SENSITIVITY OPTIMAL DESIGN OF NETWORKS OF IDENTICAL LINEAR SYSTEMS*

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Abstract. In this paper we develop a systematic sensitivity theory for homogeneous networks of identical discrete-time linear systems. For simplicity, we focus on the sensitivity with respect to input-to-state and state-to-output parameter variations. A new class of network sensitivity Gramians is introduced, whose trace measures the sensitivity of the overall network. We show that the network has minimal sensitivity if and only if the associated controllability and observability Gramians are equal. This implies that symmetric networks are sensitivity optimal. Existence and uniqueness properties of sensitivity optimal network realizations are established. We propose a locally quadradically convergent Newton algorithm to compute such realizations. In contrast to the known optimal sensitivity properties of single filters, classical balanced realizations are i.e. not sensitivity optimal for a network of systems. We show this in Section 4 for bounded real node dynamics, together with establishing bounds on the network sensitivity Gramians. This leads to some interesting perspectives on model reduction approaches via sensitivity optimal design that are briefly addressed at the end of the paper.

Key words. networked control systems, linear systems, sensitivity, Gramians

AMS subject classifications. 93B35, 93C05, 93C55

1. Introduction. Large-scale networks of interconnected dynamical systems pose a number of challenging tasks for theoretical analysis and computer simulations. For example, in diverse applications to circuit design, sensor-/actuator networks or large-scale biological networks it becomes increasingly important to be able to identify the most sensitive components or feedback interconnections. This leads to the question of designing networks of systems that are as insensitive as possible to small variations in the interconnection or component parameters.

Sensitivity analysis is a classical and well-understood topic in digital filter design. Implementations of linear systems in digital processing devices are subject to round-off and quantization errors that may drastically change the dynamics of the system. In order to minimize the effects of round-off noise one seeks for state-space realizations that optimize a sensitivity measure. Such minimal sensitivity realizations were introduced by Mullis and Roberts [8, 9] and further studied by e.g. Thiele [10], where the minimum sensitivity realization was shown to be a balanced realization. As minimum sensitivity realizations generalize the class of balanced realizations there is thus a close relation to balanced truncation and model reduction. A natural L_2-sensitivity measure was introduced and studied by Gevers and Li [2] and independently Helmke and Moore [4]. For details we refer to the books [2, 5].

The above results deal only with the case of sensitivity optimization for single systems and do not take the network structure of systems into consideration. In this paper we develop a systematic sensitivity theory for homogeneous networks of

*This work was supported by the DFG under grant HE 1858-12/2 within the SPP 1305 and the Fraunhofer Institute for Industrial Mathematics (ITWM), Kaiserslautern.

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identical discrete-time linear systems. For simplicity, we focus on the sensitivity with respect to input-to-state and state-to-output parameter variations. A new class of network sensitivity Gramians is introduced, whose trace measures the sensitivity of the overall network. We show that the network has minimal sensitivity if and only if the associated controllability and observability sensitivity Gramians are equal. This implies that symmetric networks are sensitivity optimal. Existence and uniqueness properties of sensitivity optimal network realizations are established. We propose a locally quadratically convergent Newton algorithm to compute such realizations. In contrast to the known optimal sensitivity properties of single filters, classical balanced realizations are i.g. not sensitivity optimal for a network of systems. We show this in Section 4 for bounded real node dynamics, together with establishing bounds on the network sensitivity Gramians. This leads to some interesting perspectives on model reduction approaches via sensitivity optimal design that are briefly addressed at the end of the paper.

2. Networks of Systems. We consider networks of N identical interconnected linear systems, where the dynamics of each node $i = 1, 2, \ldots, N$ are described by

$$x_i(t+1) = \alpha x_i(t) + \beta v_i(t) \tag{2.1}$$

$$w_i(t) = \gamma x_i(t). \tag{2.2}$$

Here $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n, \gamma \in \mathbb{R}^{1\times n}$ are assumed to be controllable and observable. To define a network of such identical linear systems, we fix an *interconnection structure*, defined by a matrix $A \in \mathbb{R}^{N\times N}$, and *input/output interconnection matrices* $B \in \mathbb{R}^{N\times m}$ and $C \in \mathbb{R}^{p\times N}$, respectively. Let $u = (u_1, ..., u_m)^\top \in \mathbb{R}^m$ denote the external control input applied to the whole network. Thus, the input to node i is

$$v_i(t) = \sum_{j=1}^{N} A_{ij} w_j(t) + \sum_{k=1}^{m} B_{ik} u_k(t). \tag{2.3}$$

