Uniform Ensemble Controllability for One-Parameter Families of Time-Invariant Linear Systems☆

Uwe Helmke[a], Michael Schönlein[a]

[a] *Institute for Mathematics, University of Würzburg, Emil-Fischer Straße 40, 97074 Würzburg, Germany*

Abstract

In this paper we derive necessary as well as sufficient conditions for approximate controllability of parameter-dependent linear systems in the supremum norm. Using tools from complex approximation theory, we prove the existence of parameter-independent open-loop controls that steer the zero initial state of an ensemble of linear systems uniformly to a prescribed family of terminal states. New necessary conditions for uniform ensemble controllability of single-input systems are derived. Our results extend earlier ones of Li for ensemble controllability of linear systems.

Keywords: approximate controllability, complex approximation, ensemble controllability, families of systems

1. Introduction

Spatially-invariant systems, such as the heat equation, provide interesting examples of distributed parameter systems, where control actions and measurements take place in a spatially distributed way [1]. Recent engineering applications to, e.g., smart materials, symmetrically interconnected systems [19], or the control of platoons [21], have considerably raised the interest in such systems. Using Fourier-transform techniques, spatially-invariant control systems can be identified with parameter-dependent families of linear systems; see e.g. [1] for a systematic outline of this approach. From an analytic point of view, parameter-dependent systems can be regarded as infinite-dimensional systems, defined on suitable Banach- or Hilbert spaces of functions; see e.g. [5]. We also mention the theory of systems over rings, see e.g. [13, 24], as a systematic algebraic approach to analyze parameter-dependent systems.

In many applications it is of interest to solve the corresponding problems using open-loop controls. Thus, given a family of desired terminal states, we attempt to construct a parameter-independent input function that steers the zero-state to these states, simultaneously for all parameter values. Such open-loop control issues for ensembles of bilinear systems arise e.g. in designing compensating pulse-sequences in quantum control; see e.g. Brockett, Khaneja and Li [4, 17, 18] and Becker, Bretl [2]. They are also of interest towards controlling large-scale networks, or in understanding biological systems, such as flocks of systems; see e.g., Brockett [3].

The structure of this paper is as follows. In Section 2 we formulate the problem under consideration and state the main result. Section 3 proves necessary conditions for uniform ensemble controllability. In Section 4 useful characterizations of ensemble controllability for discrete-time systems are provided. Section 5 is devoted to the proof of Theorem 1 (main achievement), which is a theoretical result, providing sufficient conditions that ensure the existence of an open-loop control function steering the zero state arbitrarily close towards the desired terminal states. The proof uses basic tools from complex approximation theory, such as the Stone-Weierstrass theorem and Mergelyan's Theorem. A constructive proof of Margelyan's Theorem can for instance be found in the original paper [20]; although there is certainly the need for more efficient computations of the approximating polynomial. Section 6 discusses the ensemble control problem for a one-parameter family of discrete-time second order systems. Given the particular system we pursue two constructive approaches to determine the control inputs. On one hand, we use Bernstein polynomials to obtain the ensemble controls. On other hand, we employ samplings of the parameter space and use Lagrange interpolation applied at equidistant interpolation points as well as applied at Chebyshev interpolation points.

2. Problem statement and main result

In this paper, we consider parameter-dependent linear time-invariant systems of the form

$$\sigma x(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \quad x(0,\theta) = 0, \tag{1}$$

where σ denotes the shift operator $\sigma x(t,\theta) := x(t+1,\theta)$ for discrete-time systems and the differential operator $\sigma x(t,\theta) := \frac{\partial}{\partial t}x(t,\theta)$ for continuous-time systems, respectively. In either case, the system matrices $A(\theta) \in \mathbb{R}^{n\times n}, B(\theta) \in \mathbb{R}^{n\times m}$ are assumed to vary continuously in a compact interval $\theta \in \mathbf{P} :=$

☆This work was partially supported by the grant HE 1858/12-2 from the German Research Foundation (DFG) under the Priority Program SPP 1305 "Control Theory for Digitally Networked Dynamical Systems", as well as by a grant from the Fraunhofer Institute for Techno-and Industrial Mathematics (ITWM)-Kaiserslautern, Germany

Email addresses: helmke@mathematik.uni-wuerzburg.de (Uwe Helmke), schoenlein@mathematik.uni-wuerzburg.de (Michael Schönlein)

$[\theta^-, \theta^+] \subset \mathbb{R}$. Let $\mathcal{T}$ denote the time-domain of the system (1), i.e. $\mathcal{T} = \mathbb{N}$ in the discrete-time case and $\mathcal{T} = [0, \infty)$ in the continuous-time case, respectively.

Definition 1. *System* (1) *is called uniformly ensemble controllable, if there exists a finite time $T > 0$ and an input function $u\colon [0, T] \cap \mathcal{T} \to \mathbb{R}^m$ that steers the initial state $x(0, \theta) = 0$ in time T into an ε-neighborhood of the desired state $x_*(\theta)$, simultaneously for all parameters $\theta \in \boldsymbol{P}$; i.e. if*

$$\sup_{\theta \in \boldsymbol{P}} \|x(T, \theta) - x_*(\theta)\| < \varepsilon. \tag{2}$$

Instead of trying to find controls that achieve the uniform ensemble controllability condition (2), one can also search for controls u that minimize the L^q-norms for $1 \le q \le \infty$

$$\left(\int_{\mathbf{P}} \|x(T, \theta) - x_*(\theta)\|^q d\theta \right)^{\frac{1}{q}} < \varepsilon. \tag{3}$$

We then say, that the system is *L^q-ensemble controllable*. If the conditions in (2) or (3) hold for $\varepsilon = 0$, then the system is called exactly ensemble controllable. It follows from a result by Triggiani [25] that parameter-dependent linear systems (1) are never exactly ensemble controllable. Thus the approximate versions of (uniform or L^q) ensemble controllability are the only meaningful ones.

The above concepts of ensemble controllability are intimately related to standard notions from infinite-dimensional systems theory, such as weak or approximate reachability and controllability. Let X denote the Banach space of $\mathbb{R}^n$-valued continuous functions on the compact interval $\mathbf{P}$, endowed with the supremum-norm. Any continuous family of continuous- or discrete-time systems $(A(\theta), B(\theta))$ then defines a linear system

$$\sigma x(t) = \mathcal{A}x(t) + \mathcal{B}u(t) \tag{4}$$

on the Banach-space X. Here $\mathcal{A}\colon X \to X$ denotes the bounded linear multiplication operator $(\mathcal{A}x)(\theta) = A(\theta)x(\theta)$ and $\mathcal{B} \in X^{1\times m}$ denotes a fixed m-tuple of Banach-space elements, defined by $(\mathcal{B})(\theta) = B(\theta)$. The parameter dependent system (1) then is uniformly ensemble controllable if and only if the infinite-dimensional system (4) is approximately reachable. In the same way, by replacing the Banach space X by the Hilbert space $H = L^2(\mathbf{P}, \mathbb{R}^n)$, we conclude that L^2-ensemble controllability of (1) becomes equivalent to approximate reachability of the infinite-dimensional system (4). See Fuhrmann [7] for characterizations of (weak) reachability of infinite-dimensional linear systems in a Hilbert space. However, these criteria are difficult to apply. We refer to Section 4.2 in [5] for more easily verifiable conditions for approximate reachability. Unfortunately, the conditions stated in [5] and [14] depend on the explicit knowledge of a Riesz basis of eigenvectors for the Hilbert space operator $\mathcal{A}$. However, except for trivial cases such as $A(\theta)$ having constant eigenvalues, the multiplication operator $\mathcal{A}$ defined by $A(\theta)$ does not have a point spectrum and therefore the results in [5] do not apply here.

