# Uniform Ensemble Controllability of Parametric Systems*

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***Abstract*— In this paper we study classes of parametric ensembles of systems which are defined by a parameter dependent family of linear and bilinear control systems. Using well-known characterizations of approximate controllability of systems in a Banach space, we present a unified approach to uniform ensemble control of parameter-varying linear and bilinear systems. Both time-invariant and time-varying linear systems are treated, leading to new necessary and sufficient conditions for ensemble controllability. We also address the issue of output controllability, thus extending recent results on average controllability of linear systems. A characterization of output controllability of families of bilinear systems is derived, together with a counterexample that shows the limitations of our approach to controlling ensembles of nonlinear systems.**

## I. Introduction

There has been recently much interest in studying motion control problems for spatio-temporal systems and infinite platoons of vehicles [2], [7], [11], [23], where control actions and measurements take place in a spatially distributed way. Such control systems are described by partial differential or partial difference equations and therefore belong to the realm of infinite-dimensional systems theory. Using Fourier-transform techniques, infinite platoons of systems, where the controls are broadcasted to all subsystems, can be identified with parameter-dependent families of control systems

$$\frac{\partial x}{\partial t}(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t).$$

Other classes of parametric ensembles include families of discrete-time linear systems [16], time-varying linear systems [19] and bilinear control systems [20], [21]. The latter class is of particular interest in applications to quantum control and the robotics [3], [4], [20], [21]. A classical and well-studied case of parametric ensembles is defined by the family of output feedback equivalent systems $(A + BKC, B, C)$, parameterized by a compact subset $\mathbf{P} \subset \mathbb{R}^{m\times p}$ of output feedback gains $K$; see [15] for stabilizability results of one-parameter families. The analysis of such families of systems relates to the classical blending problem in robust feedback controller design [18]. In stochastic control of finite-dimensional systems, a frequently considered statistical technique consists in controlling probability distributions of the state variables [12]. This leads to the control of partial differential equations, such as the Liouville transport equation [5] or, more generally, Fokker-Planck equations [10]. We refer to [6] for a recent approach to ensemble control of probability distributions based on ideas from Monge-Kantorovich optimal mass transport. We also mention recent work by [29] on average control of parametric linear systems. The problems studied in [29] are equivalent to output ensemble controllability, a topic which is discussed subsequently. In all these areas the question arises of how to approximately control, or observe, a family of systems and state variables.

A key point in controlling ensembles of systems is that the control tasks have to be achieved using an input function which is *independent* of the parameters of the systems. From an operator theoretic point of view, parametric ensembles of systems can therefore be regarded as infinite dimensional systems on Banach spaces, with the special feature that the input operator has finite-dimensional range. In this framework, ensemble controllability becomes equivalent to the classical notion of approximate controllability. We refer to [8], [13], [14], [28] for characterizations of approximate controllability of infinite-dimensional linear systems. Although functional analytic methods have been applied to ensemble control for a longer time [3], [7], [19], [20], [21], our systematic approach to ensemble controllability of linear systems seems to be new.

This paper is organized as follows. In Section II we consider ensembles of linear time-invariant systems. Using standard characterizations of approximate controllability in a separable Banach space we present new proofs of the main known results for ensemble control of linear parameter dependent time-invariant systems [19], [16], [25]. In Section III the main contributions are presented. In Theorem 4 we extend these results to time-varying systems and illustrate the effectiveness of our approach by discussing a controlled variant of the Sturm-Liouville equation. Based on [1], [28], necessary and sufficient conditions for output controllability of ensembles of linear and bilinear systems are derived in Section IV that extend recent results by [24], [29]. Finally, in Section V we present an example that shows that the approach in [20], [21], developed for a specific example of quantum control systems, cannot be extended in a straightforward way to general ensembles of bilinear systems.

## II. Ensemble Controllability of Parametric Linear Systems

In this paper our focus is on linear continuous-time systems, but most of the results hold also for discrete-time systems. We assume that the parameters $\theta$ of the system vary in a nonempty compact subset $\mathbf{P} \subset \mathbb{R}^d$. This includes the well-understood case of parallel interconnected linear systems, where $\mathbf{P}$ is a finite set.

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### A. Parametric ensembles

We consider a family of parameter-dependent linear time-invariant control systems represented by

$$\begin{aligned} \tfrac{\partial x}{\partial t}(t,\theta) &= A(\theta)x(t,\theta) + B(\theta)u(t) \\ x(0,\theta) &= x_0(\theta) \in \mathbb{R}^n. \end{aligned} \tag{1}$$

We assume that the system matrices $A(\theta) \in \mathbb{R}^{n\times n}$ and $B(\theta) \in \mathbb{R}^{n\times m}$ depend continuously on $\theta \in \mathbf{P}$, i.e. $A \in C(\mathbf{P}, \mathbb{R}^{n\times n})$, $B \in C(\mathbf{P}, \mathbb{R}^{m\times n})$. For $u \in L^1([0,T], \mathbb{R}^m)$, the solution to (1) is

$$\varphi\big(T, x_0(\theta), u\big) = e^{A(\theta)T}x_0(\theta) + \int_0^T e^{A(\theta)(T-s)}B(\theta)u(s)\,\mathrm{d}s.$$

An ensemble $\Sigma = \{\big(A(\theta), B(\theta)\big) \mid \theta \in \mathbf{P}\}$ is called *uniformly ensemble controllable on* $[0,T]$, if for any continuous families of initial states $x_0(\theta) \in \mathbb{R}^n$ and terminal states $x^*(\theta) \in \mathbb{R}^n$ and any $\varepsilon > 0$ there is an input function $u \in L^1([0,T], \mathbb{R}^m)$ such that

$$\sup_{\theta\in\mathbf{P}} \|\varphi(T, x_0(\theta), u) - x^*(\theta)\| < \varepsilon. \tag{2}$$

If $T \geq 0$ is not fixed in advance, i.e. if $T \geq 0$ may additionally depend on the intial and terminal states families $x_0(\theta)$ and $x^*(\theta)$, then $\Sigma$ is called *uniformly ensemble controllable*. We emphasize that a crucial requirement in both definitions is the independence of the input function form the parameters. Without this requirement the controllability analysis of systems (1) would be much simpler.

