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DECOMPOSITIONS OF RATIONAL FUNCTIONS OVER REAL AND COMPLEX NUMBERS AND A QUESTION ABOUT INVARIANT CURVES

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Abstract. We consider the connection of functional decompositions of rational functions over the real and complex numbers, and a question about curves in the complex plane which are invariant under a rational function.

1. Introduction

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. A *circle* in $\hat{\mathbb{C}}$ is either a usual circle in $\mathbb{C}$, or a line in $\hat{\mathbb{C}}$. So the circles in $\hat{\mathbb{C}}$ are just the curves $\Gamma = \lambda(\hat{\mathbb{R}})$, where $\lambda(z) = \frac{az+b}{cz+d} \in \mathbb{C}(z)$ is a linear fractional function (with $ad - bc \neq 0$).

Long ago Fatou suggested to study (Jordan) curves $\Gamma \subset \hat{\mathbb{C}}$ which are invariant under a rational function of degree ≥ 2. See [3] for recent progress on this. The case that Γ is a circle $\lambda(\hat{\mathbb{R}})$ is not interesting, because any rational function $r = \lambda \circ s \circ \lambda^{-1}$ with $s \in \mathbb{R}(z)$ leaves Γ invariant, and there are no other rational functions with this property.

Motivated by his results on invariant curves in [3], Alexandre Eremenko suggested to investigate the following source of invariant curves, and raised two questions about this family.

Question 1.1. Let $f, g \in \mathbb{C}(z)$ be non-constant rational functions, such that $f(g(z)) \in \mathbb{R}(z)$, so the curve $\Gamma = g(\hat{\mathbb{R}})$ is invariant under $r = g \circ f$. Assume that Γ is not contained in a circle.

(a) Is it possible that Γ is a Jordan curve? ([3], [2])
(b) Is it possible that $r : \Gamma \to \Gamma$ is injective? ([5], and special case of [4])

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Note that $\Gamma = g(\hat{\mathbb{R}})$ is contained in a circle if and only if there is a linear fractional function $\lambda \in \mathbb{C}(z)$ such that $\tilde{g} = \lambda \circ g \in \mathbb{R}(z)$. In this case, $f \circ g = \tilde{f} \circ \tilde{g}$ with $\tilde{f} = f \circ \lambda^{-1} \in \mathbb{R}(z)$. So the decomposition of $f \circ g$ over $\mathbb{C}$ essentially arises from a decomposition over $\mathbb{R}$.

There are rational functions $f \circ g \in \mathbb{R}(z)$ whose decompositions do not come from a decomposition over the reals. On the other hand, it is known that decompositions of real polynomials over the complex numbers always arise from real decompositions. See Section 5 for more about this.

The purpose of this paper is to give a positive answer to question (a), and a negative answer to a slight weakening of (b). More precisely, regarding (a), we show the following theorem.

THEOREM 1.2. *For every odd prime ℓ there are rational functions $f, g \in \mathbb{C}(z)$, both of degree ℓ, such that*

(a) $f(g(z)) \in \mathbb{R}(z)$.
(b) $g : \hat{\mathbb{R}} \to \hat{\mathbb{C}}$ *is injective, so* $g(\hat{\mathbb{R}})$ *is a Jordan curve.*
(c) $g(\hat{\mathbb{R}})$ *is not a circle.*

In order to formulate the next two results, we define a weakening of injectivity of rational functions on $\mathbb{R}$.

DEFINITION 1.3. A rational function $g \in \mathbb{R}(z)$ is said to be *weakly injective on* $\mathbb{R}$, if there exists $z_0 \in \mathbb{R}$ which is not a critical point of g, and besides z_0 there is no $y_0 \in \hat{\mathbb{R}}$ with $g(z_0) = g(y_0)$.

A partial answer to question (b) is the following.

THEOREM 1.4. *Let $f, g \in \mathbb{C}(z)$ be non-constant rational functions, such that $f \circ g \in \mathbb{R}(z)$. Assume that g is weakly injective, and that the curve $\Gamma = g(\hat{\mathbb{R}})$ is not contained in a circle. Then the map $g \circ f : \Gamma \to \Gamma$ is not injective.*

A slight variant of this theorem shows that for a fairly large class of rational functions from $\mathbb{R}(z)$, each decomposition over $\mathbb{C}$ arises from a decomposition over $\mathbb{R}$.

THEOREM 1.5. *Let $f, g \in \mathbb{C}(z)$ be non-constant rational functions such that $h(z) = f(g(z)) \in \mathbb{R}(z)$ is weakly injective. Then there is a linear fractional function $\lambda \in \mathbb{C}(z)$ such that $\lambda \circ g \in \mathbb{R}(z)$.*

The main ingredient (besides Galois theory) in the proof of the previous two theorems is the following group-theoretic result. (See Section 3 for the notation.)

PROPOSITION 1.6. *Let G be a group of permutations of the finite set Ω. Let σ be a permutation of Ω of order 2 which fixes exactly one element ω, and which normalizes G, that is $G^\sigma = G$. Let G_ω be the stabilizer of ω in G. Then $M^\sigma = M$ for each group M with $G_\omega \leq M \leq G$.*

The proof of Theorem 1.2 uses elliptic curves. The construction was motivated by a group-theoretic analysis similar to the one which led to the proofs of Theorems 1.4 and 1.5.

2. Non-circle Jordan curves invariant under a rational function

In this section, we work out our sketch from [10] and prove Theorem 1.2.

Let E be an elliptic curve given by a Weierstrass equation $Y^2 = X^3 + aX + b$ with $a, b \in \mathbb{R}$. By $E(\mathbb{C})$ and $E(\mathbb{R})$ we denote the complex and real points of E. For $p \in E(\mathbb{C})$ we let $\bar{p}$ be the complex conjugate of p. We use the structure of $E(\mathbb{C})$ as an abelian group, with neutral element 0_E the unique point at infinity. Denote by $\langle c \rangle$ the cyclic group generated by $c \in E(\mathbb{C})$.

