Estimation of Linear Positive Systems with Unknown Time-Varying Delays *

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Abstract

This paper considers the estimation problem for linear positive systems with time-varying unknown delays. Similar to set-valued estimation approaches, we provide a confident region within which the trajectory of the observed positive system always evolves. Guaranteed upper and lower estimates for the instantaneous states are characterized by means of a special kind of extended Luenberger-type interval observer. We provide constructive conditions for its existence and establish the asymptotic convergence of its associated interval error. In addition, we give an LP-based method which allows to construct the proposed interval observer solely from the data of the system.

Keywords: Delayed systems, time-varying delays, interval observer, positive systems.

1 Introduction

The reaction of real world systems to exogenous signals often involves possibly unknown delays, for instance in logistics networks the transportation of resources between various locations is subject to traffic jams and other delays. In practice, these delays may vary over time. Moreover, for such delicate estimation problems, the employed measurements may also be infected by unknown time-varying delays.

A fundamental robust estimation approach consists of obtaining on-line information on the estimated states by using set-valued observers. Such an approach ensures that the real states are always within a guaranteed predicted region, see for instance [5, 9, 19]. Mostly, the set-valued observers technique has been limited to systems

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that are not delayed. An interesting set-valued estimation technique consists of the assurance of lower and upper bounds on the estimated states. This approach makes use of an interval observer that was proposed initially in [9] for the estimation of biological systems that are subject to uncertainties, see also an extension to uncertain nonlinear systems [18]. The design of interval observers relies on the positivity notion (see [8, 11, 14] for general references). The framework of interval observers was extended in some works without considering delays on the system. In [15] it was shown recently that for detectable linear systems there exists always an interval observer which can be constructed via an adequate time-varying change of variable. In [2] the design of tight robust interval observers involving dilatation functions was introduced for general linear systems that are subject to interval uncertainties. A different approach valid for stable uncertain linear positive systems was proposed in [20]. Recently, an interesting approach for designing interval observers for nonlinear uncertain system was introduced in [17]. This approach is based on the decomposition of the nonlinear part of the system as a difference of two monotone functions, this is always possible assuming a Lipschitz condition with respect to the states.

Our aim is to handle the problem of the existence of interval observers for linear positive systems which are subject to time-varying delays. This problem is of great interest, especially when the delays are unknown. Such framework has been considered in [3, 12] for delayed positive systems subject to constant delays. In [16] linear systems with point-wise delays have been considered and the provided method for designing interval observers uses a priori bound on the delays. This bound depends implicitly on the parameters of the system via a solution to a particular Lyapunov equation. Our aim is to generalizes the known results for constant delays or point-wise delays to time-varying delays. In this paper, we assume that a common upper bound on the delays exists and the estimated delayed positive system under consideration is not necessarily stable. However, it is assumed that it can be bounded or it can have an equilibrium point. In order to construct an appropriate interval observer we take advantage of the fact that the difference of the upper (lower) interval observer trajectory and the system trajectory is nonnegative. Therefore, the question of existence of an interval observer can be answered using positive systems theory. To this end we show a comparison principle for the trajectories of linear delayed systems in terms of their delays as well as their initial conditions. Based on this fundamental result, we provide sufficient conditions for the existence of an interval observer that only incorporate an upper bound on the delays. Moreover, we state conditions so that the interval error is bounded. Following [4], we also give an efficient numerical LP-based procedure for the determination of the parameters defining the designed interval observers.

The paper is organized as follows. In Section 2, the estimation problem is formulated and the interval observers are introduced. Section 3 recalls known results on positive systems which will be used in this paper. Further, Section 4 contains a comparison principle for linear delayed positive systems with respect to the delays and the initial condition. In Section 5 we state sufficient conditions for the existence of the constructed interval observers and boundedness of the interval error. Furthermore, this section contains an LP-based method to implement the proposed interval observer uniquely from the data of the system. Section 6 presents a numerical example illustrating the presented approach. Finally, we draw conclusions in Section 7.

Notations: $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. For vector $v \in \mathbb{R}^n$, $v > 0$ means that its components v_i are positive. M^T denotes the transpose of the real matrix M. For a real matrix M, $M > 0$ means that its components are positive, that is $M_{ij} > 0$, and $M \geq 0$ means that its components are nonnegative, i.e. $M_{ij} \geq 0$.

2 Estimation under unknown delays

This section is devoted to the estimation problem under study, which involves certain intervals where the estimated states are confined. We shall explain our approach for designing guaranteed interval observers for systems that are subject to possibly unknown time-varying delays. More specifically, we deal with linear observed systems of the following form

$$\Sigma : \begin{cases} \dot{x}(t) &= A\,x(t) + \sum_{i=1}^{p} A_i\,x(t-\tau_i(t)) \\ y(t) &= C\,x(t) + \sum_{i=1}^{p} C_i\,x(t-\widetilde{\tau}_i(t)), \end{cases}$$

with $A, A_i \in \mathbb{R}^{n\times n}$, $C, C_i \in \mathbb{R}^{m\times n}$, $i = 1,\ldots,p$, and time-varying delays $\tau_1(\cdot) \geq 0, \cdots, \tau_p(\cdot) \geq 0$ and $\widetilde{\tau}_1(\cdot) \geq 0, \cdots, \widetilde{\tau}_p(\cdot) \geq 0$ that are supposed to be Lebesgue measurable functions. Without loss of generality it suffices to consider the case $\tau_i(\cdot) \equiv \widetilde{\tau}_i(\cdot)$. This situation can be achieved by enlarging p and choosing matrices $A_i = 0$ or $C_i = 0$ for appropriate indices i.

Throughout the paper the delays are considered to be unknown, however, we assume that the delays are bounded, i.e.

$$0 \leq \tau_i(t) \leq \tau_{\max} := \max_{1\leq i\leq p} \sup_{t\geq 0} \tau_i(t).$$

The initial condition $x(s) = \phi(s)$ for $s \in [-\tau_{\max}, 0]$ of the system Σ may also be unknown bounded Lebesgue measurable functions. In addition, we make the following assumption on the system Σ.

Assumption: Let $m := \sup_{-\tau_{\max}\leq s\leq 0} \phi(s)$, then there exists a vector $\bar{x}$ such that

$$(A + \sum_{i=1}^{p} A_i)\,\bar{x} \leq 0 \quad \text{and} \quad \bar{x} \geq m. \tag{1}$$

We emphasize that system Σ is not supposed to be stable as in [20]. However, it can be bounded or can have an equilibrium point ($(A+\sum_{i=1}^{p} A_i)\,\bar{x} = 0$) if it fulfills the condition (1). In fact, if the initial condition is such that $\phi(s) \leq \bar{x}$ for $s \in [-\tau_{\max}, 0]$

then the associated trajectory is bounded and satisfies $x(t) \leq \bar{x}$. This issue will be clarified further (see Remark 2).

Our aim is to estimate the instantaneous real states independently of any kind of time-varying delays that can affect the system's behavior. We shall extend the framework for interval observers in [9, 18] to delayed systems.

