Controlling Mean and Variance in Ensembles of Linear Systems ⋆

G. Dirr * **U. Helmke** * **M. Schönlein** *

* *Institute of Mathematics, University of Würzburg, Germany.*
(e-mail: {dirr,helmke,schoenlein}@mathematik.uni-wuerzburg.de).

Abstract: We investigate the possibilities of steering probability density functions of state variables in linear control systems, using a combination of open loop and time-varying output feedback control strategies. This is an intrinsically nonlinear control problem which makes contact with earlier work by Brockett (2012) on controlling the mean and variance of linear systems via time-varying state feedback transformations. We extend Brockett's work on the control of the Liouville equation to the more difficult output feedback case, as well as to parallel connected linear systems. Our methods depend on certain controllability results for bilinear systems Brockett (1976), Dirr et al. (2016), where the controls are defined by the output feedback gain.

Keywords: Ensemble Control, Bilinear Systems, Controllability, Algebraic Methods, Geometric Methods

1. INTRODUCTION

The effects of noise on the state variables in large scale networks of interconnected dynamical systems often prevent one to be able to steer individual states into each other. More realistic goals then are of interest, such as controlling the mean values or covariance of the state variables. An instance for this arises in the control of density operators in quantum systems that describe the evolution of averages of state variables, cf. Kurniawan et al. (2012). Parametric families of such quantum systems play an important role in MNR spectroscopy. Their lack of exact controllability has led to the currently very active field of ensemble control, where families of states, or probability density functions, are to be controlled rather than a single state vector only. The design of controllers that morph one probability density function into another one then becomes a natural goal of ensemble control, cf. Brockett (2012), Zeng et al. (2014), Chen et al. (2015). In contrast to familiar stochastic control approaches to random state variables, where the Fokker-Planck equation is a natural tool, our focus is on deterministic problems that are associated with controlling the Liouville transport equation.

Our starting point are time-invariant linear systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where the initial states $x(0)$ are distributed according to a given probability density function on $\mathbb{R}^n$. We assume that the system is controllable and observable in the usual sense, i.e., any two state vectors can steered into each other in finite time, using open loop controls (similarly for observability). While this ballistic definition of controllability via open loop control $u(t)$ is sufficient for controlling single states, it is not so for the tasks of controlling ensembles of states. Here it is much better to apply controllers that allow for a combination $u(t, x)$ of open loop inputs and closed loop controllers. For controlling single states only, this distinction certainly does not play a role, but it does so for controlling ensembles of states. We refer to Brockett (1997), Brockett (2012) for further motivation of using combined closed loop and open loop control. In the sequel, we focus on state–affine inputs of the form

$$u(t, x) = u(t) + K(t)Cx \tag{2}$$

by combining arbitrary open loop inputs $u(t)$ with arbitrary time–varying output feedback transformations $K(t)Cx$. Here both, $u(t)$ and the feedback gain matrix $K(t)$, are regarded as controls that act on the system. Using these control inputs we aim at steering a smooth probability density function $\rho(0)$ of initial states arbitrarily close to a desired smooth probability density function ρ^* in finite time $T > 0$.

Let $\rho(t, x)$ denote the smooth probability density function, obtained by transporting the initial density $\rho(x)$ via the linear system (1). Standard calculations reveal that $\rho(t, x)$ satisfies the Liouville transport equation

$$\begin{aligned} \frac{\partial \rho}{\partial t}(t, x) = &- \operatorname{tr}(A + BK(t)C)\rho(t, x) \\ &- \big(Ax + BK(t)Cx + Bu(t)\big)^\top \nabla\rho(t, x) \end{aligned} \tag{3}$$

with $\nabla\rho := (\frac{\partial \rho}{\partial x_1}, \dots, \frac{\partial \rho}{\partial x_n})^\top$ which is a bilinearly controlled partial differential equation. The *ensemble control* task for (1) thus becomes a bilinear control problem for the Liouville equation (3). Since the flow of (3) leaves the Euclidean group orbit $SE(n) \cdot \rho = \{\rho(g \cdot x) \mid g \in SE(n)\} \subset \mathcal{P}$ of a probability density function ρ invariant, this infinite dimensional control problem is never solvable, using controllers of the form (2). Further insight and applications

⋆ U. Helmke and M. Schoenlein are supported by the German Research Foundation HE 1858/13-1 and HE 1858/14-1.
It is with deepest sorrow that we report the demise of our friend and co-author Uwe Helmke while preparing the final manuscript.

of the controlled Liouville are provide by Brockett (2012) and Chen et al. (2015).

As a natural next step one may then proceed to the easier task of controlling the first few moments of ρ. In this paper we focus on controlling the mean and the variance

$$\begin{aligned}\mu(t) &= \int_{\mathbb{R}^n} x\rho(t,x)dx \\ Q(t) &= \int_{\mathbb{R}^n} (x-\mu(t))(x-\mu(t))^\top \rho(t,x)dx.\end{aligned} \tag{4}$$

A straightforward computation shows that the mean and variance matrices μ, Q for the probability state densities of (1) satisfy

$$\begin{aligned}\dot{\mu}(t) &= (A+BK(t)C)\mu(t) + Bu(t) \\ \dot{Q}(t) &= (A+BK(t)C)Q(t) + Q(t)(A+BK(t)C)^\top.\end{aligned} \tag{5}$$

We emphasize that this is a cascade of a linear and a bilinear control system, with independent controls $u(t)$ and $K(t)$. Of course, the equation for the mean is always controllable by suitable open loop inputs $u(t)$ and choosing, e.g., $K = 0$. However, to control the variance Q one must as well allow for arbitrary time-varying gains $K(t)$. Thus the task of simultaneously controlling mean and variance can only be achieved by combining open loop inputs together with controlling the feedback gain.