Recently, Li [16] has obtained an operator-theoretic characterization of L^2-ensemble controllability for time-varying linear multivariable systems in terms of growth rates of singular values of the input-state operator. However, these are very hard to verify, even for families of time-invariant systems (1). Moreover, the methods in [16] depend on a singular value decomposition and are therefore restricted to Hilbert spaces. In contrast, the uniform ensemble problem is stated in a Banach space and therefore the results in [16] do not apply in our context and in fact our results are stronger. It is therefore desirable to have more easily verifiable conditions for ensemble controllability. We prove a sufficient condition for uniform ensemble controllability that is even for the L^2-ensemble case more concrete than the condition in [16]. Recall that the input Hermite indices $K_1, \cdots, K_m$ are similarity invariants for linear systems (A, B), which are defined, e.g., in Kailath [15]. They sum up to n if and only if the pair (A, B) is controllable.

Theorem 1 (Main Theorem)**.** *Let $\boldsymbol{P} = [\theta^-, \theta^+]$ be a compact subset of $\mathbb{R}$. A continuous family $(A(\theta), B(\theta))$ of linear systems is uniformly ensemble controllable (more generally, L^q-ensemble controllable for any $1 \le q \le \infty$,) provided the following conditions are satisfied:*

- *(i)* *$(A(\theta), B(\theta))$ is reachable for all $\theta \in \boldsymbol{P}$.*
- *(ii)* *The input Hermite indices $K_1(\theta), \cdots, K_m(\theta)$ of $(A(\theta), B(\theta))$ are independent of $\theta \in \boldsymbol{P}$.*
- *(iii)* *For any pair of distinct parameters $\theta, \theta' \in \boldsymbol{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:*

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$

- *(iv)* *For each $\theta \in \boldsymbol{P}$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.*

The first condition is necessary for $m \ge 1$. For $m = 1$ the third condition is also necessary for uniform ensemble controllability.

One might ask whether a version of Theorem 1 holds true for higher-dimensional parameter spaces. This is left as an open problem.

Note also, that the second condition is implied by the first condition, if $m = 1$. There is an interesting situation where all these assumptions fell easily into place. Thus consider the robust output feedback control task for a fixed controllable and observable linear system (A, b, c). Let $\mathbf{P}$ be a compact interval of gain parameters. Then, for any real $\theta \in \mathbf{P}$, the closed loop characteristic polynomial is $\det(A + \theta bc) = q(z) + \theta p(z)$, with p, q coprime. It follows that given any two distinct numbers $\theta \neq \theta'$, there exists no complex number z with $q(z) + \theta p(z) = 0 = q(z) + \theta' p(z)$. The above results imply

Corollary 1. *Let (A, b, c) be controllable and observable single-input single-output. The (discrete-time or continuous-time) output feedback system $(A + \theta bc, b)$ is uniformly ensemble controllable if $A + \theta bc$ has only simple eigenvalues for all $\theta \in [\theta^-, \theta^+]$.*

3. Necessary conditions for uniform ensemble controllability

We begin by proving necessary conditions for uniform ensemble controllability. Here we are not making any assumptions on **P**. In particular, we are not assuming compactness. Let

$$(zI - A(\theta))^{-1}B(\theta) = N_\theta(z)D_\theta(z)^{-1} \tag{5}$$

be a right coprime factorization by a rectangular polynomial matrix $N_\theta(z) \in \mathbb{R}^{n\times m}[z]$ and a nonsingular polynomial matrix $D_\theta(z) \in \mathbb{R}^{m\times m}[z]$.

Lemma 1. *Assume that the family of linear systems $(A(\theta), B(\theta))_{\theta\in \boldsymbol{P}}$ is uniformly ensemble controllable. Then:*

1. *For each $\theta \in \boldsymbol{P}$ the system $(A(\theta), B(\theta))$ is reachable.*
2. *For any finite number of parameters $\theta_1, \dots, \theta_s \in \boldsymbol{P}$, the $m\times m$ polynomial matrices $D_{\theta_1}(z), \dots, D_{\theta_s}(z)$ are mutually left coprime.*
3. *For any finite number $s \geq m + 1$ of distinct parameters $\theta_1, \dots, \theta_s \in \boldsymbol{P}$, the spectra of $A(\theta)$ satisfy*

$$\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) = \emptyset. \tag{6}$$

4. *Assume $m = 1$. For any pair of distinct parameters $\theta, \theta' \in \boldsymbol{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:*

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset. \tag{7}$$

Proof. The subsequent proof is valid for both discrete-time and continuous-time systems. Consider any fixed, arbitrary parameter value $\theta \in \mathbf{P}$ and state vector $\xi \in \mathbb{R}^n$. Choose a continuous map $x_*\colon \mathbf{P} \to \mathbb{R}^n$ with $x_*(\theta) = \xi$. For any $\varepsilon > 0$ there exists by assumption a control function $u\colon [0, T] \cap \mathcal{T} \to \mathbb{R}^m$ such that

$$\sup_{\theta\in\mathbf{P}} \|x(T, \theta) - x_*(\theta)\| < \varepsilon. \tag{8}$$

In particular, we have $\|x(T, \theta) - \xi\| < \varepsilon$. Thus ξ is in the closure of the reachable set of 0; since the reachable sets of linear systems are closed in $\mathbb{R}^n$ this shows that $(A(\theta), B(\theta))$ is reachable. By the same reasoning, ensemble controllability of the family $(A(\theta), B(\theta))_\theta$ implies reachability for the parallel interconnection

$$\mathcal{A} = \begin{pmatrix} A(\theta_1) & & 0 \\ & \ddots & \\ 0 & & A(\theta_s) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B(\theta_1) \\ \vdots \\ B(\theta_s) \end{pmatrix} \tag{9}$$

of finitely many linear systems $(A(\theta_k), B(\theta_k)), k = 1, \dots, s$. Fuhrmann [8] has shown that the parallel interconnection (9) of controllable linear systems is controllable if and only if the $m \times m$ polynomial matrices $D_{\theta_1}(z), \dots, D_{\theta_s}(z)$ are mutually left coprime, see also [10] and Exercise 8.3.10 in [15]. This completes the proof of the second claim.

Controllability of (9) implies that there are at most m Jordan blocks in $\mathcal{A}$, for each eigenvalue of $\mathcal{A}$. Therefore, it is impossible that $\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) \neq \emptyset$ for $s \geq m + 1$, as otherwise any element of the intersection would be an eigenvalue of $\mathcal{A}$ with more than $m + 1$ Jordan blocks.

If $m = 1$, then the coprimeness of $D_{\theta_1}(z), D_{\theta_2}(z)$ is equivalent to the spectra of $A(\theta_1), A(\theta_2)$ being disjoint. Thus this shows the fourth claim. □

The above proof, using Fuhrmann's result [8], shows an interesting connection between ensemble controllability for *finite* parameter sets **P** and controllability for parallel interconnection schemes of single-input systems.