For our estimation purpose we consider a pair of extended Luenberger-type observers of the following form

$$\Sigma^- : \begin{cases} \dot{z}^-(t) & = (A - LC)\, z^-(t) + \sum_{i=1}^{p} (A_i - LC_i)\, z^-(t - \tau_{\max}) + L\, y(t) \\ x^-(t) & = z^-(t) \end{cases}$$

and

$$\Sigma^+ : \begin{cases} \dot{z}^+(t) & = (A - LC)\, z^+(t) + \sum_{i=1}^{p} (A_i - LC_i)\, z^+(t - \tau_{\max}) + L\, y(t) \\ x^+(t) & = z^+(t) \end{cases}$$

with constant initial conditions $\phi^-(s) = \lambda^-$ and $\phi^+(s) = \lambda^+$ which bound the unknown initial condition $\phi(s)$ of the system Σ, i.e. $\lambda^- \leq \phi(s) \leq \lambda^+$ for all $s \in [-\tau_{\max}, 0]$.

Definition 1. *A pair of systems (Σ^-, Σ^+) is called an interval observer for system Σ if for any delays such that $\tau_i(\cdot) \leq \tau_{\max}$ and any initial condition $\phi(\cdot)$, there exist constant initial conditions λ^- and λ^+ such that*

$$\lambda^- \leq \phi(s) \leq \lambda^+$$

for all $s \in [-\tau_{\max}, 0]$ implies that for all $t \geq 0$ it holds that

$$x^-(t) \leq x(t) \leq x^+(t).$$

Also, (Σ^-, Σ^+) is said to be a positive interval observer for system Σ if the lower $x^-(t)$ is nonnegative. In addition, (Σ^-, Σ^+) is said to be a convergent interval observer for system Σ if the lower and upper estimates $x^-(t)$, $x^+(t)$ converge asymptotically to the instantaneous state $x(t)$, i.e. the interval error $e(t) := x^+(t) - x^-(t)$ converges to zero.

The main idea behind the proposed interval observers design is to construct (Σ^-, Σ^+) so that the upper error $e^+(t) := x^+(t) - x(t)$ and lower error $e^-(t) := x(t) - x^-(t)$ are nonnegative. It turns out that the comparison principle given in Theorem 1 is the essential step in the verification of the proposed conditions guaranteeing the existence of the provided interval observers. In Theorem 2 we give sufficient conditions on the matrix L and the initial conditions $\lambda^-, \lambda+$ such that the pair (Σ^-, Σ^+) defines an interval observer for system Σ. The proposed conditions do not depend on the upper bound $\tau_{\max}$ on the delays. We use $\tau_{\max}$ only to implement the interval observer. In Theorem 3 we give constructive necessary and sufficient conditions for the existence of a positive interval observer for which the interval error converges asymptotically to zero. These conditions makes the estimation problem nonlinear in the parameters design λ^-, λ^+ and L. However, in Theorem 4 we provide a simple computation approach for these a priori unknown parameters, based on an efficient linear programming technique [1, 3].

3 Positivity

In this section we recall some facts on positive systems. Consider the following linear differential delayed equation

$$\dot{z}(t) = Mz(t) + \sum_{i=1}^{p} M_i z(t - \tau_i(t)), \tag{2}$$

where the given matrices $M, M_1, \ldots, M_p \in \mathbb{R}^{n\times n}$ are time-invariant and the delays are time-varying and bounded $0 \le \tau_i(\cdot) \le \tau_{\max}$. The vector $z(t) \in \mathbb{R}^n$ is the instantaneous state of the system at time t. Following [10], it can be shown that the solution to the system's equation (2) exists, is unique and totally determined by any given initial locally Lebesgue integrable vector function $\phi(\cdot)$ such that $z(s) = \phi(s)$ for all $s \in [-\tau_{\max}, 0]$.

Definition 2. *System (2) is said to be positive, if for any nonnegative initial condition $\phi(\cdot) \in \mathbb{R}^n_+$ and for any delays $\tau_1(\cdot), \ldots, \tau_p(\cdot)$, then their corresponding trajectory is nonnegative. That is, $z(t) \in \mathbb{R}^n_+$ for all $t \ge 0$.*

In the sequel, it will be shown that inherent properties from the positivity of the delayed system (2) are related to Metzler and nonnegative matrices. A real matrix M is called a *Metzler* matrix if its off-diagonal elements are nonnegative, i.e. $M_{ij} \ge 0, \;\; i \ne j$. If all the entries of M are nonnegative M is called *nonnegative.* It is well known that the matrix exponential e^{tM} is nonnegative for all $t \in \mathbb{R}_+$ if and only if the matrix M is Metzler [6]. For a more comprehensive treatment of Metzler and nonnegative matrices and their properties see e.g. [7, 8].

Further, we provide a complete characterization of the positivity of the system (2). The following lemma has been stated in [1, 13] without proof.

Lemma 1. *System* (2) *is positive if and only if the matrix M is Metzler and the matrices $M_1, ..., M_p$ are nonnegative.*

Proof. Let $\phi(\cdot)$ be a nonnegative initial condition. Assume that M is Metzler and the matrices M_i are nonnegative. Let

$$t^* := \sup\{t \ge 0 : z_k(t) \ge 0 \;\forall\; k = 1, ..., n\} \tag{3}$$

and suppose that it is finite. Let $l \in \{1, ..., n\}$ be so that $z_l(t^*) = 0$. Then, there is an $\varepsilon > 0$ such that $z_l(t^* + \varepsilon') < 0$ for all $\varepsilon' \in (0, \varepsilon)$. In particular, this implies that $\dot{z}_l(t^*) < 0$. On the other hand, since $z_l(t^*) = 0$, $M_i \ge 0$ and $M = (m_{ij})_{i,j=1}^n$ is assumed to be Metzler we have

$$\dot{z}_l(t^*) = \sum_{j=1}^{n} m_{lj} z_j(t^*) + \sum_{i=1}^{p} \big(M_i z(t - \tau_i(t))\big)_l \ge 0,$$

which yields a contradiction.

Conversely, assume that $z(t) \ge 0$ for all $t \ge 0$ and for any $\phi(\cdot) \ge 0$. First we show that M is Metzler. For constant delays $\tau_i(\cdot) \equiv \tau_i$, where $\tau_1 < \tau_2 < ... < \tau_p$, let the initial condition be $\phi(s) \equiv 0$ for all $s \in [-\tau_p, 0)$. Then, it follows that $\dot{z}(t) = Mz(t)$ for all $t \in [0, \tau_p]$ and, hence, M is Metzler.

Next, we show the nonnegativity of the matrices M_i. Let $z(0) = 0$, and notice that the positivity of the system (2) implies that $\dot{z}(0) \geq 0$ and, thus,

$$\dot{z}(0) = \sum_{i=1}^{p} M_i \phi(-\tau_i) \geq 0.$$

Choose $\phi(-\tau_k)$ arbitrary nonnegative and $\phi(-\tau_i) = 0$ for $i \neq k$. Then, the positivity of (2) implies that $M_k \phi(-\tau_k) \geq 0$ for any $\phi(-\tau_k) \geq 0$. This shows the assertion. □

Remark 1. *The forced system*

$$\dot{z}(t) = Mz(t) + \sum_{i=1}^{p} M_i z(t - \tau_i(t)) + g(t),$$

is positive for any nonnegative $g(t) \geq 0$ if and only if the free system is positive, that is, when the matrix M is Metzler and the matrices $M_1, ..., M_p$ are nonnegative.