Similarly, the output of the network is a linear combination of the individual node outputs w_i as $y(t) = Cw(t)$, with $w = (w_1, \ldots, w_N)^\top$ and $y \in \mathbb{R}^p$. Let $x = (x_1^\top, \ldots, x_N^\top)^\top \in \mathbb{R}^{nN}$ denote the global state of the network. Using the interconnection matrices A, B, C and node dynamics α, β, γ the resulting homogeneous network has the form

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u(t) \tag{2.4}$$

$$y(t) = \mathcal{C}x(t). \tag{2.5}$$

Here $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ have the tensor product representations

$$\begin{aligned} \mathcal{A} &= I_N \otimes \alpha + A \otimes \beta\,\gamma \in \mathbb{R}^{nN\times nN} \\ \mathcal{B} &= B \otimes \beta \in \mathbb{R}^{nN\times m} \\ \mathcal{C} &= C \otimes \gamma \in \mathbb{R}^{p\times nN}. \end{aligned} \tag{2.6}$$

If (α, β, γ) is controllable, it is known that the controllability and observability of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is equivalent to the fact that (A, B, C) is controllable and observable, cf. [3]. The above construction of a network is conveniently described in terms of the associated transfer functions. The *node transfer function* is the strictly proper McMillan degree n rational function

$$g(z) = \gamma(zI_n - \alpha)^{-1}\beta. \tag{2.7}$$

Throughout this paper we assume that $g(z)$ is bounded real. Thus $h(z) = 1/g(z)$ is real rational with $h(\infty) = \infty$ and $|h(z)| \geq 1$ for all $|z| \geq 1$. The *interconnection transfer function* is defined as

$$f(z) = C(zI_N - A)^{-1}B \in \mathbb{R}^{p\times m}(z). \tag{2.8}$$

We assume throughout the paper, that $f(z)$ is discrete-time stable, i.e. $f(z)$ has all its poles in the open unit disc. Then, the *global network transfer function* of (2.6)

$$F(z) := \mathcal{C}(zI_{nN} - \mathcal{A})^{-1}\mathcal{B} \tag{2.9}$$

is strictly proper and is given given by the composition of rational functions as $F(z) = (f \circ h)(z)$. Since f is stable and $g = 1/h$ is bounded real, it follows that the composition $F = f \circ h$ is discrete-time stable. Therefore, interconnecting N identical node linear systems (2.1) to a network of linear systems is equivalent to applying a frequency transformation $h(z)$ into the interconnection transfer function $f(z)$. This point of view of studying homogeneous networks of systems via compositions of transfer functions has been proposed by S. Hara and his group [3].

3. Network Sensitivity Gramians. For f, h as above (and $p = m$), the gradients of $F = f \circ h$ with respect to the input/output coupling parameters B, C are the stable transfer functions

$$\frac{\partial F}{\partial B}(z) = (h(z)I - A^\top)^{-1}C^\top, \quad \frac{\partial F}{\partial C}(z) = (h(z)I - A)^{-1}B. \tag{3.1}$$

The *network sensitivity* of F is defined as the function, that assigns to each realization (A, B, C) of f the value

$$\mathcal{S}_g(A,B,C) := \frac{1}{2\pi i}\int_{|z|=1} \left(\|(h(z)I - A)^{-1}B\,\|^2 + \|C(h(z)I - A)^{-1}\,\|^2\right)\tfrac{dz}{z}, \tag{3.2}$$

i.e. the sum of the L_2-norms of the gradients (3.1). A realization $(A_*, B_*, C_*) = (T_*AT_*^{-1}, T_*B, CT_*^{-1})$, or the associated state space coordinate transformation T_*, is said to have *minimal sensitivity*, if for all realizations (A, B, C) of f, it holds that $\mathcal{S}_g(A_*, B_*, C_*) \leq \mathcal{S}_g(A, B, C)$. Define the *network sensitivity Gramians* as