From an operator theoretic point of view, ensembles (1) can be regarded as infinite dimensional systems defined on a separable Banach space $(X, \|\cdot\|_X)$ of functions from the parameter space $\mathbf{P}$ to $\mathbb{R}^n$. Here we assume that $X$ contains the space of continuous functions. Examples include, e.g., the Banach spaces $X = C(\mathbf{P}, \mathbb{R}^n)$ and $X = L^q(\mathbf{P}, \mathbb{R}^n)$, with $q \in [1, \infty)$. Every continuous family of system matrices $\big(A(\theta), B(\theta)\big)$ induces the bounded linear operators $\mathcal{A} : X \to X$ and $\mathcal{B} : \mathbb{R}^m \to X$, where

$$(\mathcal{A}x)(\theta) = A(\theta)x(\theta), \quad \mathcal{B}u(\theta) = B(\theta)u. \tag{3}$$

The parametric ensemble of linear systems (1) then is equivalent to the linear control system on the Banach space $X$

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t). \tag{4}$$

With this identification at hand, the above notion of uniform ensemble controllability for (1) becomes equivalent to *approximate controllability* of the infinite-dimensional linear system (4) on the Banach space $C(\mathbf{P}, \mathbb{R}^n)$. Approximate controllability of (4) on another Banach space, such as $L^q(\mathbf{P}, \mathbb{R}^n)$, $q \in \mathbb{N}$, then corresponds to an equivalent notion of $L^q$-ensemble controllability.

*Remark 1:* Note that for arbitrary bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ one has the following equivalences, see [28], Thm. 3.1.1 and Remark 3.1.2:

(a) System (4) is approximately controllable.
(b) There exists $T > 0$ such that system (4) is approximately controllable on $[0,T]$.
(c) For all $T > 0$, system (4) is approximately controllable on $[0,T]$.

Hence, the parametric ensemble (1) is uniformly ensemble controllable if and only if it is uniformly ensemble controllable on some $[0,T]$. For time-dependent systems $\big(\mathcal{A}(t), \mathcal{B}(t)\big)$, however, the implications $(a) \implies (b)$ and $(a) \implies (c)$ fail in general, even if the state space $X$ is finite dimensional and $\mathcal{A}(t)$, $\mathcal{B}(t)$ depend analytically on $t$. An example follows. For time-independent, but unbounded operators $\mathcal{A}$, $\mathcal{B}$ similar problems occur, see [8].

*Example 1:* Consider the time-dependent linear system

$$\dot{x} = Ax(t) + u(t)B(t)y_0, \quad x_0 \in \mathbb{R}^2, \tag{5}$$

where $A$ and $B(t)$ are given by

$$A := \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B(t) := \mathrm{e}^{\alpha t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

with $\alpha \in \mathbb{R}$ fixed and $y_0 \neq \mathbb{R}^2 \backslash \{0\}$ can be chosen arbitrarily. Note, that $B(t) = \mathrm{e}^{At}$ holds for all $t \in \mathbb{R}$. A straightforward computation shows that (5) is never approximately controllable on any finite subinterval of $[0, \infty)$. However, for each pair $(x_0, x^*)$ of initial and final states there exists a nontrivial subinterval $[t_0, T] \subset [0, \infty)$ and a control $u : [t_0, T] \to \mathbb{R}$ which steers $x_0$ to $x^*$. Moreover, if $|\alpha| < 1$ holds, one can choose $t_0 = 0$ for all $x_0 \in \mathbb{R}^2$ and $x^* \neq 0$.

Fundamental contributions to the controllability analysis of infinite–dimensional systems (4) were obtained for Hilbert spaces by Fuhrmann [13], [14], and for general separable Banach spaces by Trigianni [28]. Two central results are proven in [28]. The first result shows that exact controllability of (4) never holds, due to the fact that the input operator $\mathcal{B} : \mathbb{R}^m \to X$ has a finite-dimensional range. This implies that ensemble controllability for systems (1) is never satisfied in an exact sense, i.e., for $\varepsilon = 0$. Second, Trigianni shows that approximate controllability of (4) is equivalent to the density condition

$$\overline{\sum_{k\in\mathbb{N}_0} \operatorname{im} \mathcal{A}^k\mathcal{B}} = X.$$

By re-interpreting this characterization in the Banach space $C(\mathbf{P}, \mathbb{R}^n)$ we arrive at the following characterization of uniform ensemble controllability.

*Theorem 1 ([25]):* Let $A \in C(\mathbf{P}, \mathbb{R}^{n\times n})$, $B \in C(\mathbf{P}, \mathbb{R}^{n\times m})$ and let $b_1, \ldots, b_m$ denote the columns of $B$. The following assertions are equivalent.

(a) The ensemble $\Sigma = \{\big(A(\theta), B(\theta)\big) \mid \theta \in \mathbf{P}\}$ is uniformly ensemble controllable.
(b) The set
$$\mathcal{L}_\Sigma := \operatorname{span}\{A^k b_j \mid 1 \leq j \leq m,\ k \in \mathbb{N}_0\}$$
is dense in $C(\mathbf{P}, \mathbb{R}^n)$ with respect to the sup-norm.