Brockett (2012) has shown that (5) is controllable if $C = I$; i.e. in the state feedback case. He raises the question under which conditions (5) is controllable using output feedback transformations (2). In this paper, we will address this problem deriving sufficient conditions for controllability of (5). In order to do so, we first have to extend some earlier work by Brockett on controllability of bilinear feedback systems. This enables us to derive sufficient conditions for controllability of (5). We also develop an alternative approach using results on pole placement by output feedback, cf. Wang (1992) and Rosenthal et al. (1995). Finally, via somewhat more delicate Lie theoretic arguments, we extend these controllability results to parallel connected systems. This may be seen as a first step towards a better understanding of ensemble control for interconnected linear systems.

2. CONTROLLABILITY OF MEAN AND VARIANCE:

2.1 Bilinear Output Feedback Control

Before starting with the controllability analysis of (5) we study controllability of the related bilinear control system

$$\dot{x}(t) = (A + BK(t)C)\, x(t), \tag{6}$$

where $(A,B,C) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}$ is real, controllable and observable and $K(t) \in \mathbb{R}^{m\times p}$ denotes the controls. Moreover, let $B = (b_1,\dots,b_m)$ and $C = \mathrm{col}(c_1,\dots,c_p)$ be of full column rank and full row rank, respectively. Let $G(s) = C(sI-A)^{-1}B$ denote the transfer function of (A,B,C) and let

$$\mathfrak{g} := \langle A, b_ic_j \mid i,j\rangle_{\mathrm{LA}}$$

denote the *system Lie algebra* of $\dot{x} = (A+BK(t)C)x$, i.e. $\mathfrak{g}$ is the real Lie algebra of $n \times n$ matrices, which is generated by A and b_ic_j, $i = 1,\dots,m, j = 1,\dots,p$.

We say that a $p \times m$ transfer function $G(s)$ is *strongly degenerate*, if there exist a constant *rank one* matrix $R \in \mathbb{R}^{p\times m}$ and a scalar transfer function $g(s)$ such that $G(s) = g(s)R$. Otherwise, we say that G is *not strongly degenerate*.

Let $\mathfrak{gl}_n(\mathbb{R}) := \mathbb{R}^{n\times n}$ and $GL_n(\mathbb{R})$ be the set of all invertible $n\times n$ matrices. Moreover, let $\mathfrak{sl}_n(\mathbb{R})$ denote the Lie algebra of real $n\times n$ matrices with trace zero and $\mathfrak{sp}_{n/2}(\mathbb{R})$ the Lie algebra of real $n\times n$ Hamiltonian matrices. The next result completes the controllability analysis in Brockett (1976) and Brockett (1983), which did not inspect all cases of multivariable transfer functions.

Theorem 1. Assume that $G(s)$ is strongly degenerate. The system Lie algebra $\mathfrak{g}$ of $\dot{x} = (A+BK(t)C)x$ is $GL_n(\mathbb{R})$-conjugate to one of the following Lie algebras:

(a) $\mathfrak{sp}_{n/2}(\mathbb{R})$ iff $G(s) = G(-s)$.
(b) $\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$ iff $G(s) = G(-s+\alpha)$ with $\alpha \in \mathbb{R}\setminus\{0\}$ suitable.
(c) $\mathfrak{sl}_n(\mathbb{R})$ iff $G(s) \neq G(-s+\alpha)$ for all $\alpha \in \mathbb{R}$ and $CB = 0, \mathrm{tr}\, A = 0$.
(d) $\mathfrak{gl}_n(\mathbb{R})$ iff none of the above.

Assume that $G(s)$ is not strongly degenerate. Then the system Lie algebra $\mathfrak{g}$ satisfies

(e) $\mathfrak{sl}_n(\mathbb{R})$ iff $CB = 0, \mathrm{tr}\, A = 0$.
(f) $\mathfrak{gl}_n(\mathbb{R})$ iff else.

Proof. Without loss of generality we assume that B and C are full column rank and full row rank matrices, respectively. The equivalences (a) - (d) are an immediate consequence of Brockett (1976). To prove the second part we note that the identity of system Lie algebras

$$\langle A, b_ic_j \mid i,j\rangle_{\mathrm{LA}} = \langle A+BK, b_ic_j \mid i,j\rangle_{\mathrm{LA}}$$

holds for every feedback matrix K. Using a result of Brasch and Pearson (1970), there exists for each i,j an output feedback transformation such that $(A+BKC, b_i, c_j)$ is controllable and observable. Thus the first part of the theorem implies that

$$\mathfrak{sp}_{n/2}(\mathbb{R}) \;\subset\; \langle A+BK, b_ic_j \mid i,j\rangle_{\mathrm{LA}} \;\subset\; \mathfrak{g}.$$

For $n = 2$ this completes the proof of (e) and (f), as $\mathfrak{sp}_1(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sp}_1(\mathbb{R}) \oplus \mathbb{R}I = \mathfrak{gl}_2(\mathbb{R})$. For $n \geq 3$, the remaining parts of the proof proceed by a case study, using the next elementary result, whose proof is in Brockett (1976).

Lemma 2. If $\mathfrak{g}$ denotes a Lie algebra of real $n\times n$ matrices which contains $\mathfrak{sp}_{n/2}(\mathbb{R})$, then $\mathfrak{g}$ is equal to $\mathfrak{sp}_{n/2}(\mathbb{R})$, $\mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I_n$, $\mathfrak{sl}_n(\mathbb{R})$, or $\mathfrak{gl}_n(\mathbb{R})$.

