Lecture Notes

Complex Analysis

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Chapter 1
The Green–Goursat Theorem

We introduce the differential operators
\[ \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \]

It is well-known that a (real) differentiable function \( f : U \to \mathbb{C} \) is holomorphic in an open set \( U \subseteq \mathbb{C} \) if and only if the Cauchy Riemann equations (CR-equations)
\[ \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{(CR\(_C\))} \]
are satisfied. Note that if we write \( f(x + iy) = u(x, y) + iv(x, y) \) with \( u = \text{Re}(f) \) and \( v = \text{Im}(f) \), then the CR-equations in complex form (CR\(_C\)) are equivalent to
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \text{(CR\(_R\))} \]
the CR-equations in real form. In the following we use the notation
\[ \partial f := \frac{\partial f}{\partial \bar{z}}, \quad \partial f := \frac{\partial f}{\partial z}. \]

We start with a version of Goursat’s theorem for not necessarily holomorphic functions.

Lemma 1.1 (The Green–Goursat theorem for a rectangle; weak version). Let \( U \subseteq \mathbb{C} \) be an open set, \( R := [a, b] \times [c, d] \subseteq U \) a closed (filled-in) rectangle and \( f : U \to \mathbb{C} \) a \( C^1 \)-function. Then
\[ \int_{\partial R} f(z) \, dz = 2i \int_R \partial f(z) \, dxdy. \]

Proof. The fundamental theorem of calculus shows that
\[ \int_R \frac{\partial f}{\partial y}(z) \, dxdy = \int_a^b \int_c^d \frac{\partial f}{\partial y}(z) \, dxdy = \int_a^b (f(x + id) - f(x + ic)) \, dx \]
\[ = - \int_{[a+ic,b+ic]} f(z) \, dz - \int_{[b+id,a+id]} f(z) \, dz. \]
In the same way, we find
\[ \int_R \frac{\partial f}{\partial x}(z) \, dxdy = i \int_{[a+ic,b+ic]} f(z) \, dz - i \int_{[a+id,a+ic]} f(z) \, dz. \]
Multiplying the first identity by \( i \) and adding the result to the second identity proves the lemma. \( \Box \)
Theorem 1.2 (The Green–Goursat theorem for a rectangle; strong version).
Let $U \subseteq \mathbb{C}$ be an open set, $R = [a, b] \times [c, d] \subseteq U$ a closed (filled-in) rectangle and $f : U \to \mathbb{C}$ a real differentiable function such that $\overline{\partial} f : U \to \mathbb{C}$ is continuous. Then
\[
\int_{\partial R} f(z) \, dz = 2i \int_{R} \overline{\partial} f(z) \, dz \, dy.
\]

Proof. Let
\[
I(R) := \int_{\partial R} f(z) \, dz - 2i \int_{R} \overline{\partial} f(z) \, dz \, dy.
\]
Divide $R$ into four rectangles of equal size and let $R_1$ be one of these rectangles for which $|I(R_1)|$ is maximal. Then $|I(R)| \leq 4|I(R_1)|$. Next repeat this process with $R_1$ and so on. We find rectangles $R \supseteq R_1 \supseteq R_2 \supseteq \ldots$ such that $|I(R)| \leq 4^n |I(R_n)|$ and $\cap R_n$ consists of one point $z_0 \in U$. Since $f$ is real differentiable in $z_0$, we have
\[
f(z) = f(z_0) + \partial f(z_0)(z - z_0) + \overline{\partial} f(z_0)(\overline{z - z_0}) + \varrho_1(z) \quad \text{with} \quad \lim_{z \to z_0} \frac{\varrho_1(z)}{z - z_0} = 0.
\]
Lemma 1.1 implies
\[
\int_{\partial R_n} f(z) \, dz = 2i \int_{R_n} \overline{\partial} f(z_0) \, dz \, dy + \int_{\partial R_n} \varrho_1(z) \, dz.
\]
Since $\overline{\partial} f$ is continuous in $z_0$, we have $\overline{\partial} f(z) = \overline{\partial} f(z_0) + \varrho_2(z)$ with $\lim_{z \to z_0} \varrho_2(z) = 0$, so
\[
2i \int_{R_n} \overline{\partial} f(z) \, dz \, dy = 2i \int_{R_n} \overline{\partial} f(z_0) \, dz \, dy + 2i \int_{R_n} \varrho_2(z) \, dz \, dy.
\]
It follows that
\[
|I(R)| \leq 4^n |I(R_n)| = 4^n \left| \int_{\partial R_n} \varrho_1(z) \, dz - 2i \int_{R_n} \varrho_2(z) \, dz \, dy \right| \leq 4^n L(\partial R_n) \max_{z \in \partial R_n} |\varrho_1(z)| + 4^n 2 \text{area}(R_n) \max_{z \in R_n} |\varrho_2(z)| \leq 4^n L(\partial R_n) \text{diam}(R_n) \max_{z \in \partial R_n} \frac{|\varrho_1(z)|}{|z - z_0|} + 2 \text{area}(R) \max_{z \in R_n} |\varrho_2(z)| = L(\partial R) \text{diam}(R) \max_{z \in \partial R_n} \frac{|\varrho_1(z)|}{|z - z_0|} + 2 \text{area}(R) \max_{z \in R_n} |\varrho_2(z)| \to 0 \quad \text{as} \quad n \to \infty.
\]

With the help of Theorem 1.2 we shall now prove variants of Cauchy's Integral Theorem and Cauchy's Integral Formula for not necessarily holomorphic functions.

Corollary 1.3.
Let $U \subseteq \mathbb{C}$ be an open set and $R, R'$ closed (filled-in) rectangles such that $R' \subseteq \text{int}(R)$ and $R \setminus \text{int}(R') \subseteq U$. If $f : U \to \mathbb{C}$ is differentiable and $\overline{\partial} f : U \to \mathbb{C}$ is continuous, then
\[
\int_{\partial R} f(z) \, dz - \int_{\partial R'} f(z) \, dz = 2i \int_{R \setminus R'} \overline{\partial} f(z) \, dz \, dy.
\]
Proof. The set $K := R \setminus \text{int}(R')$ is compact. One can find a $C^1$–function $\alpha : U \to \mathbb{C}$ such that $\alpha = 1$ on $K$ and $\alpha = 0$ outside a compact subset of $U$, see Appendix. We define $\Psi : \mathbb{C} \to \mathbb{C}$ by $\Psi(z) = \alpha(z)f(z)$ if $z \in U$ and $\Psi(z) = 0$ if $z \notin U$. Then by construction, $\Psi : \mathbb{C} \to \mathbb{C}$ is differentiable in $\mathbb{C}$ with $\overline{\partial} \Psi : \mathbb{C} \to \mathbb{C}$ continuous and $\Psi = f$ on $K$. Theorem 1.2 implies

$$2i \int_{\partial R} \overline{\partial} \Psi(z) \, dxdy = \int_{\partial R} \Psi(z) \, dz = \int_{\partial R} f(z) \, dz,$$

$$2i \int_{\partial R'} \overline{\partial} \Psi(z) \, dxdy = \int_{\partial R'} \Psi(z) \, dz = \int_{\partial R'} f(z) \, dz.$$

Subtracting the second equation from the first and using again $\Psi = f$ on $K$ completes the proof. \hfill \Box

For any function $f : U \to \mathbb{C}$ we denote by $\text{supp}(f)$ the closure of the set $\{z \in U : f(z) \neq 0\}$ in $U$. We say that $f : U \to \mathbb{C}$ has compact support, if $\text{supp}(f)$ is a compact subset of $U$.

Theorem 1.4.
Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a differentiable function with compact support. Suppose that $\overline{\partial} f : U \to \mathbb{C}$ is continuous. Then

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\overline{\partial} f(w)}{w - z} \, dudv, \quad z \in U.$$

Even though Theorem 1.4 cannot be applied directly to holomorphic functions, it plays an ubiquitous rôle in the sequel.

Proof. Fix $z \in U$. Let $R$ be a rectangle with $0 \in \text{int} R$ such that $f(z + w) = 0$ for all $w \notin \text{int}(R)$. Let $R'_\varepsilon$ be a rectangle with $0 \in \text{int}(R'_\varepsilon)$ such that the sum of the lengths of its sides is $\leq \varepsilon$. Then, by Corollary 1.3 with $f$ replaced by $w \mapsto f(z + w)/w$,

$$2i \int_{\partial R'_\varepsilon} \frac{\overline{\partial} f(z + w)}{w} \, dudv = - \int_{\partial R'_\varepsilon} \frac{f(z + w)}{w} \, dw = - \int_{\partial R'_\varepsilon} \frac{f(z + w) - f(z)}{w} \, dw - 2\pi i f(z).$$

Since $|f(z + w) - f(z)| \leq M|w|$ for some $M > 0$ and all $w \in \mathbb{C}$, the integral on the right-hand side tends to 0 as $\varepsilon \to 0$, so (note that \(\iint_{R} \frac{dudv}{|w|} < \infty\))

$$f(z) = -\frac{1}{\pi} \iint_{R} \frac{\overline{\partial} f(z + w)}{w} \, dudv = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\overline{\partial} f(w)}{w - z} \, dudv$$

by the change of variables $w \to w - z$. \hfill \Box

In the sequel we denote by $C^k_r(U)$ the set of all $k$–times (real) continuously differentiable functions $f : U \to \mathbb{C}$ on the open set $U \subseteq \mathbb{C}$ with compact support.

Theorem 1.5 (Variant of the Cauchy Integral Formula).
Let $U \subseteq \mathbb{C}$ be an open set, $K$ a compact subset of $U$ and $\alpha \in C^\infty_r(U)$ such that $\alpha \equiv 1$ on a neighborhood of $K$. Then, for any $f \in \mathcal{H}(U)$,

$$f(z) = -\frac{1}{\pi} \iint_{U} \overline{\partial} \alpha(w) \cdot \frac{f(w)}{w - z} \, dudv, \quad z \in K.$$
**Remark 1.6.**

This formula can be written

\[ f(z) = -\frac{1}{\pi} \int_C \bar{\alpha}(w) \cdot \frac{f(w)}{w-z} \, du dv, \quad z \in \Gamma, \]

where \( S \) is the strip \( \text{supp}(\alpha) \setminus \Gamma \). If we compare this with Cauchy’s integral formula, then integration along a cycle \( \Gamma \) has been replaced by integration over a strip, but the winding number \( n(\Gamma, z) \) no longer appears!

**Theorem 1.7** (The inhomogeneous Cauchy Integral Theorem; Homologous Form).

Let \( U \subset \mathbb{C} \) be an open set and let \( f : U \to \mathbb{C} \) be differentiable such that \( \overline{\partial} f : U \to \mathbb{C} \) is continuous. If \( \Gamma \) is a nullhomologous cycle in \( U \), then

\[ \int_{\Gamma} f(z) \, dz = 2i \int_C \overline{\alpha}(w) n(\Gamma, z) \, dx \, dy. \]

**Proof.** Since \( \Gamma \) is nullhomologous, the set \( K := \Gamma \cup \{ z \notin \Gamma : n(\Gamma, z) \neq 0 \} \) is a compact subset of \( U \). Let \( \psi : U \to \mathbb{R} \) be a \( C^1 \)-function with compact support in \( U \) such that \( \psi = 1 \) on a neighborhood of \( K \). Then Theorem 1.4 applied to \( g := \psi \cdot f \) implies

\[ g(z) = -\frac{1}{\pi} \int_C \overline{\beta}(w) \, du \, dv. \]

Since \( g = f \) on \( \Gamma \), we deduce

\[ \int_{\Gamma} f(z) \, dz = -\frac{1}{\pi} \int_{\Gamma} \left( \int_C \overline{\beta}(w) \, du \, dv \right) \, dz. \]

We apply Fubini’s theorem and get

\[ \int_{\Gamma} f(z) \, dz = -\frac{1}{\pi} \int_C \overline{\beta}(w) \left( \int_{\Gamma} \frac{1}{w-z} \, dz \right) \, du \, dv = 2i \int_C \overline{\alpha}(w) n(\Gamma, w) \, du \, dv. \]

\[ \square \]

**Corollary 1.8** (Green’s theorem for smoothly bounded domains).

Let \( \Omega \) be a bounded domain with piecewise \( C^1 \)-boundary and let \( U \) be an open neighborhood \( U \) of \( \overline{\Omega} \). Suppose that \( f : U \to \mathbb{C} \) is differentiable with \( \overline{\partial} f : U \to \mathbb{C} \) continuous. Then

\[ \int_{\partial \Omega} f(z) \, dz = 2i \int_{\Omega} \overline{\partial} f(z) \, dx \, dy. \]
In particular,
\[ \frac{1}{2\pi i} \oint_{\partial \Omega} \bar{z} \, dz = \iint_{\Omega} \bar{z} \, dx \, dy = \text{area}(\Omega). \]

Remark 1.9 (Green's theorem; real version).
If \( f \) in Corollary 1.8 has the form \( f = P - iQ \) for differentiable functions \( P, Q : U \to \mathbb{R} \) such that \( Q_x - P_y \) and \( P_x + Q_y \) are both continuous on \( U \), then we get
\[
\oint_{\partial \Omega} P \, dx + Q \, dy = \iint_{\Omega} (Q_x - P_y) \, dx \, dy, \quad \oint_{\partial \Omega} P \, dy - Q \, dx = \iint_{\Omega} (P_x + Q_y) \, dx \, dy.
\]

See Exercise 1.2 for a stronger version.

Theorem 1.10 (The inhomogeneous Cauchy Integral Formula; Homologous Form).
Let \( U \subseteq \mathbb{C} \) be an open set. Let \( f : U \to \mathbb{C} \) be differentiable such that \( \overline{\partial} f : U \to \mathbb{C} \) is continuous. If \( \Gamma \) is a nullhomologous cycle in \( U \), then
\[
n(\Gamma, z) f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} \, dw - \frac{1}{\pi} \iint_{\mathcal{C}} n(\Gamma, w) \cdot \overline{\partial} f(w) \frac{1}{w - z} \, dudv
\]
for every \( z \in U \setminus \Gamma \).

The inhomogeneous Cauchy Integral Formula provides a decomposition of a function \( f \) in a holomorphic part and a nonholomorphic part which depends only on \( \overline{\partial} f \). Note that Theorem 1.10 extends Theorem 1.4.

Proof. Fix a point \( z \in U \setminus \Gamma \) and \( \varepsilon > 0 \) such that the disk \( K_{\varepsilon}(z) \) is disjoint from \( \Gamma \) and compactly contained in \( U \). Then the cycle \( \Gamma_{\varepsilon} := \Gamma - n(\Gamma, z) \partial K_{\varepsilon}(z) \) is nullhomologous in \( U \setminus \{z\} \). If we apply Theorem 1.7 to the function \( w \mapsto \frac{f(w)}{w - z} \) and the cycle \( \Gamma_{\varepsilon} \), we obtain
\[
\int_{\Gamma} \frac{f(w)}{w - z} \, dw - n(\Gamma, z) \int_{\partial K_{\varepsilon}(z)} \frac{f(w)}{w - z} \, dw = 2i \int_{\mathcal{C}} \overline{\partial} f(w) \frac{1}{w - z} n(\Gamma_{\varepsilon}, w) \, dudv
\]
\[
= 2i \int_{\mathcal{C}} \frac{\overline{\partial} f(w)}{w - z} n(\Gamma, w) \, dudv - 2i n(\Gamma, z) \int_{\partial K_{\varepsilon}(z)} \overline{\partial} f(w) \frac{1}{w - z} \, dudv.
\]

Now letting \( \varepsilon \to 0 \), using the facts that
\[
w \mapsto \frac{\overline{\partial} f(w)}{w - z}
\]
is integrable over \( K_{\varepsilon}(z) \) and
\[
\int_{\partial K_{\varepsilon}(z)} \frac{f(w)}{w - z} \, dw \to 2\pi i f(z)
\]
completes the proof.
Appendix: Cutoff functions

We have repeatedly used the following result.

**Theorem A.**

Let $U \subseteq \mathbb{C}$ be an open set, $K$ a compact subset of $U$ and $V \subseteq U$ an open set that contains $K$. Then there exists a $C^\infty$-function $\alpha : U \to \mathbb{R}$ such that $0 \leq \alpha \leq 1$, $\alpha \equiv 1$ on $K$ and $\text{supp}(\alpha) \subseteq V$.

**Proof.** The function

$$f_1(x) := \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is of class $C^\infty$ and strictly positive for all $x > 0$. Then

$$f_2(x) := \frac{f_1(x)}{f_1(x) + f_1(1 - x)}$$

is of class $C^\infty$, vanishes for all $x \leq 0$, equals 1 for all $x \geq 1$, and $0 \leq f_2(x) \leq 1$ for all $x \in \mathbb{R}$. Hence

$$F(z) := f_2(2 - |z|), \quad z \in \mathbb{C},$$

is of class $C^\infty$ in $\mathbb{C}$, equals 1 for all $|z| \leq 1$, and equals 0 for all $|z| \geq 2$.

For each $q \in K$ let $K_{r_q}(q) \subseteq V$ be an open disk with radius $r_q > 0$ and center $q$. Then

$$F_q(z) := \begin{cases} F(3(z - q)/r_q), & |z - q| < r_q \\ 0, & \text{ elsewhere} \end{cases}$$

is smooth in $\mathbb{C}$, $\text{supp}(F_q) \subseteq K_{r_q}(q)$ and $F_q \equiv 1$ on $K_{r_q/3}(q)$. Since $K$ is compact, there is a finite covering $\bigcup_{j=1}^n K_{r_{q_j}/3}(q_j)$ of $K$. Let $\Psi := \sum_{j=1}^n F_{q_j}$. Then $\Psi$ is of class $C^\infty$ with $\Psi \geq 1$ on $K$ and $\text{supp}(\Psi) \subseteq V$. Hence $\alpha := f_2 \circ \Psi$ satisfies all the conditions we required. \( \square \)

**Problems**

1. Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a differentiable function.

   (a) Prove that $\overline{\partial f} = \partial \overline{f}$.

   (b) For $\alpha \in \mathbb{R}$ let

   $$\partial_\alpha f(z) := \lim_{r \to 0} \frac{f(z + re^{i\alpha}) - f(z)}{re^{i\alpha}}$$

   be the derivative of $f$ in direction $\alpha$. Prove that

   $$\partial_\alpha f = \partial f + \overline{\partial} fe^{2i\alpha}.$$

2. (Green’s theorem under weak assumptions)

   Let $U \subseteq \mathbb{R}^2$ be an open set, $R$ a filled-in closed rectangle in $U$ and $Q, P : U \to \mathbb{R}$ differentiable functions such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} : U \to \mathbb{R}$ is continuous. Prove that

   $$\int_R P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.$$
(Hint: Follow the proof of Theorem 1.2. For an even stronger version see P.J. Cohen\(^1\), On Green's theorem, Proc. Am. Math. Soc. 10, 109–112 (1959).)

3. (Green's formula)
Let \( \Omega \) be a bounded domain with piecewise \( C^1 \)-boundary and let \( v : \overline{\Omega} \rightarrow \mathbb{R} \) be a \( C^2 \)-function.

(a) Let \( \gamma(s) = (x(s), y(s)) \) denote the parametrization of \( \partial \Omega \) by arclength so that \( n := (dy/ds, -dx/ds) \) is the (outer) unit normal vector. Prove that the directional derivative

\[
\frac{\partial v}{\partial n} = \nabla v \cdot n
\]
(where \( \nabla v \) is the gradient of \( v \) and \( \cdot \) denotes the standard scalar product in \( \mathbb{R}^2 \)) has the property that

\[
\frac{\partial v}{\partial n} ds = \frac{\partial v}{\partial x} dy - \frac{\partial v}{\partial y} dx.
\]

(b) Let \( u : \overline{\Omega} \rightarrow \mathbb{R} \) be a \( C^2 \)-function and denote by \( \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) the Laplacian of \( u \). Use Green's theorem to prove

(i) (Green's first formula)

\[
\int_{\partial \Omega} u \frac{\partial v}{\partial n} ds = \iint_{\Omega} \nabla u \cdot \nabla v \, dx \, dy + \iint_{\Omega} u \Delta v \, dx \, dy,
\]

and

(ii) (Green's second formula)

\[
\int_{\partial \Omega} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \iint_{\Omega} (u \Delta v - v \Delta u) \, dx \, dy.
\]

4. Let \( U \subseteq \mathbb{C} \) be a domain, \( \Gamma \) a nullhomologous cycle in \( U \) and \( f \in \mathcal{H}(U) \). Prove that

\[
\frac{1}{2\pi i} \int_{\Gamma} f(z)f'(z) \, dz = \frac{1}{\pi} \int_{\mathbb{C}} |f'(z)|^2 n(\Gamma, z) \, dx \, dy = \frac{1}{\pi} \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} n(\Gamma, z) \, du \, dv.
\]

5. (Bergman inequality)
Let \( U \subseteq \mathbb{C} \) an open set and \( K \) a compact subset of \( U \). Prove that for every open set \( V \) such that \( K \subseteq V \) and \( \overline{V} \) is a compact subset of \( U \) and for every \( m \in \mathbb{N}_0 \) there is a constant \( C_m = C_{m, v, K} \) such that

\[
\sup_{z \in K} |f^{(m)}(z)| \leq C_m ||f||_{L^1(V)} \text{ for all } f \in \mathcal{H}(U).
\]

Here,

\[
||f||_{L^1(V)} = \iint_{V} |f(z)| \, dx \, dy
\]

denotes the \( L^1 \)-Norm of \( f \) over \( V \) (Hint: Theorem 1.5). Deduce that if \( f_j \in \mathcal{H}(U) \) and \( f : U \rightarrow \mathbb{C} \) is continuous such that \( ||f_j - f||_{L^1(K)} \rightarrow 0 \) for every compact set \( K \subseteq U \), then \( f \in \mathcal{H}(U) \).

\(^1\)Fields-Medallist 1966
6. (Argument principle and Rouché’s theorem; revisited)

Let $\Omega \subseteq \mathbb{C}$ be a domain and $K \subseteq \Omega$ compact.

(a) Let $f$ be meromorphic and nonconstant in $\Omega$ and denote by $Z$ and $P$ the number of zeros and poles of $f$ in $K$. Suppose that there is an open neighborhood $U \subseteq \Omega$ of $\partial K$ such that $f$ has neither zeros nor poles in $U$. Let $\alpha \in C^1_c(\Omega)$ with supp$(\alpha) \subseteq K \cup U$ and $\alpha \equiv 1$ in a neighborhood of $K$. Prove that

$$N - P = -\frac{1}{\pi} \int_{\Omega} \overline{\partial} \alpha(z) \frac{f'(z)}{f(z)} dx dy. $$

(b) Let $f, g \in \mathcal{H}(\Omega)$ such that $|f(z) + g(z)| < |f(z)| + |g(z)|$ for all $z \in \partial K$. Prove that $f$ and $g$ have the same number of zeros in $K$ (counting multiplicities).
Chapter 2

Runge’s theorem

Runge’s theorem belongs in every analyst’s bag of tricks.

L. Rubel [8, p. 185]

Let $K \subseteq \mathbb{C}$ be a compact set. Set $\mathcal{H}(K) := \{ f \in \mathcal{H}(U) : U \subseteq \mathbb{C} \text{ open neighborhood of } K \}$. In this chapter we are concerned with the following problem: Let $f \in \mathcal{H}(K)$. Are there polynomials $(p_n)$ such that $p_n \to f$ uniformly on $K$?

Example 2.1.
Let $K := \mathbb{D}$ and $f \in \mathcal{H}(K)$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R$ for some $R > 1$ and the convergence is uniform on $K$.

Example 2.2.
Let $K := \partial \mathbb{D}$ and $f(z) := 1/z$, which belongs to $\mathcal{H}(K)$. Then for any polynomial $p$,

$$1 = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(z) \, dz \neq \frac{1}{2\pi i} \int_{\partial \mathbb{D}} p(z) \, dz = 0.$$ 

Hence $f$ cannot be uniformly approximated on $K$ by polynomials.

In the first example $\mathbb{C} \setminus K = \{ z \in \mathbb{C} : |z| > 1 \}$ is connected, whereas in the second example $\mathbb{C} \setminus K = \mathbb{D} \cup \{ z \in \mathbb{C} : |z| > 1 \}$ has two (connected) components. Note that for any compact set $K \subseteq \mathbb{C}$ the complement $\mathbb{C} \setminus K$ is open, has at most countably many (open) components and exactly one unbounded component.

Theorem 2.3 (Runge’s theorem for compacta).
Let $K$ be a compact subset of $\mathbb{C}$ and let $\{a_k\}$ be a sequence in $\mathbb{C}$ which contains a point from each bounded component of $\mathbb{C} \setminus K$. Suppose that $f \in \mathcal{H}(K)$. Then for any $\varepsilon > 0$ there is a rational function $r$ with poles only in the set $\{a_k\}$ such that $|f - r| < \varepsilon$ on $K$.

See Figure 2.1 for an illustration of Runge’s theorem. The “poles in holes” condition in Runge’s theorem is not only sufficient, but also necessary by Example 2.2. The following beautiful proof of Runge’s theorem is due to Lars Hörmander.

Proof. We consider $\mathcal{H}(K)$ as a subspace of the normed space $C(K)$ of all continuous functions $g : K \to \mathbb{C}$ equipped with the sup-norm. By the Hahn–Banach theorem (see Appendix), we only need to show that every continuous linear functional on $\mathcal{H}(K)$ which vanishes on the
subspace of all rational functions with poles only in \( \{a_k\} \) also vanishes on \( \mathcal{H}(K) \). Let \( L \) be such a continuous linear functional on \( C(K) \) and apply \( L \) to the Cauchy kernel \( z \mapsto (z-w)^{-1} \):

\[
h(w) := L \left( \frac{1}{z-w} \right), \quad w \in \hat{C}\backslash K.
\]

(i) We claim \( h \) is holomorphic on \( \mathbb{C}\backslash K \). In order to see this, fix a point \( w_0 \in \mathbb{C}\backslash K \) and a disk \( K_r(w_0) \) compactly contained in \( \mathbb{C}\backslash K \). There is number \( \varrho < 1 \) such that \( \frac{|w-w_0|}{|z-w_0|} \leq \varrho < 1 \) for all \( z \in K \) and all \( w \in K_r(w_0) \). Therefore,

\[
\frac{1}{z-w} = \frac{1}{z-w_0} \frac{1}{1 - \frac{w-w_0}{z-w_0}} = \sum_{n=0}^{\infty} \frac{1}{(z-w_0)^{n+1}} (w-w_0)^n
\]

uniformly for \( (z,w) \in K \times K_r(w_0) \). Hence we have

\[
h(w) = \sum_{n=0}^{\infty} L \left( \frac{1}{(z-w_0)^{n+1}} \right) (w-w_0)^n, \quad w \in K_r(w_0).
\]

Thus \( h \) is holomorphic on \( \mathbb{C}\backslash K \).

(ii) We claim \( h = 0 \) on \( \mathbb{C}\backslash K \). Let \( V \) be a bounded component of \( \mathbb{C}\backslash K \) and \( a_k \) a point in \( V \). Choose a disk \( K_r(a_k) \) compactly contained in \( V \). Then

\[
\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w-a_k)^n}{(z-a_k)^{n+1}}.
\]
The convergence is uniform for \( z \in K \) for each \( w \in K_r(a_h) \). Hence

\[
h(w) = \sum_{n=0}^{\infty} (w - a_h)^n L \left( \frac{1}{(z - a_h)^{n+1}} \right) = 0, \quad w \in K_r(a_h).
\]

Therefore \( h = 0 \) on \( V \). Now let \( V \) be the unbounded component of \( C \setminus K \) and choose \( r > 0 \) such that every \( w \notin K_r(0) \) belongs to \( V \), then

\[
\frac{1}{z - w} = \frac{1}{w} \frac{1}{1 - z/w} = -\frac{1}{w} \sum_{n=0}^{\infty} \left( \frac{z}{w} \right)^n.
\]

The convergence is uniform for \( z \in K \) and \( |w| > r \). Hence

\[
L \left( \frac{1}{z - w} \right) = -\sum_{n=0}^{\infty} L(z^n) \frac{1}{w^{n+1}} = 0.
\]

Therefore, \( h = 0 \) on \( V \) and so \( h = 0 \) on \( C \setminus K \).

(iii) Now let \( f \in \mathcal{H}(K) \) so that \( f \in \mathcal{H}(W) \) for some open set \( W \) containing \( K \). Choose \( \phi \in C_c^\infty(W) \) such that \( \phi \equiv 1 \) on a neighborhood \( W_0 \) of \( K \). Then by Theorem 1.5,

\[
f(z) = \frac{1}{\pi} \int_{W \setminus W_0} \overline{\phi}(w) \cdot \frac{f(w)}{w - z} dudv, \quad z \in K,
\]

so

\[
L(f) = \frac{1}{\pi} \int_{W \setminus W_0} \overline{\phi}(w) \cdot f(w) \cdot h(w) dudv = 0,
\]

because \( h = 0 \) on \( W \setminus W_0 \subseteq C \setminus K \). \( \square \)

**Corollary 2.4.**

*Let \( K \) be a compact subset of \( C \) such that \( C \setminus K \) is connected (i.e., \( K \) has no “holes”), then any \( f \in \mathcal{H}(K) \) can be uniformly approximated on \( K \) by polynomials.*

**Example 2.5** (Pointwise, but not locally uniformly convergent polynomials).

*Runge’s theorem can be used to “construct” polynomials \( p_n \), which converge pointwise to 1 on \( \overline{D} \) and to 0 outside of \( \overline{D} \). In order to find such polynomials let

\[
K_n := \overline{D} \cup \left\{ z \in C : 1 + \frac{1}{n} \leq |z| \leq n, \quad 0 \leq \arg z \leq 2\pi - 1/n \right\}.
\]

Then \( C \setminus K_n \) is connected and

\[
f(z) := \begin{cases} 1 & \text{if } z \text{ is in a neighborhood of } \overline{D} \\
0 & \text{if } z \text{ is in a neighborhood of } K_n \setminus \overline{D}
\end{cases}
\]

is holomorphic on \( K_n \).