For general facts about elliptic curves see, for example, [12].

LEMMA 2.1. *Let $\ell \geq 3$ be a prime. Then there is a point $c \in E(\mathbb{C})$ of order ℓ, with $\bar{c} \notin \langle c \rangle$.*

Proof. Let $E[\ell] \subset E(\mathbb{C})$ be the group of ℓ-torsion points. Then $E[\ell]$ is isomorphic to the vector space $\mathbb{F}_\ell^2$, and the complex conjugation acts linearly on this space.

Suppose that the claim does not hold, so the complex conjugation fixes each 1-dimensional subspace of $E[\ell]$ setwise. Then the complex conjugation acts as a scalar map. Therefore, either $E[\ell] \subset E(\mathbb{R})$, or $\bar{c} = -c$ for each $c \in E[\ell]$. In the latter case, write $c = (u, v)$. So u is real and v is purely imaginary. Thus, upon replacing E with the twisted curve $-Y^2 = X^3 + aX + b$ (which is isomorphic over $\mathbb{R}$ to $Y^2 = X^3 + aX - b$), we obtain in either case an elliptic curve E with $E[\ell] \subseteq E(\mathbb{R})$. On the other hand, $E(\mathbb{R})$ is isomorphic to $\mathbb{R}/\mathbb{Z}$ or to $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see, e.g., [12, V. Cor. 2.3.1]). However, $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ does not have a subgroup isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$. This proves the claim. □

LEMMA 2.2. *Suppose that $X^3 + aX + b$ has three distinct real roots. Then there are elements $w \in E(\mathbb{R})$ such that there is no $\hat{w} \in E(\mathbb{R})$ with $w = 2\hat{w}$.*

Proof. If $X^3 + aX + b$ has three distinct real roots, then $E(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so any w corresponding to $(s, 1)$, $s \in \mathbb{R}/\mathbb{Z}$ arbitrary, has the property that there is no $\hat{w} \in E(\mathbb{R})$ with $w = 2\hat{w}$. (In this case, $E(\mathbb{R})$ has two connected components, and for each $\hat{w} \in E(\mathbb{R})$ the element $2\hat{w}$ is on the connected component of 0_E.) □

By an *automorphism* of an elliptic curve we mean a birational map of the curve to itself which need not fix the neutral element.

Pick $c \in E(\mathbb{C})$ of order ℓ such that $\bar{c} \notin \langle c \rangle$, and set $C = \langle c \rangle$.

Let $\Phi : E \to E' = E/C$ be the isogeny with kernel C. Let $\Phi' : E' \to E$ be the dual isogeny. Then $\Phi' \circ \Phi : E \to E$ is the multiplication by ℓ map on E.

For $w \in E(\mathbb{R})$ as in the previous lemma define involutory automorphisms

- β of E by $\beta(p) = w - p$,

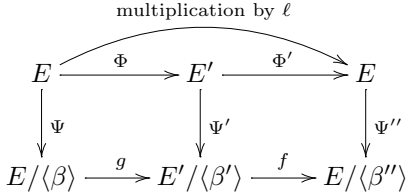
- β' of E' by $\beta'(p') = \Phi(w) - p'$, and
- β'' of E by $\beta''(p'') = \Phi'(\Phi(w)) - p'' = \ell w - p''$.

Note that $\beta'(\Phi(p)) = \Phi(w) - \Phi(p) = \Phi(w - p) = \Phi(\beta(p))$, so

$$\beta' \circ \Phi = \Phi \circ \beta \quad \text{and likewise} \quad \beta'' \circ \Phi' = \Phi' \circ \beta'. \tag{2.1}$$

In the following, we show that

- $E/\langle\beta\rangle$, $E'/\langle\beta'\rangle$, and $E/\langle\beta''\rangle$ are projective lines, and
- that there are degree 2 branched covering maps ψ, ψ', and ψ'' from the elliptic curves to these lines,
- such that ψ and ψ'' are defined over $\mathbb{R}$, and
- that after selecting uniformizing elements of the projective lines, there are unique rational functions f and g such that the following diagram commutes:



We give an algebraic rather than a geometric description of the functions f and g. This has the advantage that the method can be used to compute explicit examples, as we do at the end of this section.

Let $\mathbb{C}(E)$ and $\mathbb{C}(E')$ be the function fields of E and E', respectively. Let x and y be the coordinate functions with $x(p) = u$ and $y(p) = v$ for $p = (u, v) \in E(\mathbb{C})$. So $E(\mathbb{C}) = \mathbb{C}(x, y)$ with $y^2 = x^3 + ax + b$. The comorphism $\beta^\star$ is an automorphism of order 2 of the real function field $\mathbb{R}(E)$ (recall that $w \in E(\mathbb{R})$). We compute the fixed field of $\beta^\star$ in $\mathbb{R}(E)$: Write $w = (w_x, w_y)$, and set $z = \frac{w_y + y}{w_x - x}$ (this choice of z is taken from [8]). The addition formula for elliptic curves shows that

$$\beta^\star(x) = z^2 - w_w - x \quad \text{and} \quad \beta^\star(y) = z\big(w_x - \beta^\star(x)\big) - w_y.$$

From that, we get

$$\beta^\star(z) = \frac{w_y + \beta^\star(y)}{w_x - \beta^\star(x)} = \frac{w_y + z(w_x - \beta^\star(x)) - w_y}{w_x - \beta^\star(x)} = z,$$

so z is in the fixed field of $\beta^\star$. Clearly ,$\mathbb{R}(x, z) = \mathbb{R}(y, z) = \mathbb{R}(x, y)$. Let $F \subseteq \mathbb{R}(E)$ be the fixed field of $\beta^\star$. From $z \in F$ and $[\mathbb{R}(E) : F] = 2$, we get $F = \mathbb{R}(z)$ once we know that $[\mathbb{R}(E) : \mathbb{R}(z)] \le 3$. But this holds, as

$$\big(z(w_x - x) - w_y\big)^2 = y^2 = x^3 + ax + b,$$

so x has at most degree 3 over $\mathbb{R}(z)$. Now ψ is just the rational function $E \to \hat{\mathbb{C}}$ for which $z = \psi(x, y)$.