This result, which is an immediate consequence of Theorem 3.1.1 in [28], comprises previous characterizations of uniform ensemble controllability by [22] and [25]. We note that Theorem 1 is valid for discrete-time ensembles, too. Also we emphasize that $\mathcal{L}_\Sigma$ is dense in $C(\mathbf{P}, \mathbb{R}^n)$ if and only if for each $\varepsilon > 0$ and each $x^* \in C(\mathbf{P}, \mathbb{R}^n)$ there exist real scalar polynomials $p_1, \dots, p_m$ such that

$$\sup_{\theta \in \mathbf{P}} \| \sum_{j=1}^{m} p_j(A(\theta))\, b_j(\theta) - x^*(\theta) \| < \varepsilon. \tag{6}$$

The key issue is that this links uniform ensemble controllability to a polynomial approximation problem. The following result in [16] is a simple consequence of Theorem 1, using Mergelyan's Theorem from complex approximation theory. For the definition of the Hermite indices of a reachable pair $(A, B)$ see, e.g. [17].

*Theorem 2 ([16]):* Let $\mathbf{P} \subset \mathbb{R}$ be a finite union of disjoint compact intervals. The ensemble of linear systems $\Sigma = \{(A(\theta), B(\theta)) \mid \theta \in \mathbf{P}\}$ is uniformly ensemble controllable if the following conditions are satisfied:

(a) $(A(\theta), B(\theta))$ is reachable for all $\theta \in \mathbf{P}$.
(b) The input Hermite indices $K_1(\theta), \dots, K_m(\theta)$ of $(A(\theta), B(\theta))$ are independent of $\theta \in \mathbf{P}$.
(c) For any pair of distinct parameters $\theta, \theta' \in \mathbf{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$

(d) For each $\theta \in \mathbf{P}$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.

Concerning the necessity of the above conditions, the following in known: If $\Sigma = \{\big(A(\theta), B(\theta)\big) \mid \theta \in \mathbf{P}\}$ is uniform ensemble controllable then it is easy to show that for every $\theta \in \mathbf{P}$ the linear system $(A(\theta), B(\theta))$ has to be reachable. Moreover, for each number $s \geq m+1$ of distinct parameters $\theta_1, \dots, \theta_s \in \mathbf{P}$, the spectra of $A(\theta)$ must satisfy

$$\sigma(A(\theta_1)) \cap \dots \cap \sigma(A(\theta_s)) = \emptyset.$$

This shows that condition (a) in Theorem 2 is necessary, while condition (c) is necessary for single input systems. In contrast, one can show cf. [16], [25], that neither conditions (b) nor (d) in Theorem 2 are necessary for uniform ensemble controllability.

### B. Homogenous Ensembles

Starting from these two results we study a simple example. For fixed matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ consider the parametric ensemble

$$\tfrac{\partial}{\partial t} x(t, \theta) = \theta A x(t, \theta) + B u(t), \tag{7}$$

controlled by $m = n$ independent inputs. Let $\lambda_1, \dots, \lambda_n$ denote the (real or complex) eigenvalues of $A$. A special case is given by the family of harmonic oscillators

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By the above spectral condition, $A$ has to be invertible if $\Sigma = \{(\theta A, B) \mid \theta \in \mathbf{P}\}$ is uniformly ensemble controllable. Moreover, provided $0 \in \mathbf{P}$, then $B$ must be invertible, too. Thus, if $0 \in \mathbf{P}$ the invertibility of $A, B$ is necessary for (8) to be uniformly ensemble controllable. We next show that these two are also sufficient for uniform ensemble controllability. After linear coordinate transformations in the state space and input space one sees that $(A, B)$ is equivalent to a system $(SAS^{-1}, SBU)$ in Jordan canonical form

$$\tfrac{\partial}{\partial t} x(t, \theta) = \theta \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} x(t, \theta) + \begin{pmatrix} I & & \\ & \ddots & \\ & & I \end{pmatrix} u(t). \tag{8}$$

It is immediately seen from the decoupled form of (8) that (8) is uniformly ensemble controllable if and only if each subsystem

$$\tfrac{\partial}{\partial t} x_i(t, \theta) = \theta J_i x_i(t, \theta) + u_i(t) \tag{9}$$

is uniformly ensemble controllable. By inspection, these subsystems are uniformly ensemble controllable if and only if the last component equation $\dot{x}_{ik_i} = \theta \lambda_i x_{ik_i} + u_{ik_i}(t)$ in (9) is uniformly ensemble controllable. But this follows immediately from Theorem 2 for the special case $m = 1$, thus proving the following generalization of a result by [22] (which treated the simpler case where $\mathbf{P}$ is an interval).

*Corollary 1:* Let $m \geq n$ and let $\mathbf{P} \subset \mathbb{R}$ be the finite union of compact intervals with $0 \in \mathbf{P}$. Then $\Sigma = \{(\theta A, B) \mid \theta \in \mathbf{P}\}$ is uniformly ensemble controllable if and only if $\operatorname{rank} A = n$ and $\operatorname{rank} B = n$.

Corollary 1 deals with the situation where the number of inputs is bigger or equal to the number of states. It implies that the harmonic oscillator family with two independent inputs is always uniformly ensemble reachable; a fact that was first proven in [19] using complicated computations. If $m < n$, an application of Theorem 2 immediately yields the next result.

*Corollary 2 ([25]):* Let $m \leq n$ and let $\mathbf{P}$ be the finite union of compact real intervals with $0 \notin \mathbf{P}$. Define $\lambda \mathbf{P} := \{\lambda\, \theta \,|\, \theta \in \mathbf{P}\}$.

(a) If $\Sigma = \{(\theta A, B) \,|\, \theta \in \mathbf{P}\}$ is uniformly ensemble controllable then $(A, B)$ is controllable and $A$ is invertible.
(b) Let $(A, B)$ be controllable and let $A$ be invertible with simple eigenvalues $\lambda_k$ such that $\lambda_k \mathbf{P} \cap \lambda_l \mathbf{P} = \emptyset$ for all $k \neq l$. Then $\Sigma = \{(\theta A, B) \,|\, \theta \in \mathbf{P}\}$ is uniformly ensemble controllable.

