Interval Observers for Linear Systems with Time-Varying Delays

Mustapha Ait Rami, Jens Jordan and Michael Schönlein

***Abstract*— This paper considers linear observed systems with time-varying delays, where the state as well as the observation of the state is subject to delays. It is assumed that the delays are unknown but stay below a certain bound. Similar to the case of uncertainties in the systems parameters we aim to derive upper and lower estimates for the state of the system under consideration. A pair of estimators providing such bounds is called an interval observer. In particular, the case where the estimators converge asymptotically is of notable interest. In this case the interval observer is said to be convergent. In this paper we derive necessary and sufficient conditions for the existence of a convergent interval observer for linear observed systems with time-varying bounded delays.**

I. INTRODUCTION

Differential delay systems represent a class of infinite-dimensional systems which may ba used to model population dynamics and many physical and biological dynamical systems. As a matter of fact, the reaction of real world systems to exogenous signals is often infected by certain time delays, e.g. in logistics networks the transportation of products between different locations is subject to traffic jams etc. In practice, these delays vary over time and are frequently unknown. Further, a direct measurement of certain state variables is also subject to delays. Such phenomena can be described by a mathematical model in which the behavior of the system is described by an equation that includes information on the past evolution of the system.

A common and frequently used technique to obtain information on the unknown state of the system is to use state estimation that bounds the state from below and above. This technique, called interval observer, was introduced by [1] to obtain state estimates for biological systems that are subject to parameter uncertainties. Later the framework of interval observers was used and extended in many works, e.g. [2]. Moreover, it is of interest whether the difference between the upper and lower bound on the state of the system, called the interval error, converges. If this is the case, the interval observer is said to be convergent. In recent years the framework of interval observers is also used to derive state estimates for linear systems that are subject to time delays, see e.g. [3], [4]. In [3] the interval observers for positive linear systems with constant time-delay is based on an observer of extended Luneberger type [5]. On the other hand, [4] considers input-free linear systems that are subject to disturbances. But both works are not concerned with time-varying delays.

This work was supported by Deutscher Akademischer Austausch Dienst, DAAD, PPP Germany Spain 0811919.

M. Ait Rami is with the Departamento de Ingenieria de Sistemas y Automatica, Universidad de Valladolid, Spain aitrami@autom.uva.es

J. Jordan and M. Schönlein are with the Institute for Mathematics, University of Würzburg, Am Hubland, 97074 Würzburg, Germany {jordan,schoenlein}@mathematik.uni-wuerzburg.de

In this paper we regard the problem of the existence of convergent interval observers for linear observed systems with time-varying bounded delays. We prove necessary and sufficient conditions for the existence of a convergent interval observer. The design of interval observers relies on the theory of positive systems (see [6] for general references). In fact, enforcing the positivity of the error estimation will necessarily lead to a guaranteed bounds on the estimated states, once we start with a priori bounds on the unknown initial condition of the observed system. Here, we provide a simple and efficient way of designing observers that ensures guaranteed bounds on the estimated states.

The paper is organized as follows. In Section II the problem is formulated and main result is stated. Moreover, some preliminaries are given. Section III is devoted to the proof of the main theorem. Finally, Section IV gives some conclusions.

Notations: $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. For vector $v \in \mathbb{R}^n$, $v > 0$ means that its components v_i are positive. M^T denotes the transpose of the real matrix M. For a real matrix M, $M > 0$ means that its components are positive, that is $M_{ij} > 0$, and $M \geq 0$ means that its components are nonnegative, i.e. $M_{ij} \geq 0$.

II. PRELIMINARIES

This section is composed of two subsections. In the first subsection we introduce the objective under consideration and state the main result of this paper. The second subsection provides necessary preliminary statements about positive systems and technical keys that are primordial for the characterization and the treatment of positive systems satisfying a differential delayed equation. The introduced facts and results will be essentially used in the verification of our main result.