$$W_c^g := \frac{1}{2\pi i}\int_{|z|=1} (h(z)I - A)^{-1}B\,B^\top(\overline{h(z)}I - A^\top)^{-1}\tfrac{dz}{z} \tag{3.3}$$

$$W_o^g := \frac{1}{2\pi i}\int_{|z|=1} (\overline{h(z)}I - A^\top)^{-1}C^\top\,C\,(h(z)I - A)^{-1}\tfrac{dz}{z}. \tag{3.4}$$

Thus, the sensitivity of a network is the sum of the traces of the sensitivity Gramians

$$\mathcal{S}_g(A,B,C) = \mathrm{Tr}(W_c^g + W_o^g). \tag{3.5}$$

Since sensitivity of a realization is coordinate dependent, it is important to find state space coordinate transformations that minimize sensitivity. Since the Gramians transform as $(W_c^g, W_o^g) \mapsto (TW_c^gT^\top, (T^\top)^{-1}W_o^gT^{-1})$ under state space similarity $(A, B, C) \mapsto (TAT^{-1}, TB, CT^{-1})$, each state space transformation T (or $P := T^\top T$, respectively) gets assigned the network sensitivity cost as

$$\mathcal{S}_g(TAT^{-1}, TB, CT^{-1}) := \mathrm{Tr}(W_c^gT^\top T + W_o^gT^{-1}(T^\top)^{-1}) = \mathrm{Tr}(W_c^gP + W_o^gP^{-1}),$$

where $\mathrm{Tr}(X)$ denotes the trace of the matrix X.

Let $\mathcal{P}$ denote the convex space of all real $N \times N$ positive definite matrices P. We can thus reformulate the task of finding a network f of minimal sensitivity as the optimization problem for the convex cost function $\Phi_g : \mathcal{P} \longrightarrow \mathbb{R}, \Phi_g(P) := \mathrm{Tr}(W_c^g P + W_o^g P^{-1})$ on $\mathcal{P}$. Our first main result characterizes sensitivity optimal realizations; the proof runs closely to that of Corollary 6.4.3 in [5].

THEOREM 3.1. *Let (A, B, C) be an arbitrary, not necessarily minimal, realization of the stable strictly proper transfer function $f(z)$. Let R, O, H denote the reachability, observability and the Hankel matrix of (A, B, C), respectively. Equivalent are:*

1. *There exists an invertible coordinate transformation T_* that minimizes the sensitivity function $T \mapsto \mathcal{S}_g(TAT^{-1}, TB, CT^{-1})$.*
2. *There exists T_* such that $(A_*, B_*, C_*) = (T_* A T_*^{-1}, T_* B, C T_*^{-1})$ has equal Gramians $W_c^g(A_*, B_*) = W_o^g(C_*, A_*)$.*
3. *There exists a unique positive definite matrix $P_* = T_*^{\mathsf{T}} T_*$, that minimizes $\Phi_g(P) = \mathrm{Tr}(W_c^g P + W_o^g P^{-1})$ on $\mathcal{P}$.*
4. *There exists $P_* \in \mathcal{P}$ with $P_* W_c^g P_* = W_o^g$.*
5. $\mathrm{rk}\, R = \mathrm{rk}\, O = \mathrm{rk}\, H$.

Thus, if $\mathrm{rk}\, R = \mathrm{rk}\, O = \mathrm{rk}\, H$ holds, then a realization (A, B, C) of f is sensitivity optimal if and only if $W_c^g = W_o^g$. Moreover, if (A, B, C) is controllable and observable, then any two sensitivity optimal realizations $(A_*, B_*, C_*), (A_{**}, B_{**}, C_{**})$ are similar via a unique orthogonal coordinate transformation $T \in O_N(\mathbb{R})$.

As a trivial consequence, we conclude that any symmetric network implementation of identical bounded real systems has minimum sensitivity. In particular, this applies if A is the adjacency (or Laplacian) matrix of a graph with equal input/output weights $C^{\mathsf{T}} = B$.