By Corollary 2.4, there is a polynomial \( p_n \) such that \( |p_n - 1| < 1/n \) on \( \overline{D} \) and \( |p_n| < 1/n \) on \( K_n \setminus \overline{D} \). Hence \( p_n \) has the stated properties. Note that the pointwise limit function \( f : C \rightarrow C \) is actually holomorphic on an open and dense subset of \( C \). This is not a coincidence: If \( (f_n) \) is a sequence of holomorphic functions on a domain \( \Omega \subseteq C \) which converges pointwise to \( f : \Omega \rightarrow C \), then \( f \) is holomorphic in a dense open subset \( D \) of \( \Omega \) and the convergence is locally uniform in \( D \). This is a theorem of Osgood (see...*
W. F. Osgood, *Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, with corresponding results for definite integrals*, Ann. Math. 3 (1901), 25–34. It is nowadays usually proved in any introductory course on functional analysis as an important step from Baire’s theorem to the theorems of Banach–Steinhaus. However, in most cases Osgood’s theorem is only proved in a weaker form for continuous functions.

**Theorem 2.6** (A universal entire function; Birkhoff 1929).

There is a function \( F \in \mathcal{H}(\mathbb{C}) \) such that for any \( f \in \mathcal{H}(\mathbb{C}) \) there is a sequence \( (\tau_n) \subset \mathbb{C} \) such that \( F(z + \tau_n) \to f(z) \) locally uniformly in \( \mathbb{C} \).

Such \( F \in \mathcal{H}(\mathbb{C}) \) is called an entire universal function.

**Proof.** (i) Let \( K_n \) be pairwise disjoint, closed disks with radii converging to \( \infty \), and let \( U_n \) be closed disks with center 0 such that \( U_n \) contains \( K_1, \ldots, K_n \), but \( U_n \) is disjoint from \( K_{n+k} \) for any \( k \geq 1 \). For instance, one can take \( K_n := \overline{K_n(n^3)} \) and \( U_n := \overline{K_{n^3+n}(0)} \). Now let \( (p_n) \) be a sequence of all polynomials with rational coefficients, so \( (p_n) \) is dense in \( \mathcal{H}(\mathbb{C}) \). If \( \gamma_n \) is the center of \( K_n \), let \( q_n(z) := p_n(z - \gamma_n) \). We let \( f_1 = q_1 \) and find in view of Corollary 2.4 polynomials \( f_n \) such that

\[
|f_n - f_{n-1}| < 2^{-n} \text{ in } U_{n-1} \quad \text{and} \quad |f_n - q_n| < 2^{-n} \text{ in } K_n, \quad n \geq 2.
\]

This is possible because \( C \setminus (K_n \cup U_{n-1}) \) is connected.

Now for all \( z \in U_n \) and any \( m > n \) we get

\[
|f_m(z) - f_n(z)| \leq \sum_{k=n+1}^{m} |f_k(z) - f_{k-1}(z)| \leq \sum_{k=n+1}^{m} 2^{-k} < 2^{-n},
\]

so \( (f_n) \) converges locally uniformly to some \( F \in \mathcal{H}(\mathbb{C}) \).

(ii) Let \( f \in \mathcal{H}(\mathbb{C}) \). Then there exists a sequence \( (n_k) \) such that \( p_{n_k} \to f \) locally uniformly in \( \mathbb{C} \), i.e., \( q_{n_k}(z + \gamma_{n_k}) \to f(z) \). Now fix \( R > 0 \) and \( N \in \mathbb{N} \) such that \( K_n - \gamma_n \supseteq K_R(0) \) for all \( n \geq N \). Then for all \( z \in K_R(0) \), we have \( z + \gamma_{n_k} \in K_{n_k} \subseteq U_{n_k} \), so

\[
|F(z + \gamma_{n_k}) - f(z)| \leq |F(z + \gamma_{n_k}) - f_{n_k}(z + \gamma_{n_k})| + |f_{n_k}(z + \gamma_{n_k}) - q_{n_k}(z + \gamma_{n_k})| + |q_{n_k}(z + \gamma_{n_k}) - f(z)| < 2^{-n_k} + 2^{-n_k} + |q_{n_k}(z + \gamma_{n_k}) - f(z)| \to 0
\]

uniformly as \( k \to \infty \).

\( \square \)
Our next goal is to extend Runge’s theorem to holomorphic functions defined on open sets. This requires a number of topological considerations.

**Lemma 2.7.**
Let $\Omega \subseteq C$ be an open set and $K \subseteq \Omega$ compact. If $U$ is a connected component of $\Omega \setminus K$, which is relatively compact in $\Omega$, then $\partial U \subseteq K$.

**Proof.** Let $z_0 \in \partial U \setminus K$. Choose an open disk $K_r(z_0) \subseteq \Omega \setminus K$. Then $K_r(z_0) \cap U \neq \emptyset$, so $K_r(z_0) \cup U$ is a connected subset of $\Omega \setminus K$. Since $U$ is a component of $\Omega \setminus K$, i.e., a maximal connected open subset of $\Omega \setminus K$ it follows that $K_r(z_0) \cup U \subseteq U$. This contradicts the fact that $K_r(z_0)$ is a neighborhood of the boundary point $z_0 \in U$ and therefore contains also points outside $U$. \qed

**Definition.**
Let $\Omega \subseteq C$ be an open set and $K \subseteq \Omega$ compact. Then we let

$$\hat{K} := K_{\Omega}$$

be the union of $K$ with all (connected) components of $\Omega \setminus K$ which are compactly contained in $\Omega$.

**Lemma 2.8.**
Let $\Omega \subseteq C$ be an open set and $K \subseteq \Omega$ compact. Then

(a) $\hat{K}$ is compact and $\Omega \setminus \hat{K}$ has no relatively compact components in $\Omega$, i.e., $\hat{K} = \hat{K}$.

(b) Each bounded component of $C \setminus \hat{K}$ contains a bounded component of $C \setminus \Omega$.

**Proof.** (a) $\Omega \setminus \hat{K}$ is the union of all components of $\Omega \setminus K$, which are not relatively compact in $\Omega$, i.e., the union of open sets and therefore open. Hence $\hat{K}$ is closed. Let $\hat{K} \subseteq D := \{z \in C : |z| \leq R\}$ and let $U$ be a component of $\Omega \setminus K$, which is relatively compact in $\Omega$. Then $\partial U \subseteq K$ by Lemma 2.7 and hence $U \subseteq D$, so $\hat{K}$ is bounded and hence compact. Let $U \subseteq \Omega$ be a relatively compact component of $\Omega \setminus K$. Then $U \subseteq \Omega \setminus \hat{K} \subseteq \Omega \setminus K$, so $U$ is contained in a component $U'$ of $\Omega \setminus K$. If $U'$ is relatively compact in $\Omega$, then $U' \subseteq \hat{K}$ by definition of $\hat{K}$. Hence $U \subseteq U' \subseteq \hat{K}$, which is not possible. If $U'$ is not relatively compact in $\Omega$, then $U' \subseteq \Omega \setminus K$. Since $U \subseteq U'$ and $U$ is a component of $\Omega \setminus K$, that is, a maximal connected subset of $\Omega \setminus K$, the connected set $U'$ is also contained in $U$. Hence $U = U'$, which is also not possible, because $U$ is relatively compact in $\Omega$, but $U'$ is not.

(b) Let $Z$ be a bounded component of $C \setminus \hat{K}$. Then by Lemma 2.7 for $\Omega = C$, we have $\partial Z \subseteq K \subseteq \Omega$. Assume that $Z \subseteq \Omega$, so $Z \subseteq \Omega \setminus \hat{K}$. Since $Z$ is connected, there is a component $Z'$ of $\Omega \setminus \hat{K}$ such that $Z \subset Z'$. Now $Z' \subseteq C \setminus \hat{K}$ and $Z'$ is connected, so $Z = Z'$. Hence $Z$ is a relatively compact component of $\Omega \setminus \hat{K}$, which is not possible by (a). This shows that $Z$ intersects at least one component $S$ of $C \setminus \Omega$. In view of $C \setminus \Omega \subseteq C \setminus \hat{K}$, we see that $S$ is a connected subset of $C \setminus \hat{K}$ and therefore contained in the component $Z$. This means that $S$ is bounded. \qed

**Warning.** If $\Omega \subseteq C$ is open, then the closed set $C \setminus \Omega$ can have uncountably many nonisolated components, e.g., if $\Omega$ is the complement of a Cantor set. Hence the complement of an open set is in general much more complicated than the complement of a compact set.
Theorem 2.9 (Runge's theorem for open sets).

Let \( \Omega \subseteq \mathbb{C} \) be an open set and \( f \in \mathcal{H}(\Omega) \). Then there are rational functions with poles only in the bounded components of \( \mathbb{C} \setminus \Omega \) which converge locally uniformly on \( \Omega \) to \( f \).

Proof. Let \( K_1, K_2, \ldots \subseteq \Omega \) be compact sets such that \( K_n \subseteq \text{int}(K_{n+1}) \) and each compact set \( K \subseteq \Omega \) is contained in some \( K_n \). For instance, we can take

\[
K_n := \{ z \in \Omega : |z| \leq n \text{ and } \text{dist}(z, \partial \Omega) \geq 1/n \}.
\]

By Lemma 2.8 (b) there is for each \( n \in \mathbb{N} \) a sequence \( A_n \) of points in \( \mathbb{C} \setminus \Omega \) which contains points in each bounded component of \( \mathbb{C} \setminus K_n \). By Theorem 2.3, there is a rational function \( r_n \) with poles only in \( A_n \subseteq \mathbb{C} \setminus \Omega \) such that \( |f - r_n| < 1/n \) on \( K_n \). If \( K \) is any compact subset of \( \Omega \), we can choose \( N \in \mathbb{N} \) such that \( K \subseteq K_N \). Then \( K \subseteq K_n \subseteq K_n \) for all \( n \geq N \), so \( |f - r_n| < 1/n \) on \( K \), i.e., \( r_n \to f \) uniformly on \( K \). \( \square \)

Corollary 2.10.

Let \( \Omega \subseteq \mathbb{C} \) be a domain such that \( \mathbb{C} \setminus \Omega \) has no bounded component. Then the polynomials are dense in \( \mathcal{H}(\Omega) \).

Appendix: The theorem of Hahn–Banach

Theorem B (Hahn–Banach).

Let \( V \) be a normed (complex) vector space, \( U \) as subspace of \( V \), and \( l : U \to \mathbb{C} \) a continuous linear functional. Then there exists a continuous linear functional \( L : V \to \mathbb{C} \) such that \( L = l \) on \( U \).

For the proof of Theorem B we refer to any book on functional analysis, e.g. P. Lax, Functional Analysis, Wiley 2002. This book also contains many other applications to complex analysis.

Theorem C.

Let \( V \) be a normed (complex) vector space and \( U \) a subspace of \( V \). Then the following two conditions are equivalent.

(a) \( U \) is dense in \( V \).

(b) Every continuous linear functional \( l : V \to \mathbb{C} \) which vanishes on \( U \) also vanishes on \( V \).

Proof. "(a) \( \Rightarrow \) (b)". Let \( p \in V \) and \( l : V \to \mathbb{C} \) a continuous linear functional such that \( l(q) = 0 \) for every \( q \in U \). Since \( U \) is dense in \( V \), there exists a sequence \( (q_n) \) in \( U \) such that \( q_n \to p \). Therefore, \( l(p) = l(\lim_{n \to \infty} q_n) = \lim_{n \to \infty} l(q_n) = 0 \) by continuity of \( l \).

"(b) \( \Rightarrow \) (a)". If \( U \) is not dense in \( V \), there exists a point \( p \in V \setminus U \). Define \( l : U + \text{span}\{p\} \to \mathbb{C} \) by \( l(u + \alpha p) := \alpha \) for all \( u \in U \) and all \( \alpha \in \mathbb{C} \). Then \( l \) is linear and continuous. By Theorem B there is a continuous linear functional \( L : V \to \mathbb{C} \) such that \( L(u + \alpha p) = \alpha \) for all \( u \in U \) and all \( \alpha \in \mathbb{C} \), so \( L \) vanishes on \( U \), but not on \( V \). \( \square \)

Problems

1. Let \( \Omega \subseteq \mathbb{C} \) be an open set and \( K \subseteq \Omega \) compact.
(a) $z_0 \in \Omega \setminus K$ and $L := K \cup \{z_0\}$. Suppose that $\Omega \setminus K$ has no component, which is relatively compact in $\Omega$. Prove that $\Omega \setminus L$ has no component, which is relatively compact in $\Omega$.

(b) (A function-theoretic characterization of $\tilde{K}$)

Show that

$$\tilde{K}_\Omega = \left\{ z \in \Omega : \frac{|f(z)|}{\max_{w \in K} |f(w)|} \leq f(z) \text{ for all } f \in \mathcal{H}(\Omega) \right\}.$$

2. Let $\Omega \subseteq \mathbb{C}$ be an open set and $K \subseteq \Omega$ compact. Prove that the following statements are equivalent.

(i) $\tilde{K}_\Omega = K$.

(ii) Every $f \in \mathcal{H}(K)$ can be uniformly approximated on $K$ by rational functions with poles outside $\Omega$.

These two exercises combined give the following result.

**Theorem D.**

Let $\Omega$ be an open set in $\mathbb{C}$ and $K$ a compact subset of $\Omega$. Then the following statements are pairwise equivalent.

(a) $\mathcal{H}(\Omega)$ is dense in $\mathcal{H}(K)$. This means that for every $f \in \mathcal{H}(K)$ there is a sequence $(f_n) \subset \mathcal{H}(\Omega)$ such that $f_n \rightarrow f$ uniformly on $K$.

(b) No component of $\Omega \setminus K$ is relatively compact in $\Omega$.

(c) For any $a \in \Omega \setminus K$ there exists $f \in \mathcal{H}(\Omega)$ such that $|f(a)| > \|f\|_K := \max_{z \in K} |f(z)|$.

4. Let $\Omega \subseteq \mathbb{C}$ be a domain. Show that there are compact sets $K_n = \tilde{K}_n$ of $\Omega$ with $K_n \subset \text{int}(K_{n+1})$ for $n = 1, 2, \ldots$ such that any compact set $K \subseteq \Omega$ is contained in some $K_n$.

5. Let $\Omega \subseteq \mathbb{C}$ be an open set, $K \subseteq \Omega$ a compact set, and $V$ a component of $\Omega \setminus K$, which is not relatively compact in $\Omega$. Show that for each $a \in V$ there is a curve $\gamma$ from $a$ to some point $b \in \partial \Omega \cup \{\infty\}$ such that $\gamma \setminus \{b\} \subseteq V$. ("One can join $a$ with $\partial \Omega$ in $V"$).

6. Let $\Omega$ be a domain in $\mathbb{C}$ and $K$ a compact subset of $\Omega$ such that no component of $\Omega \setminus K$ is relatively compact in $\Omega$. Prove that $\Omega \setminus K$ has only finitely many components.

7. For a set $M \subseteq \mathbb{C}$ denote by $\mathcal{H}_{\neq 0}(M)$ the set of all $f \in \mathcal{H}(M)$ such that $f(z) \neq 0$ for every $z \in M$. Let $\Omega$ be an open set in $\mathbb{C}$ and $K$ a compact subset of $\Omega$ such that no component of $\Omega \setminus K$ is relatively compact in $\Omega$. Prove that $\mathcal{H}_{\neq 0}(\Omega)$ is dense in $\mathcal{H}_{\neq 0}(K)$.

8. Prove that there is a function $F \in \mathcal{H}(\mathbb{D})$ such that for any $f \in \mathcal{H}(\mathbb{D})$ there is a sequence $(\tau_n) \subset \mathbb{D}$ such that $F \left( \frac{z + \tau_n}{1 + \tau_n z} \right) \rightarrow f(z)$ locally uniformly in $\mathbb{D}$. (See W. Seidel & J.L. Walsh, On approximation by euclidean and non-euclidean translations of an analytic function, *Bull. Am. Math. Soc.* 47, 916-920 (1941)).
Chapter 3

\[ \partial \]

In this chapter we consider the inhomogeneous Cauchy–Riemann equation

\[ \partial f = \phi, \]

which we also call the \( \partial \)-equation. In turns out that even one is only interested in holomorphic functions, i.e., in the solutions of the homogeneous Cauchy–Riemann equation

\[ \partial f = 0, \]

it is very useful to study the corresponding inhomogeneous equation.

**Theorem 3.1** (\( \partial \)-equation for functions with compact support).

Let \( \phi \in C^1_c(C) \). Then the function

\[ f(z) = -\frac{1}{\pi} \oint C \frac{\phi(w)}{w-z} \, dw \]

belongs to \( C^1(C) \) and provides a solution to the \( \partial \)-equation

\[ \partial f = \phi. \]

If \( \phi \in C^k_c(C) \) for some \( k \geq 1 \), then \( f \in C^k(C) \).

**Proof.** Exercise for the reader. Hint: Theorem 1.4. \( \square \)

**Theorem 3.2.**

Let \( \Omega \subset C \) be a domain and \( \phi \in C^1(\Omega) \). Then there is a function \( f \in C^1(\Omega) \) such that

\[ \partial f = \phi. \]

If \( \phi \in C^k(\Omega) \), then \( f \in C^k(\Omega) \).

**Proof.** Let \( K_n \) be compact sets with \( K_n = K_{n+1} \) and \( K_n \subset \text{int}(K_{n+1}) \) such that any compact set \( K \subset \Omega \) is contained in some \( K_n \), see Exercise 2.4. For each \( n = 1, 2, \ldots \) choose a function \( \phi_n \in C^\infty_c(\text{int}(K_{n+1})) \) such that \( \phi_n \equiv 1 \) in a neighborhood of \( K_n \). By Theorem 3.1, there are \( f_n \in C^k(\Omega) \) such that \( \partial f_n = \phi_n \phi \in C^k_c(C) \). Note that \( \partial(f_{n+1} - f_n) = 0 \) in a neighborhood of \( K_n \), so \( f_{n+1} - f_n \in \mathcal{H}(K_n) \). Hence, by Theorem 2.3 and Lemma 2.8 (b), there are rational functions \( r_n \) with poles outside \( \Omega \) such that \( |f_{n+1} - f_n - r_n| < 2^{-n} \) on \( K_n \).

Let

\[ f := f_1 + \sum_{n=1}^{\infty} (f_{n+1} - f_n - r_n). \]
This series converges locally uniformly in \( \Omega \). Now fix \( j \in \mathbb{N} \). Then \( f_{n+1} - f_n - r_n \) is holomorphic in \( \text{int}(K_j) \) for each \( n \geq j \) and

\[
f - f_j = \sum_{n \geq j} (f_{n+1} - f_n - r_n) - r_1 - \ldots - r_{j-1}
\]
is uniformly convergent on \( K_j \). Hence \( f - f_j \in \mathcal{H}(\text{int}(K_j)) \), so \( f \in C^k(\text{int}(K_j)) \) and \( \overline{\partial} f = \overline{\partial} f_j = \phi_j \phi = \phi \) on \( \text{int}(K_j) \). Since \( j \geq 1 \) is arbitrary, the result follows. \( \square \)

Theorem 3.2 is interesting in its own right, but it is also an effective tool for constructing holomorphic functions with specific properties. We start with the following result.

**Theorem 3.3.**

Let \( \Omega \subseteq \mathbb{C} \) be a domain and \( f_1, f_2 \in \mathcal{H}(\Omega) \) without common zeros. Then there are \( g_1, g_2 \in \mathcal{H}(\Omega) \) such that \( f_1 g_1 + f_2 g_2 \equiv 1 \) in \( \Omega \).

This result is called *Wedderburn's lemma*. In a sense, it is the analogue of Bézout's lemma for the ring \( \mathcal{H}(\Omega) \).

**Proof.** It is obvious that

\[
\Phi_j(z) := \frac{f_j(z)}{|f_1(z)|^2 + |f_2(z)|^2}, \quad j = 1, 2,
\]

are two \( C^\infty \)-functions on \( \Omega \) such that \( f_1 \Phi_1 + f_2 \Phi_2 \equiv 1 \). Taking the \( \overline{\partial} \)-derivative, we get \( \overline{\partial} \Phi_1 \cdot f_1 + \overline{\partial} \Phi_2 \cdot f_2 \equiv 0 \) in \( \Omega \). We wish to find \( W : \Omega \rightarrow C^{2\times 2} \) such that

\[
\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + W(z) \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]
defines two holomorphic functions \( g_1, g_2 : \Omega \rightarrow \mathbb{C} \) such that \( f_1 g_1 + f_2 g_2 \equiv 1 \). This requirement is equivalent to

\[
\overline{\partial} W \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = - \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \quad \text{and} \quad (f_1, f_2) \cdot W(z) \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv 0.
\]

It therefore suffices to find an antisymmetric \( W \), that is, \( W^T = -W \), such that

\[
\overline{\partial} W \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = - \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \quad \text{(*)}
\]

Now the condition \( \overline{\partial} \Phi_1 \cdot f_1 + \overline{\partial} \Phi_2 \cdot f_2 \equiv 0 \) leads to

\[
\begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \cdot (f_1, f_2)^T \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

wheras

\[
\begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \cdot (f_1, f_2)^T \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv (|f_1|^2 + |f_2|^2) \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix}.
\]

Hence subtracting (II) from (I) and dividing the resulting expression by \( (|f_1|^2 + |f_2|^2) \) implies

\[
\frac{1}{|f_1|^2 + |f_2|^2} \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \cdot (f_1, f_2)^T - \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix} \cdot (f_1, f_2)^T \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = - \begin{pmatrix} \overline{\partial} \Phi_1 \\ \overline{\partial} \Phi_2 \end{pmatrix}.
\]

Now observe that \( F \) is antisymmetric. Hence, by Theorem 3.2, there is an antisymmetric \( W \) such that \( \overline{\partial} W = F \), so (\*) holds. The proof is complete. \( \square \)
We now apply the $\tilde{\mathcal{F}}$-technique to prove Mittag–Leffler’s theorem for arbitrary domains.

**Remark 3.4 (Invariant form of the Mittag–Leffler problem).**
Let $E$ be a discrete subset of a domain $\Omega \subseteq \mathbb{C}$ and suppose that for each $a \in E$, there is a function $p_a \in \mathcal{H}(\mathbb{C}\setminus \{a\})$. We wish to find a function $f \in \mathcal{H}(\Omega \setminus E)$ such that for each $a \in E$ the function $f - p_a$ is holomorphic in a neighborhood of $a$.

Choose an open cover $(\Omega_j)_{j \in J}$ of $\Omega$ such that each $\Omega_j$ contains at most one point $a \in E$. Fix $j \in J$ and let $z \in \Omega_j$. Then

$$f_j(z) := \begin{cases} 
    p_a(z) & \text{if } a \in \Omega_j \\
    0 & \text{if } E \cap \Omega_j = \emptyset 
\end{cases}$$

is holomorphic in $\Omega_j$ except for a possible isolated singularity at $a \in \Omega_j$. Now observe that $f_j - f_k \in \mathcal{H}(\Omega_j \cap \Omega_k)$: if $a \in E \cap \Omega_j \cap \Omega_k$, then $f_j - f_k = 0$; if $E \cap \Omega_j \cap \Omega_k = \emptyset$, then $f_j, f_k \in \mathcal{H}(\Omega_j \cap \Omega_k)$. We need to find a function $f \in \mathcal{H}(\Omega \setminus E)$ such that $f - f_j$ is holomorphic on $\Omega_j$ for each $j \in J$.

**Theorem 3.5 (Mittag–Leffler’s Theorem; abstract).**
Let $\Omega$ be a domain, $(\Omega_j)_{j \in J}$ an open cover of $\Omega$, for each $j \in J$ let $f_j$ be holomorphic on $\Omega_j$ except for isolated singularities. Suppose that $f_j - f_k$ is holomorphic in $\Omega_j \cap \Omega_k$ for all $k, j \in J$. Then there is a function $f$ holomorphic on the entire set $\Omega$ except for isolated singularities such that $f - f_j$ is holomorphic on $\Omega_j$ for each $j$.

In the proof of Theorem 3.5 we shall need the following auxiliary result. Recall that for a function $\phi : \Omega \to \mathbb{C}$ we denote by supp$(\phi)$ the closure in $\Omega(\text{!})$ of the set $\{z \in \Omega : \phi(z) \neq 0\}$.

**Lemma 3.6.**
Let $\Omega \subseteq \mathbb{C}$ be an open set and $(\Omega_j)_{j \in J}$ an open cover of $\Omega$. Then there exists a locally finite partition of unity subordinate to $(\Omega_j)$, that is, a family $(\phi_j)$ of real-valued functions $\phi_j \in C^\infty(\Omega)$ with $\phi_j \geq 0$ on $\Omega$ and supp$(\phi_j) \subseteq \Omega_j$ such that for any compact set $K \subseteq \Omega$ the set $\{j \in J : K \cap \text{supp}(\phi_j) \neq \emptyset\}$ is finite and

$$\sum_{j \in J} \phi_j(z) \equiv 1 \quad \text{for all } z \in \Omega.$$

**Proof.** There exists an open cover $(\Omega_l)_{l \in L}$ of $\Omega$ such that $\overline{\Omega}_l \subseteq \Omega$ for each $l \in L$, for any compact set $K \subseteq \Omega$ the set $\{l \in L : K \cap \overline{\Omega}_l \neq \emptyset\}$ is finite, and $\overline{\Omega}_l \subseteq \Omega_{\tau(l)}$ for some $\tau(l) \in J$ for each $l \in L$. Choose compact subsets $K_l \subseteq \overline{\Omega}_l$ such that $\bigcup_{l \in L} K_l = \Omega$. For each $l \in L$ choose $\Psi_l \in C^\infty(\Omega_l)$ with $\Psi_l > 0$ on $K_l$ (see Theorem A). Then

$$\Psi := \sum_{l \in L} \Psi_l \in C^\infty(\Omega),$$

since it is a finite sum on any compact subset of $\Omega$. Clearly $\Psi > 0$ on $\Omega$ and $\chi_l := \Psi_l / \Psi$ is a locally finite partition of unity subordinate to $(\Omega_l)$. For $j \in J$ let $L_j = \tau^{-1}(\{j\})$ and

$$\phi_j := \sum_{l \in L_j} \chi_l.$$

Then $(\phi_j)$ is a locally finite partition of unity subordinate to $(\Omega_j)$.
Proof of Theorem 3.5. Let \((\phi_j)\) be a locally finite partition of unity subordinate to \((\Omega_j)\). For each \(k\) we have \((f_k - f_j)\phi_j \in C^\infty(\Omega_j \cap \Omega_k)\) and \((f_k - f_j)\phi_j = 0\) on \(\Omega_k \setminus (\Omega_j \cap \Omega_k)\), so we think of \((f_k - f_j)\phi_j\) as a function in \(C^\infty(\Omega_k)\). Since \((\phi_j)\) is locally finite, we have

\[
g_k := \sum_j (f_k - f_j)\phi_j \in C^\infty(\Omega_k).
\]

Note that

\[
g_k - g_l = \sum_j (f_k - f_l)\phi_j = f_k - f_l \text{ in } \Omega_k \cap \Omega_l,
\]

so \(\partial g_k = \partial g_l\) in \(\Omega_k \cap \Omega_l\) for any \(k, l \in J\). Hence \(\psi := \partial g_k\) is a well-defined \(C^\infty\)-function on \(\Omega\). By Theorem 3.2, there is \(u \in C^\infty(\Omega)\) such that \(\partial u = \psi\) in \(\Omega\). Therefore, \(f := f_k - g_k + u\) is a well-defined function on \(\Omega\) except for isolated singularities with the required properties. 

Corollary 3.7 (Mittag–Leffler’s Theorem; concrete).
Let \(\Omega \subseteq \mathbb{C}\) be a domain, \(E\) a discrete subset of \(\Omega\) and for each \(a \in E\) let \(p_a \in \mathcal{H}(\mathbb{C}\setminus\{a\})\) be given. Then there exists \(f \in \mathcal{H}(\Omega\setminus E)\) such that \(f - p_a\) is holomorphic at \(a\) for every \(a \in E\). In particular, there exists \(f \in \mathcal{H}(\Omega\setminus E)\) whose principal parts at the points of \(E\) have been prescribed.

Theorem 3.8 (Weierstraß’s Theorem).
Let \(\Omega\) be a domain, \(E\) a discrete subset of \(\Omega\) and for each \(a \in E\) let \(n_a \in \mathbb{N}\). Then there exists a function \(f \in \mathcal{H}(\Omega)\) such that for any \(a \in E\) the function \(f(z)\) has a zero of order \(n_a\) at \(a\) and such that \(f\) has no zeros in \(\Omega\setminus E\).

Lemma.
Let \(\Omega \subseteq \mathbb{C}\) be a domain, \(\hat{K} = K \subseteq \Omega\) a compact subset, and \(a \in \Omega\setminus K\). Then there exists a function \(\phi_a \in C^\infty(\Omega)\) such that \(\phi_a\) does not vanish on \(\Omega\setminus\{a\}\), \(\phi_a \equiv 1\) on \(K\), and \(\phi_a\) is holomorphic in a neighborhood of \(a\) with a zero of order 1 at \(a\).

Proof. If \(\partial \Omega \neq 0\), then by Exercise 2.5 there is a curve \(\gamma\) from \(a\) to a point \(b \in \partial \Omega\) such that \(\gamma \setminus \{b\} \subseteq \Omega\setminus K\). Otherwise there is a curve \(\gamma : [0, 1] \to \Omega \setminus K\) with \(\gamma(0) = a\) and \(|\gamma(t)| \to \infty\) as \(t \to 1\). The function

\[
g(z) := \begin{cases} 
\log(z - a) & \text{if } \partial \Omega = 0 \\
\log \left(\frac{z - a}{z - b}\right) & \text{if } \partial \Omega \neq 0
\end{cases}
\]

is then well-defined on \(\mathbb{C}\setminus \gamma\). Now choose an open set \(V\) which is compactly contained in \(\Omega\setminus \gamma\) and such that \(K_n \subseteq V\), and choose a \(C^\infty\)-function \(\phi : \mathbb{C} \to [0, 1]\) such that \(\phi \equiv 0\) on \(K_n\) and \(\phi \equiv 1\) on \(\mathbb{C}\setminus V\) and let

\[
\phi_a(z) := \begin{cases} 
\frac{z - a}{z - b} & \text{if } \partial \Omega \neq 0 \text{ for } z \in \Omega \setminus V \\
z - a & \text{if } \Omega = \mathbb{C}
\end{cases}
\]

and \(\phi_a(z) := \exp(\phi(z)g(z))\) for \(z \in V\).