A. Problem formulation

In this paper we deal with linear observed systems with bounded time-varying delays $0 \leq \tau_1(\cdot), \cdots, 0 \leq \tau_p(\cdot)$ that are supposed to be Lebesgue measurable. Throughout the paper we use the notation

$$\tau_i(t) \leq \tau^+ := \sup_{1\leq i\leq p} \max_{t\geq 0} \tau_i(t).$$

ISBN 978-963-311-370-7

The systems (not necessarily positive) under consideration are of the following form

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + \sum_{i=1}^{p} A_i\,x(t-\tau_i(t)) \\ y(t) &= C\,x(t) + \sum_{i=1}^{p} C_i\,x(t-\tau_i(t)), \end{aligned} \tag{1}$$

where $\tau_i(t) \leq \tau^+$ for all $i = 1, ..., p$ and for all $t \in \mathbb{R}_+$. The initial condition for the system (1) is given by

$$x(s) = \phi(s) \qquad s \in [-\tau^+, 0].$$

Here ϕ is a given continuous function that is defined on $[-\tau^+, 0]$. The aim of this paper is to derive necessary and sufficient conditions for the existence of estimator functions that bound the state of the system from below and above. This is formalized in the following definition.

Definition 1: For any delays such that $\tau_i(t) \leq \tau^+$, an *interval observer* for system (1) is a pair of lower and upper estimator functions $(\theta^-(\cdot), \theta^+(\cdot))$ for the state $x(\cdot)$ such that

$$\theta^-(t) \leq x(t) \leq \theta^+(t)$$

for all $t \in \mathbb{R}_+$. Further, $(\theta^-(\cdot), \theta^+(\cdot))$ is said to be a *convergent interval observer* for the system (1) if the *interval error* $e(t) = \theta^+(t) - \theta^-(t)$ converges to zero.

The objective is to design convergent interval observers for systems of the form (1), in the case when the delays are not known but bounded, i.e. $\tau_i(\cdot) \leq \tau^+$. In fact, the main idea behind the proposed interval observers machinery is to reconstruct the error dynamics of the estimators in such way that it can be governed by a differential dynamical equation of the form

$$\dot{e}(t) = (A - LC)e(t) + \sum_{i=1}^{p}(A_i - LC_i)e(t-\tau^+)$$

which enforced to be inherently positive by choosing an adequate matrix gain L. That is, we have to ensure that if the initial error is nonnegative, i.e. $e(0) \geq 0$, the error will remain nonnegative over time, i.e. $e(t) \geq 0$ for all $t \geq 0$. Consequently, by using this positivity concept we will show the following necessary and sufficient condition for a convergent interval observer that ensures an estimation with guaranteed lower and upper bounds on the observed states.

Theorem 1: There exists a convergent interval observer (θ^-, θ^+) for system (1) if and only if the following conditions hold true. There is a gain matrix L such that

(i) *$A - LC$ is Metzler and $A_i - LC_i \geq 0$ for all $i = 1, ..., p$,*
(ii) *$A - LC + \sum_{i=1}^{p}(A_i - LC_i)$ is Hurwitz.*

Note that the existence of matrix L which satisfies conditions (i) and (ii) in Theorem 1 can be equivalently described by a LMI formulation, see [3], [7].

B. *Positive systems*

This section provides necessary notations and useful results from the theory of positive systems. As mentioned in the previous subsection the structure of the error dynamics will be described by a linear differential delayed equation of the following form

$$\dot{z}(t) = Mz(t) + \sum_{i=1}^{p} M_i z(t - \tau_i(t)), \tag{2}$$

where the given matrices $M, M_1, \ldots, M_p \in \mathbb{R}^{n\times n}$ are time-invariant and $0 \leq \tau_1(\cdot), \cdots, 0 \leq \tau_p(\cdot)$ are time-varying delays such that $\tau_i(t) \leq \tau^+$. The vector $z(t) \in \mathbb{R}^n$ is the instantaneous system state at time t. The whole state at time t of system (2) is infinite dimensional which is given by the set

$$\{z(s) | -\tau^+ \leq s \leq t\}.$$

Following [8], it can be shown that the solution to the system's equation (2) exists, is unique and totally determined by any given initial locally Lebesgue integrable vector function $\phi(\cdot)$ such that

$$z(s) = \phi(s) \text{ for } -\tau^+ \leq s \leq 0. \tag{3}$$

For any nonnegative initial condition $\phi(t) \in \mathbb{R}^n_+$ such that $z(t) = \phi(t)$ for $-\tau^+ \leq t \leq 0$, system (2) is said to be *positive* if the corresponding trajectory is nonnegative, that is $z(t) \in \mathbb{R}^n_+$ for all $t \geq 0$. We recall intrinsic properties of the delayed system's positivity behavior are related to Metzlerian matrices and positive matrices. A real matrix M is called a *Metzler* matrix if its off-diagonal elements are nonnegative, i.e. $M_{ij} \geq 0, \quad i \neq j$. A real matrix M is called a *positive* matrix if all its elements are nonnegative, that is $M_{ij} \geq 0$. Note that the following result from [9] shows how Metzlerian matrices are intrinsically connected to positivity. For a more detailed presentation of positive matrices and their properties see e.g. [10].

