratically convergent) and globally convergent. For simplicity, we assume $\alpha=\beta=1$ and $\gamma=0$, i.e., we focus on the task of minimizing the sensitivity function defined by

$$\mathcal{S}_{co}(A,B,C) = \operatorname{Tr}\left[\Gamma_c + \Gamma_o\right].$$

The analysis can be extended to the general sensitivity function $\operatorname{Tr}\left[\Gamma_c+\Gamma_o+\Gamma_i\right]$, but at the expense of dealing with complicated expressions for the Newton step.

Let $(\mathcal{A},\mathcal{B},\mathcal{C})$ be reachable and observable. Define the block-diagonal matrices $\overline{\mathcal{W}}_c = \operatorname{diag}\left(\mathcal{W}_c^1,\cdots,\mathcal{W}_c^N\right)$ and $\overline{\mathcal{W}}_o = \operatorname{diag}\left(\mathcal{W}_o^1,\cdots,\mathcal{W}_o^N\right)$. To optimize the cost function $\mathcal{S}_{co}$ we first consider the simplified task of optimizing the cost function on the space $\mathcal{P}(\mathbf{n})$ of block-diagonal positive definite matrices $P=\operatorname{diag}\left(P_1,\cdots,P_N\right)$ as

$$\begin{aligned}\overline{\mathcal{S}}_{co}(P) &= \operatorname{Tr}\left[\Gamma_c(P\otimes I) + \Gamma_o(I\otimes P^{-1})\right]\\ &= \sum_{k=1}^N \operatorname{Tr}\left[\|M_k\|^2\mathcal{W}_c^k P_k + \|L_k\|\mathcal{W}_o^k P_k^{-1}\right].\end{aligned}\tag{12}$$

The gradient of (12), with respect to the Riemannian metric

$$\langle\xi,\eta\rangle_P := \operatorname{Tr}\left[P^{-1}\xi P^{-1}\eta\right],\quad \xi,\eta\in T_P\mathcal{P}(\mathbf{n})$$

on $\mathcal{P}(\mathbf{n})$, is easily computed as

$$\operatorname{grad}\overline{\mathcal{S}}_{co}(P) = \operatorname{diag}\left(\nabla_1,\cdots,\nabla_N\right)$$

with
$$\nabla_k = \|M_k\|^2 P_k \mathcal{W}_c^k P_k - \|L_k\|^2 \mathcal{W}_o^k, \quad k = 1, \dots, N.$$
To simplify the expressions for the subsequent formulas we assume that $\|M_k\| = \|L_k\| = 1$ for all k. Then the gradient becomes
$$\operatorname{grad} \overline{\mathcal{S}}_{co}(P) = P\overline{\mathcal{W}}_c P - \overline{\mathcal{W}}_0.$$
For $P \in \mathcal{P}(\mathbf{n})$, let $Z = Z(P) \in \mathcal{P}(\mathbf{n})$ denote the unique positive definite solution of the Lyapunov equation
$$Z\overline{\mathcal{W}}_c P + P\mathcal{W}_c Z = P\overline{\mathcal{W}}_c P - \overline{\mathcal{W}}_o. \tag{13}$$
Since $\overline{\mathcal{W}}_c, \overline{\mathcal{W}}_o$ and P are block diagonal, the solution $Z(P) = \operatorname{diag}(Z_1, \cdots, Z_N)$ is unique and block-diagonal. Thus solving (13) is equivalent to solve the decoupled system of Lyapunov equations
$$Z_k \mathcal{W}_c^k P_k + P_k \mathcal{W}_c^k Z = P_k \mathcal{W}_c^k P_k - \mathcal{W}_o^k.$$
We use the notation $P > 0$ to denote that P is symmetric and positive definite. For any initial point $P_0 \in \mathcal{P}(\mathbf{n})$ with $P_0\overline{\mathcal{W}}_c P_0 - \overline{\mathcal{W}}_o > 0$, we consider the iterative system
$$P_{t+1} = P_t - Z(P_t), \quad t = 0, 1, \dots. \tag{14}$$
Note that $Z\overline{\mathcal{W}}_c P + P\overline{\mathcal{W}}_c Z$ is just the directional derivative of the map $P \mapsto P\overline{\mathcal{W}}_c P - \overline{\mathcal{W}}_o$ in direction Z. Thus (14) is a Newton algorithm for the strictly convex cost function $\overline{\mathcal{S}}_{co}$ on $\mathcal{P}(\mathbf{n})$. Thus we expect global quadratic convergence of the algorithm. This is confirmed by the following theorem.