For the construction of convergent interval observers we use the following reported result in [1].

Proposition 1. *Assume that system (2) is positive. Then, the following statements are equivalent.*

(i) There exist constant-time delays and a nonnegative initial condition $\phi(\cdot)$ with $\phi(0) > 0$ for which system (2) converges asymptotically to zero.

(ii) System (2) is asymptotically stable for every initial condition $\phi(\cdot)$ taking values in $\mathbb{R}^n$ and for any bounded arbitrary time-varying delays.

(iii) There exists $\lambda \in \mathbb{R}^n$ such that

$$\left(M + \sum_{i=1}^{p} M_i \right) \lambda < 0 \quad \textit{and} \quad \lambda > 0.$$

(iv) $M + \sum_{i=1}^{p} M_i$ is a Hurwitz matrix, i.e. the real part of its eigenvalues are strictly negative.

4 Comparison Principle

In this section we present a comparison principle, which will be essential for the construction of interval observers. We shall show that the trajectory of a delayed linear positive system with a forcing term can be upper bounded or lower bounded by any trajectory of a similar linear system when its delays and initial condition are chosen appropriately. For this purpose, consider the following delayed systems

$$
S_1 : \begin{cases} \dot{z}(t) & = M\, z(t) + \displaystyle\sum_{i=1}^{p} M_i\, z(t - \tau_i(t)) + g(t) \\ z(s) & = \psi(s) \qquad s \in [-\tau_{\max}, 0] \end{cases} \tag{4}
$$

and

$$
S_2 : \begin{cases} \dot{w}(t) & = M\, w(t) + \displaystyle\sum_{i=1}^{p} M_i\, w(t - \tau_{\max}) + g(t), \\ w(s) & = \eta, \qquad s \in [-\tau_{\max}, 0], \end{cases} \tag{5}
$$

where ψ is a time-varying initial condition for S_1, $\eta \in \mathbb{R}^n$ is a constant initial condition for S_2 and $g(t)$ is a locally Lebesgue integrable function.

First, we establish the following result.

Lemma 2. *Consider system* (5) *and assume that the matrix M is Metzler and the matrices $M_1, \ldots, M_p$ are nonnegative. Then, for any piecewise constant function $g(t)$ such that $g(t) \leq 0$ for all $t \geq 0$ (resp. $g(t) \geq 0$ for all $t \geq 0$) the solution $w(\cdot)$ is decreasing (resp. increasing) if and only if*

$$
\left(M + \sum_{i=1}^{p} M_i \right) \eta \leq 0 \ (\textit{resp.} \ \geq 0). \tag{6}
$$

Proof. We only show the decreasing case since the increasing case can be proved by the same line of argument. Let $w(\cdot)$ be a solution of (5) which is decreasing, i.e. for all $t \geq 0$ it holds that $\dot{w}(t) \leq 0$. In particular, for $g(0) = 0$ we have that

$$
0 \geq \dot{w}(0) = Mw(0) + \sum_{i=1}^{p} M_i\, w(-\tau_{\max}) = \left(M + \sum_{i=1}^{p} M_i \right) \eta.
$$

Conversely, let $w(\cdot)$ be the solution of (5) and suppose $\left(M + \sum_{i=1}^{p} M_i \right) \eta \leq 0$. Then, since $g(0) \leq 0$ it follows from (5) and (6) that

$$
\begin{aligned}
-\dot{w}(0) &= M(-w(0)) + \sum_{i=1}^{p} M_i(-w(-\tau_{\max})) - g(0) \\
&= -\left(M + \sum_{i=1}^{p} M_i \right) \eta - g(0) \geq 0.
\end{aligned}
$$

Moreover, since $g(t)$ is assumed to be piecewise constant then $-\dot{w}(\cdot)$ is a solution to the positive system

$$
\frac{d}{dt}[-\dot{w}(t)] = M(-\dot{w}(t)) + \sum_{i=1}^{p} M_i(-\dot{w}(t - \tau_{\max})). \tag{7}
$$

and by Lemma 1 we have that $-\dot{w}(t) \geq 0$ for all $t \geq 0$. $\square$

Next, we provide the following technical key comparison result.

Lemma 3. *Let M be Metzler and $M_i \geq 0$ for $i = 1, ..., p$. Suppose that the initial conditions of the systems S_1 and S_2 satisfy*

$$\psi(s) \leq \eta \quad (resp.\ \psi(s) \geq \eta)$$

for all $s \in [-\tau_{\max}, 0]$ and

$$\left(M + \sum_{i=1}^{p} M_i \right) \eta \leq 0 \ (resp.\ \ \geq 0). \tag{8}$$

Then, for any piecewise constant function $g(t)$ such that $g(t) \leq 0$ for all $t \geq 0$ (resp. $g(t) \geq 0$ for all $t \geq 0$) the corresponding trajectories satisfy

$$z(t) \leq w(t) \quad (resp.\ z(t) \geq w(t))\ \forall\, t \geq 0. \tag{9}$$

Proof. Assume that $\psi(s) \leq \eta$ and $g(t) \leq 0$. Observe that $w-z$ satisfies the following linear differential equation

$$\frac{d}{dt}[(w-z)(t)] = M(w(t) - z(t)) + \sum_{i=1}^{p} M_i \left(w(t - \tau_{\max}) - z(t - \tau_i(t))\right). \tag{10}$$

By adding and subtracting $\sum_{i=1}^{p} M_i w(t - \tau_i(t))$ it holds that

$$\begin{aligned} \frac{d}{dt}[(w-z)(t)] = M(w(t) - z(t)) &+ \sum_{i=1}^{p} M_i \left(w(t - \tau_i(t) - z(t - \tau_i(t))\right) \\ &+ \sum_{i=1}^{p} M_i \left(w(t - \tau_{\max}) - w(t - \tau_i(t))\right). \end{aligned} \tag{11}$$

From Lemma 2 it follows that $w(\cdot)$ is nonincreasing, i.e. for all $t \geq 0$ we have that

$$w(t - \tau_i(t)) \leq w(t - \tau_{\max}). \tag{12}$$

Hence, the last term in the right hand side in (11) is nonnegative. Due to the fact that M is Metzler and the matrices M_i are nonnegative, then by Remark 1 we have that system (10) has a nonnegative solution $w-z$. Consequently, it holds $z(t) \leq w(t)$ for all $t \geq 0$.

To show that $\psi(\cdot) \geq \eta$ implies $z(\cdot) \geq w(\cdot)$ when $g(\cdot) \geq 0$ it suffices to follow the previous line of argument and the proof is complete. □

In order to derive our main comparison result the following well-known fact will be used.

Lemma 4. *Let $g(t)$ be a locally Lebesgue integrable function such that $g(t) \leq 0$ for all $t \geq 0$. Then there exists a sequence of piecewise constant negative functions $g_n(t)$ such that for all $t \geq 0$ it holds that $\int_0^t g(s)ds = \lim_{n \to \infty} \int_0^t g_n(s)ds$.*

Now, we are in position to state the following general comparison result which can be viewed as an extension of the reported results in [1, 13].