Lemma 1: Let M be a Metzler matrix then the following holds true.

(1) *$e^{tM} \geq 0$ for all $t \in \mathbb{R}_+$ if and only if M is Metzler.*
(2) *Let $v \in \mathbb{R}^n_+$ such that $v > 0$, then $e^{tM}v > 0$ for all $t \in \mathbb{R}_+$.*

Further, we cite from [11] the following characterization of the positivity of a linear system of the form (2).

Lemma 2: System (2) is positive if and only if the matrix M is Metzler and the matrices $M_1, ..., M_p$ are positive.

Moreover, Theorem 2.1 in [3] contains a useful characterization for the stability of a positive system of the form (2) in terms of the matrices that define the system.

Proposition 1: Assume that System (2) is positive, or equivalently that the matrix M is Metzler and $M_1, \ldots, M_p$ are positive matrices. Then, the following statements are equivalent.

(i) *There exist an initial condition $\phi^*(\cdot)$ taking values in the interior of $\mathbb{R}^n_+$ such that System (2) is asymptotically stable.*

(ii) *System (2) is asymptotically stable for every initial condition $\phi(\cdot)$ taking values in $\mathbb{R}^n_+$.*
(iii) *System (2) is asymptotically stable for every initial condition $\phi(\cdot)$ taking values in $\mathbb{R}^n$ ($\phi(\cdot)$ has indefinite sign).*
(iv) *There exists $\lambda \in \mathbb{R}^n$ such that*

$$(M + \sum_{i=1}^{p} M_i)\lambda < 0, \, \lambda > 0.$$

(v) *$M + \sum_{i=1}^{p} M_i$ is a Hurwitz matrix, i.e. the real part of any eigenvalue is strictly negative.*

III. PROOF OF THE MAIN THEOREM

In this section we proof Theorem 1. In order to keep a clear presentation of the main arguments we present some auxiliary lemmas first. So, we consider a linear system with constant delay τ which has a constant initial condition λ. To be precise, we consider a system of the form

$$\begin{aligned} \dot{z}(t) &= Mz(t) + \sum_{i=1}^{p} M_i z(t-\tau), \\ z(s) &= \lambda, \;\; \text{for } s \in [-\tau, 0]. \end{aligned} \tag{4}$$

Lemma 3: *Consider the system* (4)*. Assume that the matrix M is Metzler and the matrices $M_1, \ldots, M_p$ are nonnegative. Then, the solution $z(t)$ is decreasing for all $t \geq 0$ if and only if $\lambda > 0$ and*

$$(M + \sum_{i=1}^{p} M_i)\lambda \leq 0.$$

Proof: Let $z(\cdot)$ be the solution of system (4). Then the minus derivative $-\dot{z}$ of the solution satisfies the following linear differential equation

$$\frac{d}{dt}\left[-\dot{z}(t)\right] = M(-\dot{z}(t)) + \sum_{i=1}^{p} M_i(-\dot{z}(t-\tau)). \tag{5}$$

For the initial condition it follows from (4) and by assumption that

$$\begin{aligned} -\dot{z}(0) &= M(-z(0)) + \sum_{i=1}^{p} M_i(-z(0)) \\ &= -(M + \sum_{i=1}^{p} M_i)\lambda \geq 0. \end{aligned}$$

Since M is Metzler and $M_1, \ldots, M_p$ are nonnegative the system (5) is positive and consequently $-\dot{z}(t) \geq 0$ for all $t \geq 0$.
Conversely, let $z(t)$ be a solution of (4) and assume that $\dot{z}(t) \leq 0$ for all $t \geq 0$. In particular, it holds then for $t = 0$ that

$$0 \geq \dot{z}(0) = (M + \sum_{i=1}^{p} M_i)z(0) = (M + \sum_{i=1}^{p} M_i)\lambda$$

which completes the proof. ■

Corollary 1: *Consider the positive system* (4)*. Then, the solution $z(t)$ is increasing for all $t \geq 0$ if and only if $\lambda < 0$ and*

$$(M + \sum_{i=1}^{p} M_i)\lambda \leq 0.$$

Proof: Note that the derivative of the solution also satisfies (4) and by assumption it holds for the initial condition that

$$\dot{z}(0) = (M + \sum_{i=1}^{p} M_i)z(0) = (M + \sum_{i=1}^{p} M_i)(-\lambda) \geq 0$$