Corollary 2. *Assume that $\boldsymbol{P} = \{\theta_1, \cdots, \theta_s\} \subset \mathbb{R}^d$ is finite. Then a family of single-input systems $(A(\theta), b(\theta))_{\theta\in\boldsymbol{P}}$ is uniformly ensemble controllable if and only if the following two conditions hold:*

1. *$(A(\theta_k), b(\theta_k))$ is reachable for $k = 1, \dots, s$.*
2. *The characteristic polynomials $\det(zI - A(\theta_l))$ and $\det(zI - A(\theta_k))$ are coprime for all $l \neq k$.*

In the discrete-time case, one can strengthen this result by deriving explicit polynomial formulas for the inputs that steer to a desired state. In fact, in the context of shift realizations and polynomial models [9], the minimum-time ensemble control task for finite parameter sets becomes equivalent to the Chinese remainder theorem, cf. Section 6.

4. Characterizations of ensemble controllability

It is useful to present the uniform ensemble controllability condition in a more manageable form. For simplicity we focus on the single-input discrete-time case. Corresponding characterizations in the continuous-time case are more difficult and are not needed for the subsequent analysis. The following result characterizes uniform ensemble controllability.

Proposition 1. *A family $\{(A(\theta), b(\theta)), \theta \in \boldsymbol{P}\}$ of discrete-time systems is uniformly ensemble controllable on $\boldsymbol{P}$ if and only if for all $\varepsilon > 0$ and all continuous functions $x_* : \boldsymbol{P} \to \mathbb{R}^n$ there is a real scalar polynomial $p \in \mathbb{R}[z]$ such that*

$$\sup_{\theta\in\boldsymbol{P}} \|p(A(\theta))\, b(\theta) - x_*(\theta)\| < \varepsilon. \tag{10}$$

Proof. Recall that given inputs $u(0), \dots, u(T - 1)$ the solution is given by

$$\begin{aligned} x(T, \theta) &= \sum_{k=0}^{T-1} A(\theta)^k b(\theta) u(T - 1 - k) \\ &= \left(\sum_{k=0}^{T-1} u(T - 1 - k) A(\theta)^k\right) b(\theta) \\ &= p(A(\theta)) b(\theta), \end{aligned}$$

where $p(z) = \sum_{k=0}^{T-1} u_{T-1-k} z^k$ is a parameter independent polynomial. □

Suppose that $(A(\theta), b(\theta))$ is controllable for all $\theta \in \mathbf{P}$. Then by the controller canonical form, there exists a continuous family of invertible state-space transformations $S(\theta) = R(A(\theta), b(\theta))^{-1}$ such that

$$(\tilde{A}(\theta), e_1) = (S(\theta)A(\theta)S(\theta)^{-1}, S(\theta)b(\theta))$$

is in (tall) control canonical form, where $R(A(\theta), b(\theta))$ denotes the $n\times n$ reachability matrix and $\tilde{A}(\theta)$ denotes the tall companion matrix of the characteristic polynomial $q_\theta(z) = \det(zI - A(\theta))$ and e_1 is the first standard basis vector of $\mathbb{R}^n$. Given any continuous $x_*\colon \mathbf{P} \to \mathbb{R}^n$ we consider the real polynomial u_θ in z defined by

$$u_\theta(z) := (1\, z\, \cdots\, z^{n-1})^\top R(A(\theta), b(\theta))^{-1} x_*(\theta). \tag{11}$$

Proposition 2. *Assume that the discrete-time system $(A(\theta), b(\theta))$ is reachable for any $\theta \in \boldsymbol{P}$. Then the following are equivalent.*

(1) *$(A(\theta), b(\theta))_\theta$ is uniformly ensemble controllable.*
(2) *For any continuous function $x_* \in C(\mathbf{P}, \mathbb{R}^n)$ and any $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{R}[z]$ with $\|(p - u_\theta)(A(\theta))b(\theta)\| < \varepsilon$ for all $\theta \in \mathbf{P}$.*
(3) *For any continuous function $x_* \in C(\mathbf{P}, \mathbb{R}^n)$ and any $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{R}[z]$ with $\|p(A(\theta)) - u_\theta(A(\theta))\| < \varepsilon$ for all $\theta \in \mathbf{P}$.*

Assume, that for each $\theta \in \boldsymbol{P}$, the eigenvalues of $A(\theta)$ are distinct. Let

$$C := \Big\{(z, \theta) \in \mathbb{C} \times \boldsymbol{P} \mid \det(zI - A(\theta)) = 0\Big\}.$$

Then each of the above conditions is equivalent to:

(4) *For any continuous function $x_* \in C(\mathbf{P}, \mathbb{R}^n)$ and any $\varepsilon > 0$ there is a polynomial $p \in \mathbb{R}[z]$ with*

$$|p(z) - u_\theta(z)| < \varepsilon \quad \forall (z, \theta) \in C. \tag{12}$$

Proof. Note that for any $x_* \in C(\mathbf{P}, \mathbb{R}^n)$ we have $u_\theta(A(\theta))b(\theta) = x_*(\theta)$. By Proposition 1 this shows the equivalence of (1) and (2).

Obviously, condition (3) implies (2). Conversely, assume that $\|(p - u_\theta)(A(\theta))b(\theta)\| < \varepsilon$. Then we have

$$\|(p - u_\theta)(A(\theta))\, A(\theta)^k\, b(\theta)\| < \varepsilon \sup_{\theta \in \mathbf{P}} \|A(\theta)\|^k$$

and hence it holds that

$$\|(p - u_\theta)(A(\theta))\| < c\varepsilon,$$

with $c = \sup_{\theta\in\mathbf{P}} \|R(A(\theta), b(\theta))^{-1}\| \max_{0\le k\le n-1} \|A(\theta)\|^k$. So (3) is satisfied.

Now consider any matrix X with distinct eigenvalues $\lambda_1, \cdots, \lambda_n$. Then, given any real polynomial P and any $\varepsilon > 0$, we have $\|P(X)\| < \varepsilon$ if and only if $|P(\lambda_i)| < \delta(\varepsilon)$, $i = 1, \cdots, n$. Here $\delta(\varepsilon)$ goes to zero if and only if ε goes to zero. This shows that condition (4) is equivalent to (3) and we are done. □

5. Proof of the main result

The claims in Theorem 1 concerning the necessity all follow from Lemma 1.

The proof of converse part of Theorem 1 consists mainly of the following steps: The verification of the discrete-time case is first shown for the single input systems and later on extended to the multivariable case. Second, the continuous-time case is reduced the discrete-time case using sampling arguments. A direct proof for the single-input continuous-time case is given in [22].

It is instructive to first prove the result in the one-dimensional single-input case $n = 1, m = 1$. Since $(A(\theta), B(\theta))$ is supposed to be reachable for all $\theta \in \mathbf{P}$ we consider w.l.o.g. the uniform ensemble control task for the one-dimensional system of the form

$$x(t+1) = a(\theta)x(t) + u(t), \quad x_0(\theta) = 0.$$

Here $a\colon \mathbf{P} \to \mathbb{R}$ is any continuous function on the compact Hausdorff space $\mathbf{P}$ and $x \in \mathbb{R}$. Given input signals $u(0), ..., u(T)$ the solution at time $T + 1$ is then given by

$$x(T+1, \theta) = \sum_{k=0}^{T} a(\theta)^k u(T-k),$$

which is a polynomial $p(z) = u_T + u_{T-1}z + \cdots + u_0 z^T$ in $z = a(\theta)$ having the input signals as coefficients. Thus, given the terminal states $x_*(\theta)$ it is sufficient to show that there is a polynomial $p \in \mathbb{R}[z]$ so that for any $\varepsilon > 0$ it holds that

$$|x_*(\theta) - p(a(\theta))| < \varepsilon \quad \forall\, \theta \in \mathbf{P}. \tag{13}$$

To this end, let $A := a(\mathbf{P})$ and consider the function $f_*\colon A \to \mathbb{R}$ defined by $f_*(x) := x_*(a^{-1}(x))$. By the injectivity of a this function is well-defined and continuous. Also, note that A is a compact interval. By the classical Weierstrass Theorem there exists a polynomial p so that for all $\varepsilon > 0$ it holds that $|f_*(x) - p(x)| < \varepsilon$ for all $x \in A$. This in turn implies (13) and we are done.

