BGK MODEL OF THE MULTI-SPECIES UEHLING UHLENBECK EQUATION

GI-CHAN BAE, CHRISTIAN KLINGENBERG, MARLIES PIRNER, AND SEOK-BAE YUN

ABSTRACT. In this paper, we propose a BGK model of the quantum Boltzmann equation for gas mixtures, and provide a sufficient condition that guarantees the existence of equilibrium coefficients so that the model shares the same conservation laws and H-theorem with the quantum Boltzmann equation. Compared to the proof in the case for classical BGK for gas mixtures, the existence of moment constraints has to be proven. This is due to the nature of Fermi and Bose distributions instead of Maxwellians in the relaxation operators. For this, we explicitly derive the nonlinear relations among the equilibrium coefficients of local quantum equilibriums that arise from the conservation laws and H-theorem, and verify in a unified way that the nonlinear relations uniquely determines the coefficients under certain conditions.

CONTENTS

1. INTRODUCTION

1.1. **Quantum Boltzmann equation for gas mixture.** The quantum modification of the celebrated Boltzmann equation was made in [61, 62] to incorporate the quantum effect that cannot be neglected for light molecules (such as Helium) in low temperature. Quantum Boltzmann equation is now fruitfully employed not just for low temperature gases, but in various circumstances such as the study of carrior mobility in various electronic devices. When the gas is composed of several different types of molecules (gas mixture), the quantum Boltzmann equation takes the form (For simplicity, we restricted our interest into two species case):

$$\begin{aligned} \partial_t f_1 + \frac{p}{m_1}\cdot\nabla_x f_1 &= Q_{11}(f_1,f_1)+Q_{12}(f_1,f_2),\\ \partial_t f_2 + \frac{p}{m_2}\cdot\nabla_x f_2 &= Q_{22}(f_2,f_2)+Q_{21}(f_2,f_1). \end{aligned} \tag{1.1}$$

The momentum distribution function $f_i(x,p,t)$ denotes the number density at the phase point $(x,p)\in \Omega_x\times\mathbb{R}^3_p$ at time t. The collision operator Q_{ij} $(i,j=1,2)$ takes the following form:

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- Fermion-Fermion $(-)$, Boson-Boson $(+)$.

$$Q_{ij}(f_i,f_j)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B_{ij}(|p-p_*|,w)\{f_i'f_{j,*}'(1\pm f_i)(1\pm f_{j,*})-f_if_{j,*}(1\pm f_i')(1\pm f_{j,*}')\}dwdp_*$$

- Fermion (f_1)-Boson (f_2) interaction:

$$\begin{aligned}Q_{ij}(f_i,f_j)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}&B_{ij}(|p-p_*|,w)\{f_i'f_{j,*}'(1+\tau(i)f_i)(1+\tau(j)f_{j,*})\\&-f_if_{j,*}(1+\tau(i)f_i')(1+\tau(j)f_{j,*}')\}dwdp_*.\end{aligned}$$

where $\tau(1)=-1$ and $\tau(2)=1$. We used the abbreviated notation:

$$f_i=f_i(x,p,t),\ f_{i,*}=f_i(x,p_*,t),\ f_i'=f_i(x,p',t),\ f_{i,*}'=f_i(x,p_*',t),\ i=1,2.$$

The pre-collisional momenta p' and p_*' can be derived from the local conservation laws:

$$\begin{aligned}&p'+p_*'=p+p_*,\\&\frac{|p'|^2}{2m_1}+\frac{|p_*'|^2}{2m_2}=\frac{|p|^2}{2m_1}+\frac{|p_*|^2}{2m_2},\end{aligned}\tag{1.2}$$

in the following explicit forms:

$$\begin{aligned}p'&=p-\frac{2m_1m_2}{m_1+m_2}w\left[\left(\frac{p}{m_1}-\frac{p_*}{m_2}\right)\cdot w\right],\\p_*'&=p_*+\frac{2m_1m_2}{m_1+m_2}w\left[\left(\frac{p}{m_1}-\frac{p_*}{m_2}\right)\cdot w\right].\end{aligned}$$

The collision operator has 5 collision invariants: $1,p,|p|^2$ $(k=1,2)$:

$$\begin{aligned}&\int_{\mathbb{R}^3}Q_{kk}(f_k,f_k)dp=0,\quad\int_{\mathbb{R}^3}Q_{12}(f_1,f_2)dp=\int_{\mathbb{R}^3}Q_{21}(f_2,f_1)dp=0,\\&\int_{\mathbb{R}^3}Q_{kk}(f_k,f_k)pdp=0,\quad\int_{\mathbb{R}^3}\{Q_{12}(f_1,f_2)p+Q_{21}(f_2,f_1)\}pdp=0,\\&\int_{\mathbb{R}^3}Q_{kk}(f_k,f_k)|p|^2dp=0,\quad\int_{\mathbb{R}^3}\{Q_{12}(f_1,f_2)+Q_{21}(f_2,f_1)\}|p|^2dp=0,\end{aligned}\tag{1.3}$$

which leads to the conservation of total mass, momentum and energy:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3}f_1dxdp=0,\quad\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3}f_2dxdp=0,\\&\frac{d}{dt}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}f_1pdxdp+\int_{\mathbb{T}^3\times\mathbb{R}^3}f_2pdxdp\right)=0,\\&\frac{d}{dt}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}f_1|p|^2dxdp+\int_{\mathbb{T}^3\times\mathbb{R}^3}f_2|p|^2dxdp\right)=0.\end{aligned}\tag{1.4}$$