Theorem 1 (Comparison principle)**.** *Let M be Metzler and $M_i \geq 0$ for $i = 1, ..., p$. Suppose that the initial conditions of the systems S_1 and S_2 satisfy*

$$\psi(s) \leq \eta \quad (\textit{resp. } \psi(s) \geq \eta)$$

for all $s \in [-\tau_{\max}, 0]$ and

$$\left(M + \sum_{i=1}^{p} M_i\right) \eta \leq 0 \ (\textit{resp. } \geq 0). \tag{13}$$

Then, for any locally Lebesgue integrable function $g(t)$ such that $g(t) \leq 0$ for all $t \geq 0$ (resp. $g(t) \geq 0$ for all $t \geq 0$) the corresponding trajectories for systems S_1 and S_2 satisfy

$$z(t) \leq w(t) \quad (\textit{resp. } z(t) \geq w(t))\ \forall\, t \geq 0. \tag{14}$$

Proof. Let $g(t)$ be any locally Lebesgue integrable negative function. By Lemma 4 there exists a sequence of piecewise constant negative functions $g_n(t)$ such that

$$\int_0^t g(s)ds = \lim_{n\to\infty} \int_0^t g_n(s)ds.$$

By using Lemma 3 for each $g_n(t)$ we have that the corresponding solutions $z(t)$ and $w(t)$ satisfy the inequality

$$z(t) \leq w(t) \ \text{ if } \psi(s) \leq \eta.$$

Keeping in mind that

$$z(t) = z(0) + \int_0^t Mz(s) + \sum_{i=1}^{p} M_i z(s - \tau_i(s))ds + \int_0^t g_n(s)ds,$$

$$w(t) = w(0) + \int_0^t Mw(s) + \sum_{i=1}^{p} M_i w(s - \tau_{\max})ds + \int_0^t g_n(s)ds,$$

and the fact that $\int_0^t g(s)ds = \lim_{n\to\infty} \int_0^t g_n(s)ds$, then by continuity argument it holds that $z(t) \leq w(t)$ if $\psi(s) \leq \eta$ for any locally integrable negative function $g(t)$.

The other case $z(t) \geq w(t))$ when $g(t) \geq 0$ and $\psi(s) \geq \eta$ can be shown analogously. □

5 Interval Observers Design

In this section, we investigate the existence of interval observers that provide lower and upper estimates on the instantaneous states. We shall characterize the proposed interval observers and prove their convergence. Note that, in the case of no delays our results coincide with Theorems 1 and 2 in [9]. In addition, we shall show how they can be designed numerically via an efficient LP-based method.

Existence of interval observers

Recall that we consider a delayed positive system of the form

$$\Sigma\ :\begin{cases}\dot{x}(t) & = A\,x(t)+\sum_{i=1}^{p}A_i\,x(t-\tau_i(t))\\ y(t) & = C\,x(t)+\sum_{i=1}^{p}C_i\,x(t-\tau_i(t))\\ x(s) & = \phi(s), \qquad s\in[-\tau_{\max},0].\end{cases}$$

The corresponding pair of estimators is given by

$$\Sigma^-\ :\begin{cases}\dot{x}^-(t) & = (A-LC)\,x^-(t)+\sum_{i=1}^{p}(A_i-LC_i)\,x^-(t-\tau_{\max})+Ly(t)\\ x^-(s) & = \lambda^-, \qquad s\in[-\tau_{\max},0]\end{cases}\tag{15}$$

and

$$\Sigma^+\ :\begin{cases}\dot{x}^+(t) & = (A-LC)\,x^+(t)+\sum_{i=1}^{p}(A_i-LC_i)\,x^+(t-\tau_{\max})+Ly(t)\\ x^+(s) & = \lambda^+, \qquad s\in[-\tau_{\max},0]\end{cases}\tag{16}$$

with constant initial conditions λ^-,λ^+ such that for all $s\in[-\tau_{\max},0]$ it holds that

$$\lambda^-\le\phi(s)\le\lambda^+.$$

It is more appropriate that the lower estimates $x^-(\cdot)$ given by the system Σ^- is nonnegative as well in order to provide reasonable estimates for the nonnegative states $x(\cdot)$. Since the initial condition of the system Σ is nonnegative, i.e. $\phi(s)\ge 0$ for all $s\in[-\tau_{max},0]$, then an obvious choice of the lower initial condition would be $\lambda^-=0$. However, this is not enough to ensure the nonnegativity of the lower estimates $x^-(\cdot)$. For this reason, we need extra conditions on the gain matrix L and the initial condition λ^+ for which the couple (Σ^-,Σ^+) defines an interval observer for the system Σ. Such existence conditions are exhibited in the following result.

Theorem 2. *Given a positive system Σ satisfying Assumption (1), that is, there exists a vector $\bar{x}$ such that*

$$(A+\sum_{i=1}^{p}A_i)\ \bar{x}\le 0 \quad \text{and} \quad \bar{x}\ge m:=\sup_{-\tau_{\max}\le s\le 0}\phi(s).$$

Then, if there exist a vector $\lambda\in\mathbb{R}^n$ and a matrix $L\in\mathbb{R}^{n\times m}$ satisfying

(i) $A-LC$ is Metzler and $A_i-LC_i\ge 0$, for all $i=1,...,p$,

(ii) $LC\ge 0$ and $LC_i\ge 0$ for all $i=1,...,p$,

(iii) $\left(A-LC+\sum_{i=1}^{p}(A_i-LC_i)\right)(\lambda-\bar{x})\le 0$ *and* $\lambda\ge m$.

Then, the pair (Σ^-, Σ^+) defines an interval observer for system Σ associated to the initial conditions $\lambda^- := 0$ and $\lambda^+ := \lambda$. Moreover, the lower estimate $x^-(t)$ is positive.

Proof. First, we show that $x^+(t) \geq x(t)$ for all $t \geq 0$. Let λ and L be any solution to the inequalities (i)-(ii)-(iii). Define

$$\begin{aligned} \dot{z}(t) &= (A - LC)\, z(t) + \sum_{i=1}^{p} (A_i - LC_i)\, z(t - \tau_i(t)) + Ly(t) \\ z(s) &= \lambda, \quad s \in [-\tau_{\max}, 0], \end{aligned} \tag{17}$$

and

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\, \tilde{x}(t) + \sum_{i=1}^{p} A_i\, \tilde{x}(t - \tau_{\max}) \\ \tilde{x}(s) &= \bar{x}, \; s \in [-\tau_{\max}, 0]. \end{aligned} \tag{18}$$

In order to show $x^+(t) - x(t) \geq 0$ we use the following decomposition $x^+(t) - x(t) = (x^+(t) - \tilde{x}(t)) - (z(t) - \tilde{x}(t)) + (z(t) - x(t))$. Hence, it suffices to show that both differences $(x^+(t) - \tilde{x}(t)) - (z(t) - \tilde{x}(t))$ and $z(t) - x(t)$ are nonnegative for every $t \geq 0$. The second difference satisfies the following differential equation

$$\begin{aligned} &\frac{d}{dt}(z - x)(t) = (A - LC)\,(z - x)(t) + \sum_{i=1}^{p} (A_i - LC_i)\,(z - x)(t - \tau_i(t)) \\ &\quad (z - x)(s) = \lambda - \phi(s) \geq 0, \; s \in [-\tau_{\max}, 0]. \end{aligned}$$

By assumption $A - LC$ is Metzler and the matrices $A_i - LC_i$ are positive and since $\lambda \geq \sup_{-\tau_{\max} \leq s \leq 0} \phi(s)$ then by Lemma 1 this yields the nonnegativity of $z(t) - x(t)$ for all $t \geq 0$.