Then \(\phi_a\) has the required properties. 

Proof of Theorem 3.8. (i) Let \(K_1, K_2, \ldots\) be compact sets such that \(K_n = K_n \subseteq \text{int}(K_{n+1})\) and such that every compact set \(K \subseteq \Omega\) is contained in some \(K_n\). Fix a point \(a \in E\). There exists a unique \(n = n_a \in \mathbb{N}\) with \(a \in K_{n+1} \setminus K_n\). Choose \(\phi_a\) as in the Lemma (with \(K = K_n\)). Then

\[
\phi(z) := \prod_{a \in E} \phi_a(z)^{n_a}, \quad z \in \Omega,
\]
is for each compact set $K \subseteq \Omega$ a finite product of $C^\infty$-functions with zeroes exactly in the set $E \cap K$. Hence $\phi \in C^\infty(\Omega)$ with $\phi = 0$ on $E$ and $\phi \neq 0$ in $\Omega \setminus E$. For fixed $a \in E$ choose $r > 0$ such that $K_r(a)$ is compactly contained in $\Omega$, $\phi_a$ is holomorphic in $K_r(a)$, and $K_r(a) \cap E = \{a\}$. Then $\phi = \phi_a^{n_a} \cdot \omega$ in $K_r(a)$, where $\omega \in C^\infty(K_r(a))$ is zero free. Hence $\bar{\partial} \phi = \phi_a^{n_a} \bar{\partial} \omega = \phi \cdot \bar{\partial} \omega / \omega$. This implies that $\bar{\partial} \phi / \phi \in C^\infty(K_r(a))$ and hence $\bar{\partial} \phi / \phi \in C^\infty(\Omega)$.

(ii) By Theorem 3.2, the $\bar{\partial}$-equation

$$\bar{\partial} \psi = -\frac{\bar{\partial} \phi}{\phi}$$

has a solution $\psi \in C^\infty(\Omega)$. Then $f := \phi e^\psi \in \mathcal{H}(\Omega)$ is never zero in $\Omega \setminus E$ and has a zero of order $n_a$ at each $a \in E$. \hfill \Box

**Theorem 3.9.**

Let $\Omega$ be a domain and $f$ be a meromorphic function on $\Omega$. Then there exist $g, h \in \mathcal{H}(\Omega)$ such that $f = g / h$.

**Proof.** Assume that $f$ is not constant. Then there exists $h \in \mathcal{H}(\Omega)$ such that $h$ has a zero at a point $a \in \Omega$ if and only if $f$ has a pole at $a$ (counting multiplicities). Then $g := f h \in \mathcal{H}(\Omega)$. \hfill \Box

**Problems**

1. Let $U \subseteq \mathbb{C}$ be an open set and $\phi \in L^1(\Omega) \cap C^1(\Omega)$. Prove that

$$f(z) = -\frac{1}{\pi} \iint_{\Omega} \frac{\phi(w)}{w - z} dudv$$

provides a solution to the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = \phi.$$ 

2. Let $\phi \in C^\infty_c(\mathbb{C})$. Prove that there exists $u \in C^\infty_c(\mathbb{C})$ such that $\bar{\partial} u = \phi$ if and only if

$$\iint_{\mathbb{C}} \phi(z) f(z) \, dxdy = 0 \text{ for all } f \in \mathcal{H}(\mathbb{C}).$$

3. Use Corollary 3.7 to prove Theorem 3.3.

4. (An interpolation theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain and $(z_j)_{j \in \mathbb{N}}$ a sequence of pairwise distinct points in $\Omega$ without limit point in $\Omega$. For each $j \in \mathbb{N}$ let $n_j \in \mathbb{N}$, let $a_{j, k} \in \mathbb{C}$, $k = 0, 1, \ldots, n_j - 1$, and let

$$P_j(z) := \sum_{k=0}^{n_j-1} a_{j,k} (z - z_j)^k.$$

The goal is to find $g \in \mathcal{H}(\Omega)$ such that at each $z_j$ the function $g$ has an expansion of the form $g(z) = P_j(z) +$ higher order terms. In fact, it is easy to find $w \in C^\infty(\Omega)$ such that $w(z) = P_j(z)$ in a neighborhood of $z_j$ for $j = 1, 2, \ldots$. Just choose positive numbers
\( \epsilon_j \) such that the open disks \( K_j := K_{\epsilon_j}(z_j) \) are pairwise disjoint, choose \( \phi_j \in C^\infty_c(K_j) \) with \( 0 \leq \phi_j \leq 1 \) and \( \phi_j \equiv 1 \) on \( K_{\epsilon_j}(z_j) \), and set
\[
  w(z) := \sum_{j=1}^{\infty} P_j(z) \phi_j(z).
\]

We need to modify \( w \) such that it becomes holomorphic in \( \Omega \), but without changing the first term \( P_j(z) \) of the Taylor series expansion of \( w \) at the points \( z_j \).

Show that there are functions \( f \in \mathcal{H}(\Omega) \) and \( \Psi \in C^\infty(\Omega) \) such that \( g(z) := w(z) - f(z)\Psi(z) \) belongs to \( \mathcal{H}(\Omega) \) and has the required properties.
Chapter 4

The Poisson integral

Prélude

Let \( f : U \to \mathbb{C} \) be differentiable in an open set \( U \subseteq \mathbb{C} \) with \( \overline{D} \subseteq U \) such that \( \overline{f} : U \to \mathbb{C} \) is continuous. Then the inhomogeneous Cauchy Integral formula (Theorem 1.10) shows

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw - \frac{1}{\pi} \int_D \frac{\overline{f}(w)}{w - z} \, dudv, \quad z \in D.
\]

Note that with \( w = e^{it} \in \partial D \) the integral along \( \partial D \) can be written as

\[
\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} f(e^{it}) \, dt
\]

A serious defect of this integral is the fact that its kernel

\[
\frac{e^{it}}{e^{it} - z} = \frac{w}{w - z}, \quad w = e^{it} \in \partial D, z \in D,
\]

is not positive. This deficiency can easily be remedied by multiplying this kernel by its complex conjugate

\[
\frac{\overline{w}}{\overline{w} - \overline{z}} = \frac{1}{1 - \overline{z}w}.
\]

It therefore seems to be a good idea to apply the inhomogeneous Cauchy Integral formula to the function

\[
\mathbb{D} \ni w \mapsto \frac{f(w)}{1 - \overline{z}w} \quad \text{for fixed } z \in \mathbb{D}.
\]

Doing this, we get

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{1 - |z|^2}{|1 - \overline{w}z|^2} f(w) \, dw - \frac{1}{\pi} \int_D \frac{1 - |z|^2}{(w - z)(1 - \overline{z}w)} \overline{f}(w) \, dudv. \quad (*)
\]

Now for the area integral. Its kernel is also not real. However, it can be written as

\[
\frac{1 - |z|^2}{(w - z)(1 - \overline{z}w)} = \frac{1}{w - z} + \frac{\overline{z}}{1 - \overline{z}w},
\]

and this is the \( \partial \)-derivative of a real function:

\[
\frac{1}{w - z} + \frac{\overline{z}}{1 - \overline{z}w} = \frac{\partial}{\partial w} \log \left( |w - z|^2 \right) - \frac{\partial}{\partial w} \log \left( |1 - \overline{z}w|^2 \right) = 2 \frac{\partial}{\partial w} \left( \log \left| \frac{z - w}{1 - \overline{w}z} \right| \right).
\]
Therefore, we can write (\(*\)) as
\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{1-|z|^2}{|1-wz|^2} f(w) \frac{dw}{w} - \frac{2}{\pi} \int_{D} \frac{\partial}{\partial w} \left( \log \frac{z-w}{1-wz} \right) \overline{\partial f(w)} \, du dv .
\]
We cannot resist and integrate by parts. This gives us (Check!)
\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{1-|z|^2}{|1-wz|^2} f(w) \frac{dw}{w} + \frac{2}{\pi} \int_{D} \log \left| \frac{z-w}{1-wz} \right| \partial \overline{\partial f(w)} \, du dv .
\]
Voilà, we have a real–valued kernel! If we denote by \(\Delta\) the Laplacian defined by
\[
\Delta f := 4 \partial \overline{\partial} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} ,
\]
then we have (almost) proved:

**Theorem 4.1** (Riesz decomposition of \(C^2\)–functions).

Let \(U \subseteq \mathbb{C}\) be an open set with \(\overline{D} \subseteq U\) and \(f \in C^2(U)\). Then
\[
f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it}) \, dt + \frac{1}{2\pi} \int_{\partial D} \log \left| \frac{z-w}{1-wz} \right| \Delta f(w) \, du dv .
\]

**Definition.**

The kernel function
\[
P(z, w) = P(z, e^{it}) = \frac{1-|z|^2}{|1-wz|^2} = \text{Re} \left( \frac{e^{it}+z}{e^{it}-z} \right)
\]
is called the Poisson kernel (of \(D\)). For \(f \in L^1(\partial D)\) the integral
\[
P_f(z) := \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it}) \, dt
\]
is called the Poisson integral of \(f\).

**Remark 4.2** (Harmonic functions).

A real– or complex–valued function \(f \in C^2(U)\) on an open set \(U \subseteq \mathbb{C}\) is said to be harmonic if \(\Delta f = 0\) in \(U\). Note that if \(f \in H(U)\), then \(f\) and \(\overline{f}\) are harmonic and therefore also \(\text{Re} \, f\) and \(\text{Im} \, f\) are harmonic in \(U\). In particular, for fixed \(t \in \mathbb{R}\) the Poisson kernel \(P(, e^{it})\) is a harmonic function in \(D\) and therefore \(P_t\) is harmonic in \(\overline{D}\) for every \(f \in L^1(\partial D)\). Theorem 4.1 gives a decomposition of a \(C^2\)–function \(f\) in a harmonic part and a nonharmonic part which depends only on \(\Delta f\).

**Theorem 4.3** (Poisson formula for harmonic functions).

Let \(f \in C(\overline{D})\) be harmonic in \(D\). Then
\[
f(z) = P_f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \text{Re} \left( \frac{e^{it}+z}{e^{it}-z} \right) \, f(e^{it}) \, dt , \quad z \in D .
\]

**Proof.** The result follows by applying Theorem 4.1 to the harmonic functions \(f_r(z) := f(rz)\), \(0 < r < 1\), and using the uniform convergence of \(f_r\) to \(f\) on \(\partial D\) as \(r \to 1\).
Corollary 4.4 (Mean value property).

Let $\Omega \subseteq \mathbb{C}$ be an open set and $u : \Omega \rightarrow \mathbb{C}$ harmonic. If $K_R(a)$ is compactly contained in $\Omega$, then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it}) dt.$$  

Proof. $v(z) := u(a + rz)$ is continuous on $\overline{D}$ and harmonic in $D$. Theorem 4.3 implies

$$u(a) = v(0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it}) dt.$$ 

\[\Box\]

Corollary 4.5.

Let $(f_n)$ be a sequence of harmonic functions in an open set $U \subseteq \mathbb{C}$ such that $f_n \to f$ locally uniformly in $U$. Then $f$ is harmonic in $U$.

Proof. Let $K_r(a) \subseteq U$. By considering $f_n(a + rz)$ and $f(a + rz)$ we may assume that $K_r(a) = D$. The result follows now directly from Theorem 4.3. \[\Box\]

Theorem 4.6 (Harnack’s theorem).

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $(f_n)$ be a monotonically increasing sequence of real-valued harmonic functions in $\Omega$. Then either $f_n \to +\infty$ or $f_n \to f < \infty$ locally uniformly in $\Omega$.

Proof. Let $K_R(a) \subseteq \Omega$. Again we may assume $K_R(a) = D$, and we also may assume $f_1 \geq 0$. Since $P(z, e^{it}) > 0$, $P_1 = 1$ and

$$\frac{1 - |z|}{1 + |z|} \leq P(z, e^{it}) \leq \frac{1 + |z|}{1 - |z|},$$

Theorem 4.3 shows that

$$f_n(0) \frac{1 - |z|}{1 + |z|} \leq f_n(z) \leq f_n(0) \frac{1 + |z|}{1 - |z|}, \quad z \in D.$$

Hence, if $f_n(0) \to +\infty$, then $f_n(z) \to +\infty$ locally uniformly in $D$. If $f_n(0) \to c \neq +\infty$, then $(f_n)$ is locally uniformly Cauchy, so it converges locally uniformly in $D$. Therefore, if we set $f(z) := \lim_{n \to \infty} f_n(z)$ for $z \in \Omega$, then $A := \{z \in \Omega : f(z) = +\infty\}$ is open and closed, so by connectedness $A = \emptyset$ or $A = \Omega$. \[\Box\]

Theorem 4.7 (Boundary behaviour of $P_f$).

Let $f \in L^1(\partial D)$ be continuous at the point $w \in \partial D$, then

$$\lim_{z \to w} P_f(z) = f(w).$$

This convergence is uniform on any closed set of values of $w$ at which $f$ is continuous. In particular, if $f \in C(\partial D)$, then $P_f \in C(\overline{D})$ and $P_f = f$ on $\partial D$.

Proof. Let $w = e^{it_0}$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $|f(e^{it}) - f(w)| < \varepsilon$ for all $|t - t_0| < \delta$. By Theorem 4.3 (applied to the constant function $z \mapsto f(w)$), we have

$$P_f(z) - f(w) \geq \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) \left( f(e^{it}) - f(w) \right) dt,$$
so if we write \( I_1 = \{ t \in [0, 2\pi] : |t - t_0| < \delta \} \) and \( I_2 = [0, 2\pi] \setminus I_1 \), then

\[
|P_f(z) - f(w)| \leq \frac{1}{2\pi} \int_{I_1} \operatorname{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) e^{it} dt + \frac{1}{2\pi} \int_{I_2} \frac{1 - |z|^2}{|e^{it} - z|^2} \left( |f(e^{it})| + |f(w)| \right) dt.
\]

Now, the first integral is \( \leq \varepsilon \). For the second integral let \( M := \min_{t \in I_2} |e^{it} - w| > 0 \) and observe \( |e^{it} - z|^2 \geq (M - |w - z|)^2 \), so this integral is bounded above by

\[
\frac{1 - |z|^2}{(M - |w - z|)^2} \frac{1}{2\pi} \int_{0}^{2\pi} \left( |f(e^{it})| + |f(w)| \right) dt \to 0 \quad (z \to w).
\]

\[\Box\]

**Theorem 4.8** (The Dirichlet problem for a disk).
Let \( \phi : \partial K_r(a) \to \mathbb{C} \) be continuous. Then there exists a unique function \( f \in C(\overline{K_r(a)}) \), which is harmonic in \( K_r(a) \) and satisfies \( f = \phi \) on \( \partial K_r(a) \).

**Proof.** By considering \( \phi(a + rz) \) and \( f(a + rz) \) for \( z \in \mathbb{D} \), we may assume that \( K_r(a) = \mathbb{D} \). Then \( P_f \) has the required properties. If \( f \) and \( g \) are two functions with the required properties, then \( f - g \in C(\mathbb{D}) \) is harmonic in \( \mathbb{D} \) such that \( f - g = 0 \) on \( \partial \mathbb{D} \), so \( f \equiv g \) by Theorem 4.3. \[\Box\]

**Theorem 4.9** (Harmonic functions and the mean value property).
Let \( \Omega \) be an open set in \( \mathbb{C} \) and \( u \in C(\Omega) \). Then \( u \) is harmonic in \( \Omega \) if and only if for any \( a \in \Omega \) there is \( R > 0 \) such that for any \( 0 \leq r < R \) the mean value property holds

\[
u(a) = \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{it}) dt.
\]

**Proof.** We only need to prove the “if-part” and may assume that \( u \) is real-valued. Let \( D \) be a disk which is compactly contained in \( \Omega \). It is enough to prove that \( u \) is harmonic in \( D \). Let \( h \in C(\overline{D}) \) be harmonic in \( D \) with \( h = u \) on \( \partial D \). Then \( u - h \) satisfies the mean value property in \( D \). If \( u - h \) had a positive supremum in \( D \), then it would be attained in some point \( a \in D \) with minimal distance to \( \partial D \) contradicting the mean value property. Hence \( u - h \leq 0 \). For the same reason, \( u - h \geq 0 \) and therefore \( u = h \) in \( D \). \[\Box\]

A (careful) inspection of the proof shows that we have actually established the implication “\((d) \Rightarrow (a)\)” in the following result. The other implications are obvious.

**Theorem 4.10.**
Let \( \Omega \subseteq \mathbb{C} \) be an open set and \( u : \Omega \to \mathbb{R} \) continuous. Then the following conditions are pairwise equivalent.

(a) If \( D \) is an open set which is compactly contained in \( \Omega \), \( h \in C(\overline{D}) \) is harmonic in \( D \), and \( u \leq h \) on \( \partial D \), then \( u \leq h \) in \( D \).

(b) If \( K_r(a) \) is compactly contained in \( \Omega \) and \( h \in C(\overline{K_r(a)}) \) is harmonic in \( K_r(a) \) such that \( u \leq h \) on \( \partial K_r(a) \), then \( u \leq h \) in \( K_r(a) \).

(c) If \( K_r(a) \) is compactly contained in \( \Omega \), then the submean inequality

\[
u(a) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{it}) dt
\]
holds.

(d) For each \( a \in \Omega \) the submean inequality holds for all small \( r > 0 \).

Definition.

Let \( \Omega \subseteq \mathbb{C} \) be an open set. A continuous function \( u : \Omega \to \mathbb{R} \) is called subharmonic, if for each \( a \in \Omega \) the submean inequality holds for all small \( r > 0 \).

Examples 4.11. (a) Every real-valued harmonic function is subharmonic.

(b) If \( u, v \) are subharmonic, then \( \max\{u, v\} \) is subharmonic.

(c) If \( f \) is holomorphic, then \( \log^+ |f| := \max\{|\log |f|, 0\} \) and \( |f|^a \) for every \( a > 0 \) are subharmonic.

Theorem 4.12 (Maximum principle for subharmonic functions).

Let \( \Omega \subseteq \mathbb{C} \) be a domain and let \( u : \Omega \to \mathbb{R} \) be subharmonic.

(a) If \( u \) attains a global maximum on \( \Omega \), then \( u \) is constant.

(b) If \( \limsup_{z \to w} u(z) \leq 0 \) for all \( w \in \partial \Omega \cup \{\infty\} \), then \( u \leq 0 \) on \( \Omega \).

Proof. Let \( M := \sup_{z \in \Omega} u(z) \).

(a) Let \( A := \{z \in \Omega : u(z) = M\} \). Then \( A \) is closed since \( u \) is continuous and \( A \) is open by the local submean inequality. Hence \( A = \emptyset \) or \( A = \Omega \).

(b) Let \( (z_n) \subseteq \Omega \) s.t. \( u(z_n) \to M \) and \( z_n \to z_0 \in \Omega \cup \{\infty\} \). If \( z_0 \in \Omega \), then \( u \equiv M \) by (a). If \( z_0 \in \partial \Omega \cup \{\infty\} \), then \( M \leq 0 \) by assumption. \( \square \)

Problems

1. Provide a detailed proof for Theorem 4.1.

2. Does the Riesz decomposition formula in Theorem 4.1 still hold under the weaker condition \( f \in C(\overline{\mathbb{D}}) \cap C^0(\mathbb{D}) \)? What, if \( \Delta f \geq 0 \) in \( \mathbb{D} \)?

3. Let \( \phi \in L^1(\partial \mathbb{D}) \) be real-valued and nonnegative (a.e.) and \( w_0 \in \partial \mathbb{D} \) such that \( \phi(w) \to +\infty \) as \( \partial \mathbb{D} \ni w \to w_0 \). Show that \( P_f(z) \to \infty \) as \( z \to w_0 \).

4. * (Harnack’s inequality)

   Let \( \Omega \subseteq \mathbb{C} \) be a domain and \( K \subseteq \Omega \) a compact set. Show that there is a constant \( C > 1 \) such that
   \[
   \frac{1}{C} \leq \frac{u(z)}{u(w)} \leq C
   \]
   for all positive harmonic functions \( u \) on \( \Omega \) and all points \( z, w \in K \).

5. * (‘Subharmonicity’ is conformally invariant)

   Let \( \Omega \) and \( D \) be open sets in \( \mathbb{C} \), \( u \) subharmonic in \( D \) and \( f \in \mathcal{H}(\Omega) \) such that \( f(\Omega) \subseteq D \). Prove that \( u \circ f \) is subharmonic.

6. Let \( \Omega \subseteq \mathbb{C} \) be a domain. Prove that \( \Omega \) is simply connected if and only if every harmonic function \( u : \Omega \to \mathbb{R} \) has the form \( u = \Re(f) \) for some \( f \in \mathcal{H}(\Omega) \).
7. *(Removable singularities of harmonic functions)*

Let \( \Omega \subseteq \mathbb{C} \) be an open set, \( z_0 \in \Omega \), and let \( h : \Omega \setminus \{z_0\} \to \mathbb{R} \) be a harmonic function such that

\[
\lim_{z \to z_0} \frac{h(z)}{\log |z - z_0|} = 0.
\]  

(*)

Prove that \( h \) has a harmonic extension to \( \Omega \). Note that if \( h \) is bounded at \( z_0 \), then (*) holds. This gives a "harmonic" version of the well-known result that isolated singularities of bounded holomorphic functions are always removable.

8. Let \( u : D \to \mathbb{R} \) be harmonic and positive. Show that

\[
|\partial u(z)| \leq \frac{u(z)}{1 - |z|^2}, \quad z \in D.
\]

Hint: First prove the case \( z = 0 \) using Harnack's inequality from the proof of Theorem 4.6. The general case \( z \neq 0 \) can be reduced to this special case by precomposing \( u \) with a conformal disk automorphism \( T \in \text{Aut}(D) \) which maps 0 to \( z \). Note that \( u \circ T \) is harmonic in view of Exercise 4.5.

9. Let \( u : \Omega \to \mathbb{R} \) be continuous on the open set \( \Omega \subseteq \mathbb{C} \). Suppose that for each \( a \in \Omega \) there is a sequence of positive numbers \( r_n \), converging to 0 such that

\[
u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + r_n e^{it}) dt
\]

for each \( n \in \mathbb{N} \). Prove that \( u \) is subharmonic in \( \Omega \).

This is a result of Littlewood, see J.E. Littlewood, *On the definition of a subharmonic function, J. London Math. Soc.* 2, 192–196 (1927).

10. *(The area mean value property)*

Let \( \Omega \subseteq \mathbb{C} \) be an open set and \( u : \Omega \to \mathbb{R} \) such that \( u \in L^1(K) \) for each compact set \( K \subseteq \Omega \). Prove that \( u \) is harmonic if and only if

\[
u(a) = \frac{1}{\pi r^2} \iint_{K_r(a)} u(z) dx dy
\]  

(*)

for each open disk \( K_r(a) \) which is compactly contained in \( \Omega \).

Hint: For the "if part" first prove that \( u \) is continuous and then "differentiate" equation (*) to deduce the "ordinary" mean value property, see Theorem 4.9.
Chapter 5

The Dirichlet problem

Definition.
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $\phi : \partial \Omega \to \mathbb{R}$ a continuous function. The Dirichlet problem is to find a harmonic function $h : \Omega \to \mathbb{R}$ such that

$$\lim_{z \to w} h(z) = \phi(w) \quad \text{for all } w \in \partial \Omega.$$ 

We then call $h$ the solution of the Dirichlet problem for $(\Omega, \phi)$.

By Theorem 4.12, if such a solution exists, it is unique. If $\Omega$ is a disk, then a solution always exists (Theorem 4.8).

Example 5.1 (Zaremba 1911).
Let $\Omega = \mathbb{D}\setminus\{0\}$ and $\phi(w) = 0$ for $w \in \partial \mathbb{D}$ and $\phi(0) = 1$. Assume $h$ is the solution of the Dirichlet problem for $(\Omega, \phi)$. Then $h(z) + \varepsilon \log |z|$ is harmonic in $\mathbb{D}\setminus\{0\}$ for each fixed $\varepsilon > 0$. For small $r > 0$, we then have $h(w) + \varepsilon \log |w| \leq 0$ on $|w| = r$, so $h(z) \leq -\varepsilon \log |z|$ on $D_r \setminus\{0\}$ by the maximum principle. Letting $\varepsilon \to 0$, we arrive at a contradiction.

Theorem 5.2 (Poisson modification).
Let $\Omega \subseteq \mathbb{C}$ be a domain and let $u : \Omega \to \mathbb{R}$ be subharmonic. If $D$ is an open disk with $\overline{D} \subseteq \Omega$ and $h \in C(\overline{D})$ is real-valued harmonic in $D$ with $h = u$ on $\partial D$, then the Poisson modification

$$M_Du(z) := \begin{cases} h(z) & \text{if } z \in D \\ u(z) & \text{otherwise} \end{cases}$$

is subharmonic in $\Omega$ with $u \leq M_Du$, see Figure 5.1. In addition, if $u, \hat{u} : \Omega \to \mathbb{R}$ are subharmonic and $u \leq \hat{u}$ in $\Omega$, then $M_Du \leq M_D\hat{u}$ in $\Omega$.

Proof. We have $u \leq h$ in $D$ by Theorem 4.10, so $u \leq M_Du$. If $a \in \partial D$, then

$$M_Du(a) = u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it}) \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} M_Du(a + re^{it}) \, dt$$

for all small $r > 0$. Hence $M_Du$ is subharmonic in $\Omega$. If $\hat{h} \in C(\overline{D})$ is harmonic in $D$ with $\hat{h} = \hat{u}$ on $\partial D$, then $h - \hat{h} \leq 0$ on $\partial D$, so $h \leq \hat{h}$ in $D$ by Theorem 4.12, so $M_Du \leq M_D\hat{u}$. □

Theorem 5.3 (Perron 1923).
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and let $\phi : \partial \Omega \to \mathbb{R}$ be a bounded function. Let $\mathcal{P}_\phi$ be
the family of all subharmonic functions $u : \Omega \to \mathbb{R}$ s.t.

$$\limsup_{z \to w} u(z) \leq \phi(w) \text{ for all } w \in \partial \Omega.$$  

Then

$$H_{\Omega\phi}(z) = \sup \{ u(z) : u \in \mathcal{P}_\phi \}, \quad z \in \Omega,$$

defines a harmonic function on $\Omega$.

Remark 5.4.

If $\phi : \partial \Omega \to \mathbb{R}$ is continuous and $h$ the solution of the Dirichlet problem for $(\Omega, \phi)$, then $h \in \mathcal{P}_\phi$, so $H_{\Omega\phi} \geq h$. On the other hand, if $u \in \mathcal{P}_\phi$, then $u \leq h$ by the maximum principle, so $H_{\Omega\phi} \leq h$. Hence $H_{\Omega\phi} = h$. In other words, if the Dirichlet problem for $(\Omega, \phi)$ has a solution at all, then it must be $H_{\Omega\phi}$.

Proof. First observe that $M_Du \in \mathcal{P}_\phi$ for every $u \in \mathcal{P}_\phi$ and any disk $D$ compactly contained in $\Omega$. Let $h := H_{\Omega\phi}$ and $D := K_r(z_0)$ with $D \subset \Omega$. There is $(s_n) \subset \mathcal{P}_\phi$ s.t. $s_n(z_0) \to h(z_0)$. Hence $s_n(z_0) \wedge h(z_0)$ for $S_n := \max\{s_1, \ldots, s_n\} \in \mathcal{P}_\phi$. Thus $M_DS_n(z_0) \wedge h(z_0)$ and $M_DS_n \in \mathcal{P}_\phi$ converges increasingly on $D$ to a harmonic function $H \leq h$ with $H(z_0) = h(z_0)$.

In order to prove $h = H$ in $D$ we fix $z_1 \in D$ and repeat the previous argument. There is $(v_n) \subset \mathcal{P}_\phi$ s.t. $v_n(z_1) \to h(z_1)$. Hence $w_n := \max\{s_n, v_n\}$ and $W_n := \max\{w_1, \ldots, w_n\}$ belong to $\mathcal{P}_\phi$, so $M_DW_n \in \mathcal{P}_\phi$ converges increasingly on $D$ to a harmonic function $\tilde{H} : D \to \mathbb{R}$ such that $\tilde{H}(z) \leq h(z)$ in $D$ and $\tilde{H}(z_1) = h(z_1)$. Since $\tilde{H}(z_0) = h(z_0) = H(z_0)$, the maximum principle implies $\tilde{H} = H$ on $D$, so $h(z_1) = \tilde{H}(z_1) = H(z_1)$, and $h$ is harmonic in $D$. \qed
Definition. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain. A point $w_0 \in \partial \Omega$ is called a regular point of $\partial \Omega$ if
\[
\lim_{z \to w_0} H_{\Omega} \phi(z) = \phi(w_0)
\]
for every continuous $\phi : \partial \Omega \to \mathbb{R}$. We call $\Omega$ regular, if every point $w_0 \in \partial \Omega$ is regular.

If $w_0 \in \partial \Omega$ is regular, then, taking $\phi(w) := |w - w_0|$ and $h := -H_{\Omega} \phi$, we see that $h : \Omega \to \mathbb{R}$ is harmonic and
\[
\lim_{z \to w_0} h(z) = -\phi(w_0) = 0 \quad \text{and} \quad \limsup_{z \to w} h(z) \leq -|w - w_0| < 0 \quad \text{for} \quad w \in \partial \Omega \setminus \{w_0\},
\]
because $z \mapsto |z - w_0|$ belongs to $\mathcal{P}_\phi$. We are led to the following definition.