*Proof:* (a) If $\Sigma = \{(\theta A, B) \,|\, \theta \in \mathbf{P}\}$ is uniformly ensemble controllable, then $(\theta A, B)$ is controllable for every $\theta \in \mathbf{P}$, i.e. $(A, B)$ is controllable. Suppose that zero is an eigenvalue of $A$. For distinct parameters $\{\theta_1, ..., \theta_{m+1}\} \in \mathbf{P}$ then $0 \in \sigma\big(\theta_1 A\big) \cap \dots \cap \sigma\big(\theta_{m+1} A\big)$, in contradiction to $\sigma\big(\theta_1 A\big) \cap \dots \cap \sigma\big(\theta_{m+1} A\big) = \emptyset$.

(b) We verify the sufficient conditions of Theorem 2. The reachability of the pair $(A, B)$ and the fact that $0 \notin \mathbf{P}$ implies that $(\theta\, A, B)$ is reachable for every $\theta \in \mathbf{P}$. Since $0 \notin \mathbf{P}$, the Hermite indices of $(\theta A, B)$ are independent of $\theta$. Moreover, $\lambda_k \mathbf{P} \cap \lambda_l \mathbf{P} = \emptyset$ for all $k \neq l$ implies that $\sigma\big(\theta A\big) \cap \sigma\big(\theta' A\big) = \emptyset$ for all $\theta \neq \theta' \in \mathbf{P}$. Thus the assumptions of Theorem 2 are satisfied and we are done. ■

## III. Time-variant Ensembles

In this section we consider ensembles defined by linear time-varying parameter-dependent systems

$$\begin{aligned}\tfrac{\partial x}{\partial t}(t,\theta) &= A(t,\theta)x(t,\theta) + B(t,\theta)u(t)\\ x(0,\theta) &= x_0(\theta) \in \mathbb{R}^n.\end{aligned} \tag{10}$$

We assume that $A(t,\theta)$ and $B(t,\theta)$ are of class $C^{\infty,0}$, i.e., the matrices $A(t,\theta)$ and $B(t,\theta)$ are smooth with respect to $t$ and all partial derivatives $\frac{\partial^k}{\partial t^k}A(t,\theta)$ and $\frac{\partial^k}{\partial t^k}B(t,\theta)$, $k \in \mathbb{N}_0$ are continuous on $[0,\infty) \times \mathbf{P}$. Moreover, if the $t$-expansions

$$A(t,\theta) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\partial^k}{\partial t^k}A(\tau,\theta)(t-\tau)^k$$

and

$$B(t,\theta) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\partial^k}{\partial t^k}B(\tau,\theta)(t-\tau)^k$$

converge uniformly in $\theta \in \mathbf{P}$ we say the $A(t,\theta)$ and $B(t,\theta)$ are of class $C_{\mathrm{u}}^{\omega,0}$ Let $\Phi(t,s,\theta)$ denote the transition matrix of the time-varying system $\dot{x}(t) = A(t,\theta)x(t)$. Then, for $u \in L^1([0,T],\mathbb{R}^m)$, the solution to (10) is

$$\varphi(T, x_0(\theta), u) = \Phi(T,0,\theta)x_0(\theta) + \int_0^T \Phi(T,s,\theta)B(\theta)u(s)\,\mathrm{d}s.$$

The ensemble control task for (10) then is equivalent to approximate controllability of a time-varying linear system of the from (12) on a suitable Banach space $X$ of functions $\mathbf{P} \to \mathbb{R}^n$, where $\mathcal{A}$ has to be chosen as multiplication operator $\mathcal{A}(t) : X \to X$ defined by

$$\mathcal{A}(t)x(\theta) := A(t,\theta)x(\theta),$$

and similarly for $\mathcal{B}(t)$. Ensemble controllability of (10) has been first characterized by [19] in terms of growth conditions on the non-zero singular values of the reachability operator

$$\mathcal{R}_T : u \mapsto \int_0^T \Phi(T,s,\cdot)B(s,\cdot)u(s)\,\mathrm{d}s \tag{11}$$

into $L^2(\mathbf{P},\mathbb{R}^n)$. However, these conditions are very difficult to check, even for very simple second-order time-invariant linear systems. Moreover, they are valid only for studying ensemble controllability in a Hilbert space context. Thus problems of uniform ensemble controllability are not accessible with the methods of [19]. Therefore, we aim at both simplifying and extending this approach, using a classical approximate controllability condition on Banach spaces, first derived by [28].

In a finite dimensional vector space, necessary and sufficient conditions for controllability of time-varying linear systems are well-known for a long time; see e.g. [26], [27]. Triggiani has generalized this characterization to approximate controllability of time-varying linear systems in separable Banach spaces.

Let $X$ and $U$ be separable Banach spaces. Assume that $\mathcal{A}(t) : X \to X$ and $\mathcal{B}(t) : U \to X$ are bounded operators which depend smoothly on $t \in [0,\infty)$ and consider

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t). \tag{12}$$

Moreover, define recursively the operators $\Gamma_0(t) := \mathcal{B}(t)$ and $\Gamma_{k+1}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_k(t) - \mathcal{A}(t)\Gamma_k(t)$. As a short hand notation we use

$$\Gamma_k = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - \mathcal{A}\right)^k\mathcal{B}, \qquad k \in \mathbb{N}_0. \tag{13}$$

To avoid miss interpretations, see also the subsequent example of Sturm-Liouville equation. The reachability subspace of (12) at $\tau \in [0,\infty)$ is then given by

$$\sum_{k\in\mathbb{N}_0} \operatorname{im}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - \mathcal{A}\right)^k\mathcal{B}\Big|_{t=\tau} \subset X\,.$$

*Theorem 3 ([28]):* The nonautonomous system (12) is approximately controllable on $[t_0,T] \subset [0,\infty)$ provided

$$\overline{\sum_{k\in\mathbb{N}_0} \operatorname{im}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - \mathcal{A}\right)^k\mathcal{B}\Big|_{t=\tau}} \subset X\,. \tag{14}$$

holds for some $\tau \in [t_0,T]$. If $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are analytic on $[0,\infty)$, then (12) is approximately controllable on every nontrivial subinterval of $[0,\infty)$ if and only if (14) holds for some $\tau \in [0,\infty)$.