Case 1: $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{R})$. Then A and $b_ic_j, i = 1,\dots,m, j = 1,\dots,p$ are Hamiltonian matrices and thus, for the standard symplectic form J, and all i,j one has

$$AJ = (AJ)^\top, b_ic_jJ = (b_ic_jJ)^\top.$$

For $i = j$ this implies $b_i = r_iJ^\top c_i^\top$ Since $b_i \neq 0$ and $c_j \neq 0$ we have $r_i \neq 0$. Therefore

$$r_iJ^\top c_i^\top c_jJ = J^\top c_j^\top b_i^\top = r_iJ^\top c_j^\top c_iJ,$$

which implies that $c_i^\top c_j = c_j^\top c_i$ for all i,j. But this is equivalent to $c_i = \gamma_ic$ for nonzero numbers γ_i and $c \in \mathbb{R}^{n\times 1}$. Similarly, $b_i = \beta_ib$ for $\beta_i \neq 0$ and $b \in \mathbb{R}^n$. Therefore, $G(s) = g(s)R$ has rank one, with $R = (\beta_i\gamma_j)$ a rank one matrix and $g(s) = c(sI-A)^{-1}b$. Clearly, (A,b,c)

must be controllable and observable. A straightforward computation, using the symmetry of $AJ, b_i c_j J$, shows that

$$\begin{aligned} g_{ji}(s) &= \mathrm{tr}((sI-A)^{-1} b_i c_j) = \mathrm{tr}((sJ-AJ)^{-1} b_i c_j J) \\ &= \mathrm{tr}((-sJ-AJ)^{-1} b_i c_j J) = g_{ji}(-s). \end{aligned} \tag{7}$$

Thus $G(s) = G(-s)$ and therefore $g(-s) = g(s)$.

Case 2: $\mathfrak{g} = \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$. Arguing as before we conclude that $A - \frac{\mathrm{tr}\, A}{N} I$ and $b_i c_j - \frac{c_j b_i}{N} I$ are Hamiltonian matrices. This implies

$$b_i c_j J - (b_i c_j J)^\top = \frac{2 c_j b_i}{N} J.$$

The matrix on the left hand side has rank at most two, while the matrix on the right is invertible, or zero. Since $n \geq 3$, this implies

$$b_i c_j J - (b_i c_j J)^\top.$$

Thus we can argue as before and conclude that $G(s) = g(s)R$ holds for a rank one matrix $R \in \mathbb{R}^{p\times m}$. Hence, for some $\alpha \neq 0$ it holds $g(s-\alpha) = g(s)$. This completes the proof. □

We next provide a different perspective on controllability results for (6). This rests on the assumption that the system matrices (A, B, C) are chosen generically. Recall that a complex $n \times n$ matrix M is called *strongly regular*, if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M are all distinct and satisfy

$$\lambda_i - \lambda_j \neq \lambda_k - \lambda_\ell$$

for all pairs of distinct indices $(i, j), (k, \ell)$ with $i \neq j, k \neq \ell$. We need the following result from Dirr et al. (2016) to prove Theorem 4 below.

Proposition 3. Suppose $A_1, \ldots, A_d \in \mathbb{R}^{n\times n}$ satisfy:

(a) There exist $u_1, \ldots, u_d \in \mathbb{R}$ such that $\sum_{j=1}^d u_j A_j$ is strongly regular.
(b) There exists no nontrivial common invariant subspace $V \subset \mathbb{C}^n$ of $A_1, \ldots, A_d$.

Then the real Lie algebra $\langle A_1, \ldots, A_d\rangle_{\mathrm{LA}}$ is either $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$.

Theorem 4. Assume that (A, B, C) are chosen generic and either $mp > n$, or $mp = n$ with m or p odd. Then the bilinear system (6) is controllable.

Proof. By a theorem of Eremenko and Gabrielov (2002) (see Wang (1992) and Rosenthal et al. (1995) for earlier versions), the map $\chi : \mathbb{R}^{m\times p} \to \mathbb{R}^N$ that assigns to K the coefficients of the characteristic polynomial of $A + BKC$, is surjective. This implies that there exists $K \in \mathbb{R}^{m\times p}$ such that $A + BKC$ is strongly regular. If (A, B, C) are controllable and observable, it is easily seen that there exists no nontrivial common invariant subspace of A and the matrices $BKC, K \in \mathbb{R}^{m\times p}$. By Proposition 3, the system Lie algebra is equal to either $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$. The first possibility is ruled out, as $\mathrm{tr}(BKC) \neq 0$ for some K and generic (A, B, C). This proves accessibility of (6). Moreover, by surjectivity of χ there exists K such that $A + BKC$ has only simple, purely imaginary eigenvalues. This implies almost periodicity of $e^{t(A+BKC)}$. The result follows from Sussmann and Jurdjevic (1972). □

The preceding results can be extended to feedback gains $K(t)$ that lie in a fixed linear subspace $\mathcal{K}$ of $\mathbb{R}^{m\times p}$. We omit the (difficult) details.

2.2 Controlling Mean and Variance

We next turn to a discussion of control system (5) for the mean and variance. Let

$$\mathcal{P}(n) = \{Q = Q^\top \in \mathbb{R}^{n\times n} \mid Q > 0\}$$

denote the set of positive definite $n\times n$–symmetric matrices and $\mathcal{P}_1(n) \subset \mathcal{P}(n)$ denote the submanifold of all positive definite matrices with determinant 1. Alternatively, $\mathcal{P}_1$ could denote the set of positive definite matrices with trace 1. Recall, that a nonlinear system is accessible, if the reachable sets of the initial states have interior points.

Theorem 5. Assume that the transfer function of (A, B, C) is strongly degenerate and satisfies $G(s) \neq G(s+\alpha)$ for all $\alpha \in \mathbb{R}$. Then the moment control system (5) satisfies:

(a) It is accessible in $\mathbb{R}^n \times \mathcal{P}_1(n)$.
(b) It is accessible in $\mathbb{R}^n \times \mathcal{P}(n)$, if $CB \neq 0$ or $\mathrm{tr}\, A \neq 0$.