Definition. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $w_0 \in \partial \Omega$. A barrier at $w_0$ is a harmonic function $h : \Omega \to \mathbb{R}$ such that
\[
\lim_{z \to w_0} h(z) = 0 \quad \text{and} \quad \limsup_{z \to w} h(z) < 0 \quad \text{for every} \quad w \in \partial \Omega \setminus \{w_0\}.
\]

Example 5.5 (Exterior segment condition).
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $w_0 \in \partial \Omega$. Suppose that there is a closed line segment $L = [w_0, w_1]$ such that $(w_0, w_1] \subseteq \mathbb{C} \setminus \Omega$. Then there is a barrier at $w_0$. In order to check this, observe that
\[
\Psi(z) = \sqrt{\frac{z-w_0}{z-w_1}},
\]
maps the simply connected domain $\hat{\mathbb{C}} \setminus L$ conformally onto a halfplane with the origin on the boundary, so $h = \text{Re}(\eta \Psi)$ is a barrier at $w_0$ for some $\eta \in \partial \Omega$.

Theorem 5.6.
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $w_0 \in \partial \Omega$. Then $w_0$ is a regular point of $\partial \Omega$ if and only if there is a barrier at $w_0$.

Proof. We only need to prove the “if part”, so assume $h$ is a barrier at $w_0$, choose a continuous function $\phi : \partial \Omega \to \mathbb{R}$, and let $\varepsilon > 0$. Choose $\delta > 0$ with $|\phi(w) - \phi(w_0)| \leq \varepsilon$ for all $w \in \partial \Omega$ with $|w - w_0| < \delta$, let $m := \sup\{h(z) : z \in \Omega \setminus K_\delta(w_0)\} < 0$ and $M := \max\{|\phi(w)| : w \in \partial \Omega\}$. Then
\[
v(z) := \phi(w_0) - \varepsilon - \frac{h(z)}{m}(M + \phi(w_0))
\]
is harmonic on $\Omega$ and satisfies for all $w \in \partial \Omega$
\[
\limsup_{z \to w} v(z) \leq \begin{cases} 
\phi(w_0) - \varepsilon \leq \phi(w), & \text{if} \ w \in K_\delta(w_0) \\
-M \varepsilon \leq \phi(w), & \text{if} \ w \notin K_\delta(w_0).
\end{cases}
\]
Hence $v \in \mathcal{P}_\phi$, so $v \leq H_{\Omega} \phi$ in $\Omega$, and therefore $\liminf_{z \to w_0} H_{\Omega} \phi(z) \geq \liminf_{z \to w_0} v(z) = \phi(w_0) - \varepsilon$. In a similar way, we have $\liminf_{z \to w_0} H_{\Omega} (-\phi)(z) \geq -\phi(w_0) - \varepsilon$. Now observe that $H_{\Omega} \phi + H_{\Omega} (-\phi) \leq H_{\Omega}(0) = 0$, so $H_{\Omega} \phi \leq -H_{\Omega} (-\phi)$, that is, $\limsup_{w \to w_0} H_{\Omega} \phi(w) \leq \phi(w_0) + \varepsilon$.

A (geometric) characterization of regular domains is not known. However, combining Theorem 5.6 and Example 5.5, leads to the following sufficient condition for regularity.
Corollary 5.7.
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain such that every boundary point is the endpoint of a segment with all other points in $\mathbb{C} \setminus \overline{\Omega}$. Then $\Omega$ is regular, so the Dirichlet problem $(\Omega, \phi)$ can be solved for any continuous function $\phi : \partial \Omega \to \mathbb{R}$. This is the case, in particular, if $\partial \Omega$ consists of a finite number of disjoint regular closed curves.

Theorem 5.8.
Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected domain. Then there is a barrier for every boundary point of $\Omega$. In particular, $\Omega$ is regular and the Dirichlet problem $(\Omega, \phi)$ can be solved for any continuous function $\phi : \partial \Omega \to \mathbb{R}$.

Proof. Let $w_0 \in \partial \Omega$. Since $\Omega$ is bounded, there is $w_1 \notin \Omega$ such that $A := \{z \in \mathbb{C} : |z - w_0| < \frac{1}{2}|z - w_1|\}$ contains $\Omega$. Then, since $\Omega$ is simply connected,

$$f(z) := \log \frac{z - w_0}{z - w_1} \in \mathcal{H}(\Omega)$$

and satisfies $\text{Re} \ f(z) < -1$ in $\Omega$. Hence the values of $1/f$ lie in $K_{1/2}(-1/2)$, so $h := \text{Re}(1/f)$ is harmonic and nonpositive on $\Omega$ with $\limsup_{z \to w} h(z) = 0$ if and only if $\lim_{z \to w} \text{Re} f(z) = -\infty$, which is the case if and only if $w = w_0$. \hfill $\Box$

Remark 5.9.
It can be shown that $w_0 \in \partial \Omega$ is a regular point if there exists a continuum (connected set) $L$ which contains $w_0$ such that $0 \notin L \setminus \{w_0\} \subseteq \mathbb{C} \setminus \overline{\Omega}$, see [7].

Problems

1. * Let $\Omega \subseteq \mathbb{C}$ be a domain. A nonempty family $\mathcal{F}$ of subharmonic functions on $\Omega$ is called a Perron family, if the following two conditions are satisfied:

(P1) if $u, v \in \mathcal{F}$, then $\max\{u, v\} \in \mathcal{F}$.

(P2) if $u \in \mathcal{F}$ and $D$ is a disk compactly contained in $\Omega$, then $M_D u \in \mathcal{F}$.

Prove that the upper envelope of $\mathcal{F}$,

$$u_\mathcal{F}(p) := \sup \{u(p) : u \in \mathcal{F}\}, \quad p \in \Omega,$$

is either harmonic on $\Omega$ or $u_\mathcal{F}(p) = +\infty$ for all $p \in \Omega$.

2. Let $u : \mathbb{D} \to \mathbb{R}$ be subharmonic such that $u < 0$. Prove that

$$\limsup_{r \to 1} \frac{u(r \eta)}{r} < 0$$

for each $\eta \in \partial \mathbb{D}$.

(Hint: Apply the maximum principle to $u(z) + c \log |z|$ on an annulus centered at the origin.)

3. (The Dirichlet principle)
Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piecewise $C^1$-boundary, so Green's theorem (Corollary 1.8) applies, and let $\phi : \partial \Omega \to \mathbb{R}$ be a continuous function.
(a) Assume that there is a solution \( h \in C^2(\overline{\Omega}) \) of the Dirichlet problem for \((\Omega, \phi)\) and let \( v \in C^2(\overline{\Omega}) \) be real-valued such that \( v = \phi \) on \( \partial \Omega \). Prove that

\[
\iint_{\Omega} \partial h(z) \cdot \nabla (h - v)(z) \, dx dy = 0
\]

and deduce

\[
D(h) := \iint_{\Omega} |\partial h(z)|^2 \, dx dy \leq \iint_{\Omega} |\nabla v(z)|^2 \, dx dy.
\]

(b) Suppose there is a real-valued function \( h \in C^2(\Omega) \cap C(\overline{\Omega}) \) such that \( h = \phi \) on \( \partial \Omega \) and \( D(h) \leq D(v) \) for all \( v \in C^2(\Omega) \cap C(\overline{\Omega}) \) s.t. \( v = \phi \) on \( \partial \Omega \). Prove that \( h \) is the solution of the Dirichlet problem for \((\Omega, \phi)\).

Hint: Fix \( \psi \in C^\infty_c(\Omega) \) and consider \( \frac{d}{dt} \bigg|_{t=0} D(h + t\psi) \).

Remark 5.10.
The "Dirichlet principle" claims that among all functions \( h \in C^2(\Omega) \cap C(\overline{\Omega}) \) for which \( h = \phi \) on \( \partial \Omega \) there is always a function which minimizes the "Dirichlet integral" \( D(h) \). Riemann used this principle without further justification in his proof of the Riemann mapping theorem (see Chapter 6). It is not obvious, however, that there is always such a minimizer for the Dirichlet integral. In fact, Weierstrass (1860) gave an example of a similar problem for which there is no minimizer. In 1871, Friedrich Prym (professor in Würzburg 1869–1909) constructed a striking example of a continuous function \( \phi : \partial \mathbb{D} \to \mathbb{R} \), such that there is not a single function \( v \in C^2(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \) with finite Dirichlet integral that equals \( \phi \) on the boundary. This makes it impossible even to talk about a minimizer since all functions with the correct boundary condition would have infinite Dirichlet integral. The following similar example was constructed by Hadamard (1906).

(c) For \( 0 \leq r \leq 1 \) and \( t \in [0, 2\pi] \) let

\[
u(r e^{i t}) := \sum_{n=1}^{\infty} \frac{r^n}{n^2} \sin(n!t) \cdot
\]

Show that \( \nu \) is harmonic in \( \mathbb{D} \) and continuous on \( \overline{\mathbb{D}} \), but \( D(\nu) = \infty \).

Conclusion: The Dirichlet problem for \((\mathbb{D}, \nu_{\partial \mathbb{D}})\) has a solution, which cannot be obtained by minimizing the Dirichlet integral.

Chapter 6

Green’s function and conformal mapping

Logarithmic singularities are “typical” isolated singularities of harmonic functions.

Definition.
Let $\Omega \subseteq \mathbb{C}$ be a regular domain and $z_0 \in \Omega$. A function $g(\cdot, z_0) : \Omega \setminus \{z_0\} \to \mathbb{R}$ is called Green’s function for $\Omega$ with pole at $z_0$ if

(i) $z \mapsto g(z, z_0) + \log |z - z_0|$ is harmonic in $\Omega$, and

(ii) $g(z, z_0) \to 0$ as $z$ tends to $\partial \Omega$.

For example,

$$g(z, z_0) = -\log \left| \frac{z - z_0}{1 - \overline{z_0}z} \right|$$

is a Green’s function for $\Omega = \mathbb{D}$ with pole at $z_0 \in \mathbb{D}$. Note that

$$z \mapsto \frac{z - z_0}{1 - \overline{z_0}z}$$

is a conformal map from $\Omega = \mathbb{D}$ onto $\mathbb{D}$.

Remarks (Existence, uniqueness and positivity of Green’s function).
Let $\Omega \subseteq \mathbb{C}$ be a regular domain and $z_0 \in \Omega$. If $h$ is the solution for the Dirichlet problem \((\Omega, \log |\cdot - z_0|)\), then $h(z) - \log |z - z_0|$ is a Green’s function for $\Omega$ with pole at $z_0$. The maximum principle implies that Green’s function with pole at $z_0$

(a) is uniquely determined, and

(b) satisfies $g(z, z_0) > 0$ for all $z \in \Omega \setminus \{z_0\}$, since $g(z, z_0) \to +\infty$ as $z \to z_0$.

We denote the Green’s function for $\Omega$ with pole at $z_0$ by $g_{\Omega}(\cdot, z_0)$.

Example 6.1.
Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected domain. Then, by the Riemann mapping theorem, there is a conformal map $f$ from $\Omega$ onto $\mathbb{D}$. Note that $|f(z_n)| \to 1$ for every sequence $(z_n) \in \Omega$ such that $z_n \to w \in \partial \Omega$. Therefore,

$$g_{\Omega}(z, z_0) = -\log \left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)} \right|$$

is the Green’s function for $\Omega$ with pole $z_0$. 

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This example shows that knowing a conformal map $f$ from $\Omega$ onto $\mathbb{D}$ allows the construction of Green’s function for $\Omega$. We shall now prove the converse of this observation. Note that if we put

$$f(z, z_0) := \frac{f(z) - f(z_0)}{1 - f(z_0)f(z)},$$

then $g_\Omega(z, z_0) + \log |z - z_0| = \text{Re} \, h(z)$ for

$$h(z) = -\log \left( \frac{f(z, z_0)}{z - z_0} \right),$$

so $f(z, z_0) = (z - z_0)e^{-h(z)}$.

**Theorem 6.2** (Riemann mapping theorem).

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $z_0 \in \Omega$. Then there is a conformal map $f$ from $\Omega$ onto $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

**Proof.** (a) We first assume that $\Omega$ is bounded and hence regular. Fix $z_0 \in \Omega$. Theorem 5.8 shows that Green’s function $g_\Omega(\cdot, z_0)$ for $\Omega$ with pole $z_0$ exists. In particular, $u(z) := g_\Omega(z, z_0) + \log |z - z_0|$ is harmonic in $\Omega$. Since $\Omega$ is simply connected, there is $h \in \mathcal{H}(\Omega)$ such that $u = \text{Re} \, h$. Let $f(z, z_0) := (z - z_0)e^{-h(z)}$, a holomorphic function with a simple zero at $z_0$. Since $|f(z, z_0)| \to 1$ as $z \to \partial \Omega$, we have $|f(z, z_0)| \leq 1$ by the maximum principle, so $f(\cdot, z_0)$ maps into $\mathbb{D}$. Now let $z_1 \in \Omega \setminus \{z_0\}$ and consider

$$s(z) := \frac{1}{f(z, z_1)} \frac{f(z, z_0) - f(z_1, z_0)}{1 - f(z_1, z_0)f(z, z_0)} \in \mathcal{H}(\Omega),$$

a function which satisfies $|s(z)| \leq 1$ as $z \to \partial \Omega$, so

$$|s(z_0)| = \frac{|f(z_1, z_0)|}{|f(z_0, z_1)|} \leq 1.$$  

By symmetry, $|s(z_0)| = 1$, so $|s(z)| = 1$ in $\Omega$ by the maximum principle. This implies that $f(\cdot, z_0)$ is one-to-one. Finally we show that $f(\cdot, z_0)$ is onto. Let $w \in \mathbb{D}$. Then the absolute value of

$$\tilde{f}(z) := \frac{f(z, z_0) - w}{1 + \overline{w}f(z, z_0)}$$

tends to 1 as $z \to \partial \Omega$. Since $\tilde{f}(\Omega) \subseteq \mathbb{D}$, the minimum principle implies that $\tilde{f}$ has a zero in $\Omega$.

(b) Now let $\Omega$ be possibly unbounded. In view of (a) we only need to find a conformal map from $\Omega$ onto a bounded simply connected domain $\mathbb{D}$. Let $a \in \mathbb{C} \setminus \Omega$. Then $\Psi(z) := \sqrt{z - a}$ is holomorphic and one-to-one on $\Omega$. Moreover, if $w \in \Psi(\Omega)$, then $-w \notin \Psi(\Omega)$. Since $\Psi(\Omega)$ is an open set, we have $K_r(\Psi(z_0)) \subseteq \Psi(\Omega)$ for some $r > 0$, so $|\Psi(z) + \Psi(z_0)| \geq r$ for all $z \in \Omega$ and hence $1/(\Psi(z) + \Psi(z_0))$ maps $\Omega$ conformally onto a bounded simply connected domain.

Our next goal is to define Green’s function for domains which are not necessarily regular.

**Remark 6.3** (An extremal property of Green’s function).

Let $\Omega \subseteq \mathbb{C}$ be a regular domain, $z_0 \in \Omega$ and $g_\Omega(\cdot, z_0)$ Green’s function for $\Omega$ with pole at $z_0$. Suppose that $u : \Omega \setminus \{z_0\} \to \mathbb{R}$ is subharmonic such that

(a) $u \equiv 0$ on $\Omega \setminus K$ for some compact set $K \subseteq \Omega$, and
(b) \( u(z) + \log |z - z_0| \) is subharmonic in a neighborhood of \( z_0 \).

Then \( u(z) - g_\Omega(z,z_0) \) is subharmonic on \( \Omega \) and tends to 0 at \( \partial \Omega \), so \( g_\Omega(z,z_0) \geq u(z) \) for all \( z \in \Omega \\setminus \{z_0\} \).

**Theorem 6.4.**

Let \( \Omega \) be a domain. Fix a point \( z_0 \in \Omega \) and let \( F_{z_0} \) be the family of all subharmonic functions \( u: \Omega \setminus \{z_0\} \to \mathbb{R} \) such that

(a) \( u \equiv 0 \) on \( \Omega \setminus K \) for some compact set \( K \subseteq \Omega \), and

(b) \( u(z) + \log |z - z_0| \) is subharmonic in a neighborhood of \( z_0 \).

Then \( F_{z_0} \) is a Perron family. If, in addition, \( \Omega \) is regular, then

\[
g_\Omega(z,z_0) = \sup\{u(z) : u \in F_{z_0}\}, \quad z \in \Omega \setminus \{z_0\}.
\]

**Proof.** It is immediate that \( F_{z_0} \) is a Perron family. Let \( \Omega \) be regular and \( \varepsilon > 0 \), then

\[
\max\{g_\Omega(z,z_0) - \varepsilon, 0\} \in F_{z_0}, \text{ so } g_\Omega(z,z_0) \geq \sup\{u(z) : u \in F_{z_0}\} \geq \max\{g_\Omega(z,z_0) - \varepsilon, 0\}. \quad \Box
\]

**Definition.**

Let \( \Omega \subseteq \mathbb{C} \) be a domain and \( z_0 \in \Omega \). If the “upper envelope” \( \sup\{u(z) : u \in F_{z_0}\} \) of \( F_{z_0} \) is finite on \( \Omega \setminus \{z_0\} \), then we say that Green’s function for \( \Omega \) with pole at \( z_0 \) exists, and we denote it by

\[
g_\Omega(z,z_0) := \sup\{u(z) : u \in F_{z_0}\}, \quad z \in \Omega \setminus \{z_0\}.
\]

Otherwise, we say that Green’s function for \( \Omega \) with pole \( z_0 \) does not exist.

**Theorem 6.5.**

Let \( \Omega \subseteq \mathbb{C} \) be a domain, \( z_0 \in \Omega \), and suppose that \( g_\Omega(\cdot, z_0) \) exists. Then

(a) \( g_\Omega(z,z_0) > 0 \) for all \( z \in \Omega \setminus \{z_0\} \), and

(b) \( g_\Omega(z,z_0) + \log |z - z_0| \) has a harmonic extension to \( \Omega \).

**Proof.** Let \( D = K_r(z_0) \) be compactly contained in \( \Omega \). Then it is easy to see that

\[
u(z) := \begin{cases} -\log \frac{|z - z_0|}{r} & \text{if } z \in D \\ 0 & \text{if } z \in \Omega \setminus D \end{cases}
\]

belongs to \( F_{z_0} \). Hence \( g_\Omega(z,z_0) \geq u(z) \geq 0 \) for all \( z \in \Omega \setminus \{z_0\} \) and \( g_\Omega(z,z_0) > 0 \) for all \( z \in D \setminus \{z_0\} \). By the maximum principle for the domain \( \Omega \setminus \{z_0\} \) condition (a) holds.

We now prove (b). If \( v \in F_{z_0} \), then \( v(z) + \log |z - z_0| \) is subharmonic in \( \Omega \), so we get

\[
\sup_D (v + \log |z - z_0|) \leq \sup_{\partial D} v + \log r \leq \sup_{\partial D} g_\Omega(\cdot, z_0) + \log r =: C < \infty.
\]

Hence

\[
g_\Omega(\cdot, z_0) + \log(|z - z_0|) \leq C
\]

in \( D \setminus \{z_0\} \). On the other hand, we have

\[
g_\Omega(z,z_0) + \log |z - z_0| \geq u(z) + \log |z - z_0| = \log r
\]

for all \( z \in D \setminus \{z_0\} \). This implies that \( g_\Omega(\cdot, z_0) + \log \cdot |z - z_0| \) is bounded on \( D \setminus \{z_0\} \), so it has a harmonic extension to \( D \) by Exercise 4.7. \( \Box \)
Example 6.6.
The Green’s function for $\mathbb{C}$ with pole at $z_0 = 0$ does not exist. In fact, $g_n(z) = \log n - \log |z|$ is Green’s function for $\{z \in \mathbb{C} : |z| < n\}$ with pole at 0 and $\max\{g_n, 0\} \in \mathcal{F}_0$ for each $n$, but $g_n(z) \to \infty$ as $n \to \infty$.

Problems

1. (Another extremal property of Green’s function)
Let $\Omega \subseteq \mathbb{C}$ be a domain, $z_0 \in \Omega$ and $g(z, z_0)$ Green’s function for $\Omega$ with pole $z_0$.
Suppose that $v : \Omega \setminus \{z_0\} \to \mathbb{R}$ is nonpositive and subharmonic such that $v(z) - \log |z - z_0|$ is bounded above at $z_0$. Prove that $g(z, z_0) \leq -v(z)$ for all $z \in \Omega \setminus \{z_0\}$ with equality for some $z \in \Omega \setminus \{z_0\}$ if and only if $g(\cdot, z_0) = -v$.

2. (Conformal maps of doubly connected domains)
Let $\Omega \subseteq \mathbb{C}$ be a doubly connected domain, i.e., $\mathbb{C} \setminus \Omega$ has two components.

(a) Suppose that $\partial \Omega$ consists of two disjoint closed $C^1$-curves. Fix $R > 1$. Let $\phi(w) := 1$ for $w$ on the boundary of the bounded component and $\phi(w) := R$ for $w$ on the boundary of the other component. Let $h$ be the solution for the Dirichlet problem for $(\Omega, \phi)$. Prove that if $R$ is appropriately chosen, then there is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f(z) = \exp(w(z) + iv(z))$ is a well-defined holomorphic function in $\Omega$ such that $1 < |f(z)| < e^R$ for all $z \in \Omega$. Prove that $f$ is in fact a conformal map from $\Omega$ onto the annulus $\{w \in \mathbb{C} : 1 < |w| < e^R\}$.

(b) Suppose that none of the components of $\mathbb{C} \setminus \Omega$ is a singleton. Prove that there is a unique number $R > 1$ and a conformal map $f$ from $\Omega$ onto the annulus $\{w \in \mathbb{C} : 1 < |w| < e^R\}$.

3. Let $\Omega \subseteq \mathbb{C}$ be a (bounded) domain, $z_0 \in \Omega$ and $g_\Omega(\cdot, z_0)$ Green’s function for $\Omega$ with pole at $z_0$. Let $\phi(z) := \log |z - z_0|.$

(a) Relate the Perron function $H_\Omega \phi$ with Green’s function $g_\Omega(\cdot, z_0)$.

(b) Let $w \in \partial \Omega$. Does it follow that $g_\Omega(z, z_0) \to 0$ as $z \to w$?

4. Let $\Omega$ and $D$ be two domains in $\mathbb{C}$ with Green’s functions $g_\Omega$ and $g_D$. Suppose that $f \in \mathcal{H}(\Omega)$ with $f(\Omega) \subseteq D$. Prove that

$$g_\Omega(z, z_0) \leq g_D(f(z), f(z_0))$$

for every $z \in \Omega \setminus \{z_0\}$. This is the so-called Lindelöf principle. Note that this implies in particular the following two important properties of Green’s functions:

(a) (Conformal invariance of Green’s function)
If $f$ is a conformal map from $\Omega$ onto $D$, then

$$g_\Omega(z, z_0) = g_D(f(z), f(z_0)).$$

(b) (Monotonicity of Green’s function)
If $\Omega \subseteq D$, then $g_\Omega(z, z_0) \leq g_D(z, z_0)$.

5. Let $\Omega \subseteq \mathbb{C}$ be a domain.
(a) Prove that there are regular domains $\Omega_n$ such that $\Omega_n \subseteq \overline{\Omega_{n+1}} \subseteq \Omega$ and such that every compact set $K \subseteq \Omega$ is contained in some $\Omega_n$.

(b) Prove that $\lim_{n \to \infty} g_{\Omega_n}(\cdot, z_0) = g(\cdot, z_0)$ locally uniformly in $\Omega \setminus \{z_0\}$. 
Chapter 7

Univalent functions

Definition.
A holomorphic function $f$ defined in an open set $D \subseteq \mathbb{C}$ is called univalent (or schlicht) if $f(z_1) \neq f(z_2)$ for all points $z_1, z_2 \in D$ with $z_1 \neq z_2$. We set

$$S := \{ f \in \mathcal{H}(D) \mid f \text{ univalent}, f(0) = 0, f'(0) = 1 \}.$$

Remark.
If $\Omega \subseteq \mathbb{C}$ is a simply connected domain, then the Riemann mapping theorem guarantees that there is always a conformal map (= univalent function) $\phi$ from $\mathbb{D}$ onto $\Omega$. Note that then $f(z) := (\phi(z) - \phi(0))/\phi'(0)$ belongs to the class $S$. Therefore $\{f(D) : f \in \mathcal{S}\}$ consists precisely of all simply connected domains $\Omega \subseteq \mathbb{C}$ modulo translations and dilations.

Example 7.1 (The Koebe function).
Let

$$k(z) := \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} n z^n.$$

Note that

$$k(z) = \frac{z}{(1 - z)^2} = \frac{1}{4} \left[ \left( \frac{1 + z}{1 - z} \right)^2 - 1 \right].$$

Now, the M"obius transformation $\frac{1 + z}{1 - z}$ maps $\mathbb{D}$ conformally onto the right half-plane, simply because $\text{Re} \, w > 0$ if and only if $w$ is closer to 1 than to $-1$, so

$$\text{Re} \left( \frac{1 + z}{1 - z} \right) > 0 \iff \left| \frac{1 + z}{1 - z} - 1 \right| < \left| \frac{1 + z}{1 - z} + 1 \right| \iff |z| < 1.$$

Hence, $k$ is univalent in $\mathbb{D}$ and maps the unit circle $\partial \mathbb{D}$ (minus the point 1) onto the straight half-line $L = \{ z \in \mathbb{R} : -\infty < z \leq -1/4 \}$. In particular, $k \in S$ and $k(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$.

Remark.
The class $S$ is preserved under a number of transformations.

(a) Conjugation: If $f(z) = z + a_2z^2 + a_3z^3 + \ldots \in S$ and $g(z) := \overline{f(\overline{z})} = z + \overline{a_2}z^2 + \overline{a_3}z^3 + \ldots$, then $g \in S$ with $g(\mathbb{D}) = f(\overline{\mathbb{D}})$.

(b) Rotation: If $f \in S$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in S$. We call the functions $k_\theta(z) := e^{-i\theta} k(e^{i\theta} z) \in S$, $\theta \in \mathbb{R}$, the rotations of the Koebe function.
(c) Dilation: If \( f \in \mathcal{S} \) and \( r \in (0,1) \), then \( f(rz)/r \in \mathcal{S} \).

Our next goal is to relate the Taylor coefficients \( a_2, a_3, \ldots \) of a function \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in \mathcal{S} \) with the (geometry of the) image domain \( f(D) \). For instance, the formula

\[
\text{area } f(D) = \pi \left( 1 + \sum_{n=2}^{\infty} n|a_n|^2 \right)
\]

gives the area of the image domain \( f(D) \) in terms of the coefficients \( a_n \), see Exercise 7.2. For the individual coefficients \( a_n \), the obvious inequality \( \text{area } f(D) \geq 0 \) gives nothing more but the trivial estimate \( |a_n| \geq 0 \) for each \( n \geq 2 \). However, a slight modification of this idea leads to a very useful inequality:

**Theorem 7.2 (Area Theorem; Gronwall 1914).**

*Suppose that*

\[
h(z) = \frac{1}{z} + \sum_{k=0}^{\infty} c_k z^k
\]

*is univalent in \( 0 < |z| < 1 \). Then*

\[
\sum_{k=1}^{\infty} k|c_k|^2 \leq 1.
\]

*In particular, \( |c_1| \leq 1 \) and \( |c_1| = 1 \) implies \( c_k = 0 \) for all \( k > 1 \), so*

\[
h(z) = \frac{1}{z} + c_0 + e^{i\theta} z.
\]

The proof of Theorem 7.2 will show that, roughly speaking,

\[
\text{area}(\mathbb{C}\setminus h(D)) = \pi \left( 1 - \sum_{n=1}^{\infty} n|c_n|^2 \right).
\]

This explains why Theorem 7.2 is called the Area Theorem.

**Proof.** Fix \( 0 < r < 1 \). Then the domain \( \Omega_r = \mathbb{C}\setminus h(K_r(0)) \) is positively bounded by the curve \( \gamma(t) = h(re^{it}) \), when \( t \) runs from \( 2\pi \) to \( 0 \). Corollary 1.8 implies

\[
\text{area}(\Omega_r) = -\frac{1}{2i} \int_0^{2\pi} h(re^{it}) h'(re^{it}) i re^{it} dt.
\]
Inserting the Laurent series of $h$ and taking into account that
\[ \int_0^{2\pi} e^{imt} \, dt = 0, \quad m \in \mathbb{Z}\setminus\{0\}, \]
a simple calculation gives
\[ \text{area}(\Omega_r) = \pi \left( \frac{1}{r^2} - \sum_{n=1}^{\infty} n|c_n|^2 r^{2n} \right). \]
Since $\text{area}(\Omega_r) \geq 0$, the desired result is obtained by letting $r$ tend to 1. □

Remark. If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S$, then we can apply the Area Theorem to
\[ h(z) = \frac{1}{f(z)} = \frac{1}{z + a_2z^2 + a_3z^3 + \cdots} = \frac{1}{z} - a_2 + (a_2^2 - a_3)z + \cdots, \]
and obtain $|a_3 - a_2^2| \leq 1$. This gives a necessary condition for the coefficients of the functions in $S$. However, this condition is not sufficient. To see this just note that $f(z) := z + 2z^2 + 3z^3$ does not belong to $S$. The general coefficient problem for the class $S$ was first posed by Bieberbach in 1916: Find necessary and sufficient conditions for a sequence of complex numbers $a_n$ so that the power series $\sum_{n=1}^{\infty} a_n z^n$ defines a function in $S$. This problem remains unsolved.

Theorem 7.3 (Bieberbach 1916). Suppose $f(z) = z + a_2z^2 + \ldots \in S$, then $|a_2| \leq 2$. Equality holds if and only if $f$ is a rotation of the Koebe function.