Applying the preceding result to time-variant linear ensembles immediately yields the following explicit characterization of ensemble controllability on $C(\mathbf{P},\mathbb{R}^n)$.

*Theorem 4:* Suppose that $A(t,\theta)$ and $B(t,\theta)$ are of class $C^{\infty,0}$. Then the time-varying ensemble $\Sigma := (A(t,\theta), B(t,\theta))$ is uniformly ensemble controllable on $[t_0,T]$ if

$$\mathcal{L}_\Sigma(\tau) := \operatorname{span}\left\{ \left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - A\right)^k b_j\big|_{t=\tau} \;\Big|\; 1 \le j \le m\,, k \in \mathbb{N}_0 \right\}$$

is dense in $C(\mathbf{P},\mathbb{R}^n)$ with respect to the sup-norm for some $\tau \in [t_0,T]$, where $b_j$ denotes the $j$-th column of $B$. If $A(t,\theta)$ and $B(t,\theta)$ are additionally of class $C_{\mathrm{u}}^{\omega,0}$, then $\Sigma$ is uniformly ensemble controllable on every nontrivial subinterval of $[0,\infty)$ if and only if $\mathcal{L}_\Sigma(\tau)$ is dense for some $\tau \in [0,\infty)$.

The above theorem is stated for uniform ensemble controllability, using the Banach space $C(\mathbf{P},\mathbb{R}^n)$ of continuous functions. More general results hold and can be stated for weaker notions of ensemble controllability, e.g., by working in the Hilbert space $L^2(\mathbf{P},\mathbb{R}^n)$. It is in fact easily seen that the reachability operator $\mathcal{R}_T : L^2([0,T],\mathbb{R}^m) \to L^2(\mathbf{P},\mathbb{R}^n)$ in (11) has a dense image if and only if $\mathcal{L}_\Sigma(\tau)$ is dense in $L^2(\mathbf{P},\mathbb{R}^n)$. In order to illustrate the efficiency of the preceding condition we study a simple but nontrivial example.

*Sturm-Liouville equation*

Let $\phi(t)$ be a real analytic function on $[0,\infty)$ and $\omega$ a parameter that varies in a compact interval $[\omega_-,\omega_+]$. Let $a(t,\omega) := \omega\phi(t)$ and

$$A(t,\omega) = \begin{pmatrix} 0 & 1 \\ -a(t,\omega) & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We consider the issue of controllability for the time-varying second order control system

$$\ddot{x}(t) + \omega\phi(t)x(t) = u(t). \tag{15}$$

A straighforward computation shows that the first $\Gamma_k$ are given by, where dots denote time-derivatives:

$$\begin{aligned}
\Gamma_0 &= b = e_2 \\
\Gamma_1 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)b = e_1, \\
\Gamma_2 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^2 b = -ae_2, \\
\Gamma_3 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^3 b = -ae_1 + \dot{a}e_2, \\
\Gamma_4 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^4 b = 2\dot{a}e_1 + (a^2 - \ddot{a})e_2, \\
\Gamma_5 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^5 b = (a^2 - 3\ddot{a})e_1 + (a^{(3)} - 4a\dot{a})e_2, \\
\Gamma_6 &= (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^6 b \\
&= (4a^{(3)} - 6a\dot{a})e_1 - (a^3 - 7a\ddot{a} + a^{(4)} - 4\dot{a}^2)e_2.
\end{aligned}$$

It is easily seen by induction that every term $(A - \frac{\mathrm{d}}{\mathrm{d}t}I)^k b$ is a differential polynomial in $a$ which, for fixed values of $t$, defines a polynomial of degree $\leq n$ in $\omega$. Define for $n = 0, 1, 2, \ldots$ the sequence of finite dimensional subspaces

$$\mathcal{V}_n := \operatorname{span}\{(A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)^k b \mid k = 0, \ldots, 2n\} \subset C([0,\infty) \times [\omega_-,\omega_+], \mathbb{R}^2).$$

Then for each $\tau \in [0,\infty)$, the sets

$$\mathcal{V}_n(\tau) = \operatorname{span}\{f(\tau) \mid f \in \mathcal{V}_n\}$$

and

$$\dot{\mathcal{V}}_n(\tau) = \operatorname{span}\{\dot{f}(\tau) \mid f \in \mathcal{V}_n\}$$

are finite-dimensional $\mathbb{R}$-subspaces of $\mathbb{R}[\omega]^2$. Let $\mathbb{R}[\omega]_{\leq n}$ denote the space of polynomials of degree $\leq n$. Then one has the following result.

*Lemma 1:* Let $\tau$ be any real number with $\phi(\tau) \neq 0$. Then

$$\mathcal{V}_n(\tau) = \mathbb{R}[\omega]^2_{\leq n}$$

holds for all $n \in \mathbb{N}_0$.