The collision operator Q_{ii}, Q_{ij} $(i,j\in\{1,2\})$ also satisfies the following entropy dissipation property:

$$\begin{aligned}&\int_{\mathbb{R}^3}\ln\frac{f_1}{1+\tau(1)f_1}Q_{11}(f_1,f_1)dp\le 0,\quad\int_{\mathbb{R}^3}\ln\frac{f_2}{1+\tau(2)f_2}Q_{22}(f_2,f_2)dp\le 0,\\&\int_{\mathbb{R}^3}\ln\frac{f_1}{1+\tau(1)f_1}Q_{12}(f_1,f_2)dp+\int_{\mathbb{R}^3}\ln\frac{f_2}{1+\tau(2)f_2}Q_{21}(f_2,f_1)dp\le 0.\end{aligned}\tag{1.5}$$

where $\tau(i)=-1$ when f_i denotes distribution of fermion and $\tau(i)=+1$ when f_i denotes distribution of boson.

Such dissipation implies the celebrated H-theorem for quantum mixture:

- Fermion-Fermion $(-)$, Boson-Boson $(+)$:

$$\begin{aligned}\frac{d}{dt}H(f_1,f_2) &= \int_{\mathbb{R}^3}\ln\frac{f_1}{1\pm f_1}Q_{11}(f_1,f_1)dp+\int_{\mathbb{R}^3}\ln\frac{f_2}{1\pm f_2}Q_{22}(f_2,f_2)dp\\ &\quad+\int_{\mathbb{R}^3}\ln\frac{f_1}{1\pm f_1}Q_{12}(f_1,f_2)dp+\int_{\mathbb{R}^3}\ln\frac{f_2}{1\pm f_2}Q_{21}(f_2,f_1)dp\leq 0,\end{aligned}$$

- Fermion (f_1)-Boson (f_2):

$$\begin{aligned}\frac{d}{dt}H(f_1,f_2) &= \int_{\mathbb{R}^3}\ln\frac{f_1}{1- f_1}Q_{11}(f_1,f_1)dp+\int_{\mathbb{R}^3}\ln\frac{f_2}{1+ f_2}Q_{22}(f_2,f_2)dp\\ &\quad+\int_{\mathbb{R}^3}\ln\frac{f_1}{1- f_1}Q_{12}(f_1,f_2)dp+\int_{\mathbb{R}^3}\ln\frac{f_2}{1+ f_2}Q_{21}(f_2,f_1)dp\leq 0,\end{aligned}$$

where $H(f_1,f_2)$ denotes the H-functional:

- Fermion-Fermion interaction:

$$H(f_1,f_2)=\int_{\mathbb{R}^3}f_1\ln f_1+(1-f_1)\ln(1-f_1)dp+\int_{\mathbb{R}^3}f_2\ln f_2+(1-f_2)\ln(1-f_2)dp.$$

- Boson-Boson interaction:

$$H(f_1,f_2)=\int_{\mathbb{R}^3}f_1\ln f_1-(1+f_1)\ln(1+f_1)dp+\int_{\mathbb{R}^3}f_2\ln f_2-(1+f_2)\ln(1+f_2)dp.$$

- Fermion (f_1)-Boson (f_2) interaction:

$$H_{FB}(f_1,f_2)=\int_{\mathbb{R}^3}f_1\ln f_1+(1-f_1)\ln(1-f_1)dp+\int_{\mathbb{R}^3}f_2\ln f_2-(1+f_2)\ln(1+f_2)dp.$$

The r.h.s of (1.1) vanishes if and only if f_1 and f_2 are quantum equilibrium:

- Fermion-Fermion $(+)$, Boson-Boson interaction $(-)$:

$$f_1=\frac{1}{e^{a|p-b|^2+c_1}\pm 1},\quad f_2=\frac{1}{e^{a|p-b|^2+c_2}\pm 1}.$$

- Fermion (f_1)-Boson (f_2) interaction

$$f_1=\frac{1}{e^{a|p-b|^2+c_1}+ 1},\quad f_2=\frac{1}{e^{a|p-b|^2+c_2}- 1}.$$

1.2. **Quantum BGK model for gas mixture.** In this paper, we propose the BGK type relaxation model of (1.6) :

$$\begin{aligned}\partial_t f_1+p\cdot\nabla_x f_1&=\mathcal{R}_{11}+\mathcal{R}_{12},\\ \partial_t f_2+p\cdot\nabla_x f_2&=\mathcal{R}_{21}+\mathcal{R}_{22},\end{aligned}\tag{1.6}$$

where $\mathcal{R}_{ij}$ denotes the relaxation operator for the interactions of ith and jth component. More explicitly, they are defined as follows:

- Fermion-Fermion interaction $(i\neq j)$:

$$\mathcal{R}_{ii}=\mathcal{F}_{ii}-f_i,\quad \mathcal{R}_{ij}=\mathcal{F}_{ij}-f_i,\qquad (i=1,2)$$

  where $\mathcal{F}_{ii}$ denotes the Fermi-Dirac distribution for same-species interaction:

$$\mathcal{F}_{11}=\frac{1}{e^{a_1|p-b_1|^2+c_1}+1},\quad \mathcal{F}_{22}=\frac{1}{e^{a_2|p-b_2|^2+c_2}+1},$$

  and $\mathcal{F}_{ij}$ denote Fermi-Dirac distribution for inter-species interactions:

$$\mathcal{F}_{12}=\frac{1}{e^{a|p-b|^2+c_{12}}+1},\quad \mathcal{F}_{21}=\frac{1}{e^{a|p-b|^2+c_{21}}+1}.$$