and the proof is completed. ■

Now we compare two positive linear delayed systems, where one of them is subject to time-varying bounded delays and the other is subject to a constant delay, which is given by the upper bound of the delays. That is, consider the systems

$$\begin{aligned} \dot{z}(t) &= M\, z(t) + \sum_{i=1}^{p} M_i\, z(t - \tau_i(t)) \\ z(s) &= \psi(s) \qquad s \in [-\tau^+, 0], \end{aligned} \tag{6}$$

with some positive function ψ defined on $[-\tau^*, 0]$. For the system with bounded delay

$$\begin{aligned} \dot{z}^+(t) &= M\, z^+(t) + \sum_{i=1}^{p} M_i\, z^+(t - \tau^+), \\ z^+(s) &= \lambda, \qquad s \in [-\tau^+, 0] \end{aligned} \tag{7}$$

it is additionally assumed that the constant initial condition is strictly positive, i.e. $\lambda > 0$.

Lemma 4: *If systems* (6) *and* (7) *satisfy $\psi(s) \leq \lambda$ for all $s \in [-\tau^+, 0]$ and*

$$\left(M + \sum_{i=1}^{p} M_i\right) \lambda \leq 0 \tag{8}$$

then it holds that

$$z(t) \leq z^+(t). \tag{9}$$

for all $t \in \mathbb{R}_+$.

Proof: The claim is shown by contradiction. So assume that (9) does not hold. Then there exists a maximal time t^* such that at least one component of $z(t)$ satisfies

$$z_i(t^*) > z_i^+(t^*) \quad \text{ and } \quad z(s) \leq z^+(s) \tag{10}$$

for all $s \in [-\tau^+, t^*)$. In the following we use the integral expressions for the trajectories of the system (6) at time t^*

$$z(t^*) = e^{t^*M} z(0) + \int_0^{t^*} e^{(t^*-s)M} \sum_{i=1}^{p} M_i\, z(s - \tau_i(s))\, ds \tag{11}$$

and of the system (7) at time t^*

$$\begin{aligned} z^+(t^*) &= e^{t^*M} z^+(0) \\ &\quad + \int_0^{t^*} e^{(t^*-s)M} \sum_{i=1}^{p} M_i\, z^+(s - \tau^+)\, ds. \end{aligned} \tag{12}$$

Now we are comparing the terms. Clearly it holds that

$$e^{t^* M} z(0) \le e^{t^* M} z^+(0).$$

Further since $M_i > 0$ it holds for all $t^* \ge s$ and $i = 1, ..., p$ that

$$e^{(t^*-s)M} M_i \ge 0. \quad (13)$$

As z^+ satisfies (8) we can apply Lemma 3 and since $\tau_i(s) \le \tau^+$ it follows form (10) and from Lemma 3 that for all $s \in [-\tau^+, t^*)$ it holds

$$z(s - \tau_i(s)) \le z^+(s - \tau_i(s)) \le z^+(s - \tau^+). \quad (14)$$

Because of (13) it follows

$$e^{(t^*-s)M} M_i \, z(s - \tau_i(s)) \le e^{(t^*-s)M} M_i z^+(s - \tau^+).$$

After multiplying, integrating and summing we obtain

$$z(t^*) \le z^+(t^*),$$

which is a contradiction to (10). ■

Now let $-\lambda$ be the initial condition for the system

$$\begin{aligned} \dot{z}^-(t) &= M \, z^-(t) + \sum_{i=1}^{p} M_i \, z^-(t - \tau^+), \\ z^-(s) &= -\lambda, \qquad s \in [-\tau^+, 0], \end{aligned} \quad (15)$$

By the application of Corollary 1 to the solution of (15) the following inequality, which is similar to (14), can be concluded easily

$$z^-(s - \tau^+) \le z^-(s - \tau_i(s)) \le z(s - \tau_i(s)). \quad (16)$$

Hence we obtain then.