Proof. The reachable sets of the moment system (5) coincide with those of the decoupled system

$$\begin{aligned} \dot\mu(t) &= A\mu(t) + Bu(t) \\ \dot Q(t) &= (A + BK(t)C)Q(t) + Q(t)(A + BK(t)C)^\top. \end{aligned} \tag{8}$$

Thus, accessibility (or controllability) of (5) on $\mathbb{R}^n \times \mathcal{P}_1(n)$ holds if and only if

$$\dot Q(t) = (A + BK(t)C)Q(t) + Q(t)(A + BK(t)C)^\top \tag{9}$$

is accessible (or controllable) on $\mathcal{P}_1(n)$. Moreover, the quadratic transformation $X \mapsto Q = XX^\top$ maps trajectories in $SL_n(\mathbb{R})$ of $\dot X = (A + BK(t)C)\,X$ onto those of (9). Thus (5) is accessible whenever the system Lie algebra $\mathfrak{g}$ of (6) contains $\mathfrak{sl}_n(\mathbb{R})$. By Theorem 1, the system Lie algebra $\mathfrak{g}$ is either $\mathfrak{sl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{R})$. This proves the first part; similarly one concludes the second part. □

Theorem 5 immediately leads to the following pleasing controllability result on the moment system (5).

Corollary 6. Suppose $CB \neq 0$ or $\mathrm{tr}\, A \neq 0$. Assume further that there exists K such that $A + BKC$ has only distinct purely imaginary eigenvalues. Then, any element $(\mu_0, Q_0) \in \mathbb{R}^n \times \mathcal{P}(n)$ can be steered via (5) to an arbitrary element $(\mu_*, Q_*) \in \mathbb{R}^n \times \mathcal{P}(n)$ in finite time $T > 0$.

Proof. We recall that a bilinear system on a Lie group is controllable, provided it is accessible and there exists a constant input such that the system is weakly Poisson stable. By assumption, there exists a constant matrix K such that $e^{t(A+BKC)}$ is almost periodic. Moreover, by Theorem 5, the system $\dot X = (A + BK(t)C)X$ on the Lie group $SL_n(\mathbb{R})$ is accessible, as the system Lie algebra contains $\mathfrak{sl}_n(\mathbb{R})$. This completes the proof. □

Under additional genericity assumptions we immediately deduce from Theorem 4 the following result.

Theorem 7. Assume that either $mp > n$, or $n = mp$ with m or p odd. Then, for generic choices of (A, B, C), every initial state $(\mu_0, Q_0) \in \mathbb{R}^n \times \mathcal{P}(n)$ can be steered via (5) in finite time $T > 0$ to an arbitrary element $(\mu_*, Q_*) \in \mathbb{R}^n \times \mathcal{P}(n)$.

3. PARALLEL CONNECTED MOMENT SYSTEMS

3.1 Parallel connected bilinear systems

In this subsection, we briefly sketch a general accessibility condition for parallel connected bilinear control systems, cf. Dirr (2012). Let $\mathfrak{g}_j \subset \mathfrak{gl}_{n_j}(\mathbb{R})$ for $j = 1, \ldots, r$ be Lie subalgebras and let $\mathfrak{g} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ be their direct sum represented as block diagonal matrices in $\mathfrak{gl}_{\mathbf{n}}(\mathbb{R})$ with $N = n_1 + \cdots + n_r$. Given the bilinear systems Σ_j

$$\dot{X}_j = \left(A_j + \sum_{k=1}^{m} u_k(t) B_{kj} \right) X_j \,, \tag{10}$$

with $A_j, B_{kj} \in \mathfrak{g}_j$ for $k = 1, \ldots, m$ and $j = 1, \ldots, r$. Thus Σ_j evolves on the connected matrix Lie subgroup $G_j \subset \mathrm{GL}_{n_j}(\mathbb{R})$ whose Lie algebra coincides with $\mathfrak{g}_j$. Their parallel connection Σ is given by

$$\dot{X} = \left(A + \sum_{k=1}^{m} u_k(t) B_k \right) X \tag{11}$$

with

$$A = \mathrm{diag}(A_1, \cdots, A_r) \text{ and } B_k = \mathrm{diag}(B_{k1}, \cdots, B_{kr}) \,.$$

It evolves naturally on the Lie subgroups $G \subset \mathrm{GL}_N(\mathbb{R})$, whose Lie algebra is given by $\mathfrak{g}$. Note that the controls $u_k(t) \in \mathbb{R}$ do not depend on j. Then one has the following accessibility result.

Theorem 8. Let $\mathfrak{g}_j \subset \mathfrak{gl}_{n_j}(\mathbb{R})$ be simple Lie subalgebras and assume that the system algebra of Σ_j satisfies $\langle A_j, B_{1j}, \ldots, B_{mj} \rangle_{\mathrm{LA}} = \mathfrak{g}_j$. Then the system Lie algebra $\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}}$ of Σ coincides with $\mathfrak{g}$ if, for $i \neq j$, there does not exist a Lie algebra isomorphism $\Phi : \mathfrak{g}_i \to \mathfrak{g}_j$ with $\Phi(A_i) = A_j$ and $\Phi(B_{ki}) = B_{kj}$, $k = 1, \ldots, m$.

For simplicity, we restrict to the case $r = 2$, i.e. $\mathfrak{g}$ and Σ take the explicit form $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and

$$\dot{X} = \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \sum_{k=1}^{m} u_k(t) \begin{pmatrix} B_{k1} & 0 \\ 0 & B_{k2} \end{pmatrix} \right) X \,, \tag{12}$$

respectively. We start with an auxiliary result which is of independent interest.

