Open-loop control of parameter-dependent discrete-time systems

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We consider parameter-dependent linear time-invariant discrete-time single input systems, where the system matrix and the input vector are assumed to depend continuously on a parameter varying over a compact interval. We face the problem of steering the zero state simultaneously arbitrarily close towards a given continuous family of desired terminal states with a finite parameter independent open-loop input sequence. Starting from existing sufficient conditions, which include simplicity of the eigenvalues of the system matrices, we examine the case of multiple eigenvalues.

1 Introduction

We consider parameter-dependent linear time-invariant discrete-time single input systems of the form

$$x(t+1,\theta) = A(\theta)x(t,\theta) + b(\theta)u(t), \quad x(0,\theta) = 0,$$

where $A(\theta) \in \mathbb{R}^{n\times n}$ and $b(\theta) \in \mathbb{R}^n$ are assumed to depend continuously on the parameter θ varying over a real compact interval $\mathbf{P}$. Hence, $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ defines a family of systems, which is also called an ensemble of systems. Given an input sequence $u_0, ..., u_{T-1}$ the solution at $T > 0$ is $x(T,\theta) = u_0 b(\theta) + u_1 A(\theta)b(\theta) + \cdots + u_{T-1}A(\theta)^{T-1}b(\theta)$.

Definition 1.1 A family $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ is called *uniformly ensemble controllable*, if for any continuous functions $x^* : \mathbf{P} \to \mathbb{R}^n$ and any $\varepsilon > 0$ there exists a finite input sequence $u_0, ..., u_{T-1}$ such that $\sup_{\theta\in\mathbf{P}} \|x(T,\theta) - x^*(\theta)\| < \varepsilon$.

Similar concepts have been studied in quantum control, cf. [1]. Our aim is to derive sufficient conditions for uniformly ensemble controllability. The solution formula yields immediately the following characterization.

Proposition 1.2 *A family $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ of discrete-time systems is uniformly ensemble controllable if and only if for all $\varepsilon > 0$ and all $x^* \in C(\mathbf{P}, \mathbb{R}^n)$ there is a real scalar polynomial $p \in \mathbb{R}[z]$ such that $\sup_{\theta\in\mathbf{P}} \|p(A(\theta))\, b(\theta) - x^*(\theta)\| < \varepsilon$.*

The following necessary and sufficient conditions have been derived in [2, Theorem 1].

Theorem 1.3 *Let $\mathbf{P} \subset \mathbb{R}$ be a compact interval. A continuous family $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ of linear systems is uniformly ensemble controllable provided the following conditions are satisfied:*

(i) *$(A(\theta), b(\theta))$ is reachable for all $\theta \in \mathbf{P}$.*

(ii) *For any pair of distinct parameters $\theta, \theta' \in \mathbf{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint.*

(iii) *For each $\theta \in \mathbf{P}$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.*

Moreover, the conditions (i) and (ii) are also necessary for uniform ensemble controllability.

We not that Theorem 1.3 holds also true for continuous-time systems. In the multi-input case the family $(A(\theta), B(\theta))_{\theta\in\mathbf{P}}$ is uniformly ensemble controllable if the conditions (i)-(iii) hold and additionally the input Hermite indices $K_1(\theta), \dots, K_m(\theta)$ of $(A(\theta), B(\theta))$ are independent of $\theta \in \mathbf{P}$, cf. [2]. In addition, in [3] it is shown that in the continuous-time case under these conditions it is possible to steer any continuous family of initial states towards any continuous family of terminal states.

In this paper we provide another sufficient condition for uniform ensemble controllability. Based on this we will show that simplicity of the eigenvalues is not necessary for ensemble controllability. Moreover, we point out that the set of uniform ensemble controllable systems is neither open nor closed.

2 Results

The first result provides a further sufficient condition for uniform ensemble controllability of families of linear discrete-time single-input systems.

Theorem 2.1 *Let $\mathbf{P} \subset \mathbb{R}$ be a compact interval and let $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ be a continuous family of linear systems satisfying (i) and (ii). If there are $a_1, \dots, a_{n-1} \in \mathbb{R}$ such that for all $\theta \in \mathbf{P}$ the characteristic polynomials are of the form $\chi_{A(\theta)}(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0(\theta)$, then $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ is uniformly ensemble controllable.*

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P r o o f. We can assume without loss of generality that $(A(\theta), b(\theta))$ is in controllability normal form for each $\theta \in \mathbf{P}$. Define $g(z) := z^n - a_{n-1}z^{n-1} - \cdots - a_1 z$ and for any real polynomials f_k, $k = 1, \ldots, n$, define $f(z) := \sum_{k=1}^n f_k(g(z))z^{k-1}$. As $A(\theta)$ is in controllability normal form we have $f(A(\theta))b(\theta) = \sum_{k=1}^n f_k(g(A(\theta)))A(\theta)^{k-1}e_1 = \sum_{k=1}^n f_k(g(A(\theta))e_k = \sum_{k=1}^n f_k(a_0(\theta))e_k$. The assumptions imply that $a_0 : \mathbf{P} \to \mathbb{R}$ is injective and has a continuous inverse map $a_0^{-1} : a_0(\mathbf{P}) \to \mathbf{P}$. Let $\varepsilon > 0$ and $x^* \in C(\mathbf{P}, \mathbb{R}^n)$. For any $f_k \in \mathbb{R}[z]$ we have that $\sup_{\theta\in\mathbf{P}} |f_k(a_0(\theta)) - e_k^T x^*(\theta)| < \varepsilon$ if and only if $\sup_{y\in a_0(\mathbf{P})} |f_k(y) - e_k^T x^*(a_0^{-1}(y))| < \varepsilon$. By the Weierstraß approximation theorem there are real polynomials f_k such that the latter is fullfilled. We conclude that with this choice of f_k, $k = 1, \ldots n$, we have $\sup_{\theta\in\mathbf{P}} \|f(A(\theta))b(\theta) - x^*(\theta)\|_\infty < \varepsilon$. □