- Boson-Boson interaction ($i \neq j$):
$$\mathcal{R}_{ii} = \mathcal{B}_{ii} - f_i, \quad \mathcal{R}_{ij} = \mathcal{B}_{ij} - f_i, \qquad (i = 1, 2)$$
where $\mathcal{B}_{ii}$ denotes the Bose-Einstein distribution for same-species interaction :
$$\mathcal{B}_{11} = \frac{1}{e^{a_1|p-b_1|^2+c_1} - 1}, \quad \mathcal{B}_{22} = \frac{1}{e^{a_2|p-b_2|^2+c_2} - 1},$$
while $\mathcal{B}_{ij}$ denote Bose-Einstein distribution for inter-species interactions:
$$\mathcal{B}_{12} = \frac{1}{e^{a|p-b|^2+c_{12}} - 1}, \quad \mathcal{B}_{21} = \frac{1}{e^{a|p-b|^2+c_{21}} - 1}.$$
- Fermion (f_1)-Boson (f_2) interaction:
$$\mathcal{R}_{11} = \mathcal{F}_{11} - f_1 \quad \mathcal{R}_{22} = \mathcal{B}_{22} - f_2,$$
and
$$\mathcal{R}_{12} = \mathcal{F}_{12} - f_1 \quad \mathcal{R}_{21} = \mathcal{B}_{21} - f_2,$$
where $\mathcal{F}_{11}$ denotes the Fermi-Dirac distribution and $\mathcal{B}_{22}$ denotes the Bose-Einstein distribution for i-i interaction :
$$\mathcal{F}_{11} = \frac{1}{e^{a_1|p-b_1|^2+c_1} + 1}, \quad \mathcal{B}_{22} = \frac{1}{e^{a_2|p-b_2|^2+c_2} - 1},$$
while $\mathcal{F}_{12}$ denotes Fermi-Dirac distribution and $\mathcal{B}_{21}$ denotes Bose-Einstein distribution for inter-species interactions:
$$\mathcal{F}_{12} = \frac{1}{e^{a|p-b|^2+c_{12}} + 1}, \quad \mathcal{B}_{21} = \frac{1}{e^{a|p-b|^2+c_{21}} - 1}.$$

For later convenience, and for unified proof, we introduce the following notation for quantum equilibriums:

• The quantum equilibrium $\mathcal{M}_{ij}$

Next, we will make statements on the equilibrium distributions in the relaxation operators that correspond to $\mathcal{F}_{ij}$ in the fermion case and $\mathcal{B}_{ij}$ in the boson case. In order not to list all different cases separately, we denote the equilibrium distribution by $\mathcal{M}_{ij}$ which is equal to a Fermi-Dirac or a Bose-Einstein distribution depending on the case we consider:

(1) Fermion-Fermion interaction
$$\mathcal{M}_{ij} = \mathcal{F}_{ij}. \quad (i, j = 1, 2)$$
(2) Boson-Boson interaction
$$\mathcal{M}_{ij} = \mathcal{B}_{ij}. \quad (i, j = 1, 2)$$
(3) Fermion (f_1) - Boson (f_2) interaction
$$\mathcal{M}_{1j} = \mathcal{F}_{1j}, \quad \mathcal{M}_{2j} = \mathcal{B}_{2j}. \quad (j = 1, 2)$$

The excessive computational cost has already been a very serious obstacles even for the classical Boltzmann equation. Since the difficulty mostly lies in the computation of the collision operator, various efforts to approximate the complicated collision process with a numerically more amenable model have been made. The BGK model is introduced in [9] as a result of such efforts, and now become the most popular approximate model of the Boltzmann equation because it provide a very reliable results in wide range of kinetic-fluid regime covering much of the practical problems at relatively low computational costs.

As in the classical case, the quantum BGK models are widely used in place of the quantum Boltzmann equation. However, the quantum BGK model for mixture are not rigorously studied yet. More precisely, whether the relaxation operator can be soundly defined in a rigorous manner so that it satisfies the same conservation laws and the H-theorem as the quantum Boltzmann does has never been rigorously verified in the literature. (For the relevant result for one-species quantum BGK, see [3, 4, 22, 43, **?**]), which is the main motivation of the current work.

1.3. **Determination of $\mathcal{M}_{ij}$ $(i, j = 1, 2)$.** The quantum BGK model may be far more amenable in terms of numerical computation, but the highly non-linear nature of the QBGK model gives rise to various difficulties in the analysis of the model. As such, it turns out that the requirement that the QBGK model must share the conservation laws and H-theorem with the quantum Boltzmann equation, leads to a set of very complicated nonlinear relations for the equilibrium coefficients (See Section 2.2). Moreover, they involves different conditions of solvability according to the nature of the interactions: Fermion-Fermion interaction, Fermion-Boson interaction, Boson-Boson interaction.

In this paper, we explicitly derive the nonlinear relations among the equilibrium coefficients of $\mathcal{M}_{11}$, $\mathcal{M}_{22}$, $\mathcal{M}_{12}$, $\mathcal{M}_{21}$ that arise from the physical requirement of the equation, and verify in a unified way that those nonlinear relations uniquely determined the coefficients under certain conditions.

First, we note that we need to determine the mixture local equilibrium $\mathcal{M}_{ij}$ in such way that the relaxation operator in the r.h.s of (1.6) satisfies the same cancellation properties in (1.3) and the entropy dissipation in (1.5) are determined by following conservation laws.