Theorem 5.1. *Let $\overline{\mathcal{W}}_c, \overline{\mathcal{W}}_o$ denote the block diagonal parts of the sensitivity Gramians Γ_c, Γ_o of a reachable and observable realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Assume $P_0 \in \mathcal{P}(\mathbf{n})$ is chosen such that $P_0\overline{\mathcal{W}}_c P_0 - \overline{\mathcal{W}}_o$ is positive definite. Then the Newton iterates (14) converge monotonically and locally quadratically fast to the sensitivity optimal $P_* \in \mathcal{P}(\mathbf{n})$. More precisely, we have*

(a) $(P_t)_t$ is monotonically decreasing , i.e. $P_t > P_{t+1}$ holds, and converges to the unique positive definite P_* satisfying $P_*\overline{\mathcal{W}}_c P_* = \overline{\mathcal{W}}_o$.

(b) $(P_t)_t$ converges locally quadratically fast to P_*.

A sensitivity optimal coordinate transformation $S_* \in GL(\mathbf{n})$ is obtained by computing a Cholesky factorization $S_*^\top S_* = P_*$ of P_*.

Proof. The symmetry of P_t is obvious. Assume P_t is positive definite. Then P_{t+1} is a solution of the Lyapunov equation
$$\begin{aligned} P_{t+1}\overline{\mathcal{W}}_c P_t + P_t\overline{\mathcal{W}}_c P_{t+1} &= 2P_t\overline{\mathcal{W}}_c P_t - P_{t+1}\overline{\mathcal{W}}_c Z(P_t) - Z(P_t)\overline{\mathcal{W}}_c P_t \\ &= P_t\overline{\mathcal{W}}_c P_t + \overline{\mathcal{W}}_o, \end{aligned}$$
with $P_t\overline{\mathcal{W}}_c$ having all eigenvalues with positive real part. Thus P_{t+1} is positive definite. We compute
$$\begin{aligned} P_{t+1}\overline{\mathcal{W}}_c P_{t+1} - \overline{\mathcal{W}}_o &= (P_t - Z_t)\overline{\mathcal{W}}_c (P_t - Z_t) - \overline{\mathcal{W}}_o \\ &= P_t\overline{\mathcal{W}}_c P_t - \overline{\mathcal{W}}_o - Z_t\overline{\mathcal{W}}_c P_t - P_t\overline{\mathcal{W}}_c Z_t + Z_t\overline{\mathcal{W}}_c Z_t \\ &= Z_t\overline{\mathcal{W}}_c Z_t + \overline{\mathcal{W}}_o > 0. \end{aligned}$$
Monotonicity follows from $P_t\overline{\mathcal{W}}_c P_t > 0$ and $P_t\overline{\mathcal{W}}_c Z_t + Z_t\overline{\mathcal{W}}_c P_t > 0$ and therefore $P_t - P_{t+1} = Z_t > 0$. Thus the limit $P_* = \lim_{t\to\infty} P_t$ exists. From the iteration we obtain $Z(P_*) = 0$ and therefore $P_*\overline{\mathcal{W}}_c P_* = \overline{\mathcal{W}}_o$ holds. Finally, local quadratic

convergence follows, since the iteration is a Newton algorithm and $\overline{\mathcal{S}}_{\rm co}$ is strictly convex. □

6. **Input-output sensitivity balancing of standard interconnections.**

The building blocks of interconnected systems are given by parallel, series and tree-like interconnections. To this end, we apply in this section the results of Section 4 to linear systems connected in parallel, series and via an interconnection graph being a tree.

Parallel connection. The parallel connection of N realizations (A_i, B_i, C_i) is defined by the interconnection matrices $K = 0, L = \mathrm{col}(I_m, \cdots, I_m)$ and $M = (I_p, \cdots, I_p)$. Thus, the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of the networked control systems is given by

$$\mathcal{A} = \mathrm{diag}(A_1, \cdots, A_N), \quad \mathcal{B} = \mathrm{col}(B_1, \cdots, B_N), \quad \mathcal{C} = (C_1, \cdots, C_N).$$

We assume that the parallel connected system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is reachable and observable. For necessary and sufficient conditions of reachability of parallel connected systems we refer to [4, 5]. The block-diagonal parts of the sensitivity Gramians Γ_c and Γ_o are

$$\mathrm{diag}(W_c^{(1)}, \cdots, W_c^{(N)}), \quad \mathrm{diag}(W_o^{(1)}, \cdots, W_o^{(N)}),$$

where $W_c^{(k)} = W_c(A_k, B_k, C_k)$ and $W_o^{(k)} = W_o(A_k, B_k, C_k)$ denote the standard discrete-time reachability and observability Gramians of (A_k, B_k, C_k).