Lemma 9. If the system algebra $\mathfrak{s} := \langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}}$ of (12) is given as a graph over $\mathfrak{g}_1$, i.e. if

$$\mathfrak{s} = \{ X + \Phi(X) \mid X \in \mathfrak{g}_1 \}$$

then $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ has to be an isomorphism.

Proof. Since the subalgebra $\mathfrak{s}$ is in particular a vector space, Φ has be linear. Moreover, the identity

$$[X + \Phi(X), Y + \Phi(Y)] = [X, Y] + [\Phi(X), \Phi(Y)]$$

implies $\Phi\big([X, Y]\big) = [\Phi(X), \Phi(Y)]$ for all $X, Y \in \mathfrak{g}_1$ and thus Φ is actually a Lie algebra homomorphism. Now, consider the kernel $\ker \Phi \subset \mathfrak{g}_1$. Since Φ is an Lie algebra homomorphism $\ker \Phi$ is an ideal in $\mathfrak{g}_1$ and therefore the simplicity of $\mathfrak{g}_1$ implies $\ker \Phi = \{0\}$ or $\ker \Phi = \mathfrak{g}_1$. As Φ is onto due to the fact that the system Lie algebra of Σ_2 is equal to $\mathfrak{g}_2$ we conclude that $\ker \Phi = \{0\}$ and hence Φ is an isomorphism. □

Proof of Theorem 8. (for $r = 2$) First of all, note that if the system algebra $\mathfrak{s} := \langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}}$ of (12) is the graph of some isomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$, then certainly the equalities $\Phi(A_1) = A_2$ and $\Phi(B_{k1}) = B_{k2}$ hold for $k = 1, \ldots, m$. Therefore, due to Lemma 9, it suffices to show that $\mathfrak{s}$ is a graph over $\mathfrak{g}_1$ whenever it does not coincide with $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Hence, let us assume $\mathfrak{s} \neq \mathfrak{g}_1 \oplus \mathfrak{g}_2$. To prove that $\mathfrak{s}$ is a graph over $\mathfrak{g}_1$ we only have to show $\mathfrak{s} \cap \mathfrak{g}_2 = \{0\}$ because the assumption that the system Lie algebra of Σ_1 coincides with $\mathfrak{g}_1$ guarantees that if $\mathfrak{s}$ is a graph it is defined over $\mathfrak{g}_1$.

As an intersection of two Lie subalgebras, $\mathfrak{s} \cap \mathfrak{g}_2$ is again a Lie subalgebra. To prove that $\mathfrak{s} \cap \mathfrak{g}_2$ is actually an ideal of $\mathfrak{g}_2$, we choose arbitrary $X_2 \in \mathfrak{s} \cap \mathfrak{g}_2$ and $Y_2 \in \mathfrak{g}_2$. Then, there exists $Y_1 \in \mathfrak{g}_1$ such that $Y_1 + Y_2 \in \mathfrak{s}$ holds and hence it follows

$$[X_2, Y_2] = [X_2, Y_1 + Y_2] \in \mathfrak{s} \cap \mathfrak{g}_2 \,.$$

Thus $\mathfrak{s} \cap \mathfrak{g}_2$ is an ideal of $\mathfrak{g}_2$. Since $\mathfrak{g}_2$ is simple we conclude either $\mathfrak{s} \cap \mathfrak{g}_2 = \{0\}$ or $\mathfrak{s} \cap \mathfrak{g}_2 = \mathfrak{g}_2$. By the assumption $\mathfrak{s} \neq \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we can exclude the latter equality $\mathfrak{s} \cap \mathfrak{g}_2 = \mathfrak{g}_2$ and thus we are done. □

Let $\mathfrak{g}_j = \mathfrak{z}_j \oplus \mathfrak{g}_j^0 \subset \mathfrak{gl}_{n_j}(\mathbb{R})$ be Lie subalgebras with center $\mathfrak{z}_j$ and simple ideals $\mathfrak{g}_j^0$. Let A_i^0, B_{ki}^0 and A_j^0, B_{kj}^0 denote the $\mathfrak{g}_i^0$- and $\mathfrak{g}_j^0$-components of A_i, B_{ki} and A_j, B_{kj}, respectively.

Corollary 10. Suppose that the system algebra of Σ_j satisfies $\langle A_j, B_{1j}, \ldots, B_{mj} \rangle_{\mathrm{LA}} = \mathfrak{g}_j$. Then the system algebra of Σ satisfies

$$\mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0 \subset \langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}}$$

if, for $i \neq j$, there does not exist a Lie algebra isomorphism $\Phi : \mathfrak{g}_i^0 \to \mathfrak{g}_j^0$ with $\Phi(A_i^0) = A_j^0$ and $\Phi(B_{ki}^0) = B_{kj}^0$, $k = 1, \ldots, m$.

Proof. Let $\mathfrak{s} := \langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}}$ denote the system algebra of Σ and let $\mathfrak{s}^0$ be the Lie algebra generated by $\langle A^0, B_1^0, \ldots, B_m^0 \rangle_{\mathrm{LA}}$. Since the system algebras of Σ_j coincide with $\mathfrak{g}_j$ for $j = 1, \ldots, r$, we can conclude $\langle A_j^0, B_{1j}^0, \ldots, B_{mj}^0 \rangle_{\mathrm{LA}} = \mathfrak{g}_j^0$ for $j = 1, \ldots, r$. Therefore, Theorem 8 implies $\mathfrak{s}^0 = \mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0$ and thus

$$\pi_\Sigma^0(\mathfrak{s}) = \mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0 \,,$$

where $\pi_\Sigma^0 : \mathfrak{g} \to \mathfrak{g}_1^0 \oplus \mathfrak{g}_2^0$ denotes the canonical projection onto $\mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0$. By the semisimplicity of $\mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0$, it follows

$$\mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0 = [\mathfrak{s}, \mathfrak{s}]$$

and thus $\mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_r^0 \subset \mathfrak{s}$. □