To be more specific, let N_i, P_i and E_i $(i = 1, 2)$ denotes

$$N_i = \int_{\mathbb{R}^3} f_i dp, \quad P_i = \int_{\mathbb{R}^3} f_i p dp, \quad E_i = \int_{\mathbb{R}^3} f_i |p|^2 dp.$$

Assuming that the r.h.s of (1.6) satisfies the same identities in (1.3), we arrive at the following identities:

$$\int_{\mathbb{R}^3} \mathcal{M}_{ii} dp = N_i, \qquad \int_{\mathbb{R}^3} \mathcal{M}_{ii} p dp = P_i, \qquad \int_{\mathbb{R}^3} \mathcal{M}_{ii} |p|^2 dp = E_i, \quad (i = 1, 2) \tag{1.7}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathcal{M}_{12} dp = N_1, \qquad \int_{\mathbb{R}^3} \mathcal{M}_{21} dp = N_2, \\ &\int_{\mathbb{R}^3} \mathcal{M}_{12} p dp + \int_{\mathbb{R}^3} \mathcal{M}_{21} p dp = P_1 + P_2, \\ &\int_{\mathbb{R}^3} \mathcal{M}_{12} |p|^2 dp + \int_{\mathbb{R}^3} \mathcal{M}_{21} |p|^2 dp = E_1 + E_2. \end{aligned} \tag{1.8}$$

Our goal is to show that, for each fixed N_i, P_i, E_i $(i = 1, 2)$, the relations in (1.7) and (1.8) completely and uniquely determine $\mathcal{M}_{ij}$, which is stated in Theorem 2.1.

1.4. **literature review: Quantum BGK models.** The quantum modification of the celebrated Boltzmann equation, which is often called Uehling-Uhlenbeck equation or Nordheim equation in the literature, was made in [26, 37, 61, 62] and soon recognized as a fundamental equation to describe quantum particles at mesoscopic level. But due to the complexity of the collision operator, which is a serious obstacle to practical application of the equation, and relaxation time approximations, or quantum BGK models are widely used to understand the transport phenomena and compute transport coefficients for semi-conductor device and crystal lattice [2, 21, 34, 35, 36, 45, 51] and various flow problems involving quantum effects [16, 24, 23, 35, 57, 63, 64]. For the development of numerical methods for quantum BGK model, we refer to [16, 23, 24, 47, 53, 56, 57, 63, 64, 65]. We mention that The prototype of relaxation type models in quantum theory can be traced back to the Drude model [19, 20] which successfully explained the fundamental transport property of electrons such as the Ohm's law or Hall effect. .

Mathematical results on the quantum BGK model is Nouri studied the existence of weak solutions for a stationary quantum BGK model with a discretized condensation term in [48]. Braukhoff [13, 14] established the existence of analytic solutions and studied its asymptotic behaviour for a quantum BGK type model describing the dynamics of the ultra-cold atoms in an optical lattice. Bae et al considered the existence and asymptotic stability of a fermionic quantum BGK model near a global Fermi-Dirac distribution

1.5. **BGK models for gas mixtures:** There are many BGK models for gas mixtures proposed in the literature. Examples include the model of Gross and Krook [30], the model of Hamel [32], the model of Garzo, Santos and Brey [27], the model of Greene [28], the model of Sofonea and Sekerka [58], the model of Klingenberg, Pirner and Puppo [39], the model of Haack, Hauck, Murillo [31], the model of Bobylev, Bisi, Groppi, Spiga [12], the model by Andries, Aoki and Perthame [1]. BGK models have also been extended to ES-BGK models, polyatomic molecules or chemical reactions; see for example [40, 60, 29, 41, 50, 10, 11]. BGK models are often used in applications because they give rise to efficient numerical computations as compared to models with Boltzmann collision terms [49, 25, 18, 7, 17, 8].

In the following Section 2.1, we state our main result. In Section 2.2, we derive a set of nonlinear functional relations and show that the equilibirum coefficients can be uniquely determined to satisfy the conservation laws of mass, momentum and energy. In Section 2.3, the BGK model defined with the equilibrium coefficients derived in Section 2.2, also satisfies the H-theorem.

2. Determination of the relaxation operators for quantum mixture

2.1. **Main result for general quantum-quantum interaction.** We now state our main result stating that the equilibrium coefficients, under appropriate assumptions on N_i, P_i and E_i, can be uniquely determined. To simplify the presentation, we introduce $h_{\pm1}$, $j_{\pm1}$, k by

$$h_{\pm1}(x) = \int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+x} \pm 1} dp, \qquad j_{\pm1}(x) = \frac{\int \frac{1}{e^{|p|^2+x}\pm1} dp}{\left(\int \frac{|p|^2}{e^{|p|^2+x}\pm1} dp\right)^{3/5}},$$

and

$$k_{\tau,\tau'}(x,y) = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+x}+\tau} dp}{\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+x}+\tau} dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+y}+\tau'} dp\right)^{\frac{3}{5}}},$$

where the pair (τ,τ') is chosen as follows:

$$(\tau,\tau') = \begin{cases} (+1,+1) & \text{(fermion-fermion)} \\ (-1,-1) & \text{(boson-boson)} \\ (+1,-1) & \text{(fermion-boson)} \end{cases}$$

Using h and k, we define g, which is defined as a composite function of k and h^{-1}, as follows:

$$g_{\tau,\tau'}(x) = k_{\tau,\tau'}\big(x, y(x)\big) = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+x}+\tau} dp}{\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+x}+\tau} dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+y(x)}+\tau'} dp\right)^{\frac{3}{5}}}, \tag{2.1}$$

where $y(x)$ denotes

$$y(x) = h_{\tau'}^{-1}\left(\frac{N_2}{N_1} h_\tau(x)\right),$$

Note that $h_{\pm1}^{-1}$ always exist since $h_{\pm1}$ is strictly decreasing. For simplicity of notation, we define $l : \{+1,-1\} \to \mathbb{R}$ by

$$l(x) = \begin{cases} l(+1) = -\infty \\ l(-1) = 0 \end{cases}$$

Theorem 2.1. (1) *Assume,*

$$\frac{N_1}{(E_1 - P_1^2/N_1)^{\frac{3}{5}}} \le j_\tau(l(\tau)), \qquad \frac{N_2}{(E_2 - P_2^2/N_2)^{\frac{3}{5}}} \le j_{\tau'}(l(\tau')).$$