*Proof:* We prove the assertion by induction on $n$. For $n = 0$ we have $\mathcal{V}_0 = \mathbb{R}^n = \mathbb{R}[\omega]^2_{\leq 0}$ and therefore the claim holds for $n = 0$. Now assume that the assertion holds for $\mathcal{V}_n$. Every element of $\mathcal{V}_n$ and $\dot{\mathcal{V}}_n$ is a polynomial in $\omega$ of degree $\leq n$. Therefore $\mathcal{V}_n(\tau)$ and $\dot{\mathcal{V}}_n(\tau)$ are linear subspaces of $\mathbb{R}[\omega]^2_{\leq n}$. Let $f = \operatorname{col}(u,v) \in \mathcal{V}_n$ and

$$g := (A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)f = \begin{pmatrix} v - \dot{u} \\ \omega\phi(t)u - \dot{v}. \end{pmatrix}$$

By induction, the vectors $\operatorname{col}(u,v)(\tau) \in \mathcal{V}_n(\tau)$ span $\mathbb{R}[\omega]^2_{\leq n}$. Therefore the elements $\operatorname{col}(v, \omega\phi u)(\tau)$ span $\mathbb{R}[\omega]_{\leq n} \oplus \omega\mathbb{R}[\omega]_{\leq n}$. Let

$$\mathcal{W}_n := \mathbb{R}b + (A - \tfrac{\mathrm{d}}{\mathrm{d}t})\mathcal{V}_n \subset \mathcal{V}_{n+1}.$$

Since $\operatorname{col}(\dot{u},\dot{v})(\tau) \in \mathbb{R}[\omega]^2_{\leq n}$ we conclude that the elements of $\mathcal{W}_n(\tau) \subset \mathcal{V}_{n+1}(\tau)$ span $\mathbb{R}[\omega]_{\leq n} \oplus \mathbb{R}[\omega]_{\leq n+1}$. Let $\bar{g} := \operatorname{col}(\bar{u},\bar{v}) \in \mathcal{W}_n$ be arbitrary. Then both components of $(A - \frac{\mathrm{d}}{\mathrm{d}t}I)\bar{g} = \operatorname{col}(\bar{v} - \dot{\bar{u}}, \omega\phi u - \dot{\bar{v}})$ are polynomials of degree $n+1$ in $\omega$. Moreover, by varying $\bar{g}$ one argues as before to conclude that every pair of polynomials of degree $n+1$ can be realized in

$$\mathcal{V}_{n+1} = \mathbb{R}b + \left((A - \tfrac{\mathrm{d}}{\mathrm{d}t}I)\mathcal{W}_n\right)(\tau) = \mathbb{R}[\omega]^2_{\leq n+1}.$$

This completes the proof. ■

By combining this Lemma with Theorem 4 we obtain:

*Theorem 5:* Let $\phi(t)$ be real analytic on $[0,\infty)$ with $\phi(t) \neq 0$ for some $t \in [0,\infty)$. Then the second order parametric system

$$\ddot{x}(t) + \omega\phi(t)x(t) = u(t)$$

with $\omega \in [\omega_-,\omega_+]$ is uniform ensemble controllable on every nontrivial subinterval of $[0,\infty)$.

*Proof:* Lemma 1, together with the Weierstrass approximation theorem implies that the union $\bigcup_{n\geq 0}\mathcal{V}_n$ is dense in the space of continuous functions $C([\omega_-,\omega_+],\mathbb{R}^2)$. Thus the result follows from Theorem 4. ■

In particular, the *generalized Airy equation*

$$\ddot{x}(t) + \omega t^q x(t) = u(t)$$

is uniform ensemble controllable, for any $q \in \mathbb{N}_0$ and compact parameter interval $[\omega_-,\omega_+]$. This is nontrivial even for $q = 0$ and parameter intervals $[\omega_-,\omega_+]$ that contain 0. In fact, for $\omega = 0$, the zero eigenvalue of $A(0)$ has multiplicity $> 1$; a case that has been excluded in Theorem 2.

More generally, one can consider the ensemble control task for the controlled Sturm-Liouville equation ($p > 0$)

$$p(t)\ddot{x} + \dot{p}(t)\dot{x} + (q(t) + \omega p(t)\phi(t))x = u(t,x,\dot{x}).$$

Then ensemble controllability holds via mixed open loop and state feedback controls of the form

$$u(t,x,\dot{x}) = p(t)u(t) + q(t)x + \dot{p}(t)\dot{x}.$$

## IV. Output Ensemble Controllability

In applications to, e.g. cell biology or quantum systems, a frequently met task is to extract information of the system from average measurements. This motivates the study of families of parameter-dependent systems where the measurements are given by an average output functional, i.e. the ensemble is of the form

$$\begin{aligned}
\frac{\partial x}{\partial t}(t,\theta) &= A(t,\theta)x(t,\theta) + B(t,\theta)u(t) \\
y(t) &= \int_{\mathbf{P}} C(t,\theta)x(t,\theta)\,\mathrm{d}\theta
\end{aligned} \tag{16}$$

with initial condition $x(0,\theta) = x_0(\theta) \in \mathbb{R}^n$ and $x_0 \in X$. We assume that either $X = C(\mathbf{P}, \mathbb{R}^n)$ or $X = L^q(\mathbf{P}, \mathbb{R}^n)$ with $q \in \mathbb{N}$. An ensemble $\Sigma$ is called *output ensemble controllable*, if for any $x_0 \in X$ and any $y^* \in \mathbb{R}^p$ there exist a $T > 0$ and an input function $u \in L^1([0,T], \mathbb{R}^m)$ such that

$$\int_{\mathbf{P}} C(T,\theta)\varphi(T,\theta,u)\,\mathrm{d}\theta = y^*. \qquad (17)$$

In [28, Theorem 7.3.1] Triggiani provides a characterization of output controllability for infinite dimensional systems of the form

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t) \\ y(t) &= \mathcal{C}(t)x(t), \end{aligned} \qquad (18)$$

*Theorem 6 ([28]):* The system (18) is output controllable on $[t_0, T]$ if there exist $N \in \mathbb{N}$ and $\tau \in [t_0, T]$ such that

$$\sum_{k=0}^{N} \operatorname{im} \mathcal{C}(T)\Big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - \mathcal{A}\Big)^k \mathcal{B}\Big|_{t=\tau} = \mathbb{R}^p. \qquad (19)$$

If $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are analytic on $[0,\infty)$, then the system (18) is output controllable if and only if (19) holds for some $\tau \in [0,\infty)$.