Corollary 2: If systems (6) *and* (15) *satisfy* $-\lambda \le \psi(s)$ *for all* $s \in [-\tau^+, 0]$ *and*

$$\left(M + \sum_{i=1}^{p} M_i \right) \lambda \le 0 \quad (17)$$

then for all $t \ge 0$ *it holds*

$$z^-(t) \le z(t). \quad (18)$$

In the next step we deal with the existence of an interval observer for the system

$$\begin{aligned} \dot{x}(t) &= A \, x(t) + \sum_{i=1}^{p} A_i \, x(t - \tau_i(t)) \\ y(t) &= C \, x(t) + \sum_{i=1}^{p} C_i \, x(t - \tau_i(t)) \end{aligned}$$

with initial condition

$$x(s) = \phi(s) \qquad s \in [-\tau^+, 0].$$

We define the following observers of Luenberger type

$$\begin{aligned} \dot{\theta}^-(t) &= A \, \theta^-(t) + \sum_{i=1}^{p} A_i \, \theta^-(t - \tau^+) + L(y - y^-) \\ y^-(t) &= C \, \theta^-(t) + \sum_{i=1}^{p} C_i \, \theta^-(t - \tau^-) \end{aligned}$$

and

$$\begin{aligned} \dot{\theta}^+(t) &= A \, \theta^+(t) + \sum_{i=1}^{p} A_i \, \theta^-(t - \tau^+) + L(y - y^+) \\ y^+(t) &= C \, \theta^+(t) + \sum_{i=1}^{p} C_i \, \theta^+(t - \tau^+) \end{aligned}$$

or equivalently

$$\begin{aligned} \dot{\theta}^-(t) &= (A - LC) \, \theta^-(t) \\ &\quad + \sum_{i=1}^{p} (A_i - LC_i) \, \theta^-(t - \tau^+) + Ly(t) \end{aligned} \quad (19)$$

and

$$\begin{aligned} \dot{\theta}^+(t) &= (A - LC) \, \theta^+(t) \\ &\quad + \sum_{i=1}^{p} (A_i - LC_i) \, \theta^+(t - \tau^+) + Ly(t), \end{aligned} \quad (20)$$

where the initial conditions $-\lambda$ of θ^- and λ of θ^+ satisfy

$$-\lambda \le \phi(s) \le \lambda \quad (21)$$

for all $s \in [-\tau^+, 0]$. A sufficient condition for the existence of an interval observer is the following.

Lemma 5: For the system (1) *there exists an interval observer* (θ^-, θ^+) *if the following conditions hold true. There is a gain matrix* L *such that*

(i) *$A - LC$ is Metzler and $A_i - LC_i \ge 0$ for all $i = 1, ..., p$*
(ii) *there exists a $\lambda > 0$ such that*

$$(A - LC + \sum_{i=1}^{p} A_i - LC_i)\lambda \le 0.$$

Note that the condition (ii) in the Lemma 5 is weaker than the condition (ii) in the Theorem 1, but the Lemma 5 states conditions that are sufficient for the existence of an interval observer. The conditions of Lemma 5 do not guarantee the existence of a convergent interval observer.

Proof: We prove that $\theta^+(t) - x(t) \ge 0$ for all $t \ge 0$. The lother estimate follows analogously. Define

$$\begin{aligned} \dot{\theta}(t) &= (A - LC) \, \theta(t) \\ &\quad + \sum_{i=1}^{p} (A_i - LC_i) \, \theta(t - \tau_i(t)) + Ly(t) \end{aligned}$$

with an initial condition $\theta(s) = \psi(s)$ such that

$$\phi(s) \le \psi(s) \le \lambda$$

for all $s \in [-\tau^+, 0]$. By adding and subtracting θ we obtain

$$\theta^+(t) - x(t) = \big(\theta^+(t) - \theta(t)\big) + (\theta(t) - x(t)) \, .$$

and show that both differences are positive for every $t \ge 0$. To conclude that the first difference is positive we apply

Lemma 4. The second difference satisfies the following differential equation

$$\frac{d}{dt}\left[(\theta - x)(t)\right] = (A - LC)\,(\theta - x)(t) + \sum_{i=1}^{p}(A_i - LC_i)\,(\theta - x)(t - \tau_i(t))$$

By assumption $A-LC$ is Metzler and the matrices A_i-LC_i are positive. Applying Lemma 2 yields the desired positivity. ■

Remark 1: Let $(\theta^-(\cdot), \theta^+(\cdot))$ be an interval observer for the system (1). Then the interval error $e(\cdot) = \theta^+(\cdot) - \theta^-(\cdot)$ satisfies the following positive system equation

$$\dot{e}(t) = (A - LC)\;e(t) + \sum_{i=1}^{p}(A_i - LC_i)\;e(t - \tau_i(t)) \quad (22)$$

and we conclude from Lemma 2 that necessarily condition (i) holds.

Based on all the previous lemmas and corollaries, the proof of the main theorem is given in the following. For the readers convenience we restate Theorem 1 first.