Theorem 6.1. *Assume $m = p$ and $\alpha = \beta > 0, \gamma \geq 0$.*

(a) The parallel connection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is sensitivity minimal if and only if the node realizations (A_k, B_k, C_k) are sensitivity minimal.

(b) Let (A_k, B_k, C_k) be symmetric realizations with $A_k^\top = A_k$ and $C_k^\top = B_k$ for all k. Then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is sensitivity minimal.

(c) Assume $\gamma = 0$. Then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is sensitivity minimal if and only if $W_c^{(k)} = W_o^{(k)}$ for $k = 1, \ldots, N$, i.e., if and only if (A_k, B_k, C_k) are balanced realizations.

Proof. The first assertion follows immediately from Theorem 4.3, while the other assertions follow from Corollary 1. □

Series connection. Analyzing sensitivity optimization for series connections of systems is more difficult. We focus on input and output sensitivities. The interconnection matrices for a series connection of N realizations (A_k, B_k, C_k) are

$$K = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}, \quad L = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \cdots & 0 & I \end{pmatrix}.$$

The realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of the series connection is

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ B_2 C_1 & A_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_N C_{N-1} & A_N \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & \cdots & 0 & C_N \end{pmatrix}.$$

Let $G_i(z) = C_i(zI - A_i)^{-1}B_i$. To determine the input/output sensitivity Gramians we define

$$\mathbf{G}_k^N(z) := G_N(z)\cdots G_k(z), \quad k = 1, \dots, N.$$

We compute

$$(zI - \mathcal{A})^{-1}\mathcal{B} = \begin{pmatrix} (zI - A_1)^{-1}B_1 \\ (zI - A_2)^{-1}B_2G_1(z) \\ (zI - A_3)^{-1}B_3G_2(z)G_1(z) \\ \vdots \\ (zI - A_N)^{-1}B_NG_{N-1}(z)\cdots G_1(z) \end{pmatrix}$$

and

$$\begin{aligned} &\mathcal{C}(zI - \mathcal{A})^{-1} \\ &\quad = \begin{pmatrix} \mathbf{G}_2^N(z)\, C_1(zI - A_1)^{-1} & \mathbf{G}_3^N(z)\, C_2(zI - A_2)^{-1} & \cdots & C_N(zI - A_N)^{-1} \end{pmatrix}. \end{aligned}$$

The input/output sensitivity Gramians Γ_c, Γ_o are the block-diagonal matrices

$$\Gamma_c = \operatorname{diag}(\Gamma_c^{(1)}, \cdots, \Gamma_c^{(N)}), \qquad \Gamma_o = \operatorname{diag}(\Gamma_o^{(1)}, \cdots, \Gamma_o^{(N)}),$$

respectively, where

$$\begin{aligned} \Gamma_c^{(1)} &= W_c(A_1, B_1) \\ \Gamma_c^{(k)} &= \frac{1}{2\pi i}\int\limits_{|z|=1} (zI - A_k)^{-1}B_k\mathbf{G}_1^{k-1}(z)\mathbf{G}_1^{k-1}(z^{-1})^\top B_k^\top(z^{-1}I - A_k^\top)^{-1}\tfrac{\mathrm{d}z}{z} \end{aligned}$$

for $k = 2, ..., N$, and

$$\begin{aligned} \Gamma_o^{(l)} &= \frac{1}{2\pi i}\int\limits_{|z|=1} (z^{-1}I - A_l^\top)^{-1}C_l^\top\mathbf{G}_{l+1}^N(z^{-1})^\top\mathbf{G}_{l+1}^N(z)C_l(zI - A_l)^{-1}\tfrac{\mathrm{d}z}{z} \\ \Gamma_o^{(N)} &= W_o(C_N, A_N) \end{aligned}$$

for $l = 1, ..., N-1$. Here $W_c(A_k, B_k)$ and $W_o(C_k, A_k)$ denote the standard discrete-time reachability and observability Gramians of (A_k, B_k, C_k), respectively. Recall that a transfer function G is allpass, if $G(z)G(z^{-1})^\top = I$ holds for all $|z| = 1$. The preceding computation shows.

