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A limit problem for three-dimensional ideal compressible radiation magneto-hydrodynamics

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General radiation magnetic hydrodynamics models include two main parts that are coupled: one part is the macroscopic magnetic fluid part, which is governed by the ideal compressible magnetohydrodynamic (MHD) equations with additional radiation terms; another part is the radiation field, which is described by a transfer equation. It is well known that in radiation hydrodynamics without a magnetic field there are two physical approximations: one is the so-called *P1 approximation* and the other is the so-called *gray* approximation. Starting out with a general radiation MHD model one can derive the so-called *MHD-P1* approximation model. In this paper, we study the non-relativistic type limit for this MHD-P1 approximation model since the speed of light is much larger than the speed of the macroscopic fluid. This way we achieve a rigorous derivation of a widely used macroscopic model in radiation magnetohydrodynamics.

Keywords: Radiation magnetic hydrodynamics; ideal compressible MHD equations; P1 approximation; “gray” approximation; non-relativistic type limit.

Mathematical Subject Classification 2010: 35Q35, 35Q70, 35B25

1. Introduction and Main Results

In this paper we consider a model that includes the effect of a magnetic field and radiation transport on a fully ionized compressible inviscid fluid. The radiation effects both the momentum and energy balance giving rise to the equations of radiation magnetohydrodynamics (radiation MHD). Precisely speaking, if the viscosity, heat-conductivity of macroscopic fluids and the magnetic diffusion are ignored, the general radiation magnetic hydrodynamics equations can be written as the following three-dimensional ideal compressible MHD system with additional radiation terms [16, 19, 21]:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t \left(\rho \mathbf{u} + \frac{1}{c^2} F_r\right) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}_3 + P_r) = \frac{1}{4\pi}(\nabla \times H) \times H, \tag{1.2}$$

$$\partial_t H - \nabla \times (u \times H) = 0, \quad \operatorname{div} H = 0, \tag{1.3}$$

$$\partial_t \left(\rho E + \frac{|H|^2}{8\pi} + E_r\right) + \operatorname{div}(\rho \mathbf{u} E + P\mathbf{u} + F_r) = \frac{1}{4\pi}\operatorname{div}((u \times H) \times H). \tag{1.4}$$

Here the unknowns $\rho, \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3, H = (H_1, H_2, H_3) \in \mathbb{R}^3$, and θ denote the density, the velocity, the magnetic field and the temperature of the fluid, respectively; $\mathbb{I}_3$ is the 3×3 identity matrix. The pressure $P = P(\rho, \theta)$ and the internal energy $e = e(\rho, \theta)$ are smooth functions of ρ and θ and satisfy the Gibbs relation

$$\theta \mathrm{d}S = \mathrm{d}e + P\,\mathrm{d}\left(\frac{1}{\rho}\right) \tag{1.5}$$

for some smooth function (entropy) $S = S(\rho, \theta)$, which expresses the first law of the thermodynamics. $E = e + \frac{|\mathbf{u}|^2}{2}$ denotes the total energy.

Now, we consider the radiation energy E_r, the radiation flux F_r and the radiation pressure P_r appearing in (1.1)–(1.3) which can be defined by

$$E_r = \frac{1}{c}\int_0^\infty \mathrm{d}\nu \int_{S^2} I(\nu, \omega)\mathrm{d}\omega,$$

$$F_r = \int_0^\infty \mathrm{d}\nu \int_{S^2} \omega I(\nu, \omega)\mathrm{d}\omega,$$

$$P_r = \frac{1}{c}\int_0^\infty \mathrm{d}\nu \int_{S^2} \omega \otimes \omega I(\nu, \omega)\mathrm{d}\omega.$$

Here the radiation intensity $I(\nu, \omega) = I(t, x, \nu, \omega)$, depending on the direction vector $\omega \in S^2$ and the frequency $\nu \geq 0$, is determined by solving the linear Boltzmann-type equation:

$$\begin{aligned} \frac{1}{c}\partial_t I + \omega \cdot \nabla I = S(\nu) - \sigma_a(\nu) I \\ + \int_0^\infty \mathrm{d}\nu' \int_{S^2} \Big[\frac{\nu}{\nu'}\sigma_s(\nu' \to \nu) I(\nu', \omega') \\ - \sigma_s(\nu \to \nu') I(\nu, \omega)\Big]\,\mathrm{d}\omega'. \end{aligned} \tag{1.6}$$

Here c denotes the speed of light. The emission term $S(\nu)$ can be chosen as the well-known Planck function, i.e.

$$S(\nu) = 2h\nu^3 c^{-2}(e^{h\nu/k\theta} - 1)^{-1}.$$

In general the absorbing coefficient σ_a and the scattering coefficient σ_s depend on the frequency ν, the density ρ, and the temperature θ of the macroscopic fluid.