COROLLARY 3.2. *Let (A, B, C) be a symmetric realization, i.e. $A = A^{\mathsf{T}}, C^{\mathsf{T}} = B$, of a stable transfer function $f(z)$. Then $\mathcal{S}_g(A, B, C)$ has minimum sensitivity among all realizations of f.*

Numerical algorithms for iteratively computing sensitivity optimal realizations have been proposed by [5], using either gradient steepest descent schemes or discrete-time Riccati equations. Here, following [1], we propose a simple Newton algorithm that is fast (locally quadratically convergent) and globally convergent. Given any positive definite symmetric matrices W_c^g, W_o^g and any $P \in \mathcal{P}$, let $Z = Z(P)$ denote the unique positive definite solution of the Lyapunov equation

$$Z W_c^g P + P W_c^g Z = P W_c^g P - W_o^g. \tag{3.6}$$

Given any initial point $P_0 \in \mathcal{P}$ with $P_0 W_c^g P_0 - W_o^g > 0$, we consider the iterative system

$$P_{t+1} = P_t - Z(P_t), \quad t = 0, 1, \ldots. \tag{3.7}$$

It is easily seen that this is exactly the Riemannian Newton algorithm for optimizing Φ_g, with respect to a suitable Riemannian metric on $\mathcal{P}$. The convergence properties of the algorithm are summarized in the following theorem, whose proof is omitted.

THEOREM 3.3. *Let W_c^g, W_o^g denote the network sensitivity Gramians of a controllable and observable realization (A, B, C). Assume $P_0 \in \mathcal{P}$ is given with $P_0 W_c^g P_0 - W_o^g > 0$. The Newton iterates* (3.7) *converges monotonically, i.e. $P_t > P_{t+1}$ holds, and locally quadratically fast to the sensitivity optimal P_*.*

Proof. See [1], Chapter 4, or use the following simple direct arguments.
The symmetry of P_t is obvious. Assume P_t is positive definite. Then P_{t+1} is a solution

of the Lyapunov equation

$$\begin{aligned} P_{t+1}W_c^gP_t + P_tW_c^gP_{t+1} &= 2P_tW_c^gP_t - P_{t+1}W_c^gZ(P_t) - Z(P_t)W_c^gP_t \\ &= P_tW_c^gP_t + W_o^g, \end{aligned}$$

with $P_tW_c^g$ having all eigenvalues with positive real part. Thus P_{t+1} is positive definite. We compute

$$\begin{aligned} P_{t+1}W_c^gP_{t+1} - W_o^g &= (P_t - Z_t)W_c^g(P_t - Z_t) - W_o^g \\ &= P_tW_c^gP_t) - W_o^g - Z_tW_c^gP_t - P_tW_c^gZ_t + Z_tW_c^gZ_t \\ &= Z_tW_c^gZ_t + W_o^g > 0. \end{aligned}$$

Monotonicity follows from $P_tW_c^gP_t > 0$ and $P_tW_c^gZ_t + Z_tW_c^gP_t > 0$ and therefore $P_t - P_{t+1} = Z_t > 0$. Thus the limit $P_* = \lim_{t\to\infty} P_t$ exists. From the iteration we obtain $Z(P_*) = 0$ and therefore $P_*W_c^gP_* = W_o^g$ holds. Finally, local quadratic convergence follows, since the iteration is a Newton algorithm and Φ is strictly convex. □

4. Properties of the Sensitivity Gramians. In discrete-time linear systems, the classical controllability and observability Gramians of (A, B, C) are given by the unique solutions of the following Lyapunov equations

$$AW_c(A, B)A^\top - W_c(A, B) + B\,B^\top = 0, \tag{4.1}$$

$$A^\top W_o(C, A)A - W_o(C, A) + C^\top C = 0. \tag{4.2}$$

Note that if $g(z) = \frac{1}{z}$ we have that the sensitivity Gramians W_c^g and W_o^g coincide with the classical Gramians, i.e. $W_c^g = W_c(A, B)$ and $W_o^g = W_o(C, A)$. More generally, this also true for all transfer functions g satisfying $|g(z)| = 1$ for all $|z| = 1$, as we prove in Theorem 4.2 (b). In order to achieve a better understanding of the limitations of sensitivity optimal model reduction we aim to compare the sensitivity Gramians with the classical Gramians. To this end, let $X(z) := (h(z)I - A)^{-1}B$ and $Y(z) := C(h(z)I - A)^{-1}$. Let $|\lambda_{\max}(A)|$ denote the spectral radius of A. The error between W_c^g (W_o^g) and W_c (W_o) is specified in the following lemma, which is easily established.