We now apply this result to the special situation where the bounded linear operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined as in (3) while $\mathcal{C}(t)\colon X \to \mathbb{R}^p$ denotes the finite–dimensional linear operator

$$\mathcal{C}(t)x = \int_{\mathbf{P}} C(t,\theta)x(\theta)\,\mathrm{d}\theta.$$

Applied to the ensembles of the form (16) we obtain

$$\Gamma_k = \big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A(\cdot,\theta)\big)^k B(\cdot,\theta)\big|_{t=\tau} \qquad (20)$$

and hence the following characterization of output ensemble controllability.

*Theorem 7:* Suppose that the matrices $A(t,\theta)$ and $B(t,\theta)$ are of class $C_{\mathrm{u}}^{\omega,0}$. Then the time-varying ensemble (16) is output ensemble controllable if and only if there exist $N \in \mathbb{N}$ and $\tau > 0$ such that $\mathbb{R}^p$ is spanned by the columns of the $p \times m$-matrices

$$\int_{\mathbf{P}} C(T,\theta)\big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A(\cdot,\theta)\big)^k B(\cdot,\theta)\Big|_{t=\tau} \mathrm{d}\theta, \quad k = 0, ..., N.$$

Applied to time-invariant ensembles this characterization yields the following mild improvement of a result in [29] on output ensemble controllability. Observe that in the autonomous case one has

$$\Gamma_k = \big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A(\theta)\big)^k B(\theta) = (-1)^k A(\theta)^k B(\theta).$$

*Corollary 3:* The ensemble $\Sigma = \{(A(\theta), B(\theta), C(\theta)) | \theta \in \mathbf{P}\}$ is output ensemble controllable if and only if there is a $N \in \mathbb{N}$ such that $\mathbb{R}^p$ is spanned by the columns of the $p \times m$-matrices

$$\int_{\mathbf{P}} C(\theta)A(\theta)^k B(\theta)\,\mathrm{d}\theta, \quad k = 0, ..., N.$$

## V. Bilinear Ensembles

### A. Output Controllability for Bilinear Ensembles

Similar to the linear time-invariant case, we are interested in output controllability of bilinear ensembles

$$\begin{aligned} \tfrac{\partial x}{\partial t}(t,\theta) &= \Big(A(\theta) + u(t)B(\theta)\Big)x(t,\theta) \\ y(t) &= \int_{\mathbf{P}} C(\theta)x(t,\theta)\,\mathrm{d}\theta \end{aligned} \qquad (21)$$

with initial condition $x(0,\theta) = x_0(\theta) \in \mathbb{R}^n$ and a single control function $u \in L^1([0,T], \mathbb{R})$. Let

$$\dot{x}(t) = (\mathcal{A} + u(t)\mathcal{B})x(t), \quad x(t) = x_0 \in X$$

denote the associated bilinear system on the Banach space $X = C(\mathbf{P}, \mathbb{R}^n)$, or $X = L^q(\mathbf{P}, \mathbb{R}^n)$, $q \in [1,\infty)$. Here $\mathcal{A}, \mathcal{B}$ are the multiplication operators on $X$ defined by the continuous matrix families $(A(\theta), B(\theta))$, respectively. Since the output space $\mathbb{R}^p$ is finite dimensional, we can apply a result from [1] on local controllability along the uncontrolled drift trajectory. In our situation, the result reads as follows. Let $\mathrm{ad}_A$ denote the ad-operator defined as $\mathrm{ad}_A(B) = AB - BA$.

*Theorem 8:* The rank condition

$$\operatorname{span}\left\{ \int_{\mathbf{P}} C(\theta)\,\mathrm{ad}_{A(\theta)}^k B(\theta)x_0(\theta)\,\mathrm{d}\theta \,|\, k \geq 0 \right\} = \mathbb{R}^p$$

implies local output controllability of (21) along the trajectory $t \mapsto e^{\mathcal{A}t}x_0$, i.e., for all $T > 0$ there exists $\varepsilon_T > 0$ such that for all $y^* \in \mathbb{R}^p$ with

$$\|y^* - \int_{\mathbf{P}} C(\theta)e^{A(\theta)T}x_0(\theta)\,\mathrm{d}\theta\| \leq \varepsilon_T$$

there is a control $u : [0,T] \to \mathbb{R}$ which steers the output $y(0) = \int_{\mathbf{P}} C(\theta)x_0(\theta)\,\mathrm{d}\theta$ to $y(T) = y^*$.

*Proof:* This is an immediate consequence of Theorem 4.1 in [1] and the fact that $\mathcal{A}$ is a bounded operator on $X$ and therefore the infinitesimal generator of an analytic one-parameter group. ■

### B. A Negative Result on Ensemble Controllability

The preceding results have characterized ensemble controllability for linear systems in terms of approximate reachability sets, related to the familiar Lie algebra rank condition from nonlinear control. In this section, we present a counterexample which demonstrates that for bilinear ensembles the simplest possible Lie algebraic condition does neither imply ensemble controllability nor (approximate) accessibility. Thus this example shows that it is not possible to extend the previous analysis straightforwardly to ensembles of bilinear systems. In fact, the main difficulty lies in the absense of a good Lie algebraic characterization of approximate controllability/accesibility in infinite–dimensional Banach spaces.