In the present paper, we focus on the "gray" approximation case such that the transport coefficients σ_a and σ_s are independent of the frequency ν. This hypothesis is a very crude approximation, but is motivated by the mathematical analysis. In fact, we can assume that the scattering kernel is diagonal in the following derivations see [19, p. 54]

$$\sigma_s(\nu' \to \nu) = \sigma_s \delta(\nu - \nu').$$

For simplification, we only consider the case that $\sigma_s = 0$ below.

Following [14] (also see [1, pp. 394–395] for details), we study the asymptotic regimes of the full system (1.1)–(1.6) by introducing the non-dimensional variables. Then, the full system (1.1)–(1.6) in the non-dimensional variables introduced in [1, 14] can be written as follows:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.7}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}_3) = \frac{1}{4\pi}(\nabla \times H) \times H - \mathcal{P}\vec{S}_F, \tag{1.8}$$

$$\partial_t H - \nabla \times (u \times H) = 0, \quad \operatorname{div} H = 0, \tag{1.9}$$

$$\begin{aligned} &\partial_t \left(\rho E + \frac{|H|^2}{8\pi} \right) + \operatorname{div}(\rho \mathbf{u} E + P\mathbf{u}) \\ &\quad = \frac{1}{4\pi} \operatorname{div}((u \times H) \times H) - \mathcal{C}\mathcal{P} S_E, \end{aligned} \tag{1.10}$$

$$\frac{1}{\mathcal{C}} \partial_t I + \omega \cdot \nabla I = \mathcal{L}[B(\nu, \theta) - \sigma_a I] \triangleq S_a, \tag{1.11}$$

with the same non-dimensional parameters as in [1, 13]

$$\mathcal{C} = \frac{c}{a_\infty}, \quad \mathcal{P} = \frac{a_r T_\infty^4}{\rho_\infty a_\infty^2}, \quad \mathcal{L} = \frac{1}{\lambda_a}.$$

Here a_∞, ρ_∞ and T_∞ are the characteristic values of the velocity, density and temperature of fluid, λ_a is the total mean free path of photons, and the non-dimensional quantity a_r is defined by

$$a_r = \frac{8\pi^5 k^4}{15 c^3 h^3}.$$

The first parameter $\mathcal{C}$ is a large parameter for a flow no-relativistic. The second parameter $\mathcal{P}$ measures the ratio of the radiative energy over the internal energy.

By using Eq. (1.6) with $\sigma_s = 0$ and the definitions of E_r, F_r and P_r, we have

$$S_E = \iint S_a \mathrm{d}\nu \mathrm{d}\omega, \quad \vec{S}_F = \frac{1}{c} \iint \omega S_a \mathrm{d}\nu \mathrm{d}\omega.$$

We ignore the relativistic effect in the frequency variable ν, that is, $\nu_0 = \nu' = \nu$ in [1, p. 395], which is consistent with the following non-relativistic limit considered in this work. Then, we obtain

$$B(\nu, \theta) = \frac{15\nu^3}{4\pi^5}(e^{\nu/\theta} - 1)^{-1}.$$

Consequently, the radiation quantities both I and B can be integrated on frequency. For B, we have

$$\int_0^\infty B(\nu, \theta)\mathrm{d}\nu = \int_0^\infty \frac{15\nu^3}{4\pi^5}(e^{\nu/\theta} - 1)^{-1}\mathrm{d}\nu = \bar{C}\theta^4$$

for some uniform positive constant $\bar{C}$. In this way, the equation for the integration of I with respect to frequency ν, still denoted by I, can be written as

$$\frac{1}{\mathcal{C}}\partial_t I + \omega \cdot \nabla I = \mathcal{L}[\bar{C}\theta^4 - \sigma_a I]. \tag{1.12}$$

In addition, when the distribution of photons is almost isotropic, one can take the P1 hypothesis by choosing the ansatz

$$I = I_0 + \mathbf{I}_1 \cdot \omega, \tag{1.13}$$

where I_0 and $\mathbf{I}_1$ do not depend on ω, $\mathbf{I}_1 \cdot \omega$ is regarded as a correction term of the main term I_0. Inserting the ansatz (1.13) into (1.12) yields

$$\frac{1}{\mathcal{C}}\partial_t(I_0 + \mathbf{I}_1 \cdot \omega) + \omega \cdot \nabla(I_0 + \mathbf{I}_1 \cdot \omega) = \mathcal{L}[\bar{C}\theta^4 - \sigma_a(I_0 + \mathbf{I}_1 \cdot \omega)]. \tag{1.14}$$