Then, we can define c_i $(i=1,2)$ as the unique solution of

$$j_\tau(c_1)=\frac{N_1}{(E_1-|P_1|^2/N_1)^{\frac{3}{5}}},\qquad j_{\tau'}(c_2)=\frac{N_2}{(E_2-|P_2|^2/N_1)^{\frac{3}{5}}}.$$

With c_1, c_2 obtained above, we then define a_i $(i=1,2)$ by

$$a_1=\left(\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+c_1}+\tau}dp\right)^{\frac{2}{3}}N_1^{-\frac{2}{3}},\qquad a_2=\left(\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+c_2}+\tau'}dp\right)^{\frac{2}{3}}N_2^{-\frac{2}{3}},$$

and

$$b_1=\frac{P_1}{N_1},\quad b_2=\frac{P_2}{N_2}.$$

Then, with such choice of a_i, b_i and c_i, $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ satisfies (1.7).

(2) *Assume further that*

$$\frac{N_1}{\left(E_1+E_2-\frac{|P_1+P_2|^2}{N_1+N_2}\right)^{\frac{3}{5}}}\le g\left(\max\left\{l(\tau),h_\tau^{-1}\left(\frac{N_1}{N_2}h_{\tau'}(l(\tau'))\right)\right\}\right).$$

Then c_{12}, c_{21} are defined as a unique solution of the following relations:

$$\frac{h_\tau(c_{12})}{h_{\tau'}(c_{21})}=\frac{N_1}{N_2},\quad k_{\tau,\tau'}(c_{12},c_{21})=\frac{N_1}{\left(E_1+E_2-\frac{|P_1+P_2|^2}{N_1+N_2}\right)^{\frac{3}{5}}}.$$

With such c_{12} and c_{21}, we define a and b by

$$a=\left(\frac{\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+c_{12}+\tau}}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+c_{21}+\tau'}}dp}{E_1+E_2-\frac{|P_1+P_2|^2}{N_1+N_2}}\right)^{\frac{2}{5}},\quad b=\frac{P_1+P_2}{N_1+N_2},$$

Then, with these choice our equilibrium coefficients, our quantum BGK model for gas mixture (1.6) satisfies (1.8).

(3) *With the choice of equilibrium coefficients as in (1), (2), the quantum BGK model for gas mixture (1.6) satisfies the H-theorem. The equality in the H-Theorem is characterized by f_1 and f_2 being two Fermion distributions in the Fermion-Fermion case, two Bose distributions in the Boson- Boson case and a Fermion distribution and a Bose distribution in the Fermion-Boson case. In all cases theses equilibrium distributions have the same a and b.*

2.2. **Proof of Theorem 2.1 (1), (2).** The proof for (1) can be found in [3]. Therefore, we start with the proof of (2). An explicit computation from $(1.8)_2$ gives

$$\begin{aligned}P_1(x,t)+P_2(x,t)&=\int_{\mathbb{R}^3}\frac{p}{e^{a|p-b|^2+c_{12}}+\tau}dp+\int_{\mathbb{R}^3}\frac{p}{e^{a|p-b|^2+c_{21}}+\tau'}dp\\&=\int_{\mathbb{R}^3}\frac{p+b}{e^{a|p|^2+c_{12}}+\tau}dp+\int_{\mathbb{R}^3}\frac{p+b}{e^{a|p|^2+c_{21}}+\tau'}dp\\&=b(N_1(x,t)+N_2(x,t)).\end{aligned}$$

This gives the explicit presentation of b:

$$b(x,t)=\frac{P_1(x,t)+P_2(x,t)}{N_1(x,t)+N_2(x,t)}.\tag{2.2}$$

On the other hand, we have from $(1.8)_1$ that:

$$\begin{aligned} N_1(x,t) &= \int_{\mathbb{R}^3} \frac{1}{e^{a|p-b|^2+c_{12}}+\tau}dp = a^{-\frac{3}{2}}\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+c_{12}}+\tau}dp, \\ N_2(x,t) &= \int_{\mathbb{R}^3} \frac{1}{e^{a|p-b|^2+c_{21}}+\tau'}dp = a^{-\frac{3}{2}}\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+c_{21}}+\tau'}dp, \end{aligned} \tag{2.3}$$

and from $(1.8)_3$:

$$\begin{aligned} E_1(x,t)+E_2(x,t) &= \int_{\mathbb{R}^3} \frac{|p|^2}{e^{a|p-b|^2+c_{12}}+\tau}dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{a|p-b|^2+c_{21}}+\tau'}dp \\ &= a^{-\frac{5}{2}}\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{12}}+\tau}dp + a^{-\frac{5}{2}}\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{21}}+\tau'}dp \\ &+ (N_1+N_2)b^2(x,t), \end{aligned} \tag{2.4}$$

Plugging (2.2) into (2.4), we get

$$E_1+E_2-\frac{|P_1+P_2|^2}{N_1+N_2} = a^{-\frac{5}{2}}\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{12}}+\tau}dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{21}}+\tau'}dp\right) \tag{2.5}$$