Now, let us show that $x^+(t) - \tilde{x}(t) \geq z(t) - \tilde{x}(t)$. For this task we use the fact that the differences $x^+(t) - \tilde{x}(t)$ and $z(t) - \tilde{x}(t)$ satisfy

$$\begin{aligned} \frac{d}{dt}(x^+ - \tilde{x})(t) &= (A - LC)\,(x^+ - \tilde{x})(t) + \sum_{i=1}^{p} (A_i - LC_i)\,(x^+ - \tilde{x})(t - \tau_{\max}) + g(t) \\ g(t) &:= LC\,(x(t) - \tilde{x}(t)) + \sum_{i=1}^{p} LC_i\,(x(t - \tau_i(t)) - \tilde{x}(t - \tau_{\max})) \\ (x^+ - \tilde{x})(s) &= \lambda - \bar{x}, \quad s \in [-\tau_{\max}, 0] \end{aligned} \tag{19}$$

and

$$\begin{aligned} \frac{d}{dt}(z - \tilde{x})(t) &= (A - LC)\,(z - \tilde{x})(t) + \sum_{i=1}^{p} (A_i - LC_i)\,(z - \tilde{x})(t - \tau_i(t)) + f(t) \\ f(t) &:= g(t) + \sum_{i=1}^{p} (A_i - LC_i)\; (\tilde{x}(t - \tau_i(t)) - \tilde{x}(t - \tau_{max})) \\ (z - \tilde{x})(s) &= \lambda - \bar{x}, \quad s \in [-\tau_{\max}, 0]. \end{aligned} \tag{20}$$

Next, we prove $x^+(t) \geq x(t)$ in 3 steps.

Fact 1: From Lemma 2 and Lemma 3 we have that $\tilde{x}(t)$ is decreasing and it holds $\tilde{x}(t) \geq x(t)$ for all $t \geq 0$. Hence, by taking into account the conditions $LC \geq 0$, $LC_i \geq 0$ and $A_i - LC_i \geq 0$, this leads to $g(t) \leq 0$ and $\psi(t) := \sum_{i=1}^{p}(A_i - LC_i)\,(\tilde{x}(t-\tau_i(t)) - \tilde{x}(t-\tau_{max})) \leq 0$ for all $t \geq 0$.

Fact 2: The solution $v(t)$ to the following delayed system

$$\begin{aligned} \frac{d}{dt}v(t) &= (A - LC)\,v(t) + \sum_{i=1}^{p}(A_i - LC_i)\,v(t-\tau_i(t)) + g(t) \\ v(s) &= \lambda - \bar{x}, s \in [-\tau_{\max}, 0], \end{aligned} \tag{21}$$

satisfies $v(t) \geq z(t) - \tilde{x}(t)$ for all $t \geq 0$.

This can be easily seen by applying Remark 1 for $\tilde{v} = v(t) - z(t) + \tilde{x}(t)$ which is solution to the following positive system

$$\dot{\tilde{v}}(t) = (A - LC)\tilde{v}(t) + \sum_{i=1}^{p}(A_i - LC_i)\tilde{v}(t-\tau_i(t)) - \psi(t),$$

with nonnegative forcing term $-\psi(t) = -\sum_{i=1}^{p}(A_i - LC_i)\,(\tilde{x}(t-\tau_i(t)) - \tilde{x}(t-\tau_{max})) \geq 0$ (nonnegativity of $\psi(t)$ follows from Fact 1).

Fact 3: Since $g(t) \leq 0$ and $\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)(\lambda - \bar{x}) \leq 0$ (by condition (iii)), by using the comparison result in Theorem 1 we have that $x^+(t) - \tilde{x}(t) \geq v(t)$.

Thus, from Fact 2 we can deduce that $x^+(t) - \tilde{x}(t) \geq z(t) - \tilde{x}(t)$. Consequently, it holds that $x^+(t) \geq x(t)$.

Now, the lower bounding of $x(t)$ can be proved much easier than its upper bounding. Effectively, we reconsider the intermediate observer $z(t)$ as the solution to (17) with an initial condition identically 0. As we have shown previously we have that $x(t) \geq z(t)$. Hence, it suffices to show that $z(t) \geq x^-(t)$. Since $Ly(t) \geq 0$ and $\lambda^- = 0$ satisfies the inequality condition in the comparison Theorem 1 we deduce that $z(t) \geq x^-(t)$. Henceforth, As we have established $x(t) \geq z(t)$ it holds $x(t) \geq x^-(t)$.

Finally, it remains to show that condition (ii) implies the positivity of the lower observer Σ^-. To this end, since $Ly(t) \geq 0$ it suffices to use Remark 1 or to consider the augmented system that has the augmented state $(x(t)^\top \; x^-(t)^\top)^\top$ and apply Lemma 1. □

In the light of the previous result the following remarks are in order.

Remark 2. *In Theorem 2 we have used the fact that there exists a vector $\bar{x}$ such that*

$$(A + \sum_{i=1}^{p} A_i)\,\bar{x} \leq 0.$$

Of course, the above condition is not a stability condition. It can be viewed as a boundedness condition and, if it holds with equality $(A + \sum_{i=1}^{p} A_i)\,\bar{x} = 0$ *then* $\bar{x}$ *can be viewed as an equilibrium point (see the simulation example). Effectively, the boundedness of the trajectory* $x(t)$ *is connected to the boundedness of the following system*

$$\begin{aligned}\dot{\widetilde{x}}(t) &= A\,\widetilde{x}(t) + \sum_{i=1}^{p} A_i\,\widetilde{x}(t-\tau_{\max}) \\ \widetilde{x}(s) &= \bar{x}, \quad s \in [-\tau_{\max}, 0].\end{aligned}$$

This fact steams form the results of Lemma 2 and Lemma 3 which imply that the trajectory $x(t)$ *is upper bounded by* $\widetilde{x}(t)$ *if its initial condition* $\phi(\cdot)$ *is upper bounded by* $\bar{x}$. *Since* $\widetilde{x}(t)$ *is decreasing then* $x(t)$ *is bounded by* $\bar{x}$. *Hence, if* $\bar{x}$ *is strictly positive we have global boundedness of system* Σ. *This can be clearly seen from the fact that any initial conditions can be upper bounded by* $\alpha\bar{x}$ *with* $\alpha > 0$ *which by homogeneity satisfies the above condition.*

Remark 3. *Note that, if* (Σ^-, Σ^+) *is an interval observer for the system* Σ *as defined in* (15) *and* (16) *then the interval error* $e(\cdot) = x^+(\cdot) - x^-(\cdot)$ *is nonnegative and satisfies*

$$\dot{e}(t) = (A - LC)\; e(t) + \sum_{i=1}^{p} (A_i - LC_i)\; e(t - \tau_{\max}). \tag{22}$$

We conclude that the system (22) *must be positive and then from Lemma 1 condition (i) in Theorem 2 necessarily holds.*

In order to achieve the asymptotic convergence of the interval error we will provide a simple selection the of initial conditions and the associated gain of the proposed interval observer. For this task, we will make use of the following well-known result (see for example [7]).