Proof. We use a trick first devised by Faber in 1916 and consider the square-root transform of $f$,
\[ g(z) = z \sqrt{\frac{f(z^2)}{z^2}} = z + \frac{1}{2} a_2 z^3 + \ldots \in \mathcal{H}(\mathbb{D}). \]
Note that $g$ is an odd function. In fact, $g$ is univalent since $g(z) = g(w)$ implies $f(z^2) = f(w^2)$, so $z = \pm w$. Now $z = -w$ is not possible since then $g(z) = -g(-z)$ would lead $g(z) = -g(-z) = -g(w) = -g(z)$ and hence $g(z) = 0$, which would give to $z = 0$. This means that we have $z = w$ and $g$ is injective. Therefore $g \in S$ and we can apply the Area Theorem to
\[ h(z) = \frac{1}{g(z)} = \frac{1}{z} - \frac{1}{2} a_2 z + \ldots \]
and get $|a_2| \leq 2$. If $|a_2| = 2$, then $h(z) = 1/z + e^{i\theta} z$ and a simple calculation implies $f(z) = -e^{-i\theta} k(-e^{i\theta} z)$. □

Remark 7.4. The basic coefficient estimate $|a_2| \leq 2$ has many important applications. It provides information about the second derivative of each function $f$ in $S$ at the origin. To transfer this information to any other point $z_0 \in \mathbb{D}$, we precompose $f \in S$ with the unit disk automorphism $\phi_{z_0} : \mathbb{D} \to \mathbb{D}$,
\[ \phi_{z_0}(z) = \frac{z_0 - z}{1 - z_0 \overline{z}}. \]
Now, $f \circ \phi_{z_0}$ is again univalent, but does not belong to $S$ in general. However, in view of
\[
(f \circ \phi_{z_0})(0) = f(z_0), \quad (f \circ \phi_{z_0})(0) = (1 - |z_0|^2) f'(z_0).
\]
The so-called Koebe transform of $f \in S$ w.r.t. $z_0 \in \mathbb{D}$,
\[
F(z) := \frac{f \left( \frac{z - z_0}{1 - |z_0|^2} \right) - f(z_0)}{(1 - |z_0|^2) f'(z_0)} = z + \frac{1}{2} \left( \frac{f''(z_0)}{f'(z_0)} (1 - |z_0|^2) - 2 \frac{z}{z_0} \right) z^2 + \ldots,
\]
belongs to $S$. Applying the inequality $|a_2| \leq 2$ to the Koebe transform $F(z) = z + a_2 z^2 + \ldots$
(and replacing $z_0$ by $z$ in (7.1), we get
\[
\left| \frac{f''(z)}{f'(z)} - \frac{2z}{1 - |z|^2} \right| \leq \frac{4}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]
Equality holds for $z \in \mathbb{D}$ if and only if $f$ is an appropriate rotation of the Koebe function.

**Theorem 7.5** (Distortion theorem).
Suppose $f \in S$. Then
\[
\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}
\]
for all $z \in \mathbb{D}$. Equality is only possible when $f$ is a rotation of the Koebe function.

**Proof.** Using (7.2) for $z = |z| e^{i\theta}$ we get
\[
\left| \log f'(z) - \int_0^{|z|} \frac{2r}{1 - r^2} \, dr \right| = \left| \int_0^{|z|} \frac{d}{dr} \left( \log f'(re^{i\theta}) \right) - \frac{2r}{1 - r^2} \, dr \right| \leq \int_0^{|z|} \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} - \frac{2re^{-i\theta}}{1 - r^2} \right| dr \leq \int_0^{|z|} \frac{4}{1 - r^2} \, dr,
\]
so passing to the real part,
\[
\log \frac{1}{1 - |z|^2} - 2 \log \frac{1 + |z|}{1 - |z|} \leq \log |f'(z)| \leq \log \frac{1}{1 - |z|^2} + 2 \log \frac{1 + |z|}{1 - |z|}.
\]
This is equivalent to (7.3). \hfill \square

**Theorem 7.6** (Growth theorem).
Suppose $f \in S$. Then
\[
\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}
\]
for all $z \in \mathbb{D}$. Equality for $z \neq 0$ is only possible when $f$ is a rotation of the Koebe function.

**Proof.** The upper bound in (7.4) comes from an integration of the upper bound in (7.3) along the ray from $0$ to $z = |z| e^{i\theta}$:
\[
|f(z)| = \left| \int_0^{|z|} \frac{d}{dr} \left( f(re^{i\theta}) \right) \, dr \right| \leq \int_0^{|z|} |f'(re^{i\theta})| \, dr \leq \int_0^{|z|} \frac{1 + r}{(1 - r^3)^2} \, dr = \frac{|z|}{(1 - |z|)^3}.
\]
To prove the lower bound, we fix \( z \in D \) and let \( r := |z| / (1 + r)^2 \). Note that \( K_{r+|z|}(0) \subseteq f(K_r(0)) \) and that there is a point \( \tilde{z} \in \partial K_r(0) \) such that \( |f(\tilde{z})| = m(r) \). Then \( \gamma(t) := f^{-1}(t f(\tilde{z})) \), \( 0 \leq t \leq 1 \), is a curve in \( K_r(0) \) from \( \tilde{z} \) to \( f(\tilde{z}) \). Hence

\[
|f(z)| \geq \frac{1}{m(r)} \int_0^1 |f(\gamma(t))| |\gamma'(t)| dt = \int_0^1 \frac{|f'(\gamma(t))| |\gamma'(t)|}{|\gamma'(t)|} dt \geq \int_0^1 \frac{1}{1 + |\gamma(t)|} |\gamma'(t)| dt \\
\geq \int_0^1 \frac{1}{(1 + |\gamma(t)|)^3} \frac{d}{du} \left( |\gamma(t)|^2 \right) du = \frac{|z|}{(1 + |z|)^2}.
\]

\( \square \)

**Corollary 7.7 (Koebe One-Quarter Theorem).**

Suppose \( f \in S \). Then \( f(D) \supseteq K_{1/4}(0) \).

**Proof.** Let \( r := |z| \in (0,1) \). The growth theorem gives \( |f(z)| \geq \frac{r}{(1+r)^2} \), so \( f(K_r(0)) \) contains the disk about 0 with radius \( r/(1+r)^2 \). Letting \( r \to 1 \) gives the desired result.

\( \square \)

**Remark (The univalent Bloch constant).**

The Koebe One-Quarter Theorem gives rise to the following problem. For \( f \in S \) let

\[
U(f) := \sup \{ r > 0 : f(D) \text{ contains an open disk of radius } r \},
\]

and call

\[
\mathcal{U} := \inf \{ U(f) : f \in S \}
\]

the **univalent Bloch constant**. Corollary 7.7 shows that \( \mathcal{U} \geq 1/4 \), but it is not difficult to see that \( \mathcal{U} \geq 1/2 \), see Exercise 7.9. In fact, Landau proved in 1929 that \( \mathcal{U} \geq 0.5477 \). Over the years, this lower estimate was improved. The current world record is \( \mathcal{U} > 0.570884 \), established by Xiong in 1999. The best known upper bound was found only recently by Carroll & Ortega-Cerdà (2009); they proved that \( \mathcal{U} \leq 0.6563937 \). There is not even a conjecture about the true value of the univalent Bloch constant.

**Corollary 7.8.**

The class \( S \) is compact.

**Proof.** The growth theorem implies that the class \( S \) is locally bounded. Montel’s theorem shows that \( S \) is a normal family. If \( f \) is the limit function of a locally uniformly convergent sequence \( (f_n) \subset S \), then \( f(0) = 0 \) and \( f'(0) = 1 \), so \( f \) is not constant. Hence \( f \) is one-to-one by Hurwitz’s theorem and therefore in \( S \).

\( \square \)

**Remark (The Bieberbach conjecture).**

Since \( S \) is compact, it is clear that for every fixed positive integer \( n \) there is (at least) one function \( F_n \in S \) whose \( n \)th Taylor coefficient \( A_n \) is maximal in modulus, i.e.,

\[
|a_n| \leq |A_n| \quad \text{for all } f(z) = z + \sum_{k=2}^\infty a_k z^k \in S.
\]

For the Koebe function \( k(z) = z + \sum_{n=2}^\infty n z^n \) one has \( |a_n| = n \). The Bieberbach conjecture, first proposed in 1916, asserts that \( |A_n| = n \):

“Dass \( |A_n| \geq n \) zeigt das Beispiel \( \sum n z^n \). Vielleicht ist überhaupt \( |A_n| = n \)."
Bieberbach himself established the case \( n = 2 \) (see Theorem 7.3). The case \( n = 3 \) was solved by Loewner (1923), the case \( n = 4 \) by Gambedian and Schiffer (1955), the case \( n = 6 \) by Pederson and Ozawa (1968), and the case \( n = 5 \) by Pederson and Schiffer (1972). The entire Bieberbach conjecture was finally proven in 1985 by L. de Branges.

**Theorem 7.9** (Combined Distortion– and Growth theorem).

Let \( f \in S \). Then

\[
\frac{1 - |z|}{1 + |z|} \leq \frac{|z f'(z)|}{|f(z)|} \leq \frac{1 + |z|}{1 - |z|}
\]

(7.5)

for all \( z \in \mathbb{D} \). Equality for \( z \neq 0 \) is only possible when \( f \) is a rotation of the Koebe function.

**Proof.** Using the Growth Theorem 7.6) for the Koebe transform \( F(7.1) \) of \( f \) w.r.t. the point \( z_0 \in \mathbb{D} \), we get that

\[
\frac{|z_0|}{(1 - |z_0|)^2} \leq |F(-z_0)| \leq \frac{|z_0|}{(1 + |z_0|)^2},
\]

so

\[
\frac{1 - |z_0|}{1 + |z_0|} \leq \frac{f(z_0)}{|z_0 f'(z_0)|} \leq \frac{1 + |z_0|}{1 - |z_0|}.
\]

We now think of holomorphic functions \( f : \mathbb{D} \to \mathbb{C} \) as mappings from the metric space \((\mathbb{D}, d_{\mathbb{D}})\) to the metric space \((\mathbb{C}, |\cdot|)\), that is, as maps from the hyperbolic plane to the euclidean plane.

**Definition.**

Suppose that \( f \in \mathcal{H}(\mathbb{D}) \). Then

\[
D_e f(z) := \lim_{\zeta \to z} \frac{|f(\zeta) - f(z)|}{d_{\mathbb{D}}(\zeta, z)} = \left( 1 - |z|^2 \right) |f'(z)|
\]

is called the hyperbolic–euclidean derivative of \( f \) at the point \( z \in \mathbb{D} \).

\( D_e f(z) \) measures the infinitesimal length distortion of the map \( f : (\mathbb{D}, d_{\mathbb{D}}) \to (\mathbb{C}, |\cdot|) \) at the point \( z \). Note that \( D_e f \) is invariant w.r.t. composition with the isometries of \((\mathbb{D}, d_{\mathbb{D}})\) resp. \((\mathbb{C}, |\cdot|)\):

\[
D_e (T \circ f \circ S)(z) = D_e f(S(z))
\]

for each \( S \in \text{Aut}(\mathbb{D}) \) and each (euclidean isometry) \( T \in \{ \eta z + b \mid \eta \in \partial \mathbb{D}, b \in \mathbb{C} \} \).

**Theorem 7.10** (Invariant form of the distortion theorems; Minda 1994).

Let \( f : \mathbb{D} \to \mathbb{C} \) be a univalent function. Then

\[
|f(z_1) - f(z_2)| \geq \max\{D_e f(z_1), D_e f(z_2)\} \frac{\sinh 2d_{\mathbb{D}}(z_1, z_2)}{2 \exp(2d_{\mathbb{D}}(z_1, z_2))}
\]

(7.6)

for all \( z_1, z_2 \in \mathbb{D} \). Equality holds for two distinct points \( z_1, z_2 \in \mathbb{D} \) if and only if \( f \) maps onto the complex plane slit along a ray on the line through the points \( f(z_1) \) and \( f(z_2) \).

**Proof.** Fix \( z_1 \neq z_2 \) in \( \mathbb{D} \). Replacing \( f \) by \( T \circ f \circ S \) with an appropriate \( S \in \text{Aut}(\mathbb{D}) \) and \( T \in \text{Aut}(\mathbb{C}) := \{ az + b \mid a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \} \) and using the invariance of the terms in (7.6), we may assume that \( f \in S, z_1 = 0 \) and \( z = z_2 \neq 0 \). But then (7.6) is equivalent to

\[
|f(z)| \geq \max \left\{ \frac{|z|}{(1 + |z|)^2}, \frac{1 - |z|}{1 + |z|} |z| |f'(z)| \right\}.
\]

This latter inequality now follows directly from Theorem 7.6 and Theorem 7.9. \( \square \)
1. Suppose \( f \) is holomorphic, but not injective in \( \mathbb{D} \). Show that there are points \( z_1 \neq z_2 \) in \( \mathbb{D} \) such that \( |z_1| = |z_2| \) and \( f(z_1) = f(z_2) \).

2. Let \( f \in S \) such that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( r \in (0,1) \). Prove that

\[
\iint_{|z|<r} |f'(z)|^2 \, dx \, dy = \pi \left( r^2 + \sum_{n=2}^{\infty} n|a_n|^2 r^{2n} \right)
\]

gives the area of the set \( \{ f(z) : |z| < r \} \).

3. Suppose that \( f \in \mathcal{H}(\mathbb{D}) \) and \( f_r(z) := f(rz)/z \) for \( r \in (0,1) \). Show that \( f_r \) converges locally uniformly in \( \mathbb{D} \) to \( f \) as \( r \to 1 \).

4. Prove that the polynomials in \( S \) are dense in \( S \).

5. Suppose that \( f \in S \) and \( f(\mathbb{D}) \) is a convex domain. Show that \( f(D) \supset K_{1/2}(0) \).

6. Prove that for every univalent function \( f : \mathbb{D} \to \mathbb{C} \) the following estimate holds

\[
\frac{1}{4} \text{D}_{e} f(z) \leq \text{dist}(\partial f(\mathbb{D}), f(z)) \leq \text{D}_{e} f(z), \quad z \in \mathbb{D}.
\]

(Hint: First prove that for every normalized univalent function \( h \in S \) the estimate \( 1/4 \leq \lim\inf_{|z| \to 1} |h(z)| \leq 1 \) holds.)

7. Let \( f(z) = z + a_2 z^2 + \ldots \in S \) such that \( |f(z)| \leq M < \infty \) for all \( z \in \mathbb{D} \). Prove that

\( |a_2| \leq 2(1 - 1/M) \).

(Hint: First show that \( Mk_\theta(f(z)/M) \in S \) for each rotation \( k_\theta(z) := e^{-i\theta} k(e^{i\theta} z) \) of the Koebe function.)

8. In the proof of Theorem 7.3 we made use of the transformation which associates to every \( f \in S \) the function

\[
z \sqrt{\frac{f(z)}{z^2}} \in S,
\]

the so-called square root transform of \( f \).

(a) Find the square root transform of the Koebe function and its image domain.

(b) Show that \( |c_3| \leq 1 \) for every odd function \( f(z) = z + c_3 z^3 + \ldots \) in \( S \).

9. Prove that \( \mathcal{U} \geq 1/2 \).

(Hint: If \( \gamma \notin f(\mathbb{D}) \) for some \( f \in S \), then \( f/(1 - f/\gamma) \in S \).)
Chapter 8

Covering properties of holomorphic functions

Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'(0)| = 1$. Then $f$ is not constant, so the set $f(\mathbb{D})$ is open and therefore contains an open disk. Since $f$ is univalent (=one-to-one) in a neighborhood of $z = 0$ there is a domain $G \subseteq \mathbb{D}$ which is mapped univalently by $f$ onto an open disk $D$ in $f(\mathbb{D})$.

Definition.

Let $f \in \mathcal{H}(\mathbb{D})$. A set $D \subseteq f(\mathbb{D})$ is called univalently covered by $f$ if there exists a set $G \subseteq \mathbb{D}$ such that $f$ maps $G$ in a one-to-one way onto $D$.

Theorem 8.1 (Bloch’s theorem).

Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'(0)| = 1$. Then $f(\mathbb{D})$ contains a univalently covered open disk of radius $23/100$.

Remark.

Theorem 8.1 does not say that the disk $K_{0.23}(f(0))$ is covered univalently by $f$ even though the only normalization $|f'(0)| = 1$ is at the origin! In fact, the function $f_n(z) := (e^{n^2} - 1)/n$ does not attain the value $-1/n$, so $K_{0.23}(f_n(0))$ is not entirely contained in $f_n(\mathbb{D})$ for any $n > 100/23$.

Definition.

Let $f \in \mathcal{H}(\mathbb{D})$. Then

$$B(f) := \sup\{r > 0 : f(\mathbb{D}) \text{ contains a univalently covered open disk of radius } r\}$$

is called the Bloch radius of $f$. We call $B := \inf\{B(f) : f \in \mathcal{H}(\mathbb{D}), |f'(0)| = 1\}$ Bloch’s constant.

The exact value of Bloch’s constant is unknown. Theorem 8.1 above gives the lower bound $B \geq 0.23$. Since $B(\text{id}) = 1$, we see that $B \leq 1$. In fact, it is known that

$$0.433213 \approx \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} < B \leq \sqrt{\frac{\sqrt{3} - 1}{2} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)}} \approx 0.4718617.$$

It has been conjectured by Ahlfors and Grunsky in 1937 that the upper bound is the correct value of $B$.

We start with an extension of Schwarz’ lemma due to Landau.

Lemma 8.2.

Let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$ and $f'(0) = 1$. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{D}$.
Then \( f \) covers the disk
\[
|w| < M \left( M - \sqrt{M^2 - 1} \right)^2
\]
univalently.

**Proof.** By Schwarz’ lemma we have \( M \geq 1 \). If \( M = 1 \), then \( f(z) = z \) again by Schwarz’ lemma and there is nothing to prove. We therefore need only consider the case \( M > 1 \). The function
\[
h(z) := \frac{f(z)}{Mz}
\]
maps \( \mathbb{D} \) to \( \mathbb{D} \) and \( h(0) = 1/M \), so the Schwarz–Pick lemma implies
\[
\left| \frac{h(z) - 1/M}{1 - h(z)/M} \right| \leq |z|.
\]
It follows that function
\[
w(z) := \frac{h(z) - 1/M}{1 - h(z)/M} = \frac{M}{M} \frac{f(z) - z}{M^2 z - f(z)}
\]
is holomorphic in \( \mathbb{D} \) and \( |w(z)| \leq 1 \). Since
\[
(f(z) - z)(M + w(z)z) = z^2 w(z)(M^2 - 1),
\]
the inequality \( |w(z)| \leq 1 \) leads to
\[
\max_{|z|=r} |f(z) - z| \leq \frac{M^2 - 1}{M - r} r^2, \quad r \in [0, 1).
\]
Now it is easy to see that on the interval \([0, 1]\) the function
\[
r \mapsto r - \frac{M^2 - 1}{M - r} r^2
\]
attains is global maximum \( M_1 \) at exactly one point \( r_0 \in (0, 1) \). This implies that for all \( |a| < M_1 \) and all \( |z| = r_0 \)
\[
|f(z) - a - (z - a)| = |f(z) - z| \leq \frac{M^2 - 1}{M - r_0} r_0^2 - r_0 + |z| = |z| - M_1 < |z| - |a| \leq |z - a|.
\]
Hence for each \( |a| < M_1 \) Rouché’s theorem shows that \( f(z) - a \) has exactly one zero \( z_a \) in \( |z| < r_0 \). Thus \( f \) maps the domain \( \{z_a : |a| < M_1 \} \subseteq K_{r_0}(0) \) univalently onto the disk \( |w| < M_1 \). A computation shows that \( r_0 = M - \sqrt{M^2 - 1} \) and \( M_1 = M(M - \sqrt{M^2 - 1})^2 \). \( \square \)

We now turn to the proof of Theorem 8.1.

**Exercise 8.1.**
Let \( \mathcal{H}(\mathbb{D}) := \{ f \in \mathcal{H}(U) \mid U \text{ open neighborhood of } \mathbb{D} \} \). Prove that
\[
B = \inf \{ B(f) \mid f \in \mathcal{H}(\mathbb{D}), |f'(0)| = 1 \}.
\]

**Remark 8.3.**
By Exercise 8.1 it suffices to prove Theorem 8.1 for the case that \( f \in \mathcal{H}(\mathbb{D}) \). Let \( f \in \mathcal{H}(\mathbb{D}) \) with \( |f'(0)| = 1 \). If we precompose \( f \) with the unit disk automorphism
\[
\phi_w(z) := \frac{w - z}{1 - \overline{w}z},
\]
which maps the origin onto the point \( w \in \mathbb{D} \),
\[
F_w(z) := (f \circ \phi_w)(z) := f \left( \frac{w - z}{1 - \overline{w}z} \right),
\]
we get from the Lemma of Schwarz–Pick that
\[
(1 - |z|^2) |F'_w(z)| = (1 - |z|^2) |\phi'_w(z)| |f'(\phi_w(z))| = (1 - |\phi_w(z)|^2) |f'(\phi_w(z))| \quad (*)
\]
Now, heuristically, if \( |F'_w(0)| \) is as large as possible, then \( F_w(\mathbb{D}) \) contains a largest univalently covered disk about \( F_w(0) = f(w) \). Since \( |F'_w(0)| = (1 - |w|^2)|f'(w)| \) by \((*)\), we therefore choose \( w \in \mathbb{D} \) such that
\[
(1 - |w|^2) |f'(w)| = \max_{|z| \leq 1} (1 - |z|^2) |f'(z)| =: ||f||_B \geq 1.
\]
Then \( g(z) := F_w(z)/||f||_B \) satisfies \( |g'(0)| = 1 \) and \( 1 - |z|^2 \) \( |g'(z)| \leq 1 \) for all \( z \in \mathbb{D} \) by \((*)\). Hence \( g \) is a so-called Bloch function, i.e.
\[
g \in B := \{ f \in H(\mathbb{D}) : ||f||_B < \infty \}
\]
with \( |g'(0)| = 1 \) and \( ||g||_B \leq 1 \). Note that if \( f \) univalently covers a disk of radius \( R > 0 \), then \( g \) univalently covers a disk of radius \( R/||f||_B \leq R \), so \( B(g) \leq B(f) \). This proves the following result.

**Theorem 8.4.**
If \( B \) denotes the set of all Bloch functions, then
\[
B = \inf \{ B(g) \mid g \in B, |g'(0)| = 1, ||g||_B \leq 1 \}.
\]

**Proof of Theorem 8.1.** Let \( g \in B \) with \( |g'(0)| = 1 \) and \( ||g||_B \leq 1 \). We may assume that \( g(0) = 0 \). Note that if \( z = |z|e^{i\theta} \),
\[
|g(z)| = |g(z) - g(0)| \leq \int_0^{|z|} |g'(te^{i\theta})| \, dt \leq \int_0^{|z|} \frac{dt}{1 - t^2} \, dt = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}
\]
for all \( z \in \mathbb{D} \). Fix \( r \in (0,1) \) and consider \( g_r(z) := g(rz)/r \). Then
\[
|g_r(z)| = \frac{|g(rz)|}{r} \leq \frac{1}{2r} \log \frac{1 + r}{1 - r} =: M_r, \quad z \in \mathbb{D}.
\]
Lemma 8.2 shows that
\[
B(g) \geq r B(g_r) \geq r M_r \left( M_r - \sqrt{M_r^2 - 1} \right)^2.
\]
For \( r = 6/10 \) the right-hand side has the value \( \approx 0.23 \). One can show that the maximum of the right-hand side is \( \approx 0.230838 \). \qed

**Corollary 8.5** (Valiron).
Let \( f \) be a non-constant entire function. Then \( f(\mathbb{C}) \) contains univalently covered disks of arbitrary radii.

**Proof.** Since \( f \) is not constant there is \( z_0 \in \mathbb{C} \) such that \( f'(z_0) \neq 0 \). Hence \( g(z) := f(z + z_0) \) is entire with \( g'(0) \neq 0 \). Fix \( R > 0 \). By Theorem 8.1, \( \mathbb{D} \ni z \mapsto g(Rz)/R g'(0) \) covers a disk of radius 0.23 univalently. Hence \( g \) covers a disk of radius \( R |g'(0)| 0.23 \) univalently. \qed
Lemma 8.6.
Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $f : \Omega \to \mathbb{C}\setminus\{0,1\}$ holomorphic. Then
\[
f(z) = -\exp(\pi i \cosh(2g(z))),
\]
with a function $g \in \mathcal{H}(\Omega)$ such that $g(\Omega)$ contains no disk of radius 1.

Proof. Since $f \neq 0$ there is $h \in \mathcal{H}(\Omega)$ such that $e^h = f$. Since $f \neq 1$, we see that $h \neq 0, 2\pi i$, so there exist $g_1, g_2 \in \mathcal{H}(\Omega)$ such that $g_1^2 = h/2\pi i$ and $g_2^2 = h/2\pi i - 1$. Since clearly $g_1 - g_2 \neq 0$, there is $g \in \mathcal{H}(\Omega)$ such that $g_1 - g_2 = e^g$. By construction, $1 = g_1^2 - g_2^2$, so $g_1 + g_2 = 1/(g_1 - g_2) = e^{-g}$ and hence $g_1 = (e^g + e^{-g})/2$. This implies
\[
h = 2\pi i g_1^2 = \frac{\pi i}{2} \left( e^{2g} + e^{-2g} + 2 \right) = \pi i \cosh(2g) + \pi i
\]
and hence $f = e^h = -\exp(\pi i \cosh(2g))$. Note that $g$ omits (at least) the set
\[
E := \left\{ \pm \log \left( \sqrt{n} - \sqrt{n-1} \right) + \frac{m\pi i}{2} : n \in \mathbb{N}, m \in \mathbb{Z} \right\}
\]
since if $g(z) = \pm \log(\sqrt{n} - \sqrt{n-1}) + \frac{m\pi i}{2}$, then $\cosh(2g(z)) = (-1)^m(2n - 1)$, so $f(z) = -e^{\pi i (1 - m)(2n - 1)} = 1$. Note that every $z \in \mathbb{C}$ has distance $< 1$ from $E$ because the distance of two points in $E$ is at most
\[
\sqrt{\left( \log(\sqrt{2} - 1) \right)^2 + \frac{\pi^2}{4}} < 2.
\]
Since $g$ omits $E$, $g(\Omega)$ contains no disk of radius $\geq 1$. \qed

Theorem 8.7 (The little Picard theorem).
Every entire function $f : \mathbb{C} \to \mathbb{C}\setminus\{0,1\}$ is constant.

Proof. By Lemma 8.6 there is $g \in \mathcal{H}(\mathbb{C})$ such that $f(z) = -\exp(\pi i \cosh(2g(z)))$ and such that $g(\mathbb{C})$ contains no disk of radius 1. Hence $g$ is constant by Corollary 8.5 and so is $f$. \qed

This proof of the little Picard theorem does not use the fact that the image domains of non-constant entire functions contain univalently covered disks of arbitrary radii. This led Landau to the following definition.

Definition.
For $f \in \mathcal{H}(\mathbb{D})$ the number
\[
L(f) := \sup \{ r > 0 \mid f(\mathbb{D}) \text{ contains an open disk of radius } r \}
\]
is called the Landau radius of $f$. We call
\[
L := \inf \{ L(f) \mid f \in \mathcal{H}(\mathbb{D}), |f'(0)| = 1 \}
\]
the Landau constant.

It is clear that $L \geq B$, so $L \geq 0.23$. The exact value of Landau’s constant is unknown. It is known that
\[
\frac{1}{2} + 10^{-336} < L \leq \frac{\Gamma(1/3) \Gamma(5/6)}{\Gamma(1/6)} \approx 0.543.
\]
It has been conjectured that the upper bound is the correct value of $L$. 
Supplementary Exercises

1. Let \( M > 1 \) and \( f \in \mathcal{H}(\mathbb{D}) \) with \( f(0) = 0 \) and \( f'(0) = 1 \) such that \( |f(z)| \leq M \) for every \( z \in \mathbb{D} \). Prove that
\[
|f(z)| \geq M|z| \frac{1 - M|z|}{M - |z|} \quad \text{for all } |z| < 1/M.
\]

2. Let \( f \in \mathcal{H}(\mathbb{D}) \) such that \( \|f\|_{\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty \).
Prove that \( f \in B \) with \( \|f\|_B \leq \|f\|_{\infty} \). Find an explicit example of an unbounded function in \( B \).

3. Let \( g \in \mathcal{H}(\mathbb{D}) \). Show that \( g \in B \) if and only if \( B(g) < \infty \) and also that in this case, \( \|g\|_B \leq B(g)/B \).

4. Let \( (f_n) \subseteq \mathcal{H}(\mathbb{D}) \) such that \( f_n \to f \) locally uniformly in \( \mathbb{D} \). Prove that \( \lim_{n \to \infty} B(f_n) \geq B(f) \).

5. Show that there exists \( F \in \mathcal{H}(\mathbb{D}) \) with \( |F'(0)| = 1 \) such that \( B(F) = B \).

6. Let \( \Omega \) be a simply connected domain, \( a \in \Omega \), and \( f : \Omega \to \mathbb{C} \setminus \{0,1\} \) holomorphic. Show that there is \( g \in \mathcal{H}(\Omega) \) such that
\[
f(z) = -\exp(\pi i \cosh(2g(z))) \quad \text{and} \quad |g(a)| \leq \log \left\lfloor 2 + \left| \log |f(a)|| \pi \right\rfloor \right\rfloor + \pi.
\]
(Hint: Choose \( h \in \mathcal{H}(\Omega) \) as in the proof of Lemma 8.6 such that \(-\pi \leq \text{Im} h(a) < \pi \) and assume that \( |g_1(a) - g_2(a)| \geq 1 \).

7. Suppose that \( f \) is a nonconstant entire function and \( f \) is odd (that is, \( f(-z) = -f(z) \) for all \( z \in \mathbb{C} \)). Show that \( f(\mathbb{C}) = \mathbb{C} \).