We then deduce from (2.5) and $(2.3)_1$ that

$$\frac{N_1}{\left(E_1+E_2-\frac{|P_1+P_2|^2}{N_1+N_2}\right)^{\frac{3}{5}}} = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+c_{12}+\tau}}dp}{\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{12}+\tau}}dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+c_{21}+\tau'}}dp\right)^{\frac{3}{5}}}, \tag{2.6}$$

On the other hand, we can factor out a by dividing the two relations in (2.3):

$$\frac{N_1}{N_2} = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+c_{12}+\tau}}dp}{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+c_{21}+\tau'}}dp} \tag{2.7}$$

and hence:

$$c_{21} = h_{\tau'}^{-1}\left(\frac{N_2}{N_1}h_\tau(c_{12})\right), \tag{2.8}$$

from the monotonicity of h_τ. Now, considering that a is obtained from (2.5) once c_{12} and c_{21} are chosen, it remains, under the assumption of Theorem 2.1, that (2.6) and (2.7) uniquely determine c_{12} and c_{21}. In turn, in view of (2.6) and (2.8), we see that c_{12} and c_{21} can be uniquely determined once we prove the monotonicity of g, which is stated in the following lemma.

Lemma 2.2. *Recall the definition of g given in (2.1):*

$$g_{\tau,\tau'}(x) = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{|p|^2+x}+\tau}dp}{\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+x}+\tau}dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{\frac{3}{5}}},$$

where

$$y(x) = h_{\tau'}^{-1}\left(\frac{N_2}{N_1}h_\tau(x)\right), \tag{2.9}$$

Then $g_{\tau,\tau'}(x)$ is monotone decreasing function when $x \geq \max\left\{l(\tau), h_\tau^{-1}\left(\frac{N_1}{N_2}h_{\tau'}(l(\tau'))\right)\right\}$.

Proof. **Claim :** We claim that establishing the following identity finishes the proof.

$$g'_{\tau,\tau'}(x) = 8\pi^2 \frac{D_\tau(x) + \frac{\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}D_{\tau'}(y(x))}{\left(\int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+x}+\tau}dp + \int_{\mathbb{R}^3} \frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{\frac{4}{5}}} \tag{2.10}$$

where

$$D_\tau(x)=\frac{9}{5}\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr-\int_0^\infty\frac{r^4}{e^{r^2+x}+\tau}dr\int_0^\infty\frac{1}{e^{r^2+x}+\tau}dr.$$

To see this, we first observe that $h(x)$ is strictly decreasing function on $x\in[0,\infty)$ for $\tau=-1$ and $x\in(-\infty,\infty)$ for $\tau=+1$:

$$h_\tau'(x)=-\int_{\mathbb{R}^3}\frac{e^{|p|^2+x}}{(e^{|p|^2+x}+\tau)^2}dp<0.$$

Therefore, our restriction on x: $x\geq h_\tau^{-1}\left(\frac{N_1}{N_2}h_{\tau'}(l(\tau'))\right)$ combined with the definition of y given in (2.9), leads to

$$y(x)\equiv h_{\tau'}^{-1}\left(\frac{N_2}{N_1}h_\tau(x)\right)\geq h_{\tau'}^{-1}\left(\frac{N_2}{N_1}h_\tau\left(h_\tau^{-1}\left(\frac{N_1}{N_2}h_{\tau'}(l(\tau'))\right)\right)\right)=l(\tau').$$

In conclusion, we have

$$x\geq l(\tau),\quad\text{and}\quad y(x)\geq l(\tau').$$

Therefore, we have

$$D_\tau(x)<0\quad\text{and}\quad D_{\tau'}(y(x))<0.$$

Since we already know

$$D_{+1}(x)<0\text{ on }x\in(-\infty,\infty),\qquad D_{-1}(x)<0\text{ on }x\in[0,\infty).$$

(See [43] for boson case $(+1)$ and [3, 44] for fermion case (-1)). In conclusion, we are all set if we established the identity (2.10):

• **Proof of (2.10):** By an explicit computation, we have

$$\begin{aligned}\frac{\partial g(x)}{\partial x}&=\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{-\frac{6}{5}}\\&\times\Bigg[\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{\frac{3}{5}}\partial_x\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp\\&-\frac{3}{5}\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{-\frac{2}{5}}\\&\times\partial_x\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp\Bigg].\end{aligned}$$

We then multiply $2/5$ power of

$$\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp$$

on numerator and denominator:

$$\begin{aligned}\frac{\partial g(x)}{\partial x}&=\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{-\frac{4}{5}}\\&\times\Bigg[\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\partial_x\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp\\&-\frac{3}{5}\partial_x\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp\Bigg].\end{aligned}$$

We then set the denominator to be I to write

$$\frac{\partial g(x)}{\partial x}=\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)^{-\frac{4}{5}}\times I,$$

where

$$I=\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\partial_x\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp$$
$$-\frac{3}{5}\partial_x\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp.$$