Suppose (without loss of generality) that E' has also a Weierstrass from $Y^2 = X^3 + a'X + b'$, and let x' and y' be the associated coordinate functions. Note that $\Phi((u,v)) = (A(u), B(u)v)$ for rational functions $A, B \in \mathbb{C}(z)$ and all $p = (u,v) \in E(\mathbb{C})$. Therefore, $\Phi^\star(x') = A(x)$, where $\Phi^\star : \mathbb{C}(E') \to \mathbb{C}(E)$ is the comorphism of Φ.

Pick z' in $\mathbb{C}(x', y')$ such that $\mathbb{C}(z')$ is the fixed field of $\beta'^\star$. (Here z' can not be taken from $\mathbb{R}(x', y')$, because E' is not defined over $\mathbb{R}$.) As before, let ψ' be the rational function with $z' = \psi'(x', y')$.

From (2.1), we obtain $\Phi^\star \circ \beta'^\star = \beta^\star \circ \Phi^\star$, so $\Phi^\star(z') = \beta^\star(\Phi^\star(z'))$ and hence $\Phi^\star(z') \in \mathbb{C}(z)$. So $\Phi^\star(z') = g(z)$ for a rational function $g \in \mathbb{C}(z)$. This is just the algebraic description of g from above. Similarly one computes f.

For the rest of this section, we work with the morphisms rather than the comorphisms. Recall that Ψ and Ψ'' are defined over $\mathbb{R}$. So $\overline{\Psi(p)} = \Psi(\bar{p})$ for all $p \in E(\mathbb{C})$.

We now prove the required properties of f and g. The assertion about the degrees follows from well-known facts about isogenies.

(a) As the multiplication by ℓ map is defined over $\mathbb{R}$, and so are Ψ and Ψ'', we have $f \circ g \in \mathbb{R}(x)$.

(b) We next show that g is injective on $\hat{\mathbb{R}}$. Suppose that there are distinct $z_1, z_2 \in \hat{\mathbb{R}}$ such that $g(z_1) = g(z_2)$. Pick $p, q \in E(\mathbb{C})$ such that $\Psi(p) = z_1$, $\Psi(q) = z_2$. Then

$$\Psi'\big(\Phi(p)\big) = g\big(\Psi(p)\big) = g(z_1) = g(z_2) = g\big(\Psi'(q)\big) = \Psi'\big(\Phi(q)\big),$$

so $\Phi(p) = \Phi(q)$ or $\Phi(p) = \Phi(w) - \Phi(q)$. Upon possibly replacing q with $w - q$ we may and do assume $\Phi(p) = \Phi(q)$, hence $p - q \in C$.

Recall that Ψ is defined over $\mathbb{R}$ and $\Psi(p) = z_1$ is real. So $\Psi(\bar{p}) = \Psi(p)$, and therefore $\bar{p} = p$ or $\bar{p} = w - p$. Likewise $\bar{q} = q$ or $\bar{q} = w - q$. Recall that $p - q \in C$, and that $C \cap \bar{C} = \{0_E\}$ by the choice of C. So we can't have $(\bar{p}, \bar{q}) = (p, q)$, nor $(\bar{p}, \bar{q}) = (w - p, w - q)$.

Thus, without loss of generality, $\bar{p} = p$ and $\bar{q} = w - q$. So $p \in E(\mathbb{R})$. Note that $p - q$ and $\bar{p} - \bar{q} = p - w + q$ both have order ℓ. Set $r = (p - q) + (\bar{p} - \bar{q}) = 2p - w$. Then $\ell r = 0_E$, and $r \in E(\mathbb{R})$. We obtain $w = 2(p + \frac{\ell-1}{2} r)$ with $p + \frac{\ell-1}{2} r \in E(\mathbb{R})$, contrary to the choice of w.

(c) Finally, we need to show that $g(\hat{\mathbb{R}})$ is not a circle. Suppose otherwise. Let λ be a linear fractional function which maps this circle to $\hat{\mathbb{R}}$. Then $\lambda \circ g$ maps $\mathbb{R}$ to $\hat{\mathbb{R}}$, so $\lambda \circ g \in \mathbb{R}(x)$.

Then $\lambda \circ \Psi' \circ \Phi = \lambda \circ g \circ \Psi$ is defined over $\mathbb{R}$, so $\lambda(\Psi'(\Phi(p))) = \lambda(\Psi'(\Phi(\bar{p})))$ for all $p \in E(\mathbb{C})$. As λ is bijective, Ψ' respects β', and C is the kernel of Φ, we get that for each $p \in E(\mathbb{C})$ either $p - \bar{p} \in C$, or $p + \bar{p} - w \in C$.

In the first case, note that $\bar{p} - p \in \bar{C}$, so also $p - \bar{p} = -(\bar{p} - p) \in \bar{C}$, and therefore $p - \bar{p} \in C \cap \bar{C} = \{0_E\}$. So $p \in E(\mathbb{R})$ if the first case happens.