Lemma 5. *Let* M *be a Metzler matrix, then the following statements are equivalent:*

(i) M *is Hurwitz.*

(ii) There exists a $\lambda > 0$ *such that* $M\lambda < 0$.

(iii) $M^{-1} \leq 0$.

Based on the previous considerations, we are now in place to prove our main result for the existence of a convergent positive interval observer (Σ^-, Σ^+).

Theorem 3. *Given a positive system* Σ *satisfying Assumption (1), that is, there exists a vector* $\bar{x}$ *such that*

$$(A + \sum_{i=1}^{p} A_i)\;\bar{x} \leq 0, \qquad \textit{and} \qquad \bar{x} \geq m := \sup_{-\tau_{\max} \leq s \leq 0} \phi(s).$$

Then, there exist initial conditions λ^-, λ^+ *such that the pair* (Σ^-, Σ^+) *defines a convergent positive interval observer for system* Σ *if and only if there exist a matrix* $L \in \mathbb{R}^{n\times m}$ *and a vector* $\lambda \in \mathbb{R}^n$ *such that* *(i)* $A - LC$ *is Metzler and* $A_i - LC_i \geq 0$ *for all* $i = 1, ..., p$,

(ii) $LC \geq 0$ *and* $LC_i \geq 0$ *for all* $i = 1, ..., p$,

(iii) $\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)\lambda < 0 \quad$ *and* $\quad \lambda > 0$.

In particular, λ^- *and* λ^+ *can be constructed as follows*

$$\lambda^- := 0,\ \lambda^+ := \alpha\lambda$$

with $\alpha > 0$ *is a scaling factor such that*

$$\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)\ (\alpha\lambda - \bar{x}) \leq 0. \tag{23}$$

Proof. Assume that conditions (i)-(ii)-(iii) are satisfied. First, we show how we can select the initial conditions of the interval observer such that the existence conditions in Theorem 2 are fulfilled. Assume that the vector λ satisfies condition (iii), then by homogeneity we have that $\alpha\lambda$ also satisfies this condition for any arbitrary positive scalar $\alpha > 0$. Since

$$\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)\ (\alpha\lambda) < 0,$$

then for a sufficiently large α^* it holds that

$$\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)\ (\alpha^*\lambda - \bar{x}) \leq 0. \tag{24}$$

Note that condition (i) implies that $M := A - LC + \sum_{i=1}^{p}(A_i - LC_i)$ is Metzler and condition (iii) implies that M is Hurwitz (by Lemma 5). Thus, From Lemma 5 it holds that $M^{-1} \leq 0$ which leads to $\alpha^*\lambda \geq \bar{x}$ and thus it holds $\alpha^*\lambda \geq \sup_{-\tau_{\max} \leq s \leq 0} \phi(s)$.

We conclude that for the choice $\lambda^- = 0$ and $\lambda^+ = \alpha^*\lambda$ the existence conditions in Theorem 2 are all fulfilled. Moreover, by condition (ii) the lower observer x^- is positive. Thus, the pair (Σ^-, Σ^+) defines a positive interval observer for system Σ. Hence, it remains to show that the interval error $e(\cdot)$ converges to zero. Note that the derivative of the interval error satisfies

$$\dot{e}(t) = (A - LC)\, e(t) + \sum_{i=1}^{p}(A_i - LC_i)\, e(t - \tau_{\max}) \tag{25}$$

and, by condition (i) and Lemma 1, it defines a positive system. Moreover, in view of Proposition 1 condition (iii) implies that the interval error converges to zero.

Conversely, if (Σ^-, Σ^+) is a convergent positive interval observer for the system Σ as defined in (15) and (16) the necessity of condition (i) follows from Remark 3 and the necessity of condition (ii) follows from the positivity of the lower observer x^-. To see the necessity of condition (iii) observe that the interval error satisfies (25) and defines a positive system which converges asymptotically to zero for the constant initial condition λ^+. As $e(0) = \lambda^+ > 0$ then by the first statement in Proposition 1 the condition (iii) holds true.

$\square$

Remark 4. *The parameter which is relevant for the speed of convergence of the interval error is the gain L. A decay rate $r > 0$ on the interval error $e(t)$ given by (25) can be imposed such that* $\exp(rt)e(t)$ *vanishes at infinity. By using the change of variable $v(t) = \exp(rt)e(t)$, then v satisfies the delay equation:*

$$\dot{v}(t) = (A + rI - LC)v(t) + \sum_{i=1}^{p} \exp(r\tau_{max})(A_i - LC_i)v(t - \tau_{max}). \tag{26}$$

This new system is asymptotically stable if and only if the matrix

$$M = A + rI - LC + \sum_{i=1}^{p} \exp(r\tau_{max})(A_i - LC_i)$$

is Hurwitz.

Implementation of the Interval Observers

In this part, we turn the focus to the numerical treatment for the determination of the parameters design of the interval observer (Σ^-, Σ^+). We shall provide an efficient LP-based approach for checking the solvability of our estimation problem. Note that the derived existence conditions are nonlinear and nonconvex, since they involve the mixed products $LC_i\lambda$. Thus, a direct successful computation of such parameters may not be possible. For this reason, we shall give a two step method for checking our existence conditions as well as for computing the involved parameters L and λ. Further, we shall provide a standard LP formulation for such purpose.

Next, we provide the following constructive result for positive convergent interval observers.

Theorem 4. *Assume that the system Σ is positive and satisfies Assumption (1), that is, there exists a vector $\bar{x}$ such that*

$$\left(A + \sum_{i=1}^{p} A_i\right)\bar{x} \leq 0, \qquad \text{and} \qquad \bar{x} \geq m := \sup_{-\tau_{\max} \leq s \leq 0} \phi(s).$$

Then, there exists a convergent positive interval observer of the form (Σ^-, Σ^+) for system Σ if and only if the following LP problem in the variables $\gamma \in \mathbb{R}^n$ and $Z \in \mathbb{R}^{m\times n}$ is solvable

$$\begin{cases} \left(A^\top + \displaystyle\sum_{i=1}^{p} A_i^\top\right)\gamma - \left(C^\top + \displaystyle\sum_{i=1}^{p} C_i^\top\right) Z \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} < 0 \qquad \text{and} \qquad \gamma > 0, \\ A^\top \mathbf{diag}(\gamma) - C^\top Z + I \geq 0, \\ A_i^\top \mathbf{diag}(\gamma) - C_i^\top Z \geq 0, \quad i = 1, \dots, p. \\ C^\top Z \geq 0,\ C_i^\top Z \geq 0, \quad i = 1, \dots, p. \end{cases} \tag{27}$$