Next, according to the assumptions on reachability and assuming in addition that $A(\theta)$ has only real distinct eigenvalues we consider w.l.o.g. a family of single-input discrete-time systems of the form

$$\begin{aligned} x_1(t+1) &= a_1(\theta)x_1(t) + u(t) \\ &\ \vdots \\ x_n(t+1) &= a_n(\theta)x_n(t) + u(t), \end{aligned} \tag{14}$$

with continuous and injective functions $a_1, \cdots, a_n\colon \mathbf{P} \to \mathbb{R}$. Here we apply the Stone-Weierstrass Theorem, cf. Theorem A in Section 36, Chapter 7 in [23], which is a generalization of the Weierstrass Theorem. Given input signals $u(0), ..., u(T)$ the ith component of the solution at time $T + 1$ is then given by

$$x_i(T+1, \theta) = \sum_{k=0}^{T} a_i(\theta)^k u(T-k), \quad i = 1, ..., n.$$

For each i the map $a_i\colon \mathbf{P} \to \mathbb{R}$ defines a homeomorphism onto $A_i = a_i(\mathbf{P}) \subset \mathbb{R}$. Further, by assumption (*iii*) it holds that $A_i \cap A_j = \emptyset$ for each $i \neq j$. Then, $A = A_1 \cup \cdots \cup A_n$ is a decomposition into pairwise disjoint compact sets. Given terminal states $x_*(\theta) = (x_1^*(\theta) \cdots x_n^*(\theta))^\top$ with $x_i^* \in C(\mathbf{P}, \mathbb{R})$, $i = 1, \cdots, n$, we consider the continuous function $f_*\colon A \to \mathbb{R}$ on the compact Hausdorff space A defined by $f_*|_{A_i} = x_i^* \circ a_i^{-1}$. Consider the algebra of continuous real valued functions on A as

$$\mathcal{A} = \{f \in C(A, \mathbb{R}) \mid \exists p(z) \in \mathbb{R}[z] \text{ such that } f_*|A_i = p|A_i,\ i = 1, \cdots, n\}.$$

Obviously, $\mathscr{A}$ separates points on A. Thus the Stone-Weierstrass theorem implies that $\mathscr{A}$ is uniformly dense in $C(A, \mathbb{R})$. Thus, for any $\varepsilon > 0$ there exists a real polynomial $p(z)$ with $|p(z) - f_*(z)| < \varepsilon$ for all $z \in A$. This implies $|p(a_i(\theta)) - x_i^*(\theta)| < \varepsilon$ for every $i = 1, ..., n$ and proves ensemble reachability for the system (15).

Finally, we consider the general situation where the system is of the form (15) with a self-conjugate set of complex-valued functions $a_1, \cdots, a_n : \mathbf{P} \longrightarrow \mathbb{C}$. One might consider to apply the complex version of the Stone-Weierstrass Theorem, cf. Theorem B in Section 36, Chapter 7 in [23]. However, this simple approach does not apply here and does only yield the existence of an approximating polynomial $p(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]$. Due to this we have to use a more sophisticated result from complex approximation, namely Mergelyan's Theorem, cf. Theorem 1 in Chapter II §2 in [11].

Theorem 2 (Mergelyan). *Suppose K is compact in $\mathbb{C}$ and the complement $\mathbb{C} \setminus K$ is connected; suppose further f is continuous on K and analytic in the interior of K. Then, for $\varepsilon > 0$ there exists a polynomial p such that for all $z \in K$,*

$$|f(z) - p(z)| < \varepsilon.$$

In [11] it is noted that this theorem includes the special case that the interior of K is empty. Now we turn to a proof of Theorem 1 in the discrete-time case.

Discrete-time Single-Input case: Recall that according to the assumptions on reachability and assuming in addition that $A(\theta)$ has only real distinct eigenvalues we consider w.l.o.g. a family of single-input discrete-time systems of the form

$$\begin{aligned} x_1(t+1) &= a_1(\theta)x_1(t) + u(t) \\ &\vdots \\ x_n(t+1) &= a_n(\theta)x_n(t) + u(t), \end{aligned} \tag{15}$$

with continuous and injective functions $a_1, \cdots, a_n \colon \mathbf{P} \to \mathbb{C}$. For each $i \in \{1, ..., n\}$ the map $a_i \colon \mathbf{P} \to \mathbb{C}$ defines a homeomorphism onto $A_i = a_i(\mathbf{P}) \subset \mathbb{C}$. Further, by assumption (*iii*) it holds that $A_i \cap A_j = \emptyset$ for each $i \neq j$. Then, $A = A_1 \cup \cdots \cup A_n$ is a decomposition into pairwise disjoint compact sets. Given terminal states $x_*(\theta) = (x_1^*(\theta) \cdots x_n^*(\theta))^\top$ with $x_i^* \in C(\mathbf{P}, \mathbb{R})$, $i = 1, \cdots, n$, we consider the continuous function $f_* \colon A \to \mathbb{C}$ on the compact set A defined by $f_*|_{A_i} = x_i^* \circ a_i^{-1}$. Note that the interior of A is empty. Thus, applying Mergelyan's Theorem there exists a complex polynomial q such that for all $z \in A$ we have

$$|f_*(z) - q(z)| < \varepsilon. \tag{16}$$

Thus, by considering $p(z) := \frac{1}{2}\left(q(z) + \overline{q(z)}\right)$, we can assume that (16) holds for the real polynomial $p(z)$. This implies $|p(a_i(\theta)) - x_i^*(\theta)| < \varepsilon$ for every $i = 1, ..., n$ and proves ensemble reachability for the system (15).

Discrete-time Multi-Input case: Let $(A(\theta), B(\theta))$ be a family of multivariable discrete-time systems. Based on the previous results it remains to extend the arguments to the multivariable case. By assumption (*ii*) in Theorem 1, the Hermite indices of the family $(A(\theta), B(\theta))$ are independent of parameter $\theta \in \mathbf{P}$. Hence, there exists a continuous family of invertible coordinate transformations $S(\theta)$ such that $(S(\theta)A(\theta)S(\theta)^{-1}, S(\theta)B(\theta))$ is in Hermite canonical form. Thus, w.l.o.g. we can assume that $(A(\theta), B(\theta))$ is in Hermite canonical form

$$\begin{pmatrix} A_{11}(\theta) & \cdots & A_{1m}(\theta) \\ & \ddots & \vdots \\ 0 & & A_{mm}(\theta) \end{pmatrix}, \quad \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_m \end{pmatrix} \tag{17}$$

where the m single-input subsystems $(A_{kk}(\theta), b_k) \in \mathbb{R}^{n_k \times n_k} \times \mathbb{R}^{n_k}$ are reachable and in control canonical form. Note that b_k denotes the first standard basis vector and thus is independent of θ. Partition the desired state vector as $x_*(\theta) = (x_*^1(\theta), \ldots, x_*^m(\theta))$ with $x_*^k(\theta) \in \mathbb{R}^{n_k}$. For any integer N, let $R_k^N(\theta)$ denote the $n_k \times N$ reachability matrix of the $k-$th subsystem $(A_{kk}(\theta), b_k)$. Further, Let $R^{Nm}(\theta)$ denote the reachability matrix of size $n \times Nm$ of the global system $(A(\theta), b(\theta))$. Thus $R^{Nm}(\theta)$ is block upper triangular of the form

$$R^{Nm}(\theta) = \begin{pmatrix} R_1^N(\theta) & \cdots & R_{1m}^N(\theta) \\ & \ddots & \vdots \\ 0 & & R_m^N(\theta) \end{pmatrix} \tag{18}$$