Recall that a finite dimensional bilinear control

$$\dot{x}(t) = (A + u(t)B)x(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

is accessible from $x_0$ if and only if the Lie algebra rank condition (LARC) is satisfied at $x_0$, i.e. if and only if

$$\dim\{Mx_0 \mid M \in \langle A, B\rangle_L\} = n,$$

where $\langle A, B\rangle_L$ denotes the Lie subalgebra of $\mathbb{R}^{n\times n}$ which is spanned by all finite Lie brackets of $A$ and $B$. Now, let

$$A(\theta) := \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and consider the bilinear ensemble

$$\tfrac{\partial x}{\partial t}(t,\theta) = \big(A(\theta) + u(t)B\big)x(t,\theta) \tag{22}$$

with initial condition $x(0,\theta) = x_0(\theta) \in \mathbb{R}^2$, $x_0 \in L^1([a,b],\mathbb{R}^2)$ and $0 < a < b$. We will show that this ensemble, interpreted as a bilinear system

$$\dot{x}(t) = (\mathcal{A} + u(t)\mathcal{B})x(t) \tag{23}$$

on the Banach space $X := L^1([a,b],\mathbb{R}^2)$, is not approximately accessible, in the sense that the closure of the reachable set of $x_0$ does not contain any interior points in $L^1([a,b],\mathbb{R}^2)$. However, the approximate LARC is satisfied. Thus, the most natural Lie algebraic condition does not imply approximate reachability.

*Proposition 1:* Let $0 < a < b$. System (23) is not approximately controllable within the invariant cone $\mathcal{C} := L^1([a,b],\mathbb{R}_0^+ \times \mathbb{R}_0^+)$. Morover, the reachable set of $x_0 \in \mathbb{R}^+ \times \mathbb{R}^+$ does not contain any interior points in $X = L^1([a,b],\mathbb{R}^2)$ even though the "approximate" Lie algebra condition

$$\overline{\{Mx_0 \mid M \in \langle \mathcal{A}, \mathcal{B}\rangle_L\}} = X$$

is satisfied.

*Proof:* Due to the fact that $A(\theta) + uB$ is a Metzler matrix for all $\theta \in [a,b]$ and all $u \in \mathbb{R}$ it is easy to see that the convex cone $\mathcal{C}$ is invariant under the system semigroup of (22). Since $\mathcal{C}$ is a closed convex cone with empty interior in $L^1([a,b],\mathbb{R}^2)$ it follows that the closure of the reachable set of any initial states $x_0 \in \mathcal{C}$ has empty interior in $L^1([a,b],\mathbb{R}^2)$. Hence, the bilinear ensemble (22) is neither approximately controllable nor approximately accessible. Furthermore, by exploiting the fact that $V : \mathbb{R}^2 \to \mathbb{R}$, $V(x_1,x_2) := -x_1x_2$ is a Lyapunov function, one observes that approximate controllability within $\mathcal{C}$ fails.

Next, we compute the system Lie algebra $\langle \mathcal{A}, \mathcal{B}\rangle_L$ of (22). A straightforward induction argument shows that $\langle \mathcal{A}, \mathcal{B}\rangle_L$ consists of the linear span of all parameter dependent matrices

$$\begin{pmatrix} \theta^{2k} & 0 \\ 0 & -\theta^{2k} \end{pmatrix}, \quad \begin{pmatrix} 0 & \theta^{2k+1} \\ \theta^{2k+1} & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & -\theta^{2k+1} \\ \theta^{2k+1} & 0 \end{pmatrix}, \quad k \in \mathbb{N}_0.$$

Thus the elements in the Lie algebra $\langle \mathcal{A}, \mathcal{B}\rangle_L$ are given by

$$\begin{pmatrix} p(\theta^2) & \theta q_1(\theta^2) \\ \theta q_2(\theta^2) & -p(\theta^2) \end{pmatrix},$$

where $p, q_1, q_2$ are arbitrary real polynomials. Choose for simplicity as initial value the constant function $x_0(\theta) = (1,1)^\top$. Since any polynomial can be uniquely represented as the sum of its even and odd part, we conclude that $\overline{\{Mx_0 \mid M \in \langle \mathcal{A}, \mathcal{B}\rangle_L\}}$ equals the closure of all functions of the form

$$\begin{pmatrix} f(\theta) \\ \theta g(\theta^2) - f(\theta) \end{pmatrix}$$

with arbitrary real polynomials $f, g$. In order to prove that the approximate LARC is satisfied it suffices to show that any $L^1$-function over $[a,b]$ can be approximated by a sequence of real polynomials of odd degree. This, however, follows from the following two facts: (i) the set of all continuous functions is dense in $L^1([a,b],\mathbb{R}^2)$; (ii) any continuous function $h : [a,b] \to \mathbb{R}$ can be uniformly approximated by even polynomials, cf. Stone–Weierstrass Theorem in [9]. Hence, $\hat{h} : [a,b] \to \mathbb{R}$, $\hat{h}(x) := x^{-1}h(x)$ can be approximated by even polynomials and this results in a uniform approximation of $h$ by odd polynomials. Here, the assumption $0 < a < b$ is essential. Finally, we infer $\overline{\{Mx_0 \mid M \in \langle \mathcal{A}, \mathcal{B}\rangle_L\}} = L^1([a,b],\mathbb{R}^2)$. ■

The preceding result shows that the ad hoc approximation techniques for quantum control in [20], [21] are limited to special cases and do not extend in any obvious way to general ensembles of bilinear systems.

## VI. CONCLUSIONS

We derived necessary and sufficient conditions for uniform ensemble controllability of ensembles of linear parameter-dependent systems, using well-known results from infinite–dimensional systems theory on Banach spaces. Complete characterizations of ensemble controllability for time-invariant and time-varying linear systems were obtained, together with corresponding results for output controllability of linear and bilinear parametric ensembles. A counter-example shows that the familiar approximate Lie algebra rank condition for ensembles of bilinear systems does neither imply ensemble controllability nor accessibility. Finding the appropriate sufficient condition for approximate controllability that replaces the Lie algebra rank condition is left as an open problem.

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