Then integrating both Eq. (1.14) and the resulting equation of $(1.14) \cdot \omega$ with respect to ω over S^2 lead to

$$\frac{1}{\mathcal{C}}\partial_t I_0 + \frac{1}{3|S^2|}\mathrm{div}_x \mathbf{I}_1 = \mathcal{L}[\bar{C}\theta^4 - \sigma_a I_0], \tag{1.15}$$

$$\frac{1}{\mathcal{C}}\partial_t \mathbf{I}_1 + \nabla_x I_0 = -\mathcal{L}\sigma_a \mathbf{I}_1. \tag{1.16}$$

Moreover, we have

$$S_E = \mathcal{L}|S^2|[\bar{C}\theta^4 - \sigma_a I_0], \quad \vec{S}_F = \frac{-\mathcal{L}}{\mathcal{C}a_\infty}\frac{\sigma_a}{3}\mathbf{I}_1. \tag{1.17}$$

If we take $\mathcal{C} = \frac{1}{\epsilon}, \mathcal{P} = \epsilon, \mathcal{L} = 1$ and ignore the influence of other constants, we obtain the following ideal compressible MHD-P1 approximation model via the system (1.1)–(1.3), (1.15)–(1.17):

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{1.18}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}_3) = \frac{1}{4\pi}(\nabla \times H) \times H + \epsilon^2 \mathbf{I}_1, \tag{1.19}$$

$$\partial_t H - \nabla \times (u \times H) = 0, \quad \mathrm{div}\, H = 0, \tag{1.20}$$

$$\partial_t\left(\rho E + \frac{|H|^2}{8\pi}\right) + \mathrm{div}(\rho \mathbf{u} E + P\mathbf{u}) = \frac{1}{4\pi}\mathrm{div}((u \times H) \times H) + I_0 - \theta^4, \tag{1.21}$$

$$\epsilon \partial_t I_0 + \mathrm{div}\, \mathbf{I}_1 = \theta^4 - I_0, \tag{1.22}$$

$$\epsilon \partial_t \mathbf{I}_1 + \nabla I_0 = -\mathbf{I}_1. \tag{1.23}$$

In this paper we consider the non-relativistic type limit $\epsilon \to 0$ for the system (1.18)–(1.23). Formally, letting $\epsilon = 0$ in (1.22) and (1.23), we obtain that

$$\operatorname{div} \mathbf{I}_1 = \theta^4 - I_0, \quad -\mathbf{I}_1 = \nabla I_0.$$

Hence we have

$$-\Delta I_0 = \theta^4 - I_0. \tag{1.24}$$

Taking gradient to (1.24), one gets

$$-\nabla \operatorname{div}(\nabla I_0) = \nabla\theta^4 - \nabla I_0. \tag{1.25}$$

Setting $\mathbf{q} = -\nabla I_0$, we can rewrite (1.25) as

$$-\nabla \operatorname{div} \mathbf{q} + \mathbf{q} + \nabla\theta^4 = 0.$$

Therefore, we can formally obtain the following limit system from (1.18)–(1.23) as $\epsilon \to 0$:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.26}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathbb{I}_3) = \frac{1}{4\pi}(\nabla \times H) \times H, \tag{1.27}$$

$$\partial_t H - \nabla \times (u \times H) = 0, \quad \operatorname{div} H = 0, \tag{1.28}$$

$$\partial_t \left(\rho E + \frac{|H|^2}{8\pi} \right) + \operatorname{div}(\rho \mathbf{u} E + P\mathbf{u}) = \frac{1}{4\pi}\operatorname{div}((u \times H) \times H) - \operatorname{div} \mathbf{q}, \tag{1.29}$$

$$-\nabla \operatorname{div} \mathbf{q} + \mathbf{q} + \nabla\theta^4 = 0. \tag{1.30}$$

The system (1.26)–(1.30) without magnetic fields are widely studied in [7, 8, 11–13, 18, 20, 25] to describe the dynamics of the fluid in radiation hydrodynamics. For the case that viscosity and heat-conductivity are included, one can refer to [24, 26] and references cited therein.