3.2 Parallel Connected Moment Systems

We begin with a brief discussion of controllability for parallel connected systems

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u(t) \\ &\vdots \\ \dot{x}_r &= A_r x_r + B_r u(t) \end{aligned} \tag{13}$$

of controllable pairs $(A_j, B_j) \in \mathbb{R}^{n_j \times n_j} \times \mathbb{R}^{n_j \times m}$. Choose right coprime polynomial factorizations

$$(sI - A_j)^{-1} B_j = N_j(s) D_j(s)^{-1}.$$

Deciding the controllability of (13) is of course a classical topic in control and characterizations for controllability of two parallel connected systems are well known by Fuhrmann (1975). It seems that the case of more than 2 systems has been ignored in the literature, with the exception of the following characterization in Fuhrmann and Helmke (2015).

Proposition 11. (Fuhrmann and Helmke (2015)). The parallel connection (13) is controllable if and only if the polynomial matrix

$$\begin{bmatrix} D_1(s) & D_2(s) & & \\ & \ddots & \ddots & \\ 0 & \cdots & D_{r-1}(s) & D_r(s) \end{bmatrix} \tag{14}$$

is left prime.

In the sequel, we will always assume this coprimeness condition. It will thus guarantee controllability of (13).

Assumption A. The polynomial matrix (14) is left prime. Moreover, each transfer function $G_j(s) = C_j(sI - A_j)^{-1}B_j \in \mathbb{R}(s)^{p\times m}$ is either not strongly degenerate or it satisfies $G_j(s) \neq G_j(\pm s + \alpha)$ for all $\alpha \in \mathbb{R}$.

Next, we investigate the controllability of parallel connected moment control systems (5), i.e. we consider the parallel connection

$$\begin{aligned} \dot\mu_1 &= (A_1 + B_1K(t)C_1)\mu_1 + B_1u(t) \\ &\vdots \\ \dot\mu_r &= (A_r + B_rK(t)C_r)\mu_r + B_ru(t) \end{aligned} \tag{15}$$

$$\begin{aligned} \dot Q_1 &= (A_1 + B_1K(t)C_1)Q_1 + Q_1(A_1 + B_1K(t)C_1)^\top \\ &\vdots \\ \dot Q_r &= (A_r + B_rK(t)C_r)Q_r + Q_r(A_r + B_rK(t)C_r)^\top . \end{aligned} \tag{16}$$

controlled by a vector valued input function $u(t) \in \mathbb{R}^m$ and a matrix valued input function $K(t) \in \mathbb{R}^{m\times p}$. We assume that (A_j, B_j, C_j) are controllable and observable with m inputs and p outputs, and local state spaces $\mathbb{R}^{n_j}$. Below, we establish the accessibility and controllability properties of system (15)-(16) on the state space

$$\mathbb{R}^{\mathbf{n}} \times \mathcal{P}_1(\mathbf{n}) := \mathbb{R}^{n_1+\cdots+n_r} \times \mathcal{P}_1(n_1) \times \cdots \times \mathcal{P}_1(n_r).$$

In order to apply Theorem 8 one has to determine the automorphisms of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{R})$. It can be shown, see Jacobson (1939), that the automorphisms Φ of $\mathfrak{sl}_n(\mathbb{R})$ are either of the form $\Phi(X) = \Theta X\Theta^{-1}$ or $\Phi(X) = -\Theta X^\top\Theta^{-1}$, for some $\Theta \in \mathrm{SL}_n(\mathbb{R})$.

Theorem 12. Let (A_j, B_j, C_j) satisfy Assumption A with $\operatorname{tr} A_j = 0$ and $C_jB_j = 0$ for $j = 1, \dots, r$. Then the parallel connection system (15) - (16) is accessible on $\mathbb{R}^{\mathbf{n}} \times \mathcal{P}_1(\mathbf{n})$ if $G_i(s) \neq G_j(\pm s)$ for $i \neq j$.

Proof. Consider the parallel connected bilinear control system

$$\dot X_j = (A_j + B_jK(t)C_j)X_j, \quad j = 1, \dots, r \tag{17}$$

on the Lie group $\mathrm{SL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{SL}_{n_r}(\mathbb{R})$. Once we have shown that (17) is accessible, the same argument as in the subsequent Lemma 13 will immediately imply the result. To prove accessibility, note that the assumptions on $G_j(s)$ guarantee that each of the bilinear systems in (17) is accessible. In order to invoke Theorem 8, we show that there exists no Lie algebra isomorphism Φ with $\Phi(A_i) = A_j$ and $\Phi(b_{ki}c_{li}) = b_{kj}c_{lj}$. Certainly, for $n_i \neq n_j$ such an isomorphism cannot exist. Therefore, assume $n_i = n_j$. Any automorphism Φ is either given by (i) $\Phi(X) = \Theta X\Theta^{-1}$ for some $\Theta \in \mathrm{SL}_n(\mathbb{R})$ or by (ii) $\Phi(X) = -\Theta X^\top\Theta^{-1}$. In case (i), we obtain $G_j(s) = G_i(s)$. In case (ii), for $\Theta = I_n$ we compute

$$\begin{aligned} g^j_{lk}(s) &:= c_{lj}(sI - A_j)^{-1}b_{kj} = \operatorname{tr}(sI - A_j)^{-1}b_{kj}c_{lj} \\ &= -\operatorname{tr}(sI + A_i^\top)^{-1}c_{lj}^\top b_{kj}^\top = b_{kj}^\top(-sI - A_i^\top)^{-1}c_{lj}^\top \\ &= c_{lj}(-sI - A_i)^{-1}b_{kj} = g^i_{lk}(-s). \end{aligned}$$

Thus $G_j(s) = G_i(-s)$, in contradiction to the assumption. Thus the result follows from Theorem 8. □

The preceding proof uses the following lemma.