Moreover, the gain matrix L can be determined by

$$L = Z^\top \mathbf{diag}(\gamma)^{-1},$$

where the vector γ and the matrix Z are any feasible solution to the above LP problem. In addition, the lower initial condition which defines Σ^- can be selected as $\lambda^- = 0$ and the upper initial condition λ^+ which defines Σ^+ can be determined by

$$\lambda^+ = -\alpha \left(A - LC + \sum_{i=1}^{p} (A_i - LC_i) \right)^{-1} \xi > 0, \tag{28}$$

where $\xi > 0$ is any arbitrary strictly positive vector (for instance $\xi = (1 \ \cdots \ 1)^\top$) and $\alpha > 0$ is a scaling factor to be tuned such that

$$\alpha \xi + \left(A - LC + \sum_{i=1}^{p} (A_i - LC_i) \right) \bar{x} \geq 0. \tag{29}$$

Proof. By Theorem 3 it suffices to show that the conditions

(i) $A - LC$ is Metzler and $A_i - LC_i \geq 0$ for all $i = 1, ..., p$,

(ii) $LC \geq 0$ and $LC_i \geq 0$ for all $i = 1, ..., p$,

(iii) $\left(A - LC + \sum_{i=1}^{p} (A_i - LC_i) \right) \lambda < 0 \quad \text{and} \quad \lambda > 0.$

are feasible if and only if the LP problem defined by (27) is feasible.

Now, assume that (i), (ii) and (iii) are feasible. By using Proposition 1 the above condition (ii) is equivalent to $A - LC + \sum_{i=1}^{p} (A_i - LC_i)$ is Hurwitz and thus its transpose is Hurwitz. Hence, by using again Proposition 1 we obtain the following dual equivalent condition to (iii), that is, there exists a $\gamma \in \mathbb{R}^n$ such that

$$\left(A^\top - C^\top L^\top + \sum_{i=1}^{p} (A_i^\top - C_i^\top L^\top) \right) \gamma < 0 \quad \text{and} \quad \gamma > 0.$$

Define $Z := L^\top \mathbf{diag}(\gamma)$ and observe that by this change of variable, the above inequalities (that are equivalent to (iii)) lead to the first and second inequalities in LP condition (27).

If (i) holds, then the third and fourth inequalities in (27) also hold. This can be shown as follows. Note that the matrix $A - LC$ is Metzler if and only if $(A^\top - C^\top L^\top)\mathbf{diag}(\gamma)$ is Metzler. Since all the conditions (i)-(ii)-(iii) are homogeneous in γ, then $(A^\top - C^\top L^\top)\mathbf{diag}(\gamma)$ is Metzler is equivalent to

$$(A^\top - C^\top L^\top)\mathbf{diag}(\gamma) + I \geq 0,$$

by choosing γ with sufficiently small components (by homogeneity one can consider $\alpha\gamma$ with a scalar $\alpha > 0$ sufficiently small). Thus, by our change of variable $L^\top = Z\mathbf{diag}(\gamma)^{-1}$, the above inequality is equivalent to

$$A^\top \mathbf{diag}(\gamma) - C^\top Z + I \geq 0$$

and the third inequality in (27) holds true. In the same manner we can establish the fourth and last LP inequalities

$$A_i^\top \mathbf{diag}(\gamma) - C_i^\top Z \geq 0, \quad i = 1, \ldots, p.$$

Conversely, if the conditions (27) hold then conditions (i), (ii) and (iii) are satisfied. This necessity part necessity follows from similar matrix manipulations by using the special change of variable $L^\top = Z\mathbf{diag}(\gamma)^{-1}$. Henceforth, we can conclude that the LP problem (27) is feasible if and only if conditions (i), (ii) and (iii) are feasible.

Consequently, as can be seen from the above treatment the design parameters L can be computed from any feasible solution to the LP problem (27) as

$$L = Z^\top \mathbf{diag}(\gamma)^{-1}.$$

It remains to show how one can compute the design parameter λ^+ from the precalculated gain. For this we make use of Lemma 5. Since with this gain L the matrix $A - LC + \sum_{i=1}^{p}(A_i - LC_i)$ is Metzler and Hurwitz, then its inverse is a negative matrix, so that for any arbitrary strictly positive vector $\xi > 0$ we have that

$$\lambda := -\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)^{-1} \xi > 0,$$

and obviously this λ satisfies condition (iii). Of course, one can select for example $\xi = [1 \ \ldots \ 1]^\top$ and define $\lambda^+ := \alpha\lambda$ with $\alpha > 0$ to be tuned in order to meet the existence conditions in Theorem 3.

Finally, by the choice $\lambda^+ := \lambda = -\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)^{-1} \xi > 0$ the condition (29) is nothing else than

$$\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)\left(-\alpha\left(A - LC + \sum_{i=1}^{p}(A_i - LC_i)\right)^{-1}\xi - \bar{x}\right) \leq 0,$$

which represents the existence condition (23) in Theorem 3 and the proof is complete. □

In the sequel, for the convenience of the reader, we provide a standard LP formulation of the proposed LP problem (27). For this purpose, we simply use **vec** and Kronecker product $\bigotimes$ operations and the fact that for any matrices M, N and X with appropriate size, we have that $\mathbf{vec}(MXN) = N^\top \bigotimes M\mathbf{vec}(X)$. Also, we notice that for any vector γ, the operation $\mathbf{diag}(\gamma)$ can be written as

$$\mathbf{diag}(\gamma) = \sum_{i=1}^{n} v_i \gamma^\top v_i v_i^\top, \text{ where } v_1, \ldots, v_n \text{ is the standard basis of } \mathbb{R}^n.$$

Now, equipped with the previous facts, it can be easily seen that the following standard linear inequalities represent nothing else than our previous LP problem (27), in the common used standard form

$$\begin{bmatrix} I \bigotimes (A^\top + \sum_{i=1}^{p} A_i^\top) & -\left[\ 1 \dots 1\ \right] \bigotimes (C^\top + \sum_{i=1}^{p} C_i^\top) \\ -I & 0 \end{bmatrix} w < 0, \qquad (30)$$

$$\begin{bmatrix} -\sum_{i=1}^{n} v_i v_i^\top \bigotimes A^\top v_i & I \bigotimes C^\top \\ -\sum_{i=1}^{n} v_i v_i^\top \bigotimes A_1^\top v_i & I \bigotimes C_1^\top \\ \vdots & \vdots \\ -\sum_{i=1}^{n} v_i v_i^\top \bigotimes A_p^\top v_i & I \bigotimes C_p^\top \\ 0 & -I \bigotimes C^\top \\ 0 & -I \bigotimes C_1^\top \\ \vdots & \vdots \\ 0 & -I \bigotimes C_p^\top \end{bmatrix} w \le \begin{bmatrix} \mathbf{vec}(I) \\ 0 \end{bmatrix}, \qquad (31)$$

where the unknown vector variable to be computed is given by $w := \begin{bmatrix} \gamma \\ \mathbf{vec}(Z) \end{bmatrix}$. Indeed, from w one can easily recover γ, Z and compute the parameters design λ^-, λ^+ and L as it was shown in Theorem 4.