8. Let \( n \geq 3 \) be an integer and \( f, g \in \mathcal{H}(\mathbb{C}) \) such that \( f(z)^n + g(z)^n = 1 \) for all \( z \in \mathbb{C} \). Show that \( f \) and \( g \) are constant.
Chapter 9

Conformal metrics

Conformal maps preserve angles between intersecting curves, but the euclidean length

\[ L_1(\gamma) := \int_\gamma |dz| := \int_a^b |\gamma'(t)| \, dt \]

of a curve \( \gamma : [a, b] \to \mathbb{C} \) is in general not conformally invariant. It is therefore advisable to allow more flexible ways to measure the length of curves.

Definition.

Suppose that \( \Omega \subseteq \mathbb{C} \) is a domain and \( \lambda : \Omega \to \mathbb{R} \) is a continuous non-negative function. If \( \gamma : [a, b] \to \Omega \) is a curve, then we call

\[ L_\lambda(\gamma) := \int_\gamma \lambda(z) |dz| := \int_a^b \lambda(\gamma(t)) |\gamma'(t)| \, dt \]

the \( \lambda \)-length of \( \gamma \). The expression \( \lambda(z) |dz| \) is called conformal metric, if \( \lambda \) is strictly positive; otherwise we call \( \lambda(z) |dz| \) a conformal pseudo-metric.

As an example of a conformal metric we recall the hyperbolic metric

\[ \lambda_D(z) |dz| = \frac{|dz|}{1 - |z|^2} \]

in the unit disk \( \mathbb{D} \). We call \( \lambda(z) |dz| = 1 |dz| \) the euclidean metric.

Definition.

Suppose that \( \lambda(w) |dw| \) is a conformal pseudo-metric on a domain \( \Omega \subseteq \mathbb{C} \) and \( f \) is holomorphic on the domain \( D \subseteq \mathbb{C} \) with \( f(D) \subseteq \Omega \). Then the conformal pseudo-metric

\[ (f^* \lambda)(z) |dz| := \lambda(f(z)) |f'(z)| |dz| \]

is called the pullback of \( \lambda(w) |dw| \) under the map \( f \).

This definition is motivated by the observation that

\[ L_{f^* \lambda}(\gamma) = \int_\gamma (f^* \lambda)(z) |dz| = \int_\gamma \lambda(f(z)) |f'(z)| |dz| \stackrel{w = f(z)}{=} \int_{f \circ \gamma} \lambda(w) |dw| = L_{\lambda(f \circ \gamma)}. \]
Example 9.1.

If \( f : \mathbb{D} \to \mathbb{D} \) is a holomorphic function, then the pullback \( f^* \lambda_{\mathbb{D}}(z)|dz| \) of \( \lambda_{\mathbb{D}}(z)|dz| \) under \( f \) is

\[
f^* \lambda_{\mathbb{D}}(z) = \frac{|f'(z)|}{1 - |f(z)|^2}.
\]

Note that the Schwarz–Pick lemma can be written as \( f^* \lambda_{\mathbb{D}}(z) \leq \lambda_{\mathbb{D}}(z) \). Hence we may say that the Schwarz–Pick lemma is really an inequality between conformal pseudo-metrics!

Let \( \lambda(w)|dw| \) be a conformal pseudo-metric. We wish to introduce a quantity \( T_\lambda \) which is conformally invariant in the sense that

\[
T_f^* \lambda(z) = T_\lambda(f(z))
\]

for all conformal maps \( w = f(z) \).

Consider \( f^* \lambda(z)|dz| = \lambda(f(z))|f'(z)||dz| \). We need to eliminate the conformal factor \( |f'(z)| \).
For this we note that \( \log |f'| \) is harmonic, so \( \Delta(\log |f'|) = 0 \) and therefore

\[
\Delta(\log f^* \lambda)(z) = \Delta(\log \lambda \circ f)(z) + \Delta(\log |f'|)(z) = \Delta(\log \lambda \circ f)(z) = \Delta(\log \lambda)(f(z))|f'(z)|^2.
\]

In the last step we have used the chain rule \( \Delta(u \circ f) = (\Delta u \circ f)\cdot |f'|^2 \) for the Laplace Operator \( \Delta \) and holomorphic functions \( f \). We see that the quantity \( T_\lambda(z) := \Delta(\log \lambda)(z)/\lambda(z)^2 \) is conformally invariant.

Definition.

Let \( \lambda(z)|dz| \) be a regular conformal pseudo-metric on a domain \( \Omega \subseteq \mathbb{C} \), i.e., \( \lambda \) is of class \( C^2 \) for all points \( z \in \Omega \) such that \( \lambda(z) \neq 0 \). Then

\[
\kappa_\lambda(z) := -\frac{\Delta(\log \lambda)(z)}{\lambda(z)^2}, \quad z \in \Omega \text{ s.t. } \lambda(z) \neq 0,
\]

is called the (Gauß) curvature of \( \lambda(z)|dz| \).

Note that \( \kappa_\lambda(z) \) is only defined for all points with \( \lambda(z) > 0 \). If we put \( u(z) := \log \lambda(z) \), then

\[
\Delta u = -\kappa_\lambda(z) e^{2u}.
\]

This is a nonlinear partial differential equation, the so-called Gauß curvature equation. If \( \lambda(z)|dz| \) has constant curvature 0, then the curvature equation reduces to the Laplace equation.

Our preliminary considerations can now be summarized as follows.

**Theorem 9.2** (Theorema Egregium; Gauß 1827).

For every holomorphic function \( w = f(z) \) and every regular conformal pseudo-metric \( \lambda(w)|dw| \) the relation

\[
\kappa_{f^* \lambda}(z) = \kappa_\lambda(f(z))
\]

is satisfied provided \( \lambda(f(z))|f'(z)| > 0 \).

**Example 9.3.**

A quick computation reveals that the hyperbolic metric \( \lambda_{\mathbb{D}}(z)|dz| \) has constant curvature \(-4\).
The hyperbolic metric $\lambda_D(z)|dz|$ has an important extremal property.

**Theorem 9.4** (Ahlfors Lemma; Ahlfors 1938).

Let $\lambda(z)|dz|$ be a regular conformal pseudo-metric on $\mathbb{D}$ with curvature bounded above by $-4$. Then $\lambda(z) \leq \lambda_D(z)$ for every $z \in \mathbb{D}$.

**Proof.** Fix $0 < r < 1$ and consider $\lambda_r(z) := r\lambda(rz)$. Satz 9.2 implies that $\lambda_r(z)|dz|$ has curvature $\kappa\lambda(rz) \leq -4$. Hence the function $u_r : \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$, 

$$u_r(z) := \log \left( \frac{\lambda_r(z)}{\lambda_D(z)} \right),$$

 tends to $-\infty$ as $|z| \to 1$ and therefore attains its maximal value at some interior point $z_0 \in \mathbb{D}$, where $\lambda_r(z_0) > 0$. Since $u_r$ is of class $C^2$ at $z_0$, we have

$$0 \geq \Delta u_r(z_0) = \Delta \log \lambda_r(z_0) - \Delta \log \lambda_D(z_0) \geq 4 \lambda_r(z_0)^2 - 4 \lambda_D(z_0)^2.$$ 

It follows that $u_r(z) \leq u_r(z_0) \leq 0$ for all $z \in \mathbb{D}$, i.e., $\lambda_r(z) \leq \lambda_D(z)$, $z \in \mathbb{D}$. Now let $r \to 1$.

Ahlfors' lemma includes the lemma of Schwarz–Pick as a special case: choosing $\lambda(z) := \lambda_D(f(z))|f'(z)|$ for a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ in Lemma 9.4 implies

$$\frac{|f'(z)|}{1 - |f(z)|^2} = \lambda(z) \leq \lambda_D(z) = \frac{1}{1 - |z|^2}.$$ 

**Corollary 9.5** (Abstract form of Liouville’s theorem).

Suppose that $\Omega$ is a domain in $\mathbb{C}$, which carries a regular conformal metric $\lambda(w)|dw|$ with curvature $\leq -4$. Then any holomorphic function $f : \mathbb{C} \to \Omega$ is constant.

**Proof.** Let $f_R(z) := f(Rz)$ for $z \in \mathbb{D}$. The pullback $f_R^*\lambda(z)|dz|$ is a regular conformal pseudo-metric on $\mathbb{D}$ with curvature $\leq -4$. Ahlfors' lemma implies

$$\lambda(f_R(z))|f'_R(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$ 

This is equivalent to

$$\lambda(f(z))|f'(z)| \leq \frac{R}{R^2 - |z|^2}, \quad |z| < R.$$ 

Hence $\lambda(f(z))|f'(z)| \equiv 0$. Since $\lambda$ is always positive, we deduce $f' \equiv 0$, so $f$ is constant.

**Corollary 9.6.**

The complex plane $\mathbb{C}$ and the punctured plane $\mathbb{C}\{0\}$ do not carry a regular conformal metric with curvature $\leq -4$.

It is clear that if a domain $\Omega$ carries a regular conformal metric with curvature $\leq -4$, then every subdomain of $\Omega$ has the same property. Hence there is the inevitable question: "What is the 'largest' subdomain of $\mathbb{C}$ which carries a regular conformal metric with curvature $\leq -4"
Supplementary Exercises

1. Let $\lambda(z)|dz|$ be a conformal metric on a domain $\Omega \subset \mathbb{C}$. Prove that

$$d_\lambda(z_1, z_2) := \inf \left\{ \int_\gamma \lambda(z) |dz| : \gamma \text{ curve in } \Omega \text{ from } z_1 \text{ to } z_2 \right\}$$

induces a metric space $(\Omega, d_\lambda)$.

2. Suppose that $\lambda(z)|dz|$ is a conformal metric on $\mathbb{D}$ such that $T^* \lambda(z)|dz| = \lambda(z)|dz|$ for all $T \in \text{Aut}(\mathbb{D})$. Prove that $\lambda = c \lambda_{\mathbb{D}}$ for some constant $c > 0$.

3. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $\overline{f(\mathbb{D})} \subset \mathbb{D}$. Prove that $f$ has exactly one fixed point $\xi \in \mathbb{D}$.

   (Hint: First show that for each point $z_0 \in \mathbb{D}$ and small $\varepsilon > 0$ the function

   $$g_{z_0}(z) := f(z) + \varepsilon (f(z) - f(z_0))$$

   maps $\mathbb{D}$ into itself, so $\lambda_{\mathbb{D}}(g_{z_0}(z_0)) |g_{z_0}'(z_0)| \leq \lambda_{\mathbb{D}}(z_0)$. Deduce that it is possible to Banach’s fixed point theorem to $f : (\mathbb{D}, d_{\mathbb{D}}) \to (\mathbb{D}, d_{\mathbb{D}})$.)

4. Let $\delta > 0$ and suppose that $\lambda : \mathbb{C} \to [\delta, \infty)$ is a continuous function. Show that $(\mathbb{C}, d_\lambda)$ is a complete metric space.
Chapter 10

The theorems of Picard and Huber

Corollary 9.5 shows that Picard’s little theorem follows from the following result:

**Theorem 10.1.**

\( C'' := \mathbb{C} \setminus \{0, 1\} \) carries a regular conformal metric with curvature \( \leq -4 \).

With the help of the elliptic modular function (see Chapter 13) one can show that \( C'' \) actually carries a regular conformal metric with constant curvature \( -4 \). The following proof of Theorem 10.1 is due to Minda and Schober (1987). It avoids the use of the modular function and is entirely elementary.

**Proof.** We show that

\[
\tau(z) := \varepsilon \sqrt{1 + |z|^{3\delta}} \sqrt{1 + |z - 1|^{3\delta}} \\
\frac{|z|^{\delta}}{|z - 1|^{\delta}}
\]

induces a regular conformal metric \( \tau(z) \, dz \) on \( C'' \) with the property that its curvature is bounded above by \( -4 \) if \( \varepsilon > 0 \) is sufficiently small.

To prove this we use polar coordinates for the Laplacian. For \( r := |z| \) a computation shows

\[
\Delta \log \frac{1 + |z|^{3\delta}}{|z|^{\delta}} = \Delta \left( \frac{1}{2} \log (1 + r^{3\delta}) - \frac{5}{6} \log r \right) = \frac{1}{2} \Delta \left( \log (1 + r^{3\delta}) \right) = \\
\frac{1}{2} \left( \frac{\partial^2}{\partial r^2} \log(1 + r^{3\delta}) + \frac{1}{r} \frac{\partial}{\partial r} \log(1 + r^{3\delta}) \right) = \\
\frac{1}{18} (1 + |z|^{3\delta})^{2} |z|^{-\delta}
\]

and, similarly,

\[
\Delta \log \frac{1 + |z - 1|^{3\delta}}{|z - 1|^{\delta}} = \frac{1}{18} (1 + |z - 1|^{3\delta})^{2} |z - 1|^{-\delta}
\]

This gives

\[
\kappa_r(z) = - \frac{1}{18 \varepsilon^2} \left[ \frac{|z - 1|^{5/3}}{(1 + |z|^{\delta})^3 (1 + |z - 1|)^{\delta}} + \frac{|z|^{5/3}}{(1 + |z|^{\delta})(1 + |z - 1|)^{3\delta}} \right], \quad z \in C''.
\]

Hence

\[
\lim_{z \to 0} \kappa_r(z) = - \frac{1}{36 \varepsilon^2}, \quad \lim_{z \to 1} \kappa_r(z) = - \frac{1}{36 \varepsilon^2}, \quad \lim_{|z| \to \infty} \kappa_r(z) = -\infty,
\]

so we see that \( \kappa_r(z) \leq -4 \) for all \( z \in C'' \) if \( \varepsilon > 0 \) is sufficiently small.
We now consider holomorphic functions \( f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \) in a neighborhood of \( z = 0 \). The following lemma provides us with the analogue of \( \lambda_D(z)|dz| \) for the punctured unit disk \( \mathbb{D}' := \mathbb{D} \setminus \{0\} \) (see Exercise 10.1 for the proof).

**Lemma 10.2.**

The function
\[
p(z) := \exp \left( -\frac{1 + z}{1 - z} \right)
\]
maps \( \mathbb{D} \) holomorphically onto \( \mathbb{D}' \) and is locally univalent on \( \mathbb{D} \). The conformal metric
\[
\nu(z)|dz| = \frac{|dz|}{2|z|\log(1/|z|)}
\]
satisfies
\[
\nu(p(z))|p'(z)| = \lambda_D(z), \quad z \in \mathbb{D}.
\]
In particular, \( \nu(z)|dz| \) has constant curvature \(-4\). In addition, the \( \nu \)-length of the circle \( \partial K_r := \partial K_r(0) \) is given by
\[
L_\nu(\partial K_r) = \frac{\pi}{\log(1/r)},
\]
so \( L_\nu(\partial K_r) \to 0 \) as \( r \to 0 \).

**Corollary 10.3.**

Suppose that \( D \subseteq \mathbb{C} \) is a domain which carries a regular conformal pseudo-metric \( \lambda(w)|dw| \) with curvature \( \leq -4 \) and \( f : \mathbb{D} \setminus \{0\} \to D \) is a holomorphic function. Then \( L_\lambda(f(\partial K_r)) \to 0 \) for \( r \to 0 \).

**Proof.** For \( r \in (0,1) \) consider the curves \( l_r \subset \mathbb{D} \) defined by
\[
L_r(t) := \frac{\log(r) + it + 1}{\log(r) + it - 1}, \quad t \in [0,2\pi],
\]
that is, \( p(L_r(t)) = re^{it} \). Then
\[
L_\lambda(f(\partial K_r)) = L_\lambda(f \circ p \circ \partial K_r) = L_\lambda(f \circ p \circ L_r) \leq L_\lambda_D(L_r) = L_{p'\nu}(L_r) = L_\nu(\partial K_r) \to 0, \quad (r \to 0),
\]
in view of Lemma 10.2. Note that \( L_\lambda(f \circ p \circ \partial K_r) \leq L_\lambda_D(L_r) \) follows from \( (f \circ p)'\lambda(z) \leq \lambda_D(z) \) which in turn is a consequence of Ahlfors’ lemma. \( \Box \)

**Theorem 10.4 (Huber 1953).**

Let \( G \subseteq \mathbb{C} \) be a domain which carries a regular conformal metric with curvature \( \leq -4 \) and let \( f : \mathbb{D}' \to G \) be a holomorphic function. If there is a sequence \( (z_n) \subset \mathbb{D}' \) such that \( \lim_{n \to \infty} z_n = 0 \) and such that \( \lim_{n \to \infty} f(z_n) \) exists and belongs to \( G \), then \( z = 0 \) is a removable singularity of \( f \).

Note that this is a (far reaching) extension of Riemann’s theorem on removable singularities.

**Proof.** By hypothesis \( G \) carries a conformal metric \( \lambda(w)|dw| \) with curvature \( \leq -4 \). We may assume w.l.o.g. that \( r_n := |z_n| \) is monotonically decreasing to 0 and that \( f(z_n) \to 0 \in G \).
We consider the closed curves \( \gamma_n := f(\partial K_{r_n}) \in G \). Since \( (f^* \lambda)(z)|dz| \) is a regular conformal pseudo-metric on \( \mathbb{D}' \) with curvature \( \leq -4 \), we get from Corollary 10.3 that \( L_\lambda(\gamma_n) \to 0 \) as \( n \to \infty \). Now let \( K \) be a closed disk of radius \( \varepsilon > 0 \) centered at \( w = 0 \) which is compactly contained in \( G \). We may assume \( f(z_n) \in K \) for any \( n \in \mathbb{N} \). Since \( \lambda(w)|dw| \) is a conformal
metric, its density is bounded away from zero on K, so \( \lambda(w) \geq c > 0 \) for all \( w \in K \). Hence the euclidean length of \( \gamma_n \cap K \) is bounded above by \( L_\lambda(\gamma_n) / c \). In particular, \( \gamma_n = f(\partial K_n) \subset K \), i.e., \( |f(z)| \leq \epsilon \) on \( |z| = r_n \), for all but finitely many indices \( n \). The maximum principle implies that \( |f(z)| \leq \epsilon \) in a punctured neighborhood of \( z = 0 \), so the singularity \( z = 0 \) is removable.

\[ \text{Corollary 10.5 (Picard’s Great Theorem).} \]

Suppose that \( f : \mathbb{D} \to \mathbb{C} \) is a holomorphic function with an essential singularity at \( z = 0 \). Then there exists at most one complex number \( a \) such that the equation \( f(z) = a \) has no solution.

\[ \text{Proof.} \] Assume \( a, b \not\in f(\mathbb{D}) \) for \( a \neq b \). If \( c \in G := \mathbb{C} \setminus \{a, b\} \), then the theorem of Casorati–Weierstraß tells us that there exists a sequence \( (z_n) \subseteq \mathbb{D} \) such that \( z_n \to 0 \) and \( f(z_n) \to c \in G \). This, however, contradicts Huber's Theorem 10.4, because \( G = \mathbb{C} \setminus \{a, b\} \) carries a regular conformal metric with curvature \( \leq -4 \) by Theorem 10.1.

\[ \text{Supplementary Exercises} \]

1. Prove Lemma 10.2.

2. Let \( f \) be a nonconstant entire function. Show that there exists at most one complex number \( a \in \mathbb{C} \) such that \( f \) takes on every complex number \( \neq a \) infinitely often.

3. Let \( f : K_R(0) \to \mathbb{D} \) be a holomorphic function with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( a_1 \neq 0 \). Prove that

\[ R \leq \frac{2 |a_0| \log(1/|a_0|)}{|a_1|}. \]

Show that this inequality is best possible.
Chapter 11

Analytic Continuation

Definition (Regular and singular points).
Let $D \subseteq \mathbb{C}$ be an open disk and $f \in \mathcal{H}(D)$. A point $w \in \partial D$ is called a regular point of $f$ if there exists an open disk $D_1$ with center $w$ and a function $f_1 \in \mathcal{H}(D_1)$ such that $f(z) = f_1(z)$ for all $z \in D \cap D_1$. Any boundary point of $D$ which is not a regular point of $f$ is called a singular point of $f$.

Hence a point on $\partial D$ is regular if and only if $f$ has a holomorphic extension to a neighborhood of this point. Note that the set of regular points of a function $f$ is always open and hence the set of its singular points is closed.

Theorem 11.1.
Let $f \in \mathcal{H}(D)$ such that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1. Then $f$ has at least one singular point on $\partial D$.

Proof. Otherwise, using the compactness of $\partial D$, there exist open disks $D_1, \ldots, D_n$, each centered at a point of $\partial D$ and $f_j \in \mathcal{H}(D_j)$ with $f_j = f$ on $D \cap D_j$ for each $j = 1, \ldots, n$. Since $D_j \cap D_k$ is connected, the identity principle shows that $f_j = f_k$ on $D_j \cap D_k$. Hence we may define a function $h$ on $\Omega := D \cup D_1 \cup \ldots \cup D_n$ by $h(z) = f(z)$ for $z \in D$ and $h(z) = f_j(z)$ for $z \in D_j$. Then $h$ is holomorphic on a disk $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$ and $h(z) = \sum_{n=0}^{\infty} a_n z^n$ there, so that the radius of convergence is at least $1 + \varepsilon$, contrary to our assumption.

For instance, the geometric series $\sum_{n=0}^{\infty} z^n$ has a singular point at $z = 1$, see also Exercise 11.1.

Example 11.2.
Let
$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots, \quad z \in D.$$  
Clearly, $f$ has a singular point at $z = 1$ and
$$\lim_{r \to 1^{-}} f(r) = +\infty.$$  
Since $f(z^2) = f(z) - z$, we see that for each $t \in [0,1]$ and all $k,n \in \mathbb{N},$
$$f(te^{2\pi ik/2^n}) = \sum_{j=0}^{n-1} t^{2^j} e^{2\pi ik/2^n - j} + f(t^{2^n}),$$  
so $f$ is unbounded on the radius of $D$ which ends at $e^{2\pi ik/2^n}$. These points form a dense subset of $\partial D$, so we see that every point of $\partial D$ is a singular point of $f$. 

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This example shows that it is possible that a function \( f \in \mathcal{H}(D) \) cannot be analytically continued to any boundary point of \( D \). We next consider holomorphic functions that allow analytic continuation.

Recall that a curve \( \gamma : [0, 1] \to \mathbb{C} \) is always supposed to be piecewise continuously differentiable. A continuous function \( \gamma : [0, 1] \to \mathbb{C} \) is called a path.

**Definition** (Analytic continuation).

A function element is a pair \((f, D)\), where \( D \) is an open disk in \( \mathbb{C} \) and \( f \in \mathcal{H}(D) \). Suppose that \( \gamma : [0, 1] \to \mathbb{C} \) is a path. We say that the function element \((f_0, D_0)\) can be analytically continued along \( \gamma \) if there are numbers \( 0 = t_0 < t_1 < \ldots < t_n = 1 \) and function elements \((f_j, D_j)\), \( j = 1, \ldots, n \), such that

\[
(A1) \quad D_j = K_{t_j}(\gamma(t_j)) \quad \text{and} \quad \gamma([t_j, t_{j+1}]) \subset D_j, \quad \text{and}
\]

\[
(A2) \quad f_j = f_{j+1} \quad \text{in} \quad D_j \cap D_{j+1},
\]

for each \( j = 0, \ldots, n - 1 \). We say that \((f_n, D_n)\) is defined by analytic continuation of \((f_0, D_0)\) along \( \gamma \).

The next result shows that analytic continuation along a path \( \gamma \) does not depend on the choice of the function elements \((f_j, D_j)\) along \( \gamma \).

**Lemma 11.3.**

Suppose that \((f_0, D_0)\) is a function element, \( \gamma : [0, 1] \to \mathbb{C} \) is a path, and \((f_n, D_n)\), \((g_m, D_m)\) are defined by analytic continuations of \((f_0, D_0)\) along \( \gamma \), then \( f_n \equiv g_m \) on \( D_n \cap D_m \).

Hence, we can talk about the analytic continuation of \( f_0 \) along \( \gamma \).

**Example 11.4** (Local primitives).

Suppose that \( \Omega \subseteq \mathbb{C} \) is a domain, \( D_0 := K_{t_0}(z_0) \subseteq \Omega \), \( f \in \mathcal{H}(\Omega) \) and \( F_0 \in \mathcal{H}(D_0) \) is a primitive of \( f \). Then \((F_0, D_0)\) can be analytically continued along any path \( \gamma : [0, 1] \to \Omega \) starting in \( z_0 \). If \( \gamma \) is a curve and \((F_n, D_n)\) is defined by analytic continuation of \((F_0, D_0)\) along \( \gamma \), then

\[
\int_{\gamma} f(z) \, dz = F_n(\gamma(1)) - F_0(\gamma(0)).
\]

**Proof.** Suppose that \( \gamma : [0, 1] \to \Omega \) is a path in \( \Omega \) with \( \gamma(0) = z_0 \). We can find \( 0 = t_0 < t_1 < \ldots < t_n = 1 \) and open disks \( D_j = K_{t_j}(\gamma(t_j)) \) in \( \Omega \) such that \((A1)\) holds. On each disk \( K_j \) there is a primitive \( F_j \) of \( f \) such that \( F_{j+1}(\gamma(t_{j+1})) = F_j(\gamma(t_{j+1})) \). Then \( F_{j+1} \equiv F_j \) on \( D_j \cap D_{j+1} \), so \( F \) can be analytically continued along \( \gamma \). If \( \gamma : [0, 1] \to \Omega \) is a curve, then

\[
\int_{\gamma} f(z) \, dz = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t)) \gamma'(t) \, dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} F_j(\gamma(t)) \gamma'(t) \, dt
\]

\[
= \sum_{j=0}^{n-1} \left( F_j(\gamma(t_{j+1})) - F_j(\gamma(t_j)) \right) = \sum_{j=0}^{n-1} \left( F_{j+1}(\gamma(t_{j+1})) - F_j(\gamma(t_j)) \right) = F_n(\gamma(1)) - F_0(\gamma(0)).
\]

\( \Box \)
Note that if $F_0 : D_0 \to \mathbb{C}$ and $\tilde{F}_0 : \tilde{D}_0 \to \mathbb{C}$ are two local primitives of $f$ at $z_0 \in D_0 \cap \tilde{D}_0$, then $F_0(z_0) = \tilde{F}_0(z_0) + c$ for some $c \in \mathbb{C}$. Thus $(F_0, D_0)$ and $(\tilde{F}_0 + c, \tilde{D}_0)$ have the same analytic continuation along $\gamma$ by Lemma 11.3. Hence $F_n = \tilde{F}_n + c$. This allows us to make the following definition.

**Definition (Integration of holomorphic functions along paths).**

Let $\Omega \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{H}(\Omega)$ and $\gamma : [0,1] \to \Omega$ a path in $\Omega$ from $a$ to $b$. Then

$$\int_\gamma f(z) \, dz := F_n(b) - F_n(a),$$

where $F_0$ is a primitive of $f$ in a disk $D_0 \subseteq \Omega$ centered at $\gamma(0)$ and $F_n$ is the analytic continuation of $F_0$ along $\gamma$.

We can now integrate holomorphic functions along paths (and not only along curves)! In the case of a curve $\gamma$ the definition of $\int_\gamma f(z) \, dz$ for holomorphic functions is compatible with the usual definition of a “line integral”. This follows from Example 11.4. In particular, if $\gamma : [0,1] \to \mathbb{C}$ is a path and $a \in \mathbb{C}$ is a point which does not lie on $\gamma$, then the winding number $n(\gamma, a)$ of $\gamma$ w.r.t. $a$,

$$n(a, \gamma) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a},$$

is well-defined.

We next consider the dependence of analytic continuation on the path.

**Definition (Homotopy).**

Suppose that $\Omega \subseteq \mathbb{C}$ is a domain, $a,b \in \Omega$, and $\gamma_0 : [0,1] \to \Omega$ and $\gamma_1 : [0,1] \to \Omega$ are two paths in $\Omega$ from $a$ to $b$. We say that $\gamma_0$ and $\gamma_1$ are homotopic (in $\Omega$) if there is a continuous mapping $H : [0,1] \times [0,1] \to \Omega$ such that $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$ and $H(0,s) = a$, $H(1,s) = b$ for all $t,s \in [0,1]$. We call $H$ a homotopy in $\Omega$ between $\gamma_0$ and $\gamma_1$. A closed path $\gamma : [0,1] \to \Omega$ is called null-homotopic in $\Omega$, if there exists a homotopy in $\Omega$ between $\gamma$ and the constant path $t \mapsto \gamma(0)$.

Intuitively this means that the curve $\gamma_0$ can be continuously deformed to $\gamma_1$ within $\Omega$ keeping the end points fixed.

**Theorem 11.5 (Monodromy Theorem).**

Suppose that $\Omega \subseteq \mathbb{C}$ is a domain and $H : [0,1] \times [0,1] \to \Omega$ is a homotopy between paths $\gamma_0$ and $\gamma_1$ in $\Omega$ from $a$ to $b$. Suppose that $(f_0, D_0)$ is a function element with $D_0 = K_r(a) \subseteq \Omega$, which can be analytically continued along every path $\gamma_s := H(\cdot, s)$. If $g_s$ is defined by analytic continuation of $f$ along $\gamma_s$, then $g_s \equiv g_0$ for every $s \in [0,1]$.

In particular, if $\gamma_0$ and $\gamma_1$ are two homotopic paths in $\Omega$, then analytic continuation of $(f_0, D_0)$ along $\gamma_0$ coincides with its continuation along $\gamma_1$.

**Proof.** For each fixed $s \in [0,1]$ there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ and function elements $(f_j, D_j)$ such that $\gamma_s([t_j, t_{j+1}]) \subseteq D_j$ and $f_n = g_s$. Let $\varepsilon_j > 0$ denote the distance of the sets $\gamma_s([t_j, t_{j+1}])$ and $\Omega \setminus D_j$; and let $\varepsilon := \min \{\varepsilon_j\} > 0$. Then there is a number $\delta > 0$ such that $|\gamma_s(t) - \gamma_s(u)| < \varepsilon$ for all $|s-u| < \delta$ and all $t \in [0,1]$. Let $u \in [0,1]$ such that $|s-u| < \delta$. Then $(f_n, D_n)$ is also an analytic continuation of $(f_0, D_0)$ along $\gamma_u$, so $f_n = g_u$. Thus the set $\{s \in [0,1] : g_s = f_n\}$ is open and obviously also closed, and therefore equal to $[0,1]$. \(\square\)
Corollary 11.6 (The homotopy form of Cauchy’s theorem).
Let \( \Omega \subseteq \mathbb{C} \) be a domain, \( f \in \mathcal{H}(\Omega) \) and \( \gamma_0, \gamma_1 \) two paths in \( \Omega \) from \( a \) to \( b \). If \( \gamma_0 \) and \( \gamma_1 \) are homotopic in \( \Omega \), then
\[
\int_{\gamma_0} f(z) \, dz = \int_{\gamma_1} f(z) \, dz.
\]
In particular, if \( \gamma \) is a null-homotopic closed path in \( \Omega \), then \( n(\gamma, a) = 0 \) for every \( a \in \mathbb{C} \setminus \Omega \).
In particular, every null-homotopic path is also a null-homologous path.