Lemma 13. The bilinear cascade system

$$\dot\mu = (A + BK(t)C)\mu + Du(t) \tag{18}$$

$$\dot X = (A + BK(t)C)X \tag{19}$$

is accessible (controllable) on $\mathbb{R}^n \times \mathrm{SL}_n(\mathbb{R})$ if the pair (A, D) is controllable and (19) is accessible (controllable) on $\mathrm{SL}_n(\mathbb{R})$.

Proof. Since accessibility and controllability proofs are quite similar, we restrict to the later case. Let (μ_0, X_0) and (μ_*, X_*) be arbitrary initial and final states, respectively. Choose any $\varepsilon > 0$ and let (19) evolve for ε time units without control, i.e. $K(t) = 0$ for $t \in [0, \varepsilon]$. Then, by assumption, there exists a control $K_\varepsilon(t)$ on $[0, T]$ with drives $X_\varepsilon := \mathrm{e}^{\varepsilon A}X_0$ to the desired final state X_*. Let $\Phi(0, T)$ denote the fundamental solution of the time-dependent system

$$\dot\mu = (A + BK_\varepsilon(t)C)\mu$$

and define $\mu_\varepsilon := \Phi(0, T)^{-1}\mu_*$. Due to the controllability of the pair (A, D), there exists a control $u_\varepsilon(t)$ on $[0, \varepsilon]$ which steers μ to μ_ε. Then the concatenation of the controls

$$u(t) := \begin{cases} u_\varepsilon(t) & \text{for } [0, \varepsilon], \\ 0 & \text{for } [\varepsilon, \varepsilon + T], \end{cases}$$

and

$$K(t) := \begin{cases} 0 & \text{for } [0, \varepsilon], \\ K_\varepsilon(t - \varepsilon) & \text{for } [\varepsilon, \varepsilon + T], \end{cases}$$

steers (μ_0, X_0) to (μ_*, X_*). □

Theorem 14. Assume that (A_j, B_j, C_j) satisfy Assumption A. Then the parallel connection (15) - (16) is controllable on $\mathbb{R}^{\mathbf{n}} \times \mathcal{P}_1(\mathbf{n})$, provided the following conditions hold:

(a) $G_i(s) \neq G_j(\pm s + \alpha)$ for all $\alpha \in \mathbb{R}$ and $i \neq j$.
(b) There exists a $K \in \mathbb{R}^{m\times p}$ such that $\operatorname{diag}(A_1 + B_1KC_1, \cdots, A_r + B_rKC_r)$ has distinct imaginary eigenvalues.

The proof of Theorem 14 depends on the following result.

Lemma 15. Let $n \geq 3$ and let $\Phi : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$ be either $\Phi(X) = \Theta X\Theta^{-1}$ for some fixed $\Theta \in \mathrm{SL}_n(\mathbb{R})$ or $\Phi(X) = -X^\top$. Then one has the equivalence

$$\Phi(bc) = \hat b\hat c \iff \Phi\left(bc - \frac{\operatorname{tr} cb}{n}I\right) = \hat b\hat c - \frac{\operatorname{tr}\hat c\hat b}{n}I.$$

Proof. Case 1: Assume $\Phi(X) = \Theta X\Theta^{-1}$ for some fixed $\Theta \in \mathrm{SL}_n(\mathbb{R})$. Then $\Phi(bc) = \hat b\hat c$ implies $\operatorname{tr} cb = \operatorname{tr}\hat c\hat b$ and thus

$$\Phi\left(bc - \frac{\operatorname{tr} cb}{n}I\right) = \hat b\hat c - \frac{\operatorname{tr}\hat c\hat b}{n}I\,. \tag{20}$$

Conversely, if (20) holds we conclude

$$\Theta bc\Theta^{-1} - \frac{\operatorname{tr} cb}{n} I = \hat{b}\hat{c} - \frac{\operatorname{tr} \hat{c}\hat{b}}{n} I \,.$$

Comparing the eigenvalues of the both side yields $\operatorname{tr} cb = \operatorname{tr} \hat{c}\hat{b}$ and thus $\Phi(bc) = \hat{b}\hat{c}$.

Case 2: Assume $\Phi(X) = -X^\top$. Then $\Phi(bc) = \hat{b}\hat{c}$ implies $\operatorname{tr} cb = -\operatorname{tr} \hat{c}\hat{b}$ and thus

$$\Phi\left(bc - \frac{\operatorname{tr} cb}{n} I\right) = \hat{b}\hat{c} - \frac{\operatorname{tr} \hat{c}\hat{b}}{n} I \,. \tag{21}$$

Conversely, (21) yields

$$-c^\top b^\top + \frac{\operatorname{tr} cb}{n} I = \hat{b}\hat{c} - \frac{\operatorname{tr} \hat{c}\hat{b}}{n} I \,.$$

The same eigenvalue argument as in the previous case implies $\operatorname{tr} cb = -\operatorname{tr} \hat{c}\hat{b}$ and thus $\Phi(bc) = \hat{b}\hat{c}$. □