We see that $p+\bar{p}-w \in C$ whenever $p \in E(\mathbb{C}) \setminus E(\mathbb{R})$. Recall that $w \in E(\mathbb{R})$. So $p+\bar{p}-w \in C \cap \bar{C} = \{0_E\}$, and therefore $p+\bar{p} = w$ for all $p \in E(\mathbb{C}) \setminus E(\mathbb{R})$. As $(p+q)+\overline{p+q} = 2w \neq w$ for all $p, q \in E(\mathbb{C}) \setminus E(\mathbb{R})$, we get the absurd consequence that $p+q \in E(\mathbb{R})$ whenever $p, q \in E(\mathbb{C}) \setminus E(\mathbb{R})$, so $E(\mathbb{R})$ is a subgroup of index 2 in $E(\mathbb{C})$. This final contradiction proves all the properties about the functions f and g.

REMARK 2.3. For fixed curves E, E' and isogeny Φ as in the proof of Theorem 1.2, and $w \in E(\mathbb{R})$ (which need not fulfill the property of Lemma 2.2), let $h_w = f \circ g \in \mathbb{R}(z)$ be the rational function constructed there. The case $w = 0_E$ gives a Lattès function $h_0 \in \mathbb{R}(z)$. It is easy to see that $h_w = \lambda_1 \circ h_0 \circ \lambda_2$ for linear fractional functions $\lambda_1, \lambda_2 \in \mathbb{C}(z)$. If w has the property from Lemma 2.2, then λ_1, λ_2 cannot be chosen in $\mathbb{R}(z)$. So h_w is a twist of h_0 over a quadratic field. Therefore, the relation of h_w to the Lattès map h_0 is analogous to the relation of Rédei functions (see [11]) $f_n(a,z)$ to cyclic polynomials z^n, where $f_n(z) = \lambda^{-1}(\lambda(z)^n)$ for $\lambda(z) = \frac{z+\sqrt{a}}{z-\sqrt{a}}$. Note that despite the appearance of the term $\sqrt{a}$, the coefficients of the degree n rational function $f_n(a,z)$ lie in the field generated by $\mathbb{Q}$ and $a \in \mathbb{C}$.

A construction like h_w appeared in an arithmetic context in [7].

Lattès functions, which were known before Lattès work in 1918, are classical objects in complex analysis. See [13] and [9] for the relevance of these functions in complex dynamics, and especially [9] for a lot of information about the history of these functions.

EXAMPLE 2.4. Here we explicitly compute an example for the case $\ell = 3$. We aim to find an example where the elliptic curve E is defined over $\mathbb{Q}$, $f \circ g \in \mathbb{Q}(z)$, and $f, g \in K(z)$, where K is an as small as possible number field. Let ω be a primitive third root of unity, so $\omega^2 + \omega + 1$ and $\bar{\omega} = -1 - \omega$. As $c \in E(\mathbb{C})$ is required to be a non-real point, and the coordinates of the ℓ-torsion group of an elliptic curve over $\mathbb{Q}$ generate the field of ℓ-th roots of unity, we necessarily have $\omega \in K$. Indeed, there are examples with $K = \mathbb{Q}(\omega)$.

The in terms of the conductor smallest elliptic curve E over $\mathbb{Q}$ which has a 3-torsion point in $E(K) \setminus E(\mathbb{Q})$ has the Cremona label 14a2 and Weierstrass form $Y^2 = X^3 - 46035X - 3116178$. One computes that $c = (72\omega - 33, 1080\omega - 648) \in E(\mathbb{C})$ has order 3. Set $C = \langle c \rangle$. Then $C \cap \bar{C} = \{0_E\}$. There is an isogeny $\Phi : E \to E'$ with kernel C, where E' is given by $Y^2 = X^3 + (298080\omega + 537165)X + (86819040\omega - 39204594)$.

Set $w = (-78, 0) \in E(\mathbb{Q})$. The X-coordinates of $\hat{w}$ with $2\hat{w} = w$ are roots of $X^2 + 156X + 33867 = (X+78)^2 + 27783$, so there is no $\hat{w} \in E(\mathbb{R})$ with $2\hat{w} = w$.

Thus E, C and w fulfill all the assumptions which we needed in the existence proof of $f(z)$ and $g(z)$. We now compute these functions. Let β and β' be the automorphisms of E and E' given by $\beta(p) = w - p$ and

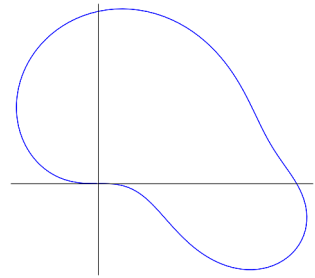


FIGURE 1. The plot shows the image of $\hat{\mathbb{R}}$ under $\frac{1}{1+g(z)}$. As expected, this curve is a Jordan curve, but not a circle.

$\beta'(p') = \Phi(w) - p'$. Write $w = (w_x, w_y)$ and $\Phi(w) = (w'_x, w'_y)$. Set $z = \frac{w_y+y}{w_x-x}$ and $z' = \frac{w'_y+y'}{w'_x-x'}$. Recall that $\Phi^\star(x') = A(x)$, where $\Phi((u,v)) = (A(u), B(u)v)$ for all $(u,v) \in E(\mathbb{C})$. From that we see also $\Phi^\star(y') = B(x)y$.

Now recall that the function $g(z)$ we are looking for fulfills $g(z) = \Phi^\star(z')$. We compute

$$g(z) = \Phi^\star\left(\frac{w'_y + y'}{w'_x - x'}\right) = \frac{w'_y + B(x)y}{w'_x - A(x)}.$$

Use this equation, and the equations $z = \frac{w_y+y}{w_x-x}$ and $y^2 = x^3 - 46035x - 3116178$ to eliminate the variables x and y. So we are left with a polynomial equation in z and the unknown function $g(z)$ which we treat as a variable. This polynomial has a factor of degree 1 with respect to $g(z)$, from which we obtain $g(z)$. Analogously we get $f(z)$. After minor linear changes over $\mathbb{Q}$ (which slightly simplify f and g) we obtain

$$f(z) = \frac{z^3 - 6(\omega+1)z}{3z^2+1},$$
$$g(z) = \frac{2z^3 + (\omega+1)z}{z^2 - \omega},$$
$$f\big(g(z)\big) = \frac{8z^9 - 24z^5 - 13z^3 - 6z}{12z^8 + 13z^6 + 12z^4 - 1} \in \mathbb{Q}(z).$$

See Figure 1 for the image of the reals under g.