Proof. Let \( D_0 \) be a disk in \( \Omega \) centered at \( a \) and suppose that \( F \in \mathcal{H}(D_0) \) is a primitive of \( f \) in \( D_0 \). Let \( F_j \) be the analytic continuation of \( F \) along \( \gamma_j \). Then \( F_0 = F_1 \), so
\[
\int_{\gamma_0} f(z) \, dz = F_0(b) - F(a) = F_1(b) - F(a) = \int_{\gamma_1} f(z) \, dz.
\]
\( \square \)

Corollary 11.7.
Suppose that \( \Omega \subseteq \mathbb{C} \) is a domain. Then the following statements are equivalent.

(a) \( \Omega \) is simply connected.

(b) Every closed path in \( \Omega \) is null-homotopic in \( \Omega \).

Condition (b) gives an "intrinsic" characterization of simply connected domains in \( \mathbb{C} \): we can decide if a domain \( \Omega \) is simply connected or not without "leaving" the domain.

Proof. "(b) \( \implies \) (a)" Suppose that \( \gamma \) is a closed curve in \( \Omega \). Then \( \gamma \) is null-homotopic in \( \Omega \). By Corollary 11.6, \( \gamma \) is also a null-homologous curve in \( \Omega \).

"(a) \( \implies \) (b)" Let \( \gamma : [0,1] \to \Omega \) be a closed path in \( \Omega \). If \( \Omega = \mathbb{C} \) we can take \( H(t,s) := (1-s)\gamma(t) + s\gamma(0) \) as a homotopy between \( \gamma \) and the constant path \( t \mapsto \gamma(0) \). If \( \Omega \neq \mathbb{C} \), then the Riemann mapping theorem shows that there is a conformal map \( f \) from \( \Omega \) onto \( \mathbb{D} \). Then \( H(t,s) := f^{-1}((1-s)f(\gamma(t)) + sf(\gamma(0))) \) is a homotopy between \( \gamma \) and the constant path \( t \mapsto \gamma(0) \).
\( \square \)

Corollary 11.8.
Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain and \((f_0,D_0)\) a function element with \( D_0 \subseteq \Omega \) that can be analytically continued along any path in \( \Omega \) starting at the center of \( D_0 \). Then there is function \( f \in \mathcal{H}(\Omega) \) such that \( f(z) = f_0(z) \) for all \( z \in D_0 \).

Proof. Let \( D_0 = K_r(a) \) and for each \( z_0 \) let \( \gamma_{z_0} \) a path in \( \Omega \) from \( a \) to \( z_0 \). Then \( (f_0,D_0) \) has an analytic continuation \((f_{z_0},D_{z_0})\) along \( \gamma_{z_0} \). We set \( f(z_0) := f_{z_0}(z_0) \). By Corollary 11.7 and Theorem 11.5, \( f_{z_0}(z_0) = f_{z_0}(\gamma_{z_0}(1)) \) does not depend on the choice of \( \gamma_{z_0} \), so we obtain a well-defined function \( f \in \mathcal{H}(\Omega) \) such that \( f(z) = f_0(z) \) for any \( z \in D_0 \).
\( \square \)

Supplementary Exercises

1. Let \((a_n)\) be a sequence of non-negative real numbers such that \( \lim_{n \to \infty} |a_n|^{1/n} = 1 \). Show that \( z = 1 \) is a singular point of the function \( f(z) := \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) \).
2. Consider 
\[ f(z) = \sum_{n=1}^{\infty} 2^{-n^2} z^{2n}. \]
(a) Show that every derivative \( f^{(k)} \) extends to a continuous function on \( \mathbb{D} \).
(b) Show that if \( n \) is a positive integer then there exists a polynomial \( p_n \) such that
\[ f(e^{2\pi i z^n}) = p_n(z) + f(z), \quad z \in \mathbb{D}, \]
and deduce that every point of \( \partial \mathbb{D} \) is a singular point of \( f \).
(c) Show that
\[ g(z) := 2z + \sum_{n=1}^{\infty} 2^{-n^2} z^{2n} \]
is univalent in \( \mathbb{D} \).

3. Let \( \Omega \subseteq \mathbb{C} \) be a domain and \( u : \Omega \to \mathbb{R} \) a harmonic function. Suppose that \( D_0 \) is an open disk in \( \Omega \) and \( f_0 \in \mathcal{H}(D_0) \) such that \( u = \text{Re}(f_0) \) in \( D_0 \). Prove that \( (f_0, D_0) \) has an analytic continuation along any path in \( \Omega \) starting in \( D_0 \).

4. Let \( \Omega \subseteq \mathbb{C} \) be a domain, \( h \in \mathcal{H}(\Omega) \), and \( w_0, w_1 \in \mathbb{C} \). We consider the initial value problem
\[ w'' + h(z)w = 0, \quad w(z_0) = w_0, \quad w'(z_0) = w_1. \]
Show that for every \( z_0 \in \Omega \) and every disk \( K_r(z_0) \) the initial value problem has a unique solution \( w \in \mathcal{H}(K_r(z_0)) \) and deduce that this solution can be analytically continued along any path in \( \Omega \). Finally, show that if \( \Omega \) is simply connected, then the initial value problem has a unique solution in \( \mathcal{H}(\Omega) \).

5. Let \( G := \{ z \in \mathbb{C} : \, r < |z| < R \} \), \( U := \{ w \in \mathbb{C} : \, \log r < \text{Re} \, w < \log R \} \) and \( p : U \to G \), \( p(w) = e^w \). Suppose that \( f \) is holomorphic on \( K_r(z_0) \subseteq G \) such that \( f \) has an analytic continuation along any path in \( G \) starting at \( z_0 \). Finally, let \( w_0 \in U \) such that \( p(w_0) = z_0 \). Show that there is a uniquely determined holomorphic function \( F : U \to \mathbb{C} \) such that \( F = f \circ p \) in a neighborhood of \( w_0 \).

6. (A converse to Corollary 11.8)
Suppose that \( \Omega \subseteq \mathbb{C} \) is a domain with the following property: For each function element \( (f_0, D_0) \) with \( D_0 \subseteq \Omega \), which can be analytically continued along any path \( \gamma : [0, 1] \to \Omega \) that starts in \( D_0 \), there exists a function \( f \in \mathcal{H}(\Omega) \) such that \( f = f_0 \) on \( D_0 \). Prove that \( \Omega \) is simply connected.
Chapter 12

The Schwarz reflection principle

Definition.
We say that a domain $\Omega \subseteq \mathbb{C}$ is symmetric about the real axis if $x \in \Omega$ whenever $z \in \Omega$.
In this case we will write
$\Omega^+: = \{z \in \Omega : \text{Im}z > 0\}$,
$\Omega^-: = \{z \in \Omega : \text{Im}z < 0\}$, and
$\Omega^0:= \Omega \cap \mathbb{R}$.

Theorem 12.1 (Schwarz reflection principle for harmonic functions).
Let $\Omega \subseteq \mathbb{C}$ be a domain that is symmetric about the real axis and let $u : \Omega^+ \to \mathbb{R}$ be a harmonic function such that
$$\lim_{z \to x} u(z) = 0 \quad \text{for each} \quad x \in \Omega^0.$$ Then $u$ extends to a function $v$ harmonic in $\Omega$; the function $v$ satisfies
$$v(\bar{z}) = -v(z), \quad z \in \Omega.$$

Proof. We set
$$v(z): = \begin{cases} u(z) & z \in \Omega^+, \\
0 & \text{if} \quad z \in \Omega^0, \\
-u(z) & z \in \Omega^-.
\end{cases}$$ We need only show that $v$ is harmonic in $\Omega$. Since $v$ is clearly continuous on $\Omega$ we need only prove that it satisfies the mean value property at each point $a \in \Omega$. If $a \in \Omega^+$, this is clear in view of Theorem 4.9 since $u$ is harmonic in $\Omega^+$. If $a \in \Omega^-$, this is also clear since
$$\Delta v(z) = -\frac{\partial^2}{\partial x^2}u(x - iy) - \frac{\partial^2}{\partial y^2}u(x - iy) = -\frac{\partial^2 u}{\partial x^2}(x - iy) - \frac{\partial^2 u}{\partial y^2}(x - iy) = -(\Delta u)(\bar{z}) = 0,$$ for all $z \in \Omega^-$ as $\bar{z} \in \Omega^+$ and $u$ is harmonic in $\Omega^+$. It remains to consider the case $a \in \Omega^0$. We have $v(a + re^{-it}) = v\left(\frac{a + re^{it}}{a + re^{-it}}\right) = -v(a + re^{it})$, so
$$\frac{1}{2\pi} \int_0^{2\pi} v(a + re^{it})dt = \frac{1}{2\pi} \int_0^{2\pi} v(a + re^{-it})dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{v(a + re^{it}) + v(a + re^{-it})}{2}dt$$
$$= \frac{1}{2\pi} \int_0^{2\pi} 0 dt = 0 = v(a).$$

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Theorem 12.2 (Schwarz reflection principle for holomorphic functions).
Let \( \Omega \subseteq \mathbb{C} \) be a domain that is symmetric about the real axis and \( f \in \mathcal{H}(\Omega^+) \) such that
\[
\lim_{z \to x^-} \Im f(z) = 0 \quad \text{for each } x \in \Omega^0.
\]
Then \( f \) extends analytically to a function in \( \mathcal{H}(\Omega) \), which we again denote by \( f \), such that
\[
f(z) = \overline{f(\overline{z})}, \quad z \in \Omega.
\]
We say that \( f \in \mathcal{H}(\Omega^+) \) has been **analytically continued by reflection** across the real line.

**Remark.**
We should note that we are only assuming that \( \Im f \) extends continuously from \( \Omega^+ \) to \( \Omega^0 \), but we are not making any assumption about \( \Re f \). If we assume that \( f \) extends continuously to \( \Omega^0 \) and takes real values there, then a somewhat simpler proof is possible using Morera's theorem.

**Proof.** By Theorem 12.1, \( v := \Im f \) is extended to a harmonic function \( v \) on \( \Omega \) by letting \( v(z) := -v(\overline{z}) \) for \( z \in \Omega^- \) and \( v(z) = 0 \) for \( z \in \Omega^0 \). If we define \( F(z) := \overline{f(\overline{z})} \) in \( \Omega^- \) and \( F := f \) in \( \Omega^+ \), then \( F \) is analytic in \( \Omega^+ \cup \Omega^- \). We show that \( F \) extends to a holomorphic function in \( \Omega \). To prove this we fix an open disk \( D \subseteq \Omega \) centered at some point in \( \Omega^0 \). Since \( D \) is simply connected, there is a function \( g \in \mathcal{H}(D) \) such that \( v = \Im g \). Since \( f-g \in \mathcal{H}(D^+) \) is real-valued, it is constant, so \( g = f + c \) for some \( c \in \mathbb{R} \). Thus \( h := g - c \in \mathcal{H}(D) \) such that \( h = f \) in \( D^+ \). Since \( h \) is real on \( D^0 \), we have \( h(z) = \overline{h(\overline{z})} \). Hence \( F = h \) in \( D^+ \cup D^- \), so \( F \) extends to a holomorphic function in \( D \).

Theorem 12.3.
Suppose that \( \Omega \subseteq \mathbb{C} \) is a domain which is symmetric about the real axis and \( f \in \mathcal{H}(\Omega^+) \) such that
\[
\lim_{z \to x^+} |f(z)| = 1 \quad \text{for every } x \in \Omega \cap \mathbb{R}.
\]
Then \( f \) has an analytic continuation to a neighborhood of \( \Omega \cap \mathbb{R} \) and a meromorphic continuation to \( \Omega \) such that
\[
f(z) = \frac{1}{f(\overline{z})}, \quad z \in \Omega.
\]

**Proof.** Fix a point \( x_0 \in \Omega^0 \). Since \( |f(z)| \to 1 \) as \( z \to x_0 \) there is a disk \( K_\epsilon(x_0) \) such that \( f \neq 0 \) in \( K_\epsilon^+(x_0) \). Then \( g(z) := i \log f(z) \) is analytic in \( K_\epsilon^+(x_0) \) and \( \Im g(z) = \Re \log f(z) = \log |f(z)| \to 0 \) as \( z \to (x_0 - \tau, x_0 + \tau) \). By the Schwarz reflection principle, \( g \) has an analytic continuation to \( K_\epsilon(x_0) \) such that \( g(\overline{z}) = \overline{g(z)} \). Hence \( f(z) := e^{-ig(z)} \) is analytic in \( K_\epsilon(x_0) \) and satisfies \( f(z) = e^{-i\overline{g(z)}} = g^{-i\overline{g(z)}} = e^{\overline{g(z)}} = 1/\overline{f(z)} \).

**Definition** (Free analytic boundary arc).
A curve \( \gamma : [a,b] \to \mathbb{C} \) is called a **free analytic boundary arc** of a domain \( \Omega \), if for each \( t_0 \in (a,b) \) the point \( \gamma(t_0) \) has an open neighborhood \( U \) for which there is a conformal map \( \varphi \) from a disk \( D \) centered at a point on the real axis onto \( U \) such that
\[
\varphi(D^+) \subseteq \Omega, \quad \varphi(D \cap \mathbb{R}) = \gamma \cap U, \quad \varphi(D^-) \subseteq \mathbb{C} \setminus \overline{\Omega}.
\]
We call \( \text{int}(\gamma) := \{ \gamma(t) : t \in (a,b) \} \) the interior of the free analytic boundary arc \( \gamma \).
Corollary 12.4.
Let \( \Omega \subseteq \mathbb{C} \) be a domain, \( \gamma \) a free analytic boundary arc of \( \Omega \), and \( f \in \mathcal{H}(\Omega) \) such that

(a) \( \text{Im } f(z) \to 0 \) as \( z \to \eta \) for every \( \eta \in \text{int}(\gamma) \),

or

(b) \( |f(z)| \to 1 \) as \( z \to \eta \) for every \( \eta \in \text{int}(\gamma) \).

Then \( f \) has an analytic continuation to \( \Omega \cup \text{int}(\gamma) \) by reflection across \( \gamma \).

Proof. We only consider the case (a). Fix a point \( z_0 \in \text{int}(\gamma) \) and a corresponding conformal map \( \varphi \) from \( D \) onto \( U \). Then \( g := f \circ \varphi \) is holomorphic in \( D^+ \) such that \( \text{Im } g(w) \to 0 \) as \( w \to D \cap \mathbb{R} \). Theorem 12.2 shows that \( g \) can be analytically continued to \( D \) by reflection across the real line. Hence \( f(z) := g \circ \varphi^{-1} \) is analytic on \( U \). \( \Box \)

The focus now shifts from general holomorphic functions to conformal maps.

Theorem 12.5 (Boundary behaviour of conformal maps).
Let \( \Omega \) and \( \Omega' \) be domains in \( \mathbb{C} \) and \( f \) a conformal map from \( \Omega \) onto \( \Omega' \). Suppose \( (z_n) \) is a sequence in \( \Omega \) such that \( z_n \to w \in \partial \Omega \). Then the limit of every convergent subsequence of \( (f(z_n)) \) belongs to \( \partial \Omega' \). In particular, if \( f \) is a conformal map from \( \Omega \) onto \( D \), then

\[
\lim_{z \to w} |f(z)| = 1 
\]

for each \( x \in \partial \Omega \).

We say that \( f(z) \) tends to the boundary of \( \Omega' \) whenever \( z \) tends to the boundary of \( \Omega \).

Proof. Let \( (f(z_n)) \) be a convergent subsequence with limit \( \alpha \in \Omega' \). Then \( K := \{f(z_n) : k \in \mathbb{N} \} \cup \{\alpha \} \) is a compact subset of \( \Omega' \). Since \( f \) is a conformal map, its inverse \( f^{-1} \) is continuous, so \( f^{-1}(K) \) is a compact subset of \( \Omega \). As \( (z_n) \) tends to the boundary, \( z_n \notin f^{-1}(K) \) for all but finitely many \( n \), so \( f(z_n) \notin K \) for all but finitely many \( k \). This is a contradiction. \( \Box \)

Remark.
The proof of Theorem 12.5 made only use of the fact that both \( f : \Omega \to \Omega' \) and its inverse \( f^{-1} : \Omega' \to \Omega \) are continuous functions, that is, \( f \) is a homeomorphism from \( \Omega' \) onto \( \Omega \).

Example 12.6.
Suppose that \( f \) is a conformal map from \( \Omega \) onto \( \Omega' \) and \( (z_n) \) is a sequence in \( \Omega \) such that \( z_n \to w \in \partial \Omega \). Then Theorem 12.8 does not say that \( (f(z_n)) \) converges to a point on \( \partial \Omega' \). In fact, if \( f \) is the inverse of the Koebe function \( k \) and \( z_n = -1 + (-1)^n i \), then \( (f(z_n)) \) does not converge.
Example 12.7 (The Cayley map; revisited).
Suppose that $f$ is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$ such that $f(i) = 0$. Then by Theorem 12.5 and Theorem 12.3, $f$ has a meromorphic continuation to $\mathbb{C}$ such that $f(z) = 1/f(z)$. In particular, $f$ has only one (simple) pole at $z = -i$ and is one-to-one on $\mathbb{C}\{i\}$. Hence $g(z) := \frac{z+i}{z-i}$ is a zero-free entire function such that $|g(z)| \to 1$ as $z \to \infty$. Since $f$ maps $\mathbb{H}$ conformally onto $\mathbb{D}$ it has a removable singularity at $\infty$, so $g$ is bounded and hence $g(z) = \eta$ for some $\eta \in \partial \mathbb{D}$ by Liouville's theorem. We see that $f(z) = \eta \frac{z+i}{z-i}$.

Theorem 12.8.
Let $\Omega$ be a simply connected domain bounded by a Jordan curve which is the union of finitely many free analytic boundary arcs with pairwise disjoint interior. Then every conformal map $f$ from $\Omega$ onto $\mathbb{D}$ has a continuous extension to $\overline{\Omega}$ which maps $\overline{\Omega}$ homeomorphically onto $\overline{\mathbb{D}}$. In fact, $f$ extends analytically across the interior of each free analytic boundary arc of $\Omega$.

Proof. Suppose that the free analytic boundary arcs are denoted by $\gamma_1, \ldots, \gamma_n$ such that for each $k = 2, \ldots, n$ the point $z_k$ is the starting point of $\gamma_k$ and the final point of $\gamma_{k-1}$ and $\{z_1\} = \gamma_1 \cap \gamma_n$.

(i) $f$ has an analytic and one-to-one extension to $\overline{\Omega}\{z_1, \ldots, z_n\}$.
Let $\gamma : [a, b] \to \mathbb{C}$ be one of the free analytic boundary arcs of $\Omega$ and let $\varphi : D \to G$ denote the corresponding conformal map. By Corollary 12.4 we see that $f$ has an analytic continuation to $\Omega \cup \text{int}(\gamma)$ such that $|f(z)| < 1$ on one side of $\gamma$ and $|f(z)| > 1$ on the other side of $\gamma$. Hence, if $z_0 \in \text{int}(\gamma)$, then $f'(z_0) \neq 0$ since otherwise every point close to $f(z_0)$ would have at least two preimages close to $z_0$. Let $z_0 \in \gamma$ and $w_0 = f(z_0)$. Choose a disk $U$ centered at $z_0$ such that $f$ is one-to-one on $U$. Then $f(U)$ includes all points of $\mathbb{D}$ in a sufficiently small disk $V$ centered at $w_0$. Since $f$ is one-to-one on $\Omega$ no point of $\Omega \setminus U$ is mapped to $V$. Hence no other point of $\gamma$ and no point lying on any other free analytic boundary arc of $\Omega$ is mapped to $z_0$. It follows that $f$ is one-to-one on $\gamma$ and that the image of $\gamma$ is disjoint from the image of any other free analytic boundary arc of $\Omega$.

(ii) $f$ extends to a homeomorphism from $\overline{\Omega}$ onto $\overline{\mathbb{D}}$.
Let $A_k$ denote the image of $\text{int}(\gamma_k)$ by $f$. Note that $A_k$ is an open subarc of $\partial \mathbb{D}$. We show that $A_{k-1}$ and $A_k$ meet at a point $w_k \in \partial \mathbb{D}$. Otherwise there would be an arc $\Gamma$ between $A_{k-1}$ and $A_k$. We may assume that $a_{k-1}$ is the starting point of $\Gamma$ and $a_k$ is the end point. Fix $\delta > 0$ and let $\gamma_\delta := \Omega \cap \partial K_\delta(z_k)$ for $0 < \delta < 1$. Then $f \circ \gamma_\delta$ is a Jordan arc from a point of $A_{k-1}$ to a point of $A_k$. By Theorem 12.5, the points of $f \circ \gamma_\delta$ tend to $\partial \mathbb{D}$ as $\delta \to 0$ and the endpoints of $f \circ \gamma_\delta$ tend to $a_{k-1}$ and $a_k$ respectively. It follows that if $w \to \Gamma$, then $f^{-1}(w)$ tends to $z_k$. Hence $f^{-1}(w) - z_k$ tends to 0 as $w \to \Gamma$. By (a variant of) Theorem 12.2, we see that $f^{-1} - z_k$ extends analytically across $\Gamma$. Since $f^{-1} - z_k = 0$ on $\Gamma$ this would imply $f^{-1} \equiv z_k$, which is not possible. Hence $A_{k-1}$ and $A_k$ meet at a point $w_k$ and $f$ extends continuously to $z_k$ such that $f(z_k) = w_k$. Moreover, $\partial \Omega = \cup_{j=1}^n \overline{A_j}$ and $f$ extends to a continuous one-to-one map from $\overline{\Omega}$ onto $\overline{\mathbb{D}}$. □

Theorem 12.9.
Let $\Omega$ be a bounded simply connected domain and $\partial \Omega$ the union of finitely many free analytic boundary arcs with pairwise disjoint interior. Suppose that $b_1, b_2, b_3$ are pairwise different and positively oriented points on $\partial \Omega$ and $a_1, a_2, a_3$ are pairwise different and positively oriented points on $\partial \mathbb{D}$. Then there exists a unique conformal map from
\( \Omega \) onto \( \mathbb{D} \) such that its homeomorphic extension to \( \overline{\Omega} \) maps \( b_j \) to \( a_j \) for \( j = 1, 2, 3 \).

**Proof.** By the Riemann mapping theorem and Theorem 12.8 there is a homeomorphism \( f \) from \( \overline{\Omega} \) onto \( \overline{\mathbb{D}} \) such that \( f \) maps \( \Omega \) conformally onto \( \mathbb{D} \) and the boundary \( \partial \Omega \) homeomorphically onto \( \partial \mathbb{D} \). Let \( b'_j := f(b_j) \) for \( j = 1, 2, 3 \). Then \( b'_1, b'_2, b'_3 \) are pairwise different and positively oriented points on \( \partial \mathbb{D} \). Exercise 12.6 shows that there exists a unique \( T \in \text{Aut}(\mathbb{D}) \) such that \( T(b'_j) = a_j \) for \( j = 1, 2, 3 \). Hence \( T \circ f \) is a conformal map from \( \Omega \) onto \( \mathbb{D} \) such that its homeomorphic extension to \( \overline{\Omega} \) maps \( b_j \) to \( a_j \) for \( j = 1, 2, 3 \). If \( g \) is another map with the same properties then \( S := T \circ f \circ g^{-1} \in \text{Aut}(\mathbb{D}) \) has three fix points \( a_1, a_2, a_3 \) on \( \partial \mathbb{D} \). Exercise 12.6 shows that \( S \) is the identity map, so \( g = T \circ f \). This proves uniqueness. \( \Box \)

In some cases it is convenient to replace the unit disk \( \mathbb{D} \) as the standard domain by the upper half-plane \( \mathbb{H} := \{ w \in \mathbb{C} : \text{Im} \ w > 0 \} \). Note that the Cayley map \( T(w) = \frac{w - 1}{w + 1} \) maps \( \mathbb{H} \) conformally onto \( \mathbb{D} \) such that \( T(\infty) = 1 \) and \( T^{-1}(1) = \infty \) in the sense that

\[
\lim_{|w| \to \infty} T(w) = 1 \quad \text{and} \quad \lim_{z \to 1} |T^{-1}(z)| = \infty.
\]

In particular, \( T \) extends to a homeomorphism of \( \overline{\mathbb{H}} := \mathbb{H} \cup \{ \infty \} \) onto \( \mathbb{D} \).

**Definition.**

Let \( \Omega \) be a bounded simply connected domain and suppose that \( T(z) = \frac{z - 1}{z + 1} \) is the Cayley map from \( \mathbb{H} \) onto \( \mathbb{D} \). We say that a conformal map \( f \) from \( \Omega \) onto \( \mathbb{H} \) has a continuous extension to \( \overline{\mathbb{H}} \) which maps \( \overline{\Omega} \) homeomorphically onto \( \overline{\mathbb{H}} \) if \( g := T \circ f \) has a continuous extension to \( \overline{\mathbb{H}} \) which maps \( \overline{\Omega} \) homeomorphically onto \( \overline{\mathbb{D}} \). If \( w = g^{-1}(1) \), then we let \( f(w) := \infty \).

The following variant of Theorem 12.8 is now self-evident.

**Theorem 12.10.**

Let \( \Omega \) be a simply connected domain bounded by a Jordan curve which is the union of finitely many free analytic boundary arcs with pairwise disjoint interior. Then every conformal map \( f \) from \( \Omega \) onto \( \mathbb{H} \) has a continuous extension to \( \overline{\Omega} \) which maps \( \overline{\Omega} \) homeomorphically onto \( \overline{\mathbb{H}} \). In fact, \( f \) extends analytically across the interior of each free analytic boundary arc of \( \Omega \).

We close this section by mentioning without proof a far-reaching extension of Theorem 12.8. A Jordan domain \( \Omega \) is a bounded simply connected domain in \( \mathbb{C} \) which is bounded by a Jordan curve.

**Theorem (Carathéodory).**

Let \( \Omega \subseteq \mathbb{C} \) be a Jordan domain. Then every conformal map from \( \Omega \) onto \( \mathbb{D} \) extends to a homeomorphism from \( \overline{\Omega} \) onto \( \overline{\mathbb{D}} \).

Supplementary Exercises

1. Suppose that \( f \in \mathcal{H}(\mathbb{D}) \) and that \( \operatorname{Re} f : \mathbb{D} \to \mathbb{R} \) has a continuous extension to \( \overline{\mathbb{D}} \). Show that
   \[
   f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left( \operatorname{Re} f(e^{it}) \right) \frac{e^{it} + z}{e^{it} - z} \, dt + i \operatorname{Im} f(0), \quad z \in \mathbb{D}.
   \]

2. For \( \alpha \in \mathbb{R} \) let \( S_\alpha := \{re^{i\alpha} : r \geq 0\} \). Prove that the following statements are equivalent.
   (a) \( \alpha/\pi \) is rational.
   (b) There exists a nonconstant entire function \( f \in \mathcal{H}(\mathbb{C}) \) such that \( \operatorname{Re} f = 0 \) on \( S_0 \) and on \( S_\alpha \).

3. Suppose that \( \Omega_1 \) and \( \Omega_2 \) are two rectangles in \( \mathbb{C} \) with positively oriented vertices \( a_1, a_2, a_3, a_4 \in \partial \Omega_1 \) and \( b_1, b_2, b_3, b_4 \in \partial \Omega_2 \). Prove that there is a conformal map \( f \) from \( Q_1 \) onto \( Q_2 \) so that its homeomorphic extension maps \( a_j \) to \( b_j \) for \( j = 1, \ldots, 4 \) if and only if
   \[
   \left| \frac{a_2 - a_1}{a_3 - a_2} \right| = \left| \frac{b_2 - b_1}{b_3 - b_2} \right|.
   \]

4. Let \( C = \partial K_R(a) \) be a circle with center \( a \in \mathbb{C} \) and radius \( R > 0 \). We call the (anticonformal) transformation
   \[
   S(z) := a + \frac{R^2}{z - a}, \quad z \in \mathbb{C} \setminus \{a\},
   \]
   the reflection across \( C \). A domain \( \Omega \) is called symmetric with respect to \( C \), if \( S(z) \in \Omega \) whenever \( z \in \Omega \). Let \( \Omega^+ := \Omega \cap K_R(a) \) and \( \Omega^- := \Omega \cap (\mathbb{C} \setminus \overline{K_R(a)}) \) and suppose that \( f \in \mathcal{H}(\Omega^+) \) such that \( \operatorname{Im} f(z) \to 0 \) as \( z \to w \) for every \( w \in C \cap \Omega \). Prove that \( f \) has an analytic continuation to \( \Omega \setminus \{a\} \) such that
   \[
   f(S(z)) = \overline{f(z)}, \quad z \in \Omega.
   \]

5. Let \( R \) be a real number, \( R > 1 \), and \( A_R := \{z \in \mathbb{C} : 1 < |z| < R\} \).
   (a) Prove that any conformal map from \( A_R \) onto \( A_{R'} \) extends to a conformal self-map of the punctured plane \( \mathbb{C} \setminus \{0\} \).
   (b) Show that \( A_R \) and \( A_{R'} \) are conformally equivalent if and only if \( R = R' \).
   (c) Find all conformal self-maps of \( A_R \).