Lemma 4.1. *Suppose that g is stable and satisfies $\max\limits_{|z|=1} |g(z)| < \frac{1}{|\lambda_{\max}(A)|}$. Then,*

$$A\,W_c^g\,A^\top - W_c^g + B\,B^\top = \frac{1}{2\pi i}\int_{|z|=1} (|h(z)|^2 - 1)\,X(z)\,X(z)^*\,\tfrac{dz}{z}, \tag{4.3}$$

$$A^\top W_o^g\,A - W_o^g + C^\top C = \frac{1}{2\pi i}\int_{|z|=1} (|h(z)|^2 - 1)\,Y(z)^*\,Y(z)\,\tfrac{dz}{z}. \tag{4.4}$$

The subsequent statement addresses the relationship between the Gramians W_c^g, W_o^g and $W_c(A, B), W_o(C, A)$, respectively. We write $X \le Y$, if $Y - X$ is positive semidefinite. Recall that g is called lossless if $|g(z)| = 1$ for all $|z| = 1$.

Theorem 4.2. (a) *Suppose that g is bounded real with $|g(z)| \not\equiv 1$ for $|z| = 1$. Then,*

$$W_c^g < W_c(A, B) \qquad \textit{and} \qquad W_o^g < W_o(C, A).$$

(b) *The transfer function g is lossless if and only if*

$$W_c^g = W_c(A,B) \qquad \text{and} \qquad W_o^g = W_o(C,A).$$

(c) *Suppose that $1 \le |g(z)| < \frac{1}{|\lambda_{\max}(A)|}$ for all $|z|=1$. Then,*

$$W_c^g \ge W_c(A,B) \qquad \text{and} \qquad W_o^g \ge W_o(C,A).$$

Proof. We only treat the case for W_c^g.

Part (a): Let g be bounded real with $|g(z)| \not\equiv 1$ for $|z|=1$. Further, we denote $Q := \frac{1}{2\pi i}\int_{|z|=1}(|h(z)|^2-1)\,X(z)\,X(z)^*\frac{dz}{z}$. Then, since $|h(z)|^2 \ge 1$ and $|h(z)|^2 \not\equiv 1$ for all $|z|=1$ it holds that $Q>0$. Hence, by (4.1) and (4.3) we have that

$$A\,(W_c^g - W_c(A,B))\,A^{\mathsf{T}} - (W_c^g - W_c(A,B)) = Q > 0.$$

Next, we regard the Lyapunov operator $\mathcal{L}_A$ defined by $\mathcal{L}_A(M) := A\,M\,A^{\mathsf{T}} - M$. Since all eigenvalues of A are contained in the unit disc, the Lyapunov operator is invertible and if $M>0$ it follows that $-\mathcal{L}_A^{-1}(M) > 0$, cf. [6, Corollary 3.3.47]. Due to the linearity of the Lyapunov operator it holds that

$$\mathcal{L}_A(W_c^g\, - \,W_c(A,B)) = Q > 0.$$

Thus, $-\mathcal{L}_A^{-1}(Q) = -(W_c^g - W_c(A,B)\,) = W_c(A,B) - W_c^g > 0$.

Part (b): If g is lossless, i.e. $|h(z)| \equiv 1$ for $|z|=1$ it holds that $Q = 0 = \mathcal{L}_A(W_c^g - W_c(A,B))$ and hence, $W_c^g = W_c(A,B)$. Conversely, assume that $W_c^g = W_c(A,B)$. Then, the right hand side in (4.3) must be identically zero. Hence, we have that $|h(z)|^2 - 1 \equiv 0$ for all $|z|=1$. That is, g is lossless.