3. Proof of Proposition 1.6

If G acts on a set Ω, then ω^g denotes the image of $\omega \in \Omega$ under $g \in G$. Furthermore, $G_\omega = \{g \in G \mid \omega^g = \omega\}$ is the stabilizer of ω in G.

For $g, h \in G$ we write g^h for the conjugate $h^{-1}gh$ of g under h. Similarly, if S is a subset or subgroup of G, then $S^h = \{s^h \mid s \in S\}$.

If G is transitive on Ω, then $\emptyset \neq \Delta \subseteq \Omega$ is called a block if $\Delta = \Delta^g$ or $\Delta \cap \Delta^g = \emptyset$ for each $g \in G$. If this is the case, then Ω is a disjoint union of

sets $\Delta_i = \Delta^{g_i}$ for g_i in a subset of G. These sets Δ_i are called a block system. Note that G acts by permuting these sets Δ_i.

We assume that Proposition 1.6 is false, so there is a group M with $G_\omega \leq M \leq G$ and $M \neq M^\sigma$. Among the counterexamples with $|G|$ minimal we pick one with $|\Omega|$ minimal. In a series of lemmas, we derive properties of such a potential counterexample, and eventually we will see that it does not exist.

Note that G_ω^σ fixes $\omega^\sigma = \omega$, hence $G_\omega^\sigma \leq G_\omega$ and therefore $G_\omega = G_\omega^\sigma$, a fact we will use frequently. Also note that the condition on σ implies that $|\Omega|$ is odd. Another trivial fact which we use throughout the proof is the following: If B is a subgroup of G, then σ normalizes $B \cap B^\sigma$ and $\langle B, B^\sigma \rangle$.

LEMMA 3.1. *G is transitive on Ω.*

Proof. Set $\Delta = \omega^G$. If $\Delta = \Omega$ then we are done. So assume that $\Delta \subsetneq \Omega$. If $\Delta = \{\omega\}$, then $G_\omega = G$, and therefore of course $M = G = M^\sigma$.

Thus $\{\omega\} \subsetneq \Delta \subsetneq \Omega$. Note that $\Delta^\sigma = \omega^{G\sigma} = \omega^{\sigma G} = \omega^G = \Delta$. By the assumption of a minimal counterexample, we obtain that the proposition holds for the action of $\langle G, \sigma \rangle$ on Δ, hence $M = M^\sigma$, a contradiction. □

LEMMA 3.2. *$G = \langle M, M^\sigma \rangle$.*

Proof. Set $H = \langle M, M^\sigma \rangle$. Then $H^\sigma = H$, so if H is a proper subgroup of G, then $M = M^\sigma$ by the minimality assumption of a counterexample. □

LEMMA 3.3. *$M \cap M^\sigma = G_\omega$.*

Proof. Set $W = M \cap M^\sigma$. Note that $G_\omega \leq W$ and $W^\sigma = W$. Therefore, $\Delta = \omega^W$ is a block for the action of $\langle G, \sigma \rangle$ on Ω, and $\Delta^\sigma = \Delta$. Let $\bar{\Omega}$ be the block system which contains Δ, so $\langle G, \sigma \rangle$ acts on $\bar{\Omega}$. By the transitivity of G all blocks in $\bar{\Omega}$ have the same size, and this size divides the odd number $|\Omega|$. So the blocks have odd size, therefore σ has a fixed point in each block which is fixed setwise. Thus Δ is the only block fixed by σ. For $g \in G$ let $\bar{g}$ be the induced permutation on $\bar{\Omega}$. The stabilizer of Δ in $\bar{G}$ is $\bar{W}$.

Now suppose that $W > G_\omega$, hence $|\Delta| > 1$ and therefore $|\bar{\Omega}| < |\Omega|$. Note that $\bar{W} \leq \bar{M}$. So the proposition applies and yields $\bar{M}^\sigma = \bar{M}$. But the kernel of the map $g \mapsto \bar{g}$ is contained in $W = M \cap M^\sigma$, so $M^\sigma = M$, a contradiction. □

LEMMA 3.4. *If $B \leq M$, then either $G = \langle B, B^\sigma \rangle$, or $B \leq G_\omega$.*

Proof. Set $H = \langle B, B^\sigma \rangle$, and suppose that $H < G$. Hence, the proposition holds for H, in particular $\langle B, H_\omega \rangle^\sigma = \langle B, H_\omega \rangle$. Thus, $B \leq \langle B, H_\omega \rangle^\sigma \leq \langle M, G_\omega \rangle^\sigma = M^\sigma$. Together with the previous lemma we get $B \leq M \cap M^\sigma = G_\omega$. □

LEMMA 3.5. *G has even order.*

Proof. Suppose that the order of G is odd. Pick $g \in M \setminus G_\omega$. Note that

$$\left(gg^{-\sigma}\right)^\sigma = g^\sigma g^{-1} = \left(gg^{-\sigma}\right)^{-1},$$

so σ acts on $\langle gg^{-\sigma}\rangle$ by inverting the elements. As $\langle gg^{-\sigma}\rangle$ has odd order, there is $h \in \langle gg^{-\sigma}\rangle$ with $gg^{-\sigma} = h^2$. Set $c = hg^\sigma$. First, note that c is fixed under σ:

$$c^\sigma = \left(hg^\sigma\right)^\sigma = h^\sigma g = h^{-1}g = h^{-1}h^2g^\sigma = hg^\sigma = c$$

From this, we obtain that c permutes the fixed points of σ, so $c \in G_\omega$ because ω is the only fixed point of σ.