Theorem 1: There exists a convergent interval observer (θ^-, θ^+) *for system* (1) *if and only if the following conditions hold true. There is a gain matrix* L *such that*

(i) $A-LC$ *is Metzler and* $A_i-LC_i \geq 0$ *for all* $i = 1, ..., p$,

(ii) $A - LC + \sum_{i=1}^{p}(A_i - LC_i)$ *is Hurwitz.*

Proof: First we show that conditions (i) and (ii) are sufficient for the existence of a convergent interval observer. As condition (ii) implies condition (ii) in Lemma 5 the existence of an interval observer follows from Lemma 5. Hence it remains to prove that $e(t)$ is convergent. The derivative of the interval error satisfies

$$\begin{aligned}\dot{e}(t) &= \dot{\theta}^+(t) - \dot{\theta}^-(t)\\ &= (A - LC)\;(\theta^+(t) - \theta^-(t))\\ &\quad + \sum_{i=1}^{p}(A_i - LC_i)\;\left(\theta^+(t-\tau^+) - \theta^-(t-\tau^+)\right)\\ &= (A - LC)\,e(t) + \sum_{i=1}^{p}(A_i - LC_i)\,e(t-\tau^+).\end{aligned}$$

Since $A - LC + \sum_{i=1}^{p} A_i - LC_i$ is Hurwitz it follows from Proposition 1 that the interval error is asymptotically stable. Hence the interval error is convergent.

Conversely, to conclude that conditions (i) and (ii) are also necessary consider the following observer

$$\dot{\theta}(t) = (A - LC)\,\theta(t) + \sum_{i=1}^{p}(A_i - LC_i)\,\theta(t - \tau_i(t)) + Ly(t).$$

Since $e(\cdot)$ is convergent it holds also that $\theta^+(t) - \theta^-(t)$ converges to 0 as t goes to infinity as well as $\theta(t) - x(t)$ converges to 0 as t goes to infinity. This means that

$$\frac{d}{dt}\left[(\theta - x)(t)\right] = (A - LC)(\theta - x)(t) + \sum_{i=1}^{p}(A_i - LC_i)(\theta - x)(t - \tau_i(t))$$

is asymptotically convergent and this is by Proposition 1 equivalent to the fact that $A - LC + \sum_{i=1}^{p} A_i - LC_i$ is Hurwitz. The necessity of (i) follows from Remark 1. ■

IV. CONCLUSIONS

This paper presents a necessary and sufficient condition for the existence of convergent interval observers for linear systems with time-varying bounded delays. Our approach is based on observers of extended Luenberger type. This framework yields to positive systems with subject to the error between the state and the observation of the system. Thus the analysis of the evolution of the error uses techniques from positive systems.

REFERENCES

[1] J. Gouzé, A. Rapaport, and M. Hadj-Sadok, "Interval observers for uncertain biological systems," *Ecological modelling*, vol. 133, no. 1-2, pp. 45–56, 2000.

[2] A. Rapaport and J. Gouzé, "Parallelotopic and practical observers for non-linear uncertain systems," *International Journal of Control*, vol. 76, no. 3, pp. 237–251, 2003.

[3] M. Ait Rami, U. Helmke, and F. Tadeo, "Positive observation problem for time-delays linear positive systems." in *Proc. of the 15th IEEE Med. Conference Athens*, 2007, pp. 1–6.

[4] F. Mazenc, S. Niculescu, and O. Bernard, "Interval observers for linear systems with delay," in *Proc. of the 48th IEEE Conference on Decision and Control, 2009, Shanghai, China*, 2009, pp. 1860–1865.

[5] D. Luenberger, *Introduction to dynamic systems: theory, models, and applications.* Wiley New York, 1979.

[6] L. Farina and S. Rinaldi, *Positive linear systems. Theory and applications.* New York, NY: Wiley, 2000.

[7] M. Ait Rami, "Stability analysis and synthesis for linear positive systems with time-varying delays." Bru, R. and Romero Vivo (eds.), Positive Systems. LNCIS 389, 205-215., pp. 205–215, 2009.

[8] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations.* Applied Mathematical Sciences. 99. New York, NY: Springer-Verlag, 1993.

[9] R. Bellman, *Introduction to matrix analysis. 2nd ed., Reprint of the 1970 Orig.* Classics in Applied Mathematics. 19. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 1997.

[10] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences.* Classics in Applied Mathematics. 9. Philadelphia, PA: SIAM, 1994.

[11] W. M. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51, no. 5, pp. 355 – 361, 2004.