6. Let \( b_1, b_2, b_3 \) be pairwise different and positively oriented points and \( a_1, a_2, a_3 \) pairwise different and positively oriented points on \( \partial \mathbb{D} \). Then there exists a unique \( T \in \operatorname{Aut}(\mathbb{D}) \) such that \( T(b_j) = a_j \) for \( j = 1, 2, 3 \).
Chapter 13

The elliptic modular function

In this section we apply the Schwarz reflection principle to construct a particularly neat example of a holomorphic function, the so-called elliptic modular function, which has been introduced by Dedekind around 1877. The elliptic modular function is one of the single most important holomorphic functions and has a variety of applications in number theory and algebra. In combination with the Monodromy Theorem, the elliptic modular function leads to a transparent proof of Picard's theorem.

Tiling the upper half–plane by hyperbolic triangles

Suppose that $\Delta^0$ is the domain in the upper half–plane $\mathbb{H} = \{w \in \mathbb{C} : \text{Im} \, w > 0\}$ which is bounded by the lines $\text{Im} \, w = 0$ and $\text{Im} \, w = 1$ and by the circle $\partial K_{1/2}(1/2)$. By reflection across these three boundary arcs, we obtain three domains $\Delta^1$. If we reflect again across the newly generated lines and circles, we obtain six more domains denoted by $\Delta^2$. We continue this process and obtain pairwise disjoint domains, each bounded by lines or circular arcs, see Figure 13.1.

![Figure 13.1: Tiling the upper half–plane by hyperbolic triangles](image)

We claim that

$$\left( \bigcup_{j=0}^{\infty} \Delta^j \right) \cap \mathbb{H}$$
covers the entire upper half-plane. In fact, this follows from the observation that the boundary points of $\bigcup_j \Delta^j$ in $\mathbb{R}$ are dense in $\mathbb{R}$, see Exercise 13.1.

Tiling the unit disk by hyperbolic triangles

We now map the upper half-plane $\mathbb{H}$ onto $\mathbb{D}$ by the Möbius transform

$$T := T_2 \circ T_1, \quad T_1(z) = \frac{z - i}{z + i}, \quad T_2(z) = \frac{\left(1 + \frac{i}{2 + \sqrt{3}}\right) \left(z - \frac{i}{2 + \sqrt{3}}\right)}{\left(1 - \frac{i}{2 + \sqrt{3}}\right) \left(\frac{i z}{2 + \sqrt{3}} + 1\right)},$$

in such a way that $A := T(0) = 1$, $B := T(1) = e^{2\pi i / 3}$ and $C := T(\infty) = e^{\pi i / 3}$. In this way, we obtain a “tiling” of $\mathbb{D}$ with hyperbolic triangles, which we also denote by $\Delta^k$.

Figure 13.2: Hyperbolic tilings; level $k = 3$ (left) and level $k = 11$ (right)

Recall from the results of Chapter 12 that there is a unique conformal map $f : \Delta^0 \to \mathbb{H}$ with a homeomorphic extension from $\overline{\Delta^0}$ to $\overline{\mathbb{H}}$ such that $f(A) = 0$, $f(B) = 1$, $f(C) = \infty$, and such that the circular arc $AB$ is mapped to $[0, 1]$, the arc $BC$ is mapped to $[1, \infty]$ and the arc $CA$ is mapped to $[\infty, 0]$. Repeated application of the Schwarz reflection principle shows that $f$ extends to a holomorphic function $\pi : \mathbb{D} \to C \setminus \{0, 1\}$, which maps each white triangle conformally onto the upper half-plane and each black triangle conformally onto the lower half-plane. This map $\pi : \mathbb{D} \to C \setminus \{0, 1\}$ is called the elliptic modular function.

**Theorem 13.1.**

The elliptic modular function $\pi : \mathbb{D} \to C \setminus \{0, 1\}$ has the following properties.

(a) $\pi : \mathbb{D} \to C \setminus \{0, 1\}$ is onto (surjective) and locally univalent (locally injective).

(b) $\pi$ maps each white triangle conformally onto the upper half-plane and each black triangle conformally onto the lower half-plane.

(c) $\pi$ is real-valued on the boundary of each triangle and takes the values 0, 1 and $\infty$ in the three corners of each triangle.
Corollary 13.2.
Let \( \pi : \mathbb{D} \to \mathbb{C}\backslash\{0, 1\} \) be the elliptic modular function. Suppose that \( D_0 \subseteq \mathbb{C}\backslash\{0, 1\} \) is a disk which is univalently covered by \( \pi \) and let \( f_0 \in \mathcal{H}(D_0) \) such that \( \pi \circ f_0 = \text{id}|_{D_0} \). Then \( (f_0, D_0) \) can be analytically continued along any path in \( \mathbb{C}\backslash\{0, 1\} \) starting in \( D_0 \). If \( (f_n, D_n) \) is defined by analytic continuation of \( (f_0, D_0) \) along such a path, then \( \pi \circ f_n = \text{id} \) on \( D_n \). In particular, \( f_n \) maps \( D_n \) into \( \mathbb{D} \).

Proof. This follows directly from the mapping properties of \( \pi \).

Theorem 13.3 (Liftings).
Suppose that \( \Omega \subseteq \mathbb{C} \) is a simply connected domain and \( f : \Omega \to \mathbb{C}\backslash\{0, 1\} \) is a holomorphic function. Then there exists a holomorphic function \( g : \Omega \to \mathbb{D} \) such that \( \pi \circ g = f \).

Proof. Fix \( z_0 \in \Omega \) and suppose that \( f(z_0) \in \mathbb{H} \). Then the elliptic modular function \( \pi : \mathbb{D} \to \mathbb{C}\backslash\{0, 1\} \) has a local inverse in a neighborhood of \( f(z_0) \) which we denote by \( \pi^{-1} \), so \( g := \pi^{-1} \circ f \) is holomorphic at \( z_0 \). If \( \gamma \) is a path in \( \Omega \) starting at \( z_0 \), then \( f \circ \gamma \) is a path in \( \mathbb{C}\backslash\{0, 1\} \), so \( \pi^{-1} \) can be analytically continued along \( f \circ \gamma \), see Corollary 13.2. Hence \( g \) can be analytically continued along \( \gamma \). Corollary 11.8. shows that there is a function \( g \in \mathcal{H}(\Omega) \) such that \( \pi \circ g = f \) in \( \Omega \).

Theorem 13.4 (The little Picard theorem).
Every holomorphic function \( f : \mathbb{C} \to \mathbb{C}\backslash\{0, 1\} \) is constant.

Proof. By Theorem 13.3 there is a holomorphic function \( g : \mathbb{C} \to \mathbb{D} \) such that \( \pi \circ g = f \). But \( g \) is constant by Liouville’s theorem, so \( f \) is constant.

This proof is essentially the original proof of Picard’s theorem due to Picard in 1879.

Supplementary Exercises

1. The Farey sequence \( F_N \) of order \( N \) is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to \( N \), arranged in order of increasing size.

   (a) Show that \( F_{N+1} \) can be obtained from \( F_N \) by inserting between two successive terms \( a/b \) and \( c/d \) of \( F_N \) the fraction \( \frac{a+c}{b+d} \) reduced to lowest terms provided that the resulting denominator is \( \leq N + 1 \).

   (b) Suppose that \( C \) is a circle orthogonal to \( \mathbb{R} \) and passing through the points \( m/n \) and \( p/q \), which are successive elements of \( F_N \). Then, on reflecting over \( C \), show that the point \( \frac{m}{n} : = \frac{\pm m}{\pm n} \) is mapped to the point \( \frac{m+p}{n+q} \).

2. Let \( z_0, z_1 \in \mathbb{D} \) such that \( \pi(z_0) = \pi(z_1) \). Prove that there exists \( T \in \text{Aut}(\mathbb{D}) \) such that \( T(z_0) = z_1 \) and \( (\pi \circ T)(z) = \pi(z) \) for all \( z \in \mathbb{D} \).

   (Hint: First define \( T \) locally in a neighborhood \( D_0 \) of \( z_0 \). Show then that \( (f_0, D_0) \) can be analytically continued along any path \( \gamma : [0, 1] \to \mathbb{D} \) with \( \gamma(0) = z_0 \).

3. \( (The \ hyperbolic \ metric \ on \ \mathbb{C}' := \mathbb{C}\backslash\{0, 1\}) \)
Let \( \mathbb{C}' := \mathbb{C}\backslash\{0, 1\} \) and let \( \pi : \mathbb{D} \to \mathbb{C}' \) be the elliptic modular function. Use Exercise 13.2 to show that there exists a regular conformal metric \( \lambda_{\mathbb{C}'}(z)|dz| \) on \( \mathbb{C}' \) with constant
curvature $-4$ such that

$$\pi^*\lambda_C'(z) |dz| = \lambda_D(z) |dz|.$$  

Deduce that $\lambda_C(f(z)) |f'(z)| \leq \lambda_D(z)$ for all $z \in \mathbb{D}$ and every holomorphic function $f : \mathbb{D} \to \mathbb{C}''$.  


Chapter 14

The theorems of Landau, Picard, Schottky, Montel and Julia

The main tool in this chapter is the hyperbolic metric $\lambda_{C''}(z)|dz|$ of the twice-punctured plane $C'' := C\setminus\{0,1\}$. It is defined with the help of the elliptic modular function $\pi : D \to C''$ as follows.

**Theorem 14.1** (The hyperbolic metric on $C'' := C\setminus\{0,1\}$).
There exists a (uniquely determined) regular conformal metric $\lambda_{C''}(z)|dz|$ on $C''$ with constant curvature $-4$ such that

$$\pi^* \lambda_{C''}(z)|dz| = \lambda_D(z)|dz|.$$  

In addition, the Schwarz-Pick-type inequality $\lambda_{C''}(f(z))|f'(z)| \leq \lambda_D(z)$ holds for all $z \in D$ and every holomorphic function $f : D \to C''$.

**Proof.** Fix $w \in C''$. Suppose that $z_0, z_1 \in D$ such that $\pi(z_0) = \pi(z_1) = w$. By Exercise 13.2 we see that there is $T \in \text{Aut}(D)$ such that $T(z_0) = z_1$ and $\pi \circ T = \pi$. Hence

$$\lambda(\pi(z_1)) = \frac{\lambda_D(z_1)}{\pi'(z_1)} = \frac{\lambda_D(T(z_0))}{\pi'(z_0)} = \frac{\lambda_D(z_0)}{\pi'(z_0)} = \lambda(\pi(z_0)).$$

This shows that we can define $\lambda(w) := \frac{\lambda_D(z)}{\pi'(z)}$. Note that $\lambda_{C''}(z)|dz|$ has constant curvature $-4$, so $\lambda_{C''}(f(z))|f'(z)||dz| = f^* \lambda_{C''}(z)|dz|$ is a regular conformal pseudo-metric on $D$ with curvature $-4$. Hence, Ahlfors’ lemma implies $\lambda_{C''}(f(z))|f'(z)| \leq \lambda_D(z)$ for all $z \in D$. □

**Corollary 14.2** (Landau’s theorem, 1904).
Let $R > 0$ and $f : K_R(0) \to C''$ holomorphic with $f'(0) \neq 0$ Then $R \leq \frac{1}{\lambda_{C''}(f(0))|f'(0)|}$.

**Proof.** Consider $f_R(z) := f(Rz)$, $z \in D$. Applying Theorem 14.1 to $f = f_R$ and $z = 0$ gives $\lambda_{C''}(f_R(0))|f'(0)||R = \lambda_{C''}(f_R(0))|f'_R(0)| \leq \lambda_D(0) = 1$. □

**Remark 14.3** (Landau implies Little Picard).
If $g : C \to C''$ is an entire function, then we can apply Landau’s theorem to $f(z) := g(z_0 + z) = g(z_0) + g'(z_0)z + \ldots$ for any $R > 0$ and any $z_0 \in C$ and obtain $|g'(z_0)| \leq \frac{1}{\lambda_{C''}(g(z_0))} \to 0$ as $R \to \infty$. Hence $g' \equiv 0$ and $g$ is constant.

Landau has proved his result in 1904: “Er hat lange mit der Publikation gezögert, da der Beweis richtig, aber der Satz zu unwahrscheinlich schien”. Already in 1905, Carathéodory found the above sharp version of Landau’s theorem.
The Schwarz–Pick-type inequality $\lambda_{C^v}(f(z))|f'(z)| \leq \lambda_D(z)$ of Theorem 14.1 is difficult to apply as long as we do not know an explicit formula or at least an explicit lower bound for $\lambda_{C^v}$. In order to find such a lower bound for $\lambda_{C^v}$ we first prove an extension of Ahlfors' lemma due to David Minda (1987).

**Theorem 14.4** (Minda's extension of Ahlfors' lemma).

Let $D = \mathbb{D}$ or $D = C'$. Let $\lambda(z)|dz|$ be a conformal pseudo–metric on $D$ such that for each $a \in D$ either

(i) $\lambda(a) \leq \lambda_D(a)$

or

(ii) $\lambda$ is a regular conformal metric with curvature bounded above by $-4$ in a neighborhood of $a$.

Then $\lambda(z) \leq \lambda_D(z)$ for all $z \in D$.

**Proof.** We first consider the case $D = \mathbb{D}$. Refer to the proof of Theorem 9.4. Fix $0 < r < 1$ and let $\lambda_D(z) := \frac{r}{|z|}$. It suffices to show

$$u_r(z) := \log \frac{\lambda(z)}{\lambda_D(z)} \leq 0$$

for every $|z| < r$. Observe that $u_r : \mathbb{D} \to [-\infty, \infty)$ is continuous and tends to $-\infty$ as $|z| \to r$. Consequently, the function $u_r$ must attain a maximum value at some point $a$ with $|a| < r$. Now, there are two cases to consider. If (i) holds, then $u(a) < 0$, since $\lambda_D(a) > \lambda_D(a) \geq \lambda(a)$. If (ii) holds, then $u_r$ is of class $C^2$ in a neighborhood of $a$, so

$$0 \geq \Delta u_r(z_0) = \Delta \log \lambda(a) - \Delta \log \lambda_D(a) \geq 4\lambda(a)^2 - 4\lambda_D(a)^2$$

since $\kappa_{\lambda_D} \equiv -4$. Thus, $\lambda(a) \leq \lambda_D(a)$ or $u_r(a) \leq 0$. This completes the proof for the case $D = \mathbb{D}$.

Now let $D = C'$. Let $\mu(z) := \pi^*(\lambda)(z)$. Then $\mu(z)|dz|$ is conformal pseudo–metric on $\mathbb{D}$. We claim that for each $b \in \mathbb{D}$ either

(i) $\mu(b) \leq \lambda_D(b)$

or

(ii) $\mu(z)|dz|$ is a regular conformal metric with curvature bounded above by $-4$ in a neighborhood of $b$.

To prove this, fix $b \in \mathbb{D}$ and let $a := \pi(b)$. If $\lambda(a) \leq \lambda_C(a)$, then we see that $\mu(b) = (\pi^* \lambda)(b) = \lambda(a)|\pi'(b)| \leq \lambda_C(a)|\pi'(b)| = \lambda_D(b)$, so (i) holds. Otherwise, $\lambda(z)|dz|$ is a regular conformal metric with curvature bounded above by $-4$ in a neighborhood of $a$, so Theorem 9.2 implies that (ii) holds. By what we have proved before, $\mu(z) \leq \lambda_D(z)$ for every $z \in \mathbb{D}$. This implies

$$\lambda(\pi(z))|\pi'(z)| = (\pi^* \lambda)(z) = \mu(z) \leq \lambda_D(z) = \lambda_C(\pi(z))|\pi'(z)|, \quad z \in \mathbb{D},$$

so $\lambda \leq \lambda_C$ on $C'$ since $\pi : \mathbb{D} \to C'$ is onto. \qed
Corollary 14.5.
\[ \lim_{z \to 1} \lambda_{C''}(z) = +\infty. \]

**Proof.** Let \( \tau(z) \, |dz| \) be the regular conformal metric on \( C'' \) employed in the proof of Theorem 10.1. Since \( \kappa_r \leq -4 \), we get from Lemma 14.4 that \( \tau(z) \leq \lambda_{C''}(z) \) for each \( z \in C'' \). Now just observe that \( \lim_{z \to 1} \tau(z) = +\infty. \)

In particular, since \( \lambda_{C''} \) is a continuous and positive function on \( \partial \mathbb{D}\setminus\{1\} \), it has a positive minimal value on \( \partial \mathbb{D}\setminus\{1\} \). We can now derive a simple sharp and (semi)explicit lower bound for \( \lambda_{C''} \) even though there is no simple explicit formula for \( \lambda_{C''} \) itself.

**Theorem 14.6.**

Let

\[ K := \frac{1}{2 \min_{|z|=1} \lambda_{C''}(z)}. \]

Then

\[ \lambda_{C''}(z) \geq \frac{1}{2|z|(|\log |z|| + K)} \]

for all \( z \in C'' \) and this estimate is sharp.

**Proof.** Define

\[ \lambda(z) := \frac{1}{2|z|(|\log |z|| + K)}, \quad z \in C''. \]

Then \( \lambda(z) \, |dz| \) is a conformal metric on \( C'' \) such that for \( |z| = 1 \) we have

\[ \lambda(z) = \frac{1}{2K} = \min_{|z|=1} \lambda_{C''}(z) \leq \lambda_{C''}(z). \]

For \( 0 < |z| < 1 \) and \( r := e^K > 1 \),

\[ \lambda(z) = \frac{1}{2|z|(|\log(r/|z|)|)} = \nu(z/r)/r, \]

where \( \nu(z) \, |dz| \) is the conformal metric of Lemma 10.2. Hence \( \lambda(z) \, |dz| \) has constant curvature \(-4\) on \( |z| < 1 \). Similarly, \( \lambda(z) \, |dz| \) has constant curvature \(-4\) on \( |z| > 1 \), since there

\[ \lambda(z) = \frac{1}{2|z|\log(|z|/r')}, \quad r' := e^{-K} < 1. \]

Lemma 14.4 gives \( \lambda \leq \lambda_{C''} \) on \( C'' \).

**Remark.**

One can show that \( \lambda_{C''} \) attains its minimal value on \( |z| = 1 \) precisely at the point \( z = -1 \) and

\[ \lambda_{C''}(-1) = \frac{\Gamma\left(\frac{3}{4}\right)^4}{2\pi^2} \approx 0.114237. \]

Hence the constant \( K \) has the value \( \frac{\pi^2}{\Gamma(3/4)^3} \approx 4.37687. \)

We can now prove a sharp quantitative form of Schottky's theorem. In the sequel, we use the standard notation

\[ \log^+ x := \max\{0, \log x\} \quad \text{for} \quad x \in \mathbb{R}^+. \]
Theorem 14.7.
Suppose that \( f : \mathbb{D} \to \mathbb{C}' \) is holomorphic. Then
\[
\log |f(z)| \leq \left[ K + \log^+ |f(0)| \right] \frac{1 + |z|}{1 - |z|} - K
\]
for all \( z \in \mathbb{C}' \).

**Proof.** Since the right side of the inequality is always nonnegative, the inequality holds for any \( z \in \mathbb{D} \) such that \( |f(z)| \leq 1 \). Take \( z_0 \in \mathbb{D} \) such that \( |f(z_0)| > 1 \) and write \( z_0 = |z_0|\eta \) with \( \eta \in \partial \mathbb{D} \). Consider the curve \( \gamma(t) := f(t\eta) \), \( t \in [0,|z_0]| \) from \( f(0) \) to \( f(z_0) \). Let
\[
\tau := \inf \{ s : \gamma(t) \in \mathbb{C}\setminus\mathbb{D} \text{ for all } t \in (s,|z_0|) \},
\]
so \( \gamma(\tau) \) is the “last” point on \( \gamma \) inside \( \mathbb{D} \). We estimate the \( \lambda_{\mathbb{C}'} \)-length of the curve \( \hat{\gamma} := \gamma|_{\tau,|z_0]} \) from above
\[
\int_{\hat{\gamma}} \lambda_{\mathbb{C}'}(w) |dw| = \int_{\tau}^{\frac{|z_0|}{1 - |z_0|}} \lambda_{\mathbb{C}'}(f(\eta t)) |f'(\eta t)| \, dt \leq \int_{\tau}^{\frac{|z_0|}{1 - |z_0|}} \frac{dt}{1 - t^2} \leq \frac{1}{2} \log \frac{1 + |z_0|}{1 - |z_0|},
\]
where we have used \( \lambda_{\mathbb{C}'}(f(z_0)|f'(z_0)| \leq \lambda_{\mathbb{D}}(z) \), see Theorem 14.1. We next estimate the \( \lambda_{\mathbb{C}'} \)-length of the curve \( \hat{\gamma} \) from below using Theorem 14.6 and obtain
\[
\int_{\hat{\gamma}} \lambda_{\mathbb{C}'}(w) |dw| = \int_{\tau}^{\frac{|z_0|}{1 - |z_0|}} \lambda_{\mathbb{C}'}(f(\eta t)) |f'(\eta t)| \, dt \geq \int_{\tau}^{\frac{|z_0|}{2|f'(\eta t)||\log |f(\eta t)| + K|}} \frac{|f'(\eta t)|}{2|f(\eta t)||\log |f(\eta t)| + K|} \, dt.
\]
Writing \( g(t) := |f(\eta t)| \) and using \( g'(t) \leq |f'(\eta t)| \), we arrive at
\[
\int_{\tau}^{\frac{|z_0|}{1 - |z_0|}} \frac{g'(t)}{2g(t)(\log g(t) + K)} \, dt = \int_{\frac{|z_0|}{|f'(\eta t)|}}^{\frac{|f(z_0)|}{2(\log x + K)}} \frac{1}{x} \, dx = \frac{1}{2} \log \left( \frac{\log |f(z_0)| + K}{\log |f(\eta \tau)| + K} \right),
\]
and hence
\[
\log |f(z_0)| \leq \left[ \log |f(\eta \tau)| + K \right] \frac{1 + |z_0|}{1 - |z_0|} - K.
\]
If \( \gamma \subseteq \mathbb{C}\setminus\mathbb{D} \), then \( \tau = 0 \) and \( |f(0)| \geq 1 \), so
\[
\log |f(z_0)| \leq \left[ \log^+ |f(0)| + K \right] \frac{1 + |z_0|}{1 - |z_0|} - K.
\]
Otherwise, \( |f(\eta \tau)| = 1 \), and therefore
\[
\log |f(z_0)| \leq K \frac{1 + |z_0|}{1 - |z_0|} - K \leq \left[ \log^+ |f(0)| + K \right] \frac{1 + |z_0|}{1 - |z_0|} - K.
\]
\[\square\]

Schottky’s theorem allows a quick proof of a celebrated result of Montel. Note that we call a family \( \mathcal{F} \subseteq \mathcal{H}(\mathbb{D}) \) a normal family if every sequence in \( \mathcal{F} \) has a subsequence which converges locally uniformly in \( \mathbb{D} \) either to some \( f \in \mathcal{H}(\mathbb{D}) \) or to \( \infty \).
Theorem 14.8 (Montel’s fundamental normality test).

The family $\mathcal{F} := \{ f \in \mathcal{H}(D) : f(D) \subseteq \mathbb{C} \}$ is a normal family.

Proof. Consider $\mathcal{G} := \{ f \in \mathcal{F} : |f(0)| \leq 1 \}$ and let $(f_n)$ be a sequence in $\mathcal{F}$.
1. Case: A subsequence of $(f_n)$ is contained in $\mathcal{G}$.
In view of Schottky’s Theorem 14.7 a subsequence of $(f_n)$ is locally bounded and hence $\mathcal{F}$ is a normal family by Montel’s theorem.
2. Case: $|f_n(0)| > 1$ for (almost) all $n \in \mathbb{N}$.
Then $g_n := 1/f_n \in \mathcal{G}$, so $(g_n)$ is locally bounded and by Montel’s theorem we may assume that $(g_n)$ converges locally uniformly in $D$ to a holomorphic function $g : D \rightarrow \mathbb{C}$. If $g$ is zerofree in $D$, then $f_n = 1/g_n \rightarrow 1/g =: f \in \mathcal{H}(D)$ locally uniformly in $D$. If $g$ has zeros in $D$, then $g \equiv 0$ by Hurwitz’ Theorem, so $f_n$ converges locally uniformly in $D$ to $\infty$.

Lemma 14.9.

Suppose that $f \in \mathcal{H}(D')$ has an essential singularity at $z = 0$. Then the family $\{f_n\}$ defined by $f_n(z) := f(z/2^n)$ is not normal in $1/8 < |z| < 1$.

Proof. Otherwise there is a subsequence $(f_{n_k})$ that converges locally uniformly in the open annulus $A := \{ z \in D : |z| > 1/8 \}$, in particular, the convergence is uniform in the closed annulus $A_0 := \{ z \in C : 1/2^2 \leq |z| \leq 1/2^1 \}$.
The limit function $F$ of $(f_{n_k})$ must be identically infinite, since otherwise, $F$ would be bounded on $|z| = q \in (1/4, 1/2)$, say, so $(f_{n_k})$ would be uniformly bounded by a number $M > 0$ on $|z| = q$. But then $|f(z)| \leq M$ on $|z| = q/2^n$ for each $k \in \mathbb{N}$, so the maximum principle implies that $f$ is bounded by $M$ on $0 < |z| < q/2^n$, which is not possible as $z = 0$ is an essential singularity of $f$.

Since $(f_n)$ is a normal family, this reasoning also shows that the entire sequence would converge to $\infty$ locally uniformly in $A$. It follows that for any $M > 0$ there is an $n_0$ such that

$$|f_n(z)| > M, \quad n \geq n_0, \: z \in A_0.$$  

But the values that $f_n$ takes in $A_0$ coincide with the values that $f$ takes in $A_n := \{ z \in C : 1/2^{n+2} \leq |z| \leq 1/2^{n+1} \}$. This implies that

$$|f(z)| > M, \quad 0 < |z| < 1/2^{n_0+1}.$$  

In other words, $\lim_{z \to 0} f(z) = \infty$, contradicting the fact that $z = 0$ is an essential singularity.

The lemma permits the following extension of Picard’s great theorem.

Theorem 14.10 (Julia’s Theorem; Julia 1924).

Suppose that $f \in \mathcal{H}(D')$ has an essential singularity at $z = 0$. Then there is at least one direction $\theta \in \mathbb{R}$ such that in every sector

$$S_\varepsilon(\theta) := \{ z = e^{i\phi} |z| : \theta - \varepsilon < \phi < \theta + \varepsilon \}, \quad \varepsilon > 0,$$

$f$ assumes, infinitely often, every complex value with at most one exception.

Proof. Consider $f_n(z) := f(z/2^n)$. Lemma 14.9 shows that there is a point $z_0 \in D'$ such that $(f_n)$ is not normal in any neighborhood of $z_0$. Let $z_0 = |z_0| e^{i\theta}$, fix $\varepsilon > 0$ and suppose that $f$ assumes $0$ and $1$ only finitely often in $S_\varepsilon(\theta)$. Choose $r > 0$ such that $\sin \varepsilon > r/|z_0|$.
and $D_0 := K_r(z_0) \subseteq \mathcal{D}$. Then $(f_n)$ is not normal in $D_0$. On the other hand, if we set $D_n := K_{r/2^n}(z_0/2^n)$, then we see that $D_n \subseteq S_\epsilon(\theta)$ and $f_n(D_0) = f(D_n) \subseteq \mathbb{C}^n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. By Theorem 14.8, $(f_n)$ is normal in $D_0$, a contradiction. \[ \Box \]

**Notes**

In 1904 Schottky showed that for each $\alpha_0 \in \mathbb{C}$ and each $0 \leq r < 1$ there is a number $M(\alpha_0, r) > 0$ such that $|f(z)| \leq M(f(0), r)$ for every holomorphic function $f : \mathcal{D} \to \mathbb{C}^n$ and every $z \in \mathcal{D}$ with $|z| = r$. Schottky did not give an explicit expression for the upper bound $M(f(0), r)$ for $|f(z)|$. Subsequently, explicit bounds were given by Ostrowski (1931, 1933), Ahlfors (1938), Robinson (1939), Hayman (1947), Jenkins (1955) and Hempel (1980). Lemma 14.4 was proved by Minda in 1987 and Theorem 14.6 by Hempel (1980). Our presentation follows an approach devised by Minda (1987). Schottky’s theorem in the form of Theorem 14.7 is a more recent result due to Li and Qi (2007).

Montel proved his “Critère fondamental” (Theorem 14.8) in 1912. However, the statement is already implicit in a 1911 paper of Carathéodory and Landau, see Exercise 14.5. Julia published Theorem 14.10 in his lecture notes *Leçons sur les fonctions uniformes à point singulier essential isolé*, Gauthiers-Villars, Paris, 1924.

**Supplementary Exercises**

1. Let $\lambda(z)|dz|$ be a regular conformal pseudo-metric on $\mathcal{D}' := \mathcal{D}\setminus\{0\}$ with curvature $\leq -4$.

   (a) Prove that
   \[ \lambda(z) \leq \frac{1}{2|z|\log(1/|z|)} \]
   for all $z \in \mathcal{D}'$.

   (b) Show that
   \[ \lim_{z \to 0} 2|z|\log(1/|z|) \lambda_{\mathcal{C}'}(z) = 1. \]

2. Show that the estimate in Theorem 14.7 are sharp.

3. Let $f : \mathcal{D} \to \mathbb{C}\setminus\overline{\mathcal{D}}$ be a holomorphic function. Show that
   \[ \log |f(z)| \leq \frac{1+|z|}{1-|z|} \log |f(0)|, \quad z \in \mathcal{D}. \]

4. Does Montel’s theorem follow from Montel’s fundamental normality test?

5. Prove the following extension of Vitali’s theorem due to Carathéodory and Landau (1911): *Let $D \subseteq \mathbb{C}$ be a domain, $(f_n)$ a sequence of holomorphic functions in $D$ omitting 0 and 1, $(z_n)$ a sequence of points in $D$ with a limit point in $D$ such that $(f_n(z_j))$ converges in $\mathbb{C}$ for each $j = 1, 2, \ldots$. Show that $(f_n)$ converges locally uniformly in $D$ to a holomorphic function.*