Another calculation shows

$$cg^{-\sigma}c = hg^\sigma g^{-\sigma} hg^\sigma = h^2 g^\sigma = g,$$

hence

$$g \in \left\langle G_\omega, g^{-\sigma}\right\rangle \leq M^\sigma,$$

contrary to $M \cap M^\sigma = G_\omega$ and the choice of g. □

LEMMA 3.6. *M contains at least one involution which is not contained in G_ω.*

Proof. Let J be the set of involutions of M. Then $J^m = J$ for all $m \in M$. Suppose that $J \subseteq G_\omega$. Then J is the set of involutions of G_ω, and together with $G_\omega^\sigma = G_\omega$ we obtain $J^\sigma = J$. So J is normalized by $\langle M, \sigma\rangle$. But $\langle M, \sigma\rangle = G$ by Lemma 3.2. In particular, $\langle J\rangle \leq G_\omega$ is a normal subgroup of G, hence $|\langle J\rangle| = 1$ as G acts faithfully.

We obtain $J = \emptyset$, contrary to $|G|$ being even by the previous lemma, together with $[G : G_\omega] = |\Omega|$ being odd. □

We now obtain the final contradiction: Let $a \in M \setminus G_\omega$ be an involution. Set $b = a^\sigma \in M^\sigma$ and let D be the dihedral group generated by a and b. From Lemma 3.4, with $B = \langle a\rangle$, we get $D = G$.

Set $C = \langle ab\rangle$. Then $[G : C] = 2$ (because $G = C \cup Ca$). We claim that C is transitive on Ω. If this were not the case, then, by the transitivity of G and $C \triangleleft G$, C would have exactly two orbits of equal size, so $|\Omega|$ were even, a contradiction.

So $G = CG_\omega$, and $C \cap G_\omega = 1$, because transitive abelian groups act regularly. The modular law yields $M = (C \cap M)G_\omega$ and $M^\sigma = (C \cap M^\sigma)G_\omega$.

From $|M| = |M^\sigma|$ we get $|C \cap M| = |C \cap M^\sigma|$. But the subgroups of the cyclic group C are determined uniquely by their order, hence $C \cap M = C \cap M^\sigma$ and finally $M = M^\sigma$.

4. Proof of Theorems 1.4 and 1.5

For the rational function $g(z) \in \mathbb{C}(z)$, let $\bar{g}(z)$ be the function with complex conjugate coefficients. Recall that $g(\hat{\mathbb{R}})$ is a circle in $\hat{\mathbb{C}}$ if and only if there is a linear fractional function $\lambda \in \mathbb{C}(z)$ such that $\lambda \circ g \in \mathbb{R}(z)$. The following lemma gives a useful necessary and sufficient criterion for this to hold. By $\mathbb{C}(g(z))$ we mean the field of rational functions in $g(z)$.

LEMMA 4.1. *Let $g(z) \in \mathbb{C}(z)$. Then $\lambda \circ g \in \mathbb{R}(z)$ for some linear fractional function $\lambda \in \mathbb{C}(z)$ if and only if $\mathbb{C}(g(z)) = \mathbb{C}(\bar{g}(z))$.*

Proof. If $\lambda \circ g \in \mathbb{R}(z)$, then $\lambda \circ g = \overline{\lambda \circ g} = \bar{\lambda} \circ \bar{g}$, hence $\bar{g}(z) = \bar{\lambda}^{-1}(\lambda(g(z)))$, and therefore $\mathbb{C}(\bar{g}(z)) = \mathbb{C}(g(z))$.

To prove the other direction, suppose that $\mathbb{C}(g(z)) = \mathbb{C}(\bar{g}(z))$. This assumption is preserved upon replacing g with $\mu \circ g$ for a linear fractional function $\mu \in \mathbb{C}(z)$. Thus, without loss of generality, we may assume that there are $r_1, r_2, r_3 \in \mathbb{R}$ with $g(r_1) = \infty$, $g(r_2) = 0$, $g(r_3) = 1$. From $\mathbb{C}(g(z)) = \mathbb{C}(\bar{g}(z))$, we get $\bar{g} = \rho \circ g$ for a linear fractional function $\rho \in \mathbb{C}(z)$. Evaluating in r_1, r_2, and r_3 yields that ρ fixes ∞, 0 and 1, hence $\rho(z) = z$. So $\bar{g} = g$, and therefore $g \in \mathbb{R}(z)$. □

REMARK 4.2. The lemma holds more generally if we replace $\mathbb{R}$ with a field K and $\mathbb{C}$ with a Galois extension E of K, and $\mathbb{C}(g(z)) = \mathbb{C}(\bar{g}(z))$ by the condition $E(g(z)) = E(g^\sigma(z))$ for all $\sigma \in \mathrm{Gal}(E/K)$. Indeed, if K is an infinite field, then we find $r_1, r_2, r_3 \in K$ such that the values $g(r_1)$, $g(r_2)$, and $g(r_3)$ are distinct and therefore without loss of generality equal to ∞, 0 and 1. So, as above, $g = g^\sigma$ for all $\sigma \in \mathrm{Gal}(E/K)$. Thus, the coefficients of g are fixed under $\mathrm{Gal}(E/K)$ and therefore contained in K.

If K is finite, we can argue as follows: We may assume that $g(\infty) = \infty$, so $g(z) = p(z)/q(z)$ for relatively prime polynomials $p, q \in E[z]$ with $\deg p > \deg q$. In addition, we may assume that p and q are monic. Let σ be a generator of the cyclic group $\mathrm{Gal}(E/K)$. From $g^\sigma(z) \in E(g(z))$ and $g^\sigma(z) = \frac{p^\sigma(z)}{q^\sigma(z)}$ we obtain $g^\sigma = g + b$ for some $b \in E$. Repeated application of σ shows that $\mathrm{Trace}_{E/K} b = 0$. So by the additive Hilbert's theorem 90 there is $c \in E$ with $c - c^\sigma = b$, hence $(g + c)^\sigma = g + c$ and therefore $g + c \in K(z)$. (The same argument, except that Hilbert's theorem 90 is a trivial fact for the extension $\mathbb{C}/\mathbb{R}$, works as an alternative proof of the lemma too.)