We then carry out the following two integrations

$$\begin{aligned}\partial_x\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp&=\int_{\mathbb{R}^3}\frac{-e^{|p|^2+x}}{(e^{|p|^2+x}+\tau)^2}dp\\&=4\pi\int_0^\infty\frac{-r^2e^{r^2+x}}{(e^{r^2+x}+\tau)^2}dr\\&=-2\pi\int_0^\infty\frac{1}{e^{r^2+x}+\tau}dr\end{aligned}\tag{2.11}$$

where we used the following integration by parts : $u'=\frac{2re^{r^2+x}}{(e^{r^2+x}+\tau)^2}$, $v=\frac{1}{2}r$, and

$$\begin{aligned}&\partial_x\left(\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+x}+\tau}dp+\int_{\mathbb{R}^3}\frac{|p|^2}{e^{|p|^2+y(x)}+\tau'}dp\right)\\&=\int_{\mathbb{R}^3}\frac{-|p|^2e^{|p|^2+x}}{(e^{|p|^2+x}+\tau)^2}dp+\frac{\partial y(x)}{\partial x}\int_{\mathbb{R}^3}\frac{-|p|^2e^{|p|^2+y(x)}}{(e^{|p|^2+y(x)}+\tau')^2}dp\\&=4\pi\int_0^\infty\frac{-r^4e^{r^2+x}}{(e^{r^2+x}+\tau)^2}dr+4\pi\frac{\partial y(x)}{\partial x}\int_0^\infty\frac{-r^4e^{r^2+y(x)}}{(e^{r^2+y(x)}+\tau')^2}dr\\&=-6\pi\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr-6\pi\frac{\partial y(x)}{\partial x}\int_0^\infty\frac{r^2}{e^{r^2+y(x)}+\tau'}dr,\end{aligned}\tag{2.12}$$

where we used similar integration by parts : $u'=\frac{2re^{r^2+c}}{(e^{r^2+c}+\tau)^2}$, $v=\frac{1}{2}r^3$ for

$$\int_0^\infty\frac{r^4e^{r^2+c}}{(e^{r^2+c}+\tau)^2}dr=\frac{3}{2}\int_0^\infty\frac{r^2}{e^{r^2+c}+\tau}dr.$$

Using (2.11) and (2.12), we rewrite I as

$$\begin{aligned}I=&-8\pi^2\left(\int_0^\infty\frac{r^4}{e^{r^2+x}+\tau}dr+\int_0^\infty\frac{r^4}{e^{r^2+y(x)}+\tau'}dr\right)\int_0^\infty\frac{1}{e^{r^2+x}+\tau}dr\\&+\frac{72\pi^2}{5}\left(\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr+\frac{\partial y(x)}{\partial x}\int_0^\infty\frac{r^2}{e^{r^2+y(x)}+\tau'}dr\right)\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr\end{aligned}\tag{2.13}$$

We then recall

$$D_\tau(x)=-\int_0^\infty\frac{r^4}{e^{r^2+x}+\tau}dr\int_0^\infty\frac{1}{e^{r^2+x}+\tau}dr+\frac{9}{5}\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr<0,$$

and express (2.13) as follows: So subtracting $D_\tau(x)$ on each sides gives

$$\begin{aligned}\frac{I}{8\pi^2}-D_\tau(x)=&-\int_0^\infty\frac{r^4}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty\frac{1}{e^{r^2+x}+\tau}dr\\&+\frac{9}{5}\frac{\partial y(x)}{\partial x}\int_0^\infty\frac{r^2}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty\frac{r^2}{e^{r^2+x}+\tau}dr.\end{aligned}\tag{2.14}$$

Now we compute $\partial y(x)/\partial x$. Recall

$$y(x)=h_{\tau'}^{-1}\left(\frac{N_2}{N_1}h_\tau(x)\right),$$

and compute

$$\frac{dy(x)}{dx} = (h_{\tau'}^{-1})' \left(\frac{N_2}{N_1}h_\tau(x)\right) \times \frac{d}{dx}\frac{N_2}{N_1}h_\tau(x).$$

Then, since the differentiation rule for inverse function gives

$$(h_{\tau'}^{-1})' \left(\frac{N_2}{N_1}h_\tau(x)\right) = \frac{1}{h'_{\tau'}(y(x))},$$

we get

$$\frac{dy(x)}{dx} = \frac{N_2}{N_1}\frac{h'_\tau(x)}{h'_{\tau'}(y(x))}.$$

Finally, we use

$$h'_\tau(x) = \int_{\mathbb{R}^3} \frac{-e^{|p|^2+x}}{(e^{|p|^2+x}+\tau)^2}dp = 4\pi\int_0^\infty \frac{-r^2e^{r^2+x}}{(e^{r^2+x}+\tau)^2}dr = -2\pi\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr,$$

to obtain the following expressions for $\partial y/\partial x$:

$$\frac{\partial y(x)}{\partial x} = \frac{N_2}{N_1}\frac{\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}.$$

Inserting this into (2.14)

$$\begin{aligned}
\frac{I}{8\pi^2} - D_\tau(x) &= -\int_0^\infty \frac{r^4}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr \\
&\quad + \frac{9}{5}\frac{N_2}{N_1}\frac{\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{r^2}{e^{r^2+x}+\tau}dr \\
&= -\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr\left(\int_0^\infty \frac{r^4}{e^{r^2+y(x)}+\tau'}dr - \frac{9}{5}\frac{N_2}{N_1}\frac{\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{r^2}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}\right)
\end{aligned}$$

Finally, we use

$$\frac{N_2}{N_1} = \frac{h_{\tau'}(y(x))}{h_\tau(x)} = \frac{\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+y(x)}+\tau'}dp}{\int_{\mathbb{R}^3}\frac{1}{e^{|p|^2+x}+\tau}dp} = \frac{\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr}{\int_0^\infty \frac{r^2}{e^{r^2+x}+\tau}dr}$$

to derive

$$\begin{aligned}
&\frac{I}{8\pi^2} - D_\tau(x) \\
&\quad= -\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr\left(\int_0^\infty \frac{r^4}{e^{r^2+y(x)}+\tau'}dr - \frac{9}{5}\frac{\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}\right) \\
&\quad= \frac{\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr} \\
&\quad\times\left(\frac{9}{5}\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{r^2}{e^{r^2+y(x)}+\tau'}dr - \int_0^\infty \frac{r^4}{e^{r^2+y(x)}+\tau'}dr\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr\right) \\
&\quad= \frac{\int_0^\infty \frac{1}{e^{r^2+x}+\tau}dr}{\int_0^\infty \frac{1}{e^{r^2+y(x)}+\tau'}dr}D_{\tau'}(y(x)),
\end{aligned}$$

which complete the proof. □

2.3. **Proof of Theorem 2.1 (3).** It remains to prove the H-theorem to conclude Theorem 2.1 (3).