Theorem 6.2. *Assume that the transfer functions G_i are allpass for $i = 1, \dots, N$. Let $\gamma = 0$. The series connection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is sensitivity minimal if and only if the node realizations (A_k, B_k, C_k) are balanced, i.e. $W_c(A_k, B_k) = W_o(C_k, A_k)$ for all $k = 1, \dots, N$.*

Tree interconnections. Coupling N state space systems (A_k, B_k, C_k) in form of a tree, leads, after a suitable permutation, to interconnection matrices

$$K = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ K_{12} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ K_{1N} & \cdots & K_{N-1,N} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{pmatrix}, \quad M = \begin{pmatrix} M_1 & M_2 & \cdots & M_N \end{pmatrix}.$$

Here K_{ij} is zero, whenever the node system i is not connected with node j, and $L_i = I$ if the node i has a nonzero in-degree, $L_i = 0$ otherwise. Similarily, $M_i = I$ if node i has a nonzero out-degree, $M_i = 0$ otherwise. For simplicity, we consider the example of three systems in series/parallel connection, see Figure 1.

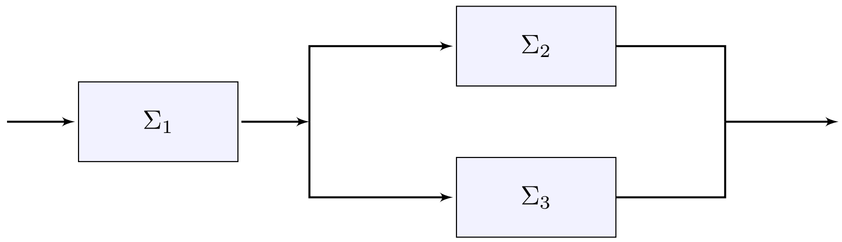

FIGURE 1. Network of three systems in series/parallel connection.

Then, the interconnection matrices are

$$K = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ I & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & I & I \end{pmatrix}$$

and we obtain the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of the networked control system

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 & 0 \\ B_2C_1 & A_2 & 0 \\ B_3C_1 & 0 & A_3 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & C_2 & C_3 \end{pmatrix}.$$

A calculation yields

$$(zI - \mathcal{A})^{-1}\mathcal{B} = \begin{pmatrix} (zI - A_1)^{-1}B_1 \\ (zI - A_2)^{-1}B_2G_1(z) \\ (zI - A_3)^{-1}B_3G_1(z) \end{pmatrix}$$

and

$$\mathcal{C}(zI - \mathcal{A})^{-1} = \\ \begin{pmatrix} (G_2(z) + G_3(z))C_1(zI - A_1)^{-1} & C_2(zI - A_2)^{-1} & C_3(zI - A_3)^{-1} \end{pmatrix}.$$

Consequently, the diagonal entries of the sensitivity Gramians are

$$\begin{aligned} \Gamma_c^{(1)} &= W_c(A_1, B_1) \\ \Gamma_c^{(k)} &= \frac{1}{2\pi i} \int\limits_{|z|=1} (zI - A_k)^{-1}B_kG_1(z)G_1(z^{-1})^\top B_k^\top (z^{-1}I - A_k^\top)^{-1} \tfrac{\mathrm{d}z}{z} \end{aligned}$$

and

$$\begin{aligned} \Gamma_o^{(1)} &= \tfrac{1}{2\pi i} \int\limits_{|z|=1} (\tfrac{1}{z}I - A_1^\top)^{-1}C_1^\top \big(G_2(\tfrac{1}{z})^\top + G_3(\tfrac{1}{z})^\top\big)\big(G_2(z) + G_3(z)\big)C_1(zI - A_1)^{-1} \tfrac{\mathrm{d}z}{z} \\ \Gamma_o^{(k)} &= W_o(C_k, A_k) \end{aligned}$$

for $k = 2, 3$. Consequently, we obtain the following

Theorem 6.3. *Assume that the transfer functions $G_1(z)$ and $G_2(z) + G_3(z)$ are allpass. Then the series/parallel connection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is sensitivity minimal if and only if (A_k, B_k, C_k) are balanced realizations for $k = 1, 2, 3$.*

7. Conclusions. In this paper we consider heterogeneous networks of N linear systems, where the node dynamics are given by multi-input-multi-output (MIMO) linear systems Σ_i, represented by strictly proper transfer functions $G_i(z)$. The interconnection structure of the networked system is regarded as fixed and known. We extend the sensitivity analysis from single systems to interconnected systems where we considered the sensitivity of the network transfer function with respect to the realizations (A, B, C) of the node systems. To this end, we adapt the well-known L^2-sensitivity measure for single linear systems to the class of heterogeneous networks of linear MIMO systems. A new class of network sensitivity Gramians is introduced whose traces provide the sensitivity measure. Existence and uniqueness of sensitivity optimal realizations is shown. Also we discuss the linear systems connected in series and in parallel as well as networks with tree structure.