The above LP problem can be solved by using any existing LP softwares such the well-known linprog in Matlab or Sedumi which is a free public software.

6 Numerical Simulation

Let the system Σ be given by the matrices

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},$$

$$C = 0 \text{ (no instantaneous measurements)}, \quad C_1 = C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Observe that this system has an equilibrium point $\bar{x} = [1 \ \ 1]^\top$ since we have

$$(A + A_1 + A_2)\bar{x} = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0,$$

and hence Assumption (1) is satisfied.

The time-varying delays are assumed to be unknown and given by

$$\tau_1(t) = 4\delta(\cos(2\,t) + 1) \text{ and } \tau_2(t) = 4\delta(\sin(3\,t) + 1),$$

where $0 \leq \delta \leq 1$ represents an uncertain parameter. Obviously, one may choose $\tau_{max} = 8$ as a bound on these unknown delays. Also, the components of the initial conditions of the system Σ are assumed to be unknown and bounded. For simulation purpose we choose the uncertain parameter $\delta = 1$ and use the nonnegative initial conditions $\phi_1(s) = |\cos(1.5\,s + 1) + 1|$ and $\phi_2(s) = |\sin(s) + 1|$ for all $s \in [-8, 0]$.

In the first simulation no decay rate was imposed. We have solved in standard form the LP problem (27) by using linprog Matlab function. The computed design gain L of the interval observer (Σ^-, Σ^+) that satisfies the existence conditions in Theorem 4 is given by

$$L = \begin{bmatrix} 0.5066 & 0.0000 \\ 0.0000 & 0.7118 \end{bmatrix}.$$

From this gain we have calculated the initial condition of the upper observer according to the following formulas given by Theorem 4 with $\alpha = 3$, $\xi = [1 \;\; 1]^\top$ and $\bar{x} = 2[1 \;\; 1]^\top$

$$\lambda^+ = -\alpha \left(A - LC + \sum_{i=1}^{p} (A_i - LC_i) \right)^{-1} \xi,$$

$$\alpha\xi + \left(A - LC + \sum_{i=1}^{p} (A_i - LC_i) \right) \bar{x} \geq 0.$$

We have obtained $\lambda^+ = \begin{bmatrix} 2.6099 \\ 2.4310 \end{bmatrix}$. The convergent interval observer (Σ^-, Σ^+) is initialized with $\lambda^- = 0$ and $\lambda^+ = [2.6099 \;\; 2.4310]^\top$. The simulation results are shown in Figures 1.

In the second simulation a decay rate $r = 0.2$ was imposed. In order to compute the second interval observer it suffices to apply Remark 4 by replacing A by $A + rI$, A_i (resp. C_i) by $\exp(r\tau_{max})A_i$ (resp. $\exp(r\tau_{max})C_i$). With these data we have solved in standard form the LP problem (27). The computed design gain L of the interval observer (Σ^-, Σ^+) that satisfies the existence conditions in Theorem 4 is given by

$$L = \begin{bmatrix} 0.9544 & 0.0000 \\ 0.0000 & 0.9885 \end{bmatrix}.$$

As previously the initial condition of the upper observer was computed with $\alpha = 1.2$, $\xi = [1 \;\; 1]^\top$ and $\bar{x} = 2[1 \;\; 1]^\top$ and is given by

$$\lambda^+ = \begin{bmatrix} 3.4349 \\ 5.7996 \end{bmatrix}.$$

The second convergent interval observer (Σ^-, Σ^+) is initialized with $\lambda^- = 0$ and $\lambda^+ = [3.4349 \;\; 5.7996]^\top$. The simulation results are shown in Figures 2. Indeed, we see from the depicted trajectories that this second interval observer has much better convergence and provides tight lower and upper estimates.

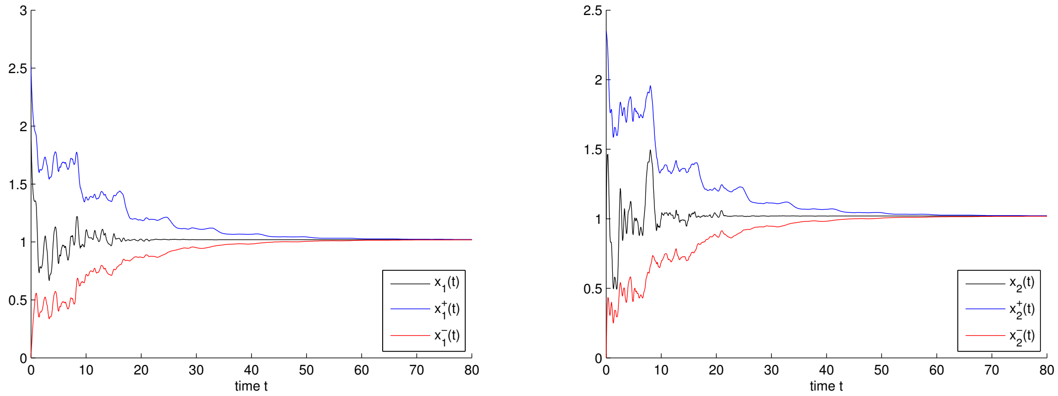

Figure 1: Components of the system and its convergent interval observer with no decay rate $r = 0$.

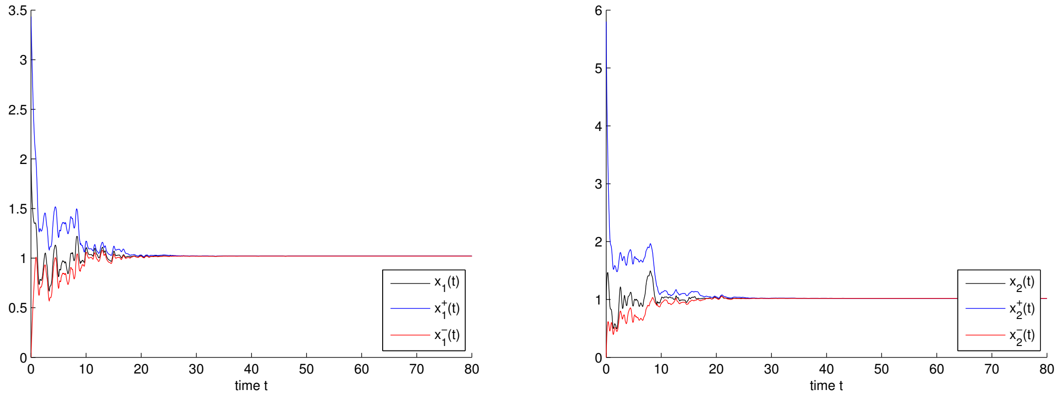

Figure 2: Components of the system and its convergent interval observer with decay rate $r = 0.2$.

7 Conclusions

We have provided a comprehensive and constructive approach for the estimation problem for linear positive systems with unknown time-varying delays. It turns out that the existence of the provided interval observers of extended Luenberger type involves similar positivity condition for linear delayed systems together with an adequate property on the initial conditions of such interval observers. Moreover, their convergence is connected to a stability condition for linear delayed positive systems. Our results are complemented by an efficient numerical LP-based procedure for the determination of the parameters design for the proposed interval observers.

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