Part (c): Suppose the transfer function g satisfies $1 \le |g(z)| \le 1/|\lambda_{\max}|$ and $|g(z)| \not\equiv 1$ for all $|z|=1$. Then, it holds that $1/|\lambda_{\max}| < |h(z)| \le 1$ and $|h(z)| \not\equiv 1$ for all $|z|=1$. Hence, we have that $Q<0$ and the assertion follows from the same line of reasoning as in part (a). □

Let $\sigma_1 \ge \ldots \ge \sigma_n$ and $\sigma_1^g \ge \ldots \ge \sigma_n^g$ denote the singular values of W_cW_o and $W_c^gW_o^g$, respectively. Then standard eigenvalue inequalities show for $k=1,\ldots,n$ $\sum_{i=1}^k(\sigma_{n-i+1}^g)^2 \le \sum_{i=1}^k \sigma_{n-i+1}^2$.

COROLLARY 4.3. *Assume g is bounded real. Then for $k=1,\ldots,n$*

$$\sum_{i=1}^{k}(\sigma_{n-i+1}^g)^2 \le \sum_{i=1}^{k}\sigma_{n-i+1}^2$$

Proof. From Theorem 4.2 we obtain $W_c^gW_o^g \le W_cW_o$ and both matrices are similar to positive definite symmetric matrices. Thus the result follows from from standard estimates for the k smallest eigenvalues of symmetric matrices $A \le B$. □

The Gramians W_c^g and W_o^g are defined by complex contour integrals, which are of course nontrivial to evaluate. In order to compute the sensitivity Gramians there are different possibilities. First, one can use numerical integration methods to evaluate the contour integrals. Second, one can try to analytically evaluate the integrals. For example, the first order system $g(z) = \frac{a}{z-b}$ is bounded real if and only if $|a|+|b| \le 1$. Then, $W_c^g = \frac{1}{|a|^2}W_c(aA+bI,B), W_o^g = \frac{1}{|a|^2}W_o(C,aA+bI)$. Thus, a realization

(A,B,C) is g-sensitivity optimal, if and only if $(aA+bI,B,C)$ is balanced. Unfortunately, such elegant characterizations are not easily available for higher order node systems. Third, we note that, as shown in [9], the Gramians are related as follows

$$(h(z)I-A)^{-1}B=(I\otimes\gamma)\,(zI-\mathcal{A})^{-1}\mathcal{B},\quad C(h(z)I-A)^{-1}=\mathcal{C}\,(zI-\mathcal{A})^{-1}(I\otimes\beta).$$

Hence, it holds that

$$W_c^g=(I\otimes\gamma)\,W_c(\mathcal{A},\mathcal{B})\,(I\otimes\gamma^\mathsf{T}),\qquad W_o^g=(I\otimes\beta^\mathsf{T})\,W_o(\mathcal{C},\mathcal{A})\,(I\otimes\beta).$$

This implies that the sensitivity Gramians can be effectively computed as $W_c^g=(I\otimes\gamma)\,X\,(I\otimes\gamma^\mathsf{T})$ via the linear matrix inequality

$$\mathcal{A}X\mathcal{A}^\mathsf{T}-X+\mathcal{B}\mathcal{B}^\mathsf{T}=0\,,\qquad X>0.$$

In [7, Lemma 5] it is shown that, if the transfer function g is bounded real, then for the controllability and observability Gramians for $(\mathcal{A},\mathcal{B},\mathcal{C})$ and (A,B,C) the following inequalities hold

$$W_c(\mathcal{A},\mathcal{B})\leq W_c(A,B)\otimes P^{-1},\qquad W_o(\mathcal{C},\mathcal{A})\leq W_o(C,A)\otimes P,$$

where P is the unique positive definite solution to the Riccati equation

$$\alpha P\alpha^\mathsf{T}-P+\frac{\alpha^\mathsf{T}P\beta\beta^\mathsf{T}P\alpha}{1-\beta^\mathsf{T}P\beta}+\gamma^\mathsf{T}\gamma=0.$$

Note that $\beta^\mathsf{T}P\beta\leq 1$ and $\gamma^\mathsf{T}P^{-1}\gamma\leq 1$. This yields the following corollary.