Proof of Theorem 14. The assumptions on $G_j(s)$ guarantee that the system Lie algebras of the subsystems in (17) are either $\mathfrak{sl}_{n_j}(\mathbb{R})$ or $\mathfrak{gl}_{n_j}(\mathbb{R})$. Now, assume that for $n_i = n_j$ there exists a Lie algebra automorphism Φ with

$$\Phi(A_i - \frac{\operatorname{tr} A_i}{n} I) = A_j - \frac{\operatorname{tr} A_j}{n} I$$

and

$$\Phi(b_{ki}c_{li} - \frac{\operatorname{tr} b_{ki}c_{li}}{n} I) = b_{kj}c_{lj} - \frac{\operatorname{tr} b_{kj}c_{lj}}{n} I \,.$$

Lemma 15 implies

$$\Phi(b_{ki}c_{li}) = b_{kj}c_{lj}$$

and thus, for $\Phi(X) = \Theta X \Theta^{-1}$, a straightforward computation shows $G_j(s) = G_i(s - \alpha)$ for some $\alpha \in \mathbb{R}$. For $\Phi(X) = -X^\top$, we obtain

$$\begin{aligned} g^j_{lk}(s) &:= c_{lj}(sI - A_j)^{-1} b_{kj} = \operatorname{tr}(sI - A_j)^{-1} b_{kj} c_{lj} \\ &= -\operatorname{tr}(sI + A_i^\top + \alpha I)^{-1} c_{lj}^\top b_{kj}^\top \\ &= b_{kj}^\top(-(s+\alpha)I - A_i^\top)^{-1} c_{lj}^\top \\ &= c_{lj}(-(s+\alpha)I - A_i)^{-1} b_{kj} = g^i_{lk}(-s - \alpha) \end{aligned}$$

Therefore, $G_j(s) = G_i(-s - \alpha)$ with $\alpha := \frac{\operatorname{tr} A_i + A_j}{n}$. Both cases, however, are excluded by Assumption A. Thus Corollary 10 implies that $\mathfrak{sl}_{n_1}(\mathbb{R}) \times \cdots \times \mathfrak{sl}_{n_r}(\mathbb{R})$ is contained in the system algebra of the parallel connection (16). Moreover, the assumption that there exists a constant matrix K such that $\operatorname{diag}(A_1 + B_1KC_1, \ldots, A_r + B_rKC_r)$ has distinct imaginary eigenvalues implies that the system group and the system semigroup of (16) coincide. Therefore, the $\mathrm{SL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{SL}_{n_r}(\mathbb{R})$-orbit of any initial point $Q(0)$ is contained in the reachable set of $Q(0)$. Controllability of (15)-(16) follows from Lemma 13. □

4. CONCLUSIONS

We derived sufficient conditions for controlling the mean and variance in linear systems, using a mixture of open loop and output feedback controllers. Our results use Lie algebraic techniques for controllability of bilinear systems as well as pole placement results for static linear output feedback. We also established controllability results for the mean and variance in parallel connected linear systems. These results present the first step towards extensions to interconnected linear systems, as well as towards ensemble control of parameter dependent families of linear systems.

REFERENCES

F.M. Brasch and J. Pearson. Pole placement using dynamic compensators. *IEEE Trans. Automat. Contr.*, 15 (1), 34–43, 1970.

R. Brockett. Notes on the control of the Liouville equation. In F. Alabau-Boussouira et al. (Eds.), *Control of Partial Differential Equations*, pp. 101–129. Springer, 2012.

R. W. Brockett. The Lie groups of simple feedback systems. *Proc. of 15th IEEE Conf. Decision and Control, including the 15th Symposium on Adaptive Systems*, 1189–1193 (1976).

R. W. Brockett. Minimum attention control. *Proc. of 36th IEEE Conf. Decision and Control*, 2628–2632, (1997).

R.W. Brockett. Linear feedback systems and the groups of Galois and Lie. *Linear Algebra Appl.*, 50, 45–60 (1983).

Y. Chen, T. Georgiou, and M. Pavon. Optimal transport over a linear dynamical system. *arXiv:1502.01265v1*, February 4, 2015.

G. Dirr. Ensemble controllability of bilinear systems. *Oberwolfach Reports*, 12, 674–676, (2012).

G. Dirr, U. Helmke and F. Rüppel. Accessibility of bilinear networks of systems: control by interconnections. *Math. Control Signals*, in press, (2016).

A. Eremenko and A. Gabrielov. Pole placement by static output feedback for generic linear systems. *SIAM J. Control Optim.*, 41 (1), 303–312, (2012).

P.A. Fuhrmann. On controllability and observability of systems connected in parallel. *IEEE Trans. Circuits Syst.*, 22 (1975), 57.

P.A. Fuhrmann and U. Helmke. *The Mathematics of Networks of Linear Systems.* Universitext, Springer International Publishing, Switzerland, 2015.

I. Kurniawan, G. Dirr and U. Helmke. Controllability aspects of open quantum dynamics: A unfied approach for closed and open quantum systems. *IEEE Trans. Automat. Contr.*, 57 (8), 1984–1996, (2012).

N. Jacobson. Structure and automorphisms of semi–simple Lie groups in the large. *Ann. Math.*, 40 (4), 750–763, (1939).

J. Rosenthal, J.M. Schumacher and J.C. Willems. Generic eigenvalue assignment by memoryless real output feedback. *Syst. Control Lett.*, 26, 253–260, 1995.

H.J. Sussmann and V. Jurdjevic. Controllability of nonlinear systems. *J. Differ. Equations*, 12, 95–116 (1972).

H. Wang. Robust control of the output probability density functions for multivariable stochastic systems with guaranteed stability. *IEEE Trans. Automat. Contr.*, 44, 2103–2107, 1999.

X. Wang. Pole placement by static output feedback. *J. Math. Systems Estim. Control*, 2, 202–218, 1992.

S. Zeng, S. Waldherr, and F. Allgöwer. An inverse problem of tomographic type in population dynamics. *2014 IEEE 53rd Conf. on Decision and Control (CDC)*, Los Angeles, 1643–1648, 2014.