REFERENCES

[1] J.B. Cruz and W.R. Perkins. A new approach to the sensitivity problem in multivariable feedback system design. *IEEE T. Automat. Contr.*, **9** (1964), 216–223.

[2] D.F. Delchamps. New geometric approaches to parameter sensitivity in feedback systems. In *C.I. Byrnes and A. Lindquist (Eds.), Modelling, Identification and Robust Control*, pp. 445–456, Elsevier Science Publishers B.V. (North-Holland), 1986.

[3] R. Fornaro. Numerical evaluation of integrals around simple closed curves. *SIAM J. Numer. Anal.*, **10** (1973), 623–634.

[4] P.A. Fuhrmann and U. Helmke. Reachability, observability and strict equivalence of networks of linear systems. *Math. Control Signal.*, **25** (2013), 437–471.

[5] P.A. Fuhrmann and U. Helmke. *The Mathematics of Networks of Linear Systems.* Cham: Springer, 2015.

[6] M. Gevers and G. Li. *Parametrizations in control, estimation and filtering problems: accuracy aspects.* Springer, Berlin, 1993.

[7] S. Hara. A unified approach to decentralized cooperative control for large-scale networked dynamical systems. In *J.C. Willems et al. (Eds.), Perspectives in mathematical system theory, control, and signal processing*, pp. 61–72, Springer, Berlin, 2010.

[8] S. Hara, T. Hayakawa, and H. Sugata. LTI systems with generalized frequency variables: A unified framework for homogeneous multi-agent dynamical systems. *SICE Journal of Control, Measurement, and Systems Integration*, **2** (2009), 299–306.

[9] U. Helmke, I. Kurniawan, P. Lang and M. Schönlein. Sensitivity optimal design of networks of identical linear systems. In *Proc. Mathematical Theory of Networks and Systems (MTNS2012)*, Melbourne, Australia, 5-9 July 2012, paper 0283.

[10] U. Helmke and J. B. Moore. L^2 sensitivity minimization of linear system representations via gradient flows. *J. Math. Syst. Estim. Control*, **5**(1) (1995), 79–98.

[11] U. Helmke and J. B. Moore. *Optimization and Dynamical Systems.* Springer, London, 1994.

[12] R.A. Horn, R. Mathias. Block-matrix generalizations of Schur's basic theorems on Hadamard products. *Linear Algebra Appl.*, **172** (1992), 337–346.

[13] S. Koshita, M. Abe, and M. Kawamata. Analysis of second-order modes of linear discrete-time systems under bounded-real transformations. *IEICE T. Fund. Electr.*, **90** (2007), 2510–2515.

[14] S. Liu. Matrix results on the Khatri-Rao and Tracy-Singh products. *Linear Algebra Appl.*, **289** (1998), 267–277.

[15] J. Lunze. *Control theory of digitally networked dynamic systems.* Cham: Springer, 2014.

[16] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks.* Princeton University Press, 2010.

[17] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE T. Automat. Contr.*, **26** (1981), 17–32.

[18] C. T. Mullis and R. A. Roberts. Roundoff noise in digital filters: frequency transformations and invariants. *IEEE T. Acoust. Speech*, **24** (1976), 538–550.

[19] C. T. Mullis and R. A. Roberts. Synthesis of minimum roundoff noise fixed point digital filters. *IEEE T. Circuits Syst.*, **23** (1976), 551–562.

[20] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, **95** (2007), 215–233.

[21] L. Pernebo and L. M. Silverman. Model reduction via balanced state space representations. *IEEE T. Automat. Contr.*, **27** (1982), 382–387.

[22] V. Tavsanoglu and L. Thiele. Optimal design of state-space digital filters by simultaneous minimization of sensitivity and roundoff noise. *IEEE T. Automat. Contr.*, **31** (1984), 884–888.

[23] L. Thiele. On the sensitivity of linear state-space systems. *IEEE T. Circuits Syst.*, **33** (1986), 502–510.

[24] W.-Y. Yan, J. B. Moore and U. Helmke. Recursive algorithms for solving a class of nonlinear matrix equations with applications to certain sensitivity optimization problems. *SIAM J. Control Optim.*, **32** (1994), 1559–1576.

[25] G. Zames. Functional analysis applied to nonlinear feedback systems. *IEEE T. Circuits Syst.*, **10** (1963), 392–404.

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