COROLLARY 4.4. *Suppose that g is bounded real and P as above. Then,*

$$W_c^g\leq\gamma^\mathsf{T}P^{-1}\gamma\cdot W_c(A,B),\qquad W_o^g\leq\beta^\mathsf{T}P\beta\cdot W_o(C,A).$$

We say a realization (A,B,C) is *g-sensitivity balanced* if $W_c^g=W_o^g=\Sigma^g$, where $\Sigma^g=\operatorname{diag}(\sigma_1^g,\ldots,\sigma_N^g)$ is diagonal. The entries $\sigma_1^g\geq\ldots\geq\sigma_N^g$ are called the *g-singular values* of the network. From Theorem 4.2 (b) we obtain.

THEOREM 4.5. *Suppose that g is lossless. Then the network is g-sensitivity balanced if and only if (A,B,C) is balanced, i.e. $W_c(A,B)=W_o(C,A)=\Sigma$. In particular, the g-singular values coincide with the singular values of (A,B,C).*

Finally, we investigate the problem of networked sensitivity based model reduction. Assume that $\Sigma^g=\operatorname{diag}(\Sigma_1^g,\Sigma_2^g)$, where Σ_1^g is $k\times k$. Let (A,B,C) be g-sensitivity balanced and be partitioned accordingly as

$$A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix},\qquad B=\begin{bmatrix}B_1\\B_2\end{bmatrix},\qquad C=\begin{bmatrix}C_1&C_2\end{bmatrix}.$$

Note, that truncations of sensitivity optimal realizations are no longer optimal in general. This is true even for lossless systems g. The transfer function of the reduced interconnection (A_{11},B_1,C_1) is $f_r(z)=C_1(zI-A_{11})^{-1}B_1$. If g is bounded real, the maximum principle implies

$$\|F\|_\infty:=\max_{|z|=1}|f(h(z))|=\max_{|z|\geq 1}|f(h(z))|\leq\max_{|z|\geq 1}|f(z)|=\max_{|z|=1}|f(z)|=\|f\|_\infty,$$

In particular, for any stable f_r and $\widehat{F}=f_r\circ h$ we obtain $\|F-\widehat{F}\|_\infty\leq\|f-f_r\|_\infty$, where equality holds if g is lossless. We conclude

THEOREM 4.6. *Let g satisfy $1\leq|g(z)|<\frac{1}{|\lambda_{\max}(A)|}$ and let f_r denote the sensitivity reduced transfer function.*

1. *Then* f_r *is stable and satisfies* $\|f - f_r\|_\infty \leq 2\sum_{i=1}^{k} \sigma^g_{n-i+1}$.

2. *If* g *is lossless, then* $\mathcal{S}_g(A_{11}, B_1, C_1) \leq 2\sum_{i=1}^{k} \sigma_i^g = 2\sum_{i=1}^{k} \sigma_i$.

5. Conclusions. In this paper we considered homogeneous networks of identical discrete-time linear systems. The dynamics in the nodes are described by a strictly proper bounded real transfer function g. The network structure is given by a discrete-time stable transfer function f. In this paper we addressed the sensitivity of the networked control system $F = f \circ g$ with respect to realizations (A, B, C) of the network transfer function f. To this end, we introduced a new class of Gramians, called sensitivity Gramians, whose traces measure the sensitivity of the overall network. We showed that a sensitivity minimal realization with respect to the input-to-state matrix B and the state-to-output matrix C is characterized by the equality of the controllability and observability sensitivity Gramian. In addition, we provide a convergent Newton algorithm to compute such realizations. Furthermore, we investigated properties of the sensitivity Gramians related to the usual Gramians associated with (A, B, C). In particular, it turns out that if the node transfer function g of the individual systems is lossless then the sensitivity Gramians are equal to usual Gramians. Also, we stated conditions on the node transfer function g that allow for estimates of the sensitivity Gramians in terms of the classical Gramians of (A, B, C). Finally, we presented a perspective of sensitivity optimal design that provides an interesting relation to model reduction for networked control systems. In the future research this work will be extended to heterogenous networks.

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