We emphasize that constructive proofs for the Weierstraß approximation theorem. are available which might provide algorithms for the constructions of the polynomials in this case. Theorem 2.1 applies only to certain special cases, but it gives examples of families of systems with eigenvalues of higher multiplicity which are uniformly ensemble controllable.

Example 2.2 Consider the discrete-time ensemble of harmonic oscillators

$$A(\theta) = \begin{pmatrix} 0 & -\theta^2 \\ 1 & 0 \end{pmatrix}, \quad b(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \theta \in \mathbf{P} := [-1, 1],$$

with the characteristic polynomial $\chi(z) = z^2 + \theta^2$. The prerequisites of Theorem 2.1 are fulfilled and $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ is uniformly ensemble controllable. We emphasize that $A(0)$ has the double eigenvalue 0. Consequently, condition (iv) in Theorem 1.3 is not necessary for uniform ensemble controllability. Also, we note that Theorem 2.1 allows the parameter set to be an arbitrary compact real interval. In particular, the restriction $\mathbf{P} \subset (0, \infty)$ in Theorem 3 in [2] is not needed.

However, if the family of system matrices has eigenvalues of higher multiplicity on a non-degenerate interval, this prevents uniform ensemble controllability. To be more precise, we have the following theorem:

Theorem 2.3 *Let $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ be a continuous family satisfying (i) and (ii). If there exists a non-degenerate compact interval $I \subset \mathbf{P}$, a natural number $k > 1$ and a continuously differentiable function $\lambda : I \to \mathbb{C}$ such that $\lambda(\theta)$ is an eigenvalue of $A(\theta)$ of multiplicity precisely k for all $\theta \in \mathbf{P}$, then $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ is not uniformly ensemble controllable.*

The essential step of the proof is outlined in the following example. For a detailed proof see [4].

Example 2.4 Let $\mathbf{P} := [0, 1]$,

$$A(\theta) := \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}, \ b(\theta) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ fulfils the necessary conditions (i) and (ii) for uniform ensemble controllability we have examined in Theorem 1.3. Further $A(\theta)$ has a double eigenvalue in the whole parameter set, a non-degenerate compact interval.

Now assume that $(A(\theta), b(\theta))_{\theta\in\mathbf{P}}$ is uniformly ensemble controllable. By Proposition 1.2, for all $\varepsilon > 0$ and all $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ there is a polynomial $p \in \mathbb{R}[z]$ such that $\sup_{\theta\in\mathbf{P}} \|p(A(\theta))b(\theta) - x^*(\theta)\|_\infty < \varepsilon$. Therefore, since the structure of $A(\theta)$ yields $p(A(\theta))b(\theta) = \big(p'(\theta), p(\theta)\big)^T$, it follows

$$\sup_{\theta\in\mathbf{P}} |p'(\theta) - x_1^*(\theta)| < \varepsilon \qquad \text{and} \qquad \sup_{\theta\in\mathbf{P}} |p(\theta) - x_2^*(\theta)| < \varepsilon.$$

Thus, using the triangle inequality we conclude that for all $\theta \in \mathbf{P}$ we have

$$\left| \int_0^\theta x_1^*(\tilde\theta)d\tilde\theta + x_2^*(0) - x_2^*(\theta) \right| \le \left| \int_0^\theta x_1^*(\tilde\theta) - p'(\tilde\theta)d\tilde\theta \right| + \left| \int_0^\theta p'(\tilde\theta)d\tilde\theta + p(0) - x_2^*(\theta) \right| + |x_2^*(0) - p(0)| \le 3\varepsilon.$$

As this is true for all $\varepsilon > 0$, we can conclude that there exists a $c \in \mathbb{R}$ such that for all $\theta \in \mathbf{P}$ we have $x_2^*(\theta) = \int_0^\theta x_1^*(\tilde\theta)d\tilde\theta + c$. However, this highly restricts the states which can be reached. For example $x^*(\theta) := \big(1\ , \ 2\theta\big)$ cannot be reached, as otherwise $c = 0$ if $\theta = 0$ and $c = 1$ if $\theta = 1$.

It is well known that the set of reachable pairs $(A, b) \in \mathbb{R}^{n\times(n+1)}$ is open. However, with Theorem 2.1 and Theorem 2.3 it is possible to construct examples (see [4]) that show:

Theorem 2.5 *In the set of families of single-input systems $C(\mathbf{P}, \mathbb{R}^{n\times(n+1)})$, equipped with the supremum norm, the set of families of uniformly ensemble controllable systems is neither open nor closed.*

References

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