For simplicity now assume that $m = 2$. The argument is easily extended to the general case by induction. Applying the proven part of Theorem 1 to the single-input system

$$\begin{aligned} x_2(t+1, \theta) &= A_{22}(\theta)x_2(t, \theta) + b_2 u(t), \\ x_2(0, \theta) &= 0 \end{aligned} \tag{19}$$

shows the existence of a finite input sequence $u = (u_0, \cdots, u_N)^\top$ such that

$$\|R_2^N(\theta)u - x_*^2(\theta)\| < \varepsilon$$

holds uniformly for all θ. Similarly, applying Theorem 1 to the single-input system

$$\begin{aligned} x_1(t+1, \theta) &= A_{11}(\theta)x_1(t, \theta) + b_2 u(t), \\ x_1(0, \theta) &= 0 \end{aligned} \tag{20}$$

with the modified terminal state condition $x_*^1(\theta) + R_{12}^N(\theta)u$, shows the existence of an input sequence $v = (v_0, \cdots, v_N)^\top$ with

$$\max_{\theta \in \mathbf{P}} \|R_1^N(\theta)v - x_*^1(\theta) - R_{12}^N(\theta)u\| < \varepsilon.$$

Therefore the length N input sequence $w = (w_0, \cdots, w_N)$, $w_i := (v_i, u_i) \in \mathbb{R}^2$ leads to the desired estimate

$$\max_{\theta \in \mathbf{P}} \|R^{2N}(\theta)w - x_*(\theta)\| < \varepsilon.$$

This completes the proof of Theorem 1 in the discrete-time case.

Continuous-time case: For continuous-time systems

$$\dot{x} = A(\theta)x(t, \theta) + B(\theta)u(t), \quad x_0(\theta) = 0, \tag{21}$$

by sampling the system with any positive sampling period $\tau > 0$, we obtain the discrete-time system

$$\begin{aligned} x(t+1,\theta) &= F(\theta)x(t,\theta) + G(\theta)u(t) \\ x_0(\theta) &= 0, \end{aligned} \tag{22}$$

where

$$F(\theta) := e^{\tau A(\theta)}, \quad G(\theta) := \left(\int_0^\tau e^{sA(\theta)} ds\right) B(\theta). \tag{23}$$

In a first step we show that if $(A(\theta), B(\theta))$ satisfy the assumptions of Theorem 1, then the sampled system (22) satisfies the assumptions, too.

To see this, we recall that controllability of a continuous-time linear system (A, B) implies reachability of the sampled discrete-time system (F, G), if the sampling period is sufficiently small. Thus the pointwise reachability condition (i) follows for $(F(\theta), G(\theta))$. For the second condition assume that $(A(\theta), B(\theta))$ are in Hermite-canonical form with constant Hermite indices $K_1, \dots, K_m$. For $0 < \tau < \tau_*$ sufficiently small, $S = \int_0^\tau e^{sA(\theta)} ds$ is invertible. Note that S commutes with A. Thus the Hermite indices of F, G coincide with those of $(e^{\tau A}, B) = (S^{-1}FS, S^{-1}G)$. By the upper-triangular structure of the Hermite-canonical form it is easily seen that $(e^{\tau A}, B)$ and (A, B) have the same Hermite-indices, whenever τ is small enough. This shows (ii). The other conditions follow by continuity of the eigenvalues and local injectivity of the matrix exponential function.

Second, note that under sampling, the continuous and discrete-time solutions coincide at the sampling points. So, the finite input sequence u_k for uniform ensemble controllability of $(F(\theta), G(\theta))$ induces a finite piecewise constant control function $u^\tau \colon [0, T] \to \mathbb{R}^m$ that achieves the uniform ensemble control task for the continuous-time system (21) and this completes the proof of Theorem 1. □

Theorem 1 is an existence result and thus a theoretic achievment. The original proof of Mergelyan is constructive [20], but there is certainly the need for more efficient computations of the approximating polynomial p. Another natural approach is to use samplings $\mathbf{P}_{\text{samp}} \subset \mathbf{P}$. The ensemble for $\mathbf{P}_{\text{samp}}$ then can be interpreted as a system of parallel connected identical systems. The task then is to compute the open-loop control for the parallel connection system. In the SISO case this can be done e.g. using the Chinese Remainder Theorem as illustrated in Section 6. For multivariable systems this can in principle be done using the pseudo-inverse of the reachability matrix. However, this is not a very efficient method. Thus, even using singular-value decompositions there is need for efficient numerical algorithms for open loop control. We leave this as an open problem.

6. Example: Discrete-time harmonic oscillators

In this section we present a construction procedures for parameter-independent controls that steer an ensemble of discrete-time systems to a family of target states. Specifically, we consider an ensemble of second order discrete-time linear systems. For $\theta \in \mathbf{P} = [\theta^-, \theta^+] \subset \mathbb{R}$ we consider the system

$$x(t+1) = A(\theta)x(t) + b(\theta)u(t)$$

defined by

$$A(\theta) := \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b(\theta) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{24}$$

Using the state space transformation $S = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ it follows that

$$SA(\theta)S^{-1} := \theta \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Sb(\theta) := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Define $z^*(\theta) := x_1^*(\theta) + i\, x_2^*(\theta)$. Then

$$S\, x_*(\theta) = \begin{pmatrix} x_1^*(\theta) - i\, x_2^*(\theta) \\ x_1^*(\theta) + i\, x_2^*(\theta) \end{pmatrix} = \begin{pmatrix} \overline{z^*(\theta)} \\ z^*(\theta) \end{pmatrix}.$$

The uniform ensemble controllability condition now reads as follows:

For any $\varepsilon > 0$ and any continuous function $z^*(\theta)$, $\theta \in \mathbf{P}$ there exists a real polynomial $p \in \mathbb{R}[z]$ with

$$\sup_{\theta \in \mathbf{P}} |p(i\,\theta) - z^*(\theta)| < \varepsilon.$$

Let $p_1, p_2 \in \mathbb{R}[z]$ such that $p(iz) = p_1(z) + ip_2(z)$. Note that the polynomial $p(z)$ is real if and only if p_1 is even and p_2 is odd. Finally, this leads to the following formulation of uniform ensemble controllability:

For any $\varepsilon > 0$ and any continuous functions x_1^*, x_2^* there are real polynomials p_1, p_2 such that p_1 is even and p_2 is odd with

$$\max_{\theta \in \mathbf{P}} |p_1(\theta) - x_1^*(\theta)| < \varepsilon, \quad \max_{\theta \in \mathbf{P}} |p_2(\theta) - x_2^*(\theta)| < \varepsilon.$$

A necessary condition for uniform ensemble controllability is that $0 \notin [\theta^-, \theta^+]$. To see this, consider the constant function $f \colon [\theta^-, \theta^+] \to \mathbb{R}, f(\theta) := 1$ with $\theta^- < 0 < \theta^+$. It is easy to observe that p cannot be approximated arbitrary well by an odd polynomial.