Proposition 2.1. *Let $f_i \le 1$ only when f_i is the distribution function for fermion components, then we have*

$$\ln \frac{f_1}{1-\tau f_1}\big\{(\mathcal{M}_{11}-f_1)+(\mathcal{M}_{12}-f_1)\big\} + \ln \frac{f_2}{1-\tau' f_2}\big\{(\mathcal{M}_{22}-f_2)+(\mathcal{M}_{21}-f_2)\big\} \le 0.$$

Proof. The proof for

$$\int_{\mathbb{R}^3} \ln \frac{f_1}{1-f_1}(\mathcal{M}_{11}-f_1)dp + \int_{\mathbb{R}^3} \ln \frac{f_2}{1-f_2}(\mathcal{M}_{22}-f_2)dp \le 0, \tag{2.15}$$

can be found in [63]. So we only prove

$$S \equiv \int_{\mathbb{R}^3} \ln \frac{f_1}{1-\tau f_1}(\mathcal{M}_{12}-f_1)dp + \int_{\mathbb{R}^3} \ln \frac{f_2}{1-\tau' f_2}(\mathcal{M}_{21}-f_2)dp \le 0.$$

First, we observe that

$$I = \int_{\mathbb{R}^3} \ln \frac{\mathcal{M}_{12}}{1-\tau \mathcal{M}_{12}}(\mathcal{M}_{12}-f_1)dp + \int_{\mathbb{R}^3} \ln \frac{\mathcal{M}_{21}}{1-\tau' \mathcal{M}_{21}}(\mathcal{M}_{21}-f_2)dp = 0,$$

which follows from an explicit computation using the conservation laws (1.8):

$$\begin{aligned} I &= -\int_{\mathbb{R}^3} \left(a|p-b|^2 + c_{12}\right)(\mathcal{M}_{12}-f_1)dp - \int_{\mathbb{R}^3} \left(a|p-b|^2 + c_{21}\right)(\mathcal{M}_{21}-f_2)dp \\ &= a\int_{\mathbb{R}^3} |p|^2 \left(f_1+f_2-\mathcal{M}_{12}-\mathcal{M}_{21}\right)dp - 2ab\cdot \int_{\mathbb{R}^3} p\left(f_1+f_2-\mathcal{M}_{12}-\mathcal{M}_{21}\right)dp \\ &= 0. \end{aligned}$$

From this, we find

$$\begin{aligned} S - I = &\int_{\mathbb{R}^3} \left(\ln \frac{f_1}{1-\tau f_1} - \ln \frac{\mathcal{M}_{12}}{1-\tau \mathcal{M}_{12}}\right)(\mathcal{M}_{12}-f_1)dp \\ &+ \int_{\mathbb{R}^3} \left(\ln \frac{f_2}{1-\tau' f_2} - \ln \frac{\mathcal{M}_{21}}{1-\tau' \mathcal{M}_{21}}\right)(\mathcal{M}_{21}-f_2)dp \le 0, \end{aligned}$$

since $\ln \frac{x}{1+x}$ is an increasing function for $x \in [0,\infty)$, and $\ln \frac{x}{1-x}$ is an increasing function when $0 < x < 1$. Here, we have equality if and only if $f_1 = M_{12}$ and $f_2 = M_{21}$. This completes the proof. □

Remark 2.3. The equality in the H-Theorem is characterized by two distributions with the same value for a and b. Due to the fact that b is equal to pressure over the density, this leads to $P_1 = \frac{N_1}{N_2}P_2$.

Therefore, to complete the proof of Theorem 2.1 (3), it remains to prove that $f_i < 1$ in the case of fermions.

Lemma 2.4. *Let f_i be a distribution function for fermions and $f_i(x,p,0) < 1$. Then we have $f_i(x,p,t) < 1$ for $t \ge 0$.*

Proof. Integrating (1.6) along the characteristic, we get the mild form :

$$f_i(x,p,t) = e^{-2t} f_i(x-pt,p,0) + \int_0^t e^{2(\tau-t)}(\mathcal{F}_{ii}+\mathcal{F}_{ij})(x+(\tau-t)p,p,\tau)d\tau,$$

for $j \neq i$. Since $\mathcal{F}_{ii} < 1$ and $\mathcal{F}_{ij} < 1$ for all (x,p,t) by definition, we have

$$\begin{aligned} f_i(x,p,t) &\le e^{-2t} f_i(x-pt,p,0) + \int_0^t 2e^{2(\tau-t)}d\tau \\ &= e^{-2t} f_i(x-pt,p,0) + (1-e^{-2t}) \\ &< 1. \end{aligned}$$

□

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Department of mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea
Email address: gcbae02@skku.edu

Department of mathematics, Würzburg University, Emil Fischer Str. 40, 97074 Würzburg, GERMANY
Email address: klingen@mathematik.uni-wuerzburg.de

Department of mathematics, Vienna University, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
Email address: marlies.pirner@mathematik.uni-wuerzburg.de

Department of mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea
Email address: sbyun01@skku.edu