Theorem 1.4 is a direct consequence of Theorem 1.5. For if g is weakly injective, and $g \circ f$ is injective on $g(\hat{\mathbb{R}})$, then $g \circ f \circ g$ is weakly injective, so $f \circ g$ is weakly injective even more.

Thus, we only need to prove Theorem 1.5.

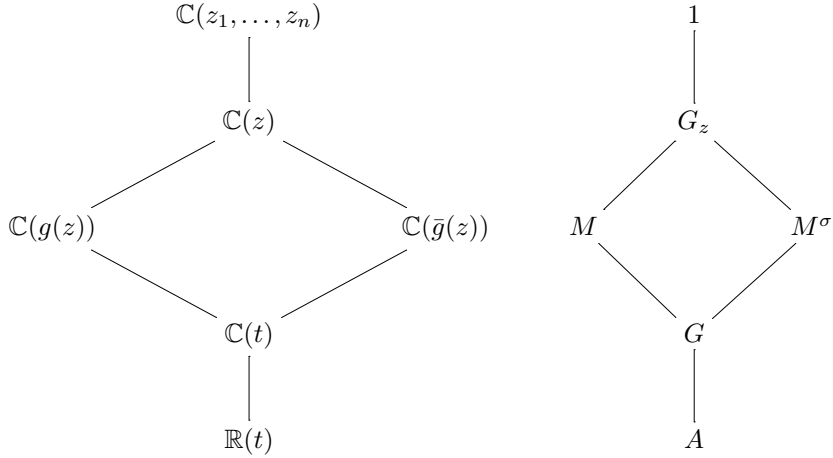
Let t be a variable over $\mathbb{C}$, and Z be another variable over the field $\mathbb{C}(t)$ of rational functions in t.

If $h(\infty) \neq \infty$, then upon replacing $h(Z)$ with $\frac{1}{h(Z)-h(\infty)}$ (and τ with $\frac{1}{\tau-h(\infty)}$) we may assume that $h(\infty) = \infty$. By further replacing $h(Z)$ with $h(Z) - \tau$, we may and do assume that $h(Z) = \frac{p(Z)}{q(Z)}$, where $p(Z), q(Z) \in \mathbb{R}[Z]$ are relatively prime polynomials, $\deg p(Z) = n > \deg q(Z)$, and $p(Z) = \prod_{i=1}^{n}(Z - \alpha_i)$, where the α_i are pairwise distinct, $\alpha_1 \in \mathbb{R}$, and $\alpha_i \notin \mathbb{C} \setminus \mathbb{R}$ for $i \geq 2$.

By Hensel's lemma, $p(Z) - tq(Z) = \prod_{i=1}^{n}(Z - z_i)$, where $z_i \in \mathbb{C}[[t]]$ has constant term α_i. As $\alpha_1 \in \mathbb{R}$ and $p, q \in \mathbb{R}[Z]$, we actually have $z_1 \in \mathbb{R}[[t]]$. Write $z = z_1$. The complex conjugation acts on the coefficients of the formal Laurent series $\mathbb{C}((t))$ and fixes t. Under this action, z is fixed, and the z_i''s for $i \geq 2$ are flipped in pairs. Note that $t = h(z_i)$ for all i.

The field $\mathbb{C}(z_1, z_2, \ldots, z_n)$ is a Galois extension of $\mathbb{R}(t)$ and a subfield of $\mathbb{C}((t))$. So the restriction to $\mathbb{C}(z_1, z_2, \ldots, z_n)$ of the complex conjugation action on the coefficients of $\mathbb{C}((t))$ is an involution σ in the Galois group $A = \mathrm{Gal}(\mathbb{C}(z_1, z_2, \ldots, z_n)/\mathbb{R}(t))$ which fixes $z = z_1$, and moves all z_i with $i > 1$.

Now write $h(z) = f(g(z))$ as in Theorem 1.5. Then also $t = h(z) = f(g(z)) = \bar{f}(\bar{g}(z))$. This yields the following inclusion of fields and the corresponding subgroups of A by Galois correspondence:



Here G_z is the stabilizer of $z = z_1$ in G. As M is the stabilizer of $g(z)$ in G, and σ maps $g(z)$ to $\bar{g}(z)$, the stabilizer of $\bar{g}(z)$ is $\sigma^{-1}M\sigma = M^\sigma$.

By construction, $\mathbb{C}(z_1, z_2, \ldots, z_n)$ is the splitting field of $p(Z) - tq(Z)$ over $\mathbb{C}(t)$, hence G acts faithfully on $\{z = z_1, z_2, \ldots, z_n\}$. Now G is normal in A, so $\sigma \in A$ normalizes G. Furthermore, σ fixes exactly one of the z_i. So $M = M^\sigma$ by Proposition 1.6. Thus $\mathbb{C}(g(z)) = \mathbb{C}(\bar{g}(z))$ by the Galois correspondence, and finally $\lambda(g(z)) \in \mathbb{R}(z)$ for some linear fractional $\lambda \in \mathbb{C}(z)$ by Lemma 4.1. This proves Theorem 1.5.