Bernstein approximation

From now on we assume that $0 < \theta^- < \theta^+$. Under this assumption, we give a constructive proof of ensemble controllability that also shows a qualitative error bound.

Theorem 3. *Let $0 < \theta^- < \theta^+$ and $\boldsymbol{P} = [\theta^-, \theta^+]$. Let $\theta \mapsto x_*(\theta)$ be any Lipschitz continuous function with Lipschitz constant L^*. Then there exist an input sequence $u_0, \cdots, u_T$, $T \geq 3$ such that the zero state of*

$$(A(\theta), b(\theta)) = \left(\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_{\theta \in \boldsymbol{P}}$$

is steered uniformly to $x_*(\theta)$ with approximation error

$$\sup_{\theta \in \mathbf{P}} \|x^*(\theta) - x(T, \theta)\| \leq K^* \sqrt{\frac{\ln(T)}{T}},$$

where $K^* = 4 \sup_{\theta \in \mathbf{P}} \|x^*(\theta)\| + 2|\theta^+ - \theta^-| L^*$.

Proof. To show the assertion we consider *Bernstein polynomials*, cf. for instance [6]. Due to the fact that the eigenvalues of $A(\theta)$ are conjugate to each other, we extend $\mathbf{P}$ to a symmetric interval around zero. That is, let $\widehat{\mathbf{P}} := [-\theta^+, \theta^+]$. For any continuous function $f \in C(\widehat{\mathbf{P}})$, the nth order Bernstein polynomial is given by

$$B_n(f)(x) := \frac{1}{(2\theta_+)^n} \sum_{l=0}^{n} \binom{n}{l} f(-\theta^+ + \tfrac{l}{n} 2\theta^+)\,(x+\theta^+)^l\,(\theta^+ - x)^{n-l}.$$

It is immediately seen by inspection, that the Bernstein polynomial of an even (or odd) function $f\colon \widehat{\mathbf{P}} \to \mathbb{R}$ is even (or odd). We consider $z^*(\theta) = x_1^*(\theta) + i\,x_2^*(\theta)$ and define the even and odd extensions x_1^{*e}, x_2^{*e} of the real and imaginary parts x_1^*, x_2^*, respectively defined by

$$x_1^{*e}(\theta) := \begin{cases} x_1^*(-\theta) & \theta \in [-\theta^+, -\theta^-] \\ x_1^*(\theta_-) & \theta \in [-\theta^-, \theta^-] \\ x_1^*(\theta) & \theta \in [\theta^-, \theta^+] \end{cases},$$

and

$$x_2^{*e}(\theta) := \begin{cases} -x_2^*(-\theta) & \theta \in [-\theta^+, -\theta^-] \\ \frac{x_2^e(\theta^-)}{\theta^-}\,\theta & \theta \in [-\theta^-, \theta^-] \\ x_2^*(\theta) & \theta \in [\theta^-, \theta^+] \end{cases}.$$

Then, by the large deviation result of Gzyl and Palacios [12, Theorem 1] there are Bernstein polynomials $B_T(x_1^{*e})$, $B_T(x_2^{*e})$ with $T \geq 3$ such that for all $\theta \in \widehat{\mathbf{P}}$ it holds

$$|x_1^{*e}(\theta) - B_T(x_1^{*e})(\theta)| \leq K^* \sqrt{\frac{\ln(T)}{T}}$$
$$|x_2^{*e}(\theta) - B_T(x_2^{*e})(\theta)| \leq K^* \sqrt{\frac{\ln(T)}{T}}.$$

Moreover, we define

$$p(z) := B_T(x_1^{*e})(iz) + iB_T(x_2^{*e})(iz)$$

Note that p is a real polynomial since for $x \in \mathbb{R}$ it holds

$$\begin{aligned}\overline{p(x)} &= \overline{B_T(x_1^{*e}(-ix) + iB_T(x_2^{*e}(-ix)} = B_T(x_1^{*e}(ix) - iB_T(x_2^{*e}(ix) \\ &= B_T(x_1^{*e}(-ix) + iB_T(x_2^{*e}(-ix) = p(x).\end{aligned}$$

Furthermore, we have

$$\sup_{\theta\in\mathbf{P}} |p(i\theta) - z^*(\theta)| \leq K^* \sqrt{\frac{\ln(T)}{T}}.$$

It remains to determine the control input sequence. As $B_T(x_1^{*e})$ and $B_T(x_2^{*e})$ are even and odd, respectively, let

$$B_T(x_1^{*e})(x) = \sum_{k=0}^{T/2} p_{2k}^1 x^{2k}, \quad B_T(x_2^{*e})(x) = \sum_{k=0}^{(T-1)/2} p_{2k+1}^2 x^{2k+1}.$$

Then, we have

$$p(z) := \sum_{k=0}^{n} p_k z^k = \sum_{k=0}^{T/2} p_{2k} z^{2k} + \sum_{k=0}^{(T-1)/2} p_{2k+1} z^{2k+1},$$

where $p_{2k} = (-1)^k p_{2k}^1$ and $p_{2k+1} = (-1)^k p_{2k+1}^2$. For $T \in \mathbb{N}$, $t \geq 3$, the sequence of approximating controls then is given by

$$\boxed{u_0 = p_T \quad u_1 = p_{T-1} \quad \ldots \quad u_T = p_0} \tag{25}$$

This completes the proof of Theorem 3. □

Lagrange interpolation

On other hand, given the interval $\mathbf{P} = [\theta^-, \theta^+]$ another approach is to consider sampling points $\theta^- \leq \theta_1 < \theta_2 < \cdots < \theta_s \leq \theta^+$. This can be interpreted as the parallel connection of s harmonic oscillators $(A(\theta_k), b(\theta_k))_{k=1,\ldots,s}$. In this case it is very easy to carry out the calculations. Similarly, the transformation

$$S_c := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\theta} \end{pmatrix} \tag{26}$$

transforms (A, b) into controller canonical form

$$S_c A(\theta) S_c^{-1} := \begin{pmatrix} 0 & -\theta^2 \\ 1 & 0 \end{pmatrix}, \quad S_c b(\theta) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{27}$$

Thus, an input sequence $u_0, \ldots u_N$ steers the zero state of (24) into $x^* := (x_1^*(\theta), x_2^*(\theta))^\top$ if and only if it steers the zero state of the system (27) into $\xi = (\xi_1(\theta), \xi_2(\theta))^\top := (x_1^*(\theta), x_2^*(\theta)/\theta)^\top$. For θ_k let

$$A(\theta_k) := \begin{pmatrix} 0 & -\theta_k^2 \\ 1 & 0 \end{pmatrix}, \quad b(\theta_k) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{28}$$

Uniform ensemble controllability of the finite family $(A(\theta_k), b(\theta_k))$ is equivalent to the systems $(A_k, b_k) := (A(\theta_k), b(\theta_k))_{k=1,\ldots,s}$ being reachable, with pairwise coprime characteristic polynomials

$$q_k(z) = \det(zI - A_k) = z^2 + \theta_k^2.$$

Define $\hat{q}_k(z) := \prod_{l\neq k} q_l(z)$ and $q(z) := \prod_{k=1}^{s} q_k(z)$. Without loss of generality we can assume that (A_k, b_k) are in controllability canonical form with local state spaces

$$X_{q_k} := \{p \in \mathbb{R}[z] \mid \deg p < \deg q_k = 2\}. \tag{29}$$

Consider the parallel connection system

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix} \tag{30}$$

The state space of this global system is

$$X_q := \{p \in \mathbb{R}[z] \mid \deg p < \deg q = 2s\} \tag{31}$$

with direct sum decomposition

$$X_q := \hat{q}_1(z) X_{q_1} \oplus \cdots \oplus \hat{q}_s(z) X_{q_s}. \tag{32}$$

Assume that local target state vectors $x_k^* = (x_1^*(\theta_k), x_2^*(\theta_k)^\top \in \mathbb{R}^2$ are given. These vectors uniquely define polynomial elements $r_k(z) \in X_{q_k}$ of degree < 2, via $x_k^* = (x_{1,k}^*, x_{2,k}^*)$ and

$$r_k(z) = x_{1,k}^* + x_{2,k}^*\, z. \tag{33}$$

Thus the components of x_k^* are just the coefficients of the polynomial r_k. The ensemble control goal then is to find a polynomial $f(z) \in X_q$ such that its remainder modulo q_k is r_k. In fact, the coefficients

$$\boxed{u_0, \ldots, u_{2s-1} \quad \text{of} \quad f(z) = \sum_{l=0}^{2s-1} u_l\, z^l} \tag{34}$$

are then just the desired inputs that steer the system from zero to the local states x_k^*. In particular, the minimum length of such an ensemble control is $2s = \sum_{k=1}^{s} 2$, as it should be.

To compute f, we apply the Bezout identity. Thus, by coprimeness of $q_k, \hat{q}_k$, there exist unique polynomials $a_k(z)$ of degree < 2 and $b_k(z)$ with

$$a_k(z)\hat{q}_k(z) + b_k(z)q_k(z) = 1. \tag{35}$$

Define

$$f(z) = \sum_{k=1}^{s} r_k(z)\, a_k(z)\, \hat{q}_k(z). \tag{36}$$

The *Chinese Remainder Theorem* then asserts that f is the unique polynomial of degree $2s$ that has r_k as remainder modulo q_k. The coefficients of f thus give the desired controls for (A, b). The Bezout equation (35) is easily solved by the constant polynomial

$$a_k(z) = \prod_{l \neq k} \left(\theta_l^2 - \theta_k^2\right)^{-1}, \quad k = 1, \ldots, s.$$

For the local states $r_l(z) = x_1^*(\theta_l) + z\, x_2^*(\theta_l)$, the formula (36) for the Chinese Remainder Polynomial then is

$$f(z) = \sum_{l=1}^{s} \left(x_1^*(\theta_l) + z x_2^*(\theta_l)\right) \prod_{k \neq l} \frac{z^2 + {\theta_k}^2}{\theta_k^2 - \theta_l^2}. \tag{37}$$

Note, that this is exactly the degree $2s-1$ *Lagrange interpolation polynomial* satisfying

$$f(\pm\, i\theta_k) = r_k(\pm\, i\theta_k), \qquad k = 1, ..., s$$

whose coefficients determine the controls.

To illustrate the procedures we consider the parameter interval $\mathbf{P} = [1, 2]$. The family of target states is supposed to be

$$x^*(\theta) := (x_1^*(\theta)\ x_2^*(\theta))^\top = \left(\tfrac{1}{1+(2\theta-3)^2}\ \ 0\right)^\top.$$

We apply the Lagrange interpolation, where the parameters are chosen

- equidistant, i.e. $\theta_k = 1 + \frac{k-1}{s-1}$ for $k = 1, ..., s$.
- as Chebyshev points, i.e. for $k = 1, ..., s$,

$$\theta_k = 1 + \tfrac{1}{2}\Big(\cos\left(\tfrac{2k-1}{2s}\pi\right) + 1\Big).$$

If parameters are chosen equidistantly, then the approximation errors tends to grow unlimited at the boundary of the parameter interval $\mathbf{P} = [1, 2]$. In Figure 1 this phenomenon is depicted

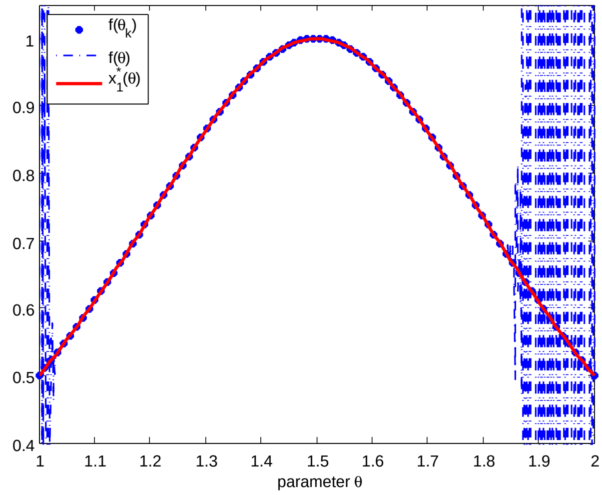

Figure 1: This figure shows the approximation of the first component of the target states $x_1(\theta) = \frac{1}{1+(2\theta-3)^2}$ using inputs as given in (34) for the polynomial defined by (37) with $s = 90$ equidistant sampling points in the interval $\mathbf{P} = [1, 2]$.

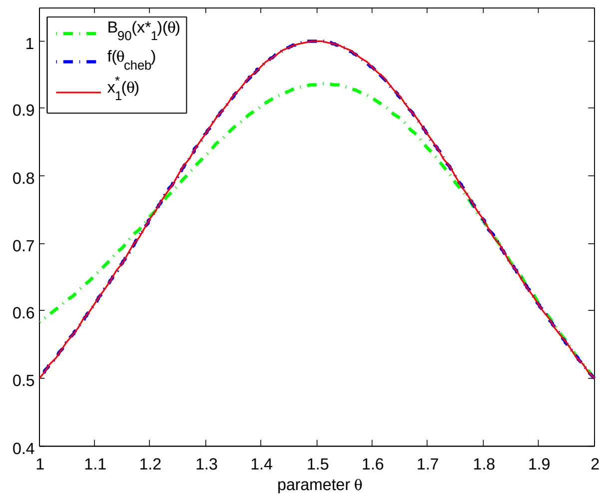

Figure 2: This figure shows the approximation of the first component of the target state $x_1^*(\theta) = \frac{1}{1+(2\theta-3)^2}$ using inputs defined by (25) via the 90th order Bernstein polynomial. Also, this figure shows the approximation of $x_1^*(\theta)$ using inputs as given in (34) for the polynomial defined by (37) with $s = 90$ Chebyshev sampling points.