5. Some more examples

If $h = f \circ g$ for polynomials $f, g \in \mathbb{C}[z]$, and $h \in \mathbb{R}[z]$, then it is well known that there is a linear polynomial $\lambda \in \mathbb{C}[z]$ such that $\lambda \circ g \in \mathbb{R}[z]$. See [6, Theorem 3.5], or [14, Prop. 2.2] for a down to earth proof. A less elementary but more conceptual proof can be based on the fact that the Galois group of $h(Z) - t$ over $\mathbb{C}(t)$ contains an element which cyclically permutes the roots of $h(Z) - t$, and the other fact that subgroups of cyclic groups are uniquely determined by their orders.

Note that if $h = f \circ g$ for a polynomial h and rational functions f, g, then there is a linear fractional function $\rho \in \mathbb{C}(z)$ such that $h = (f \circ \rho^{-1}) \circ (\rho \circ g)$, and $f \circ \rho^{-1}$ and $\rho \circ g$ are polynomials. (This follows from looking at the fiber $h^{-1}(\infty)$.)

So in order to get examples of rational functions $h \in \mathbb{R}(z)$ which decompose as $h = f \circ g$ with $f, g \in \mathbb{C}(z)$ such that there is no linear fractional function $\lambda \in \mathbb{C}(z)$ with $\lambda \circ g \in \mathbb{R}(z)$, one has to assume that h is not a polynomial.

One also has to assume that g is not a polynomial, as the following easy result shows.

LEMMA 5.1. *Suppose that $f \circ g \in \mathbb{R}(z)$ where $f \in \mathbb{C}(z)$ and $g \in \mathbb{C}[z]$ are not constant. Then $\lambda \circ g \in \mathbb{R}[z]$ for a linear polynomial $\lambda \in \mathbb{C}[z]$.*

Proof. Assume without loss of generality that g is monic. Write $f = \frac{p}{q}$ with $p, q \in \mathbb{C}[z]$ relatively prime and p monic. From $f \circ g \in \mathbb{R}(z)$, we obtain

$$\frac{\bar{p}(\bar{g}(z))}{\bar{q}(\bar{g}(z))} = \frac{p(g(z))}{q(g(z))}.$$

Clearly, both fractions are reduced, and the numerators of both sides are monic. Therefore, $\bar{p}(\bar{g}(z)) = p(g(z))$, hence $p \circ g \in \mathbb{R}[z]$, and the claim follows from the polynomial case. □

Now we give some examples of rational functions $h \in \mathbb{R}(z)$ with a decomposition $h = f \circ g$ with $f, g \in \mathbb{C}(z)$ which is not equivalent to a decomposition over $\mathbb{R}$. Recall that this is equivalent to $g(\hat{\mathbb{R}})$ not being a circle. Of course, Theorem 1.2 gives many example for this. But these examples are quite complicated and not explicit. However, if one drops the requirement that $g(\hat{\mathbb{R}})$ is a Jordan curve, then there are quite simple examples. We give two series.

EXAMPLE 5.2 (*Attributed to Pakovich by Eremenko in* [3]). Let $T_n \in \mathbb{R}[z]$ be the polynomial with $T_n(z + \frac{1}{z}) = z^n + \frac{1}{z^n}$ (so T_n is essentially a Chebychev polynomial.) Set $g(z) = \zeta z + \frac{1}{\zeta z}$ for an n-th root of unity ζ. Then $T_n(g(z)) = z^n + \frac{1}{z^n} \in \mathbb{R}(z)$, while $g(\hat{\mathbb{R}})$ is not a circle if $\zeta^4 \neq 1$.

EXAMPLE 5.3. Pick $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, and set

$$F = z^k(1 - z)^{n-k}, \qquad G = \frac{1 - \zeta z^k}{1 - \zeta z^n}, \qquad \mu(z) = \frac{z + i}{z - i}$$

for $1 \le k < n$. A straightforward calculation shows that

$$F(\bar{G}\left(\frac{1}{z}\right) = \frac{1}{\zeta^{n-k}} F\big(G(z)\big). \tag{5.1}$$

Pick $\rho \in \mathbb{C}$ with $\rho^2 = \frac{1}{\zeta^{n-k}}$, and set

$$f(z) = \rho F(z), \qquad g(z) = G\big(\mu(z)\big).$$

From $\bar{\mu}(z) = \mu(\frac{1}{z})$ and (5.1) we get $\overline{f \circ g} = f \circ g$, hence $f \circ g \in \mathbb{R}(z)$.

On the other hand, it is easy to see that, except for a some degenerate cases, $g(\hat{\mathbb{R}})$ is not a circle. Furthermore, we see that $g(\hat{\mathbb{R}})$ isn't even a Jordan curve (unless it is a circle), for if z runs through $\hat{\mathbb{R}}$, $\mu(z)$ runs through the unit circle, so the numerator and denominator of $g(z) = G(\mu(z))$ vanish k and n times, respectively, so $g(\hat{\mathbb{R}})$ has several self intersections.

Originally I had only found the cases $\zeta = -1$, $k = n - 1$. Mike Zieve observed the strong similarity of these examples with functions which turned up in work of Avanzi and Zannier. In [1], they classify triples $F \in \mathbb{C}[z]$, $G_1, G_2 \in \mathbb{C}(z)$ such that $F \circ G_1 = F \circ G_2$. One of their cases ([1, Prop. 4.7(3)]) is the above series with $\zeta = 1$, and the series [1, Prop. 5.6(4)] is essentially our series from above.

The connection with the work by Avanzi and Zannier is not a surprise: If we look for polynomials $F \in \mathbb{R}[z]$ such that there is $G \in \mathbb{C}(z) \setminus \mathbb{R}(z)$ with $F \circ G \in \mathbb{R}(z)$, then $F \circ \bar{G} = F \circ G$ with $G \neq \bar{G}$. Furthermore, note that if ζ is an m-th root of unity, then $f(z)^{2m} \in \mathbb{R}(z)$. So upon setting $\tilde{f} = f^{2m} = F^{2m}$, we have $\tilde{f} \circ g = \tilde{f} \circ \bar{g}$.

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