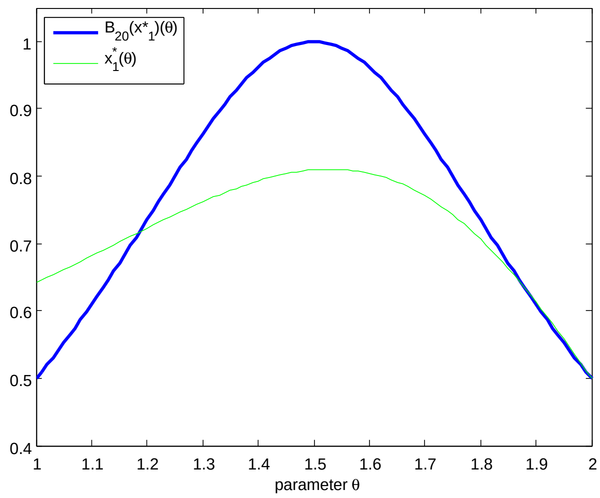

Figure 3: This figure shows the approximation of the first component of the target state $x_1^*(\theta) = \frac{1}{1+(2\theta-3)^2}$ using inputs defined by (25) via the 20th order Bernstein polynomial.

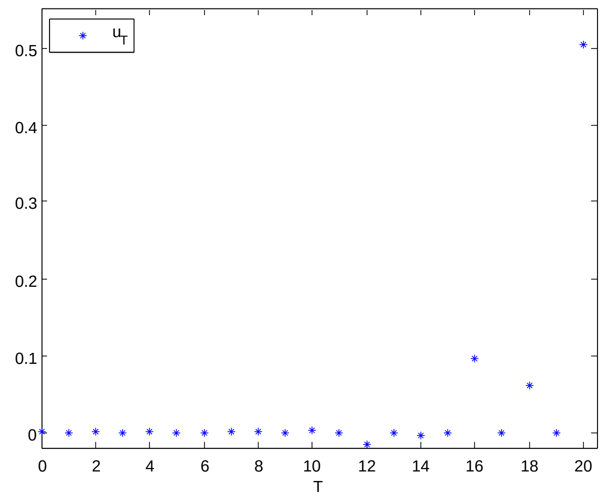

Figure 4: This figure shows the inputs defined by (25) for the 20th order Bernstein polynomial.

for the first component $x_1^*(\theta) = \frac{1}{1+(2\theta-3)^2}$ of the terminal state. Figure 2 illustrates the well-known fact that *Chebyshev* parameters enjoy the advantages compared to equidistant ones. It turns out that then the oscillations near the boundary of $[1, 2]$ do not appear and so the approximation is comparable to the one by the Bernstein polynomial. In Figure 6 we display the approximation of the first component $x_1^*(\theta)$ of the terminal state using the Bernstein polynomial with $n = 20$. Figure 6 shows the corresponding sequence of input values.

We emphasize that the preceeding technique based on sampling and computing (36) using the Chinese Remainder Theorem provides a general construction method to obtain the inputs. A discussion of other computational approaches for continuous-time systems can be found in [22].

7. Conclusions

In this paper we considered one parameter families of linear time-invariant systems and investigated controllability of an ensemble of linear systems simultaneously by a parameter independent input function. We obtained conditions that are necessary for uniform ensemble controllability as well as characterizations of uniform ensemble controllability for discrete-time single input systems. Based on Mergelyan's Theorem from complex approximation we also provide sufficient conditions for discrete-time and continuous-time linear systems time-invariant systems. The focus of this paper is not on construction procedures for the explicite input function; we rather presented a theoretical result showing that under the assumptions of Theorem 1 uniform ensemble controlllability is generally possible. Moreover, for the particular case of a second order discrete-time linear system we displayed procedures to determine the open-loop controls. The controls are explicitely computed by the coefficients of a Bernstein polynomial and a Lagrange polynomial, respectively.

Acknowledgements

The authors thank Barbara Scherlein for pointing out some errors in an early version of this paper.

References

[1] B. Bamieh, F. Paganini, M. Dahleh, Distributed control of spatially invariant systems, IEEE Transactions on Automatic Control 47 (7) (2002) 1091-1107.

[2] A. Becker, T. Bretl, Approximate steering of a unicycle under bounded model perturbation using ensemble control. IEEE Trans. on Robotics 28 (3) (2012) 580-591.

[3] R.W. Brockett, On the control of a flock by a leader, Proc. Steklov Institute of Mathematics 268 (1) (2010) 49-57.

[4] R. Brockett, N. Khaneja, On the stochastic control of quantum ensembles, In: Djaferis, T., Schick, I. (Eds), System Theory: Modeling, Analysis and Control, The Springer International Series in Engineering and Computer Science (2000), pp 75-96.

[5] R.F. Curtain, H.J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Texts in Applied Mathematics Vol. 21, Springer, New York (1995).

[6] P.J. Davis, Interpolation and approximation, Blaisdell Pub. Co., New York (1963).

[7] P.A. Fuhrmann, On weak and strong reachability and controllability of infinite-dimensional linear systems, J. Optimization Theory and Appl. 9 (2) (1972) 77-89.

[8] P.A. Fuhrmann, On controllability and observability of systems connected in parallel, IEEE Trans. Circuits and System 22 (1975) 57.

[9] P.A. Fuhrmann, A Polynomial Approach to Linear Algebra, second ed., Springer, New York (2012).

[10] P.A. Fuhrmann, U. Helmke, Reachability, observability and strict equivalence of networks of linear systems, Math Control Signals Syst, published online 3.01.2013.

[11] D. Gaier, Lectures on Complex Approximation, Birkhäuser, Boston, Inc. (1987).

[12] H. Gzyl, J.L. Palacios, The Weierstrass approximation theorem and large deviations, Amer. Math. Monthly 104 (1997) 650-653.

[13] M. Hazewinkel, A partial survey of the uses of algebraic geometry in systems and control theory, Istituto Nazionale di Alta Matematica Francesco Severi Symposia Methematica XXIV (1981), 245-292.

[14] B. Jacob, J.R. Partington, On controllability of diagonal systems with one-dimensional input space, Systems & Control Letters 55 (2006) 321-328.

[15] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs Publ., N.J. (1980).
[16] J.-S. Li, Ensemble control of finite-dimensional time-varying linear systems, IEEE Trans. Automatic Control 56 (2) (2011) 345-357.
[17] J.-S. Li, N. Khaneja, Control of inhomogeneous quantum ensembles, Phys. Rev. A 73 (2006) 030302.
[18] J.-S. Li, N. Khaneja, Ensemble Control of Bloch Equations, IEEE Trans. Automatic Control 54 (3) (2009) 528-536.
[19] P. Massioni, M. Verhaegen, Distributed control for identical dynamically coupled systems: a decomposition approach, IEEE Trans. Automatic Control 54 (1) (2009) 124-135.
[20] S.N. Mergelyan, On the representation of functions by series of polynomials on closed sets (Russian), Dokl. Ackad. Nauk SSSR 78, 405-408. Translation: Translations Amer. Math. Soc. 3 (1962) 287-293.
[21] J.A. Rogge, D., Aeyels, Vehicle platoons through ring coupling, IEEE Trans. Automatic Control 53 (6) (2008) 1370-1377.
[22] M. Schönlein, U. Helmke. Control of ensembles of single-input continuous-time linear systems, Proc. of the 4th IFAC workshop on Distributed estimation and control in networked systems, 25-26 September, Koblenz, Germany (2013).
[23] G. F. Simmons, Introduction to Topology and Modern Analysis, International series in pure and applied mathematics, McGraw-Hill, Inc. New York (1963).
[24] E.D. Sontag, Linear systems over commutative rings: A survey, Ricerche di Automatica 7 (1976) 1–34.
[25] R. Trigianni, On the lack of exact controllability for mild solutions in Banach spaces, J. Math. Analysis and Appl. 50 (1975) 438-446.