The purpose of this paper is to give a rigorous derivation of the system (1.26)–(1.30) from the MHD-P1 approximation model (1.18)–(1.23) as ϵ tends to zero. For the sake of simplicity and clarity of presentation, we shall focus on the fluids obeying the perfect gas relations

$$P = \mathfrak{R}\rho\theta, \quad e = c_V\theta, \tag{1.31}$$

where the parameters $\mathfrak{R} > 0$ and $c_V > 0$ are the gas constant and the heat capacity at constant volume. We consider the system (1.18)–(1.23) in the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3 = (\mathbb{R}/(2\pi\mathbb{Z}))^3$, which will be denoted by Ω.

In what follows, for simplicity of presentation, we take the physical constants $\mathfrak{R}$ and c_V to be one. To emphasize the unknowns depending on the small parameter

ϵ, we rewrite the system (1.18)–(1.23), (1.5), (1.31) as

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) = 0, \tag{1.32}$$

$$\rho^\epsilon(\partial_t \mathbf{u}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon) + \nabla(\rho^\epsilon \theta^\epsilon) = \frac{1}{4\pi}(\nabla \times H^\epsilon) \times H^\epsilon + \epsilon^2 \mathbf{I}_1^\epsilon, \tag{1.33}$$

$$\partial_t H^\epsilon - \nabla \times (u^\epsilon \times H^\epsilon) = 0, \quad \operatorname{div} H^\epsilon = 0, \tag{1.34}$$

$$\rho^\epsilon(\partial_t \theta^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \theta^\epsilon) + \rho^\epsilon \theta^\epsilon \operatorname{div} \mathbf{u}^\epsilon = I_0^\epsilon - (\theta^\epsilon)^4 - \epsilon \mathbf{I}_1^\epsilon \cdot \mathbf{u}^\epsilon, \tag{1.35}$$

$$\epsilon \partial_t I_0^\epsilon + \operatorname{div} \mathbf{I}_1^\epsilon = (\theta^\epsilon)^4 - I_0^\epsilon, \tag{1.36}$$

$$\epsilon \partial_t \mathbf{I}_1^\epsilon + \nabla I_0^\epsilon = -\mathbf{I}_1^\epsilon, \tag{1.37}$$

the following identity is used to derive (1.35) from (1.19)–(1.21):

$$\operatorname{div}((\mathbf{u}^\epsilon \times H^\epsilon) \times H^\epsilon) = (\nabla \times H^\epsilon) \times H^\epsilon \cdot \mathbf{u}^\epsilon + \nabla \times (\mathbf{u}^\epsilon \times H^\epsilon) \cdot H^\epsilon. \tag{1.38}$$

The system (1.32)–(1.37) is supplemented with initial data

$$\begin{aligned}(\rho^\epsilon, \mathbf{u}^\epsilon, H^\epsilon, \theta^\epsilon, I_0^\epsilon, \mathbf{I}_1^\epsilon)|_{t=0}& \\ &= (\rho_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), H_0^\epsilon(x), \theta_0^\epsilon(x), I_{00}^\epsilon(x), \mathbf{I}_{10}^\epsilon(x)), \quad x \in \Omega. \end{aligned}\tag{1.39}$$

We also rewrite the limit equations (1.26)–(1.30), (1.5), (1.31) (recall that $\mathfrak{R} = c_V = 1$) as

$$\partial_t \rho^0 + \operatorname{div}(\rho^0 \mathbf{u}^0) = 0, \tag{1.40}$$

$$\rho^0(\partial_t \mathbf{u}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) + \nabla(\rho^0 \theta^0) = \frac{1}{4\pi}(\nabla \times H^0) \times H^0, \tag{1.41}$$

$$\partial_t H^0 - \nabla \times (u^0 \times H^0) = 0, \quad \operatorname{div} H^0 = 0, \tag{1.42}$$

$$\rho^0(\partial_t \theta^0 + \mathbf{u}^0 \cdot \nabla \theta^0) + \rho^0 \theta^0 \operatorname{div} \mathbf{u}^0 = -\operatorname{div} \mathbf{q}^0, \tag{1.43}$$

$$-\nabla \operatorname{div} \mathbf{q}^0 + \mathbf{q}^0 + \nabla(\theta^0)^4 = 0. \tag{1.44}$$

The system (1.40)–(1.44) is equipped with initial data

$$(\rho^0, \mathbf{u}^0, H^0, \theta^0)|_{t=0} = (\rho_0^0(x), \mathbf{u}_0^0(x), H_0^0(x), \theta_0^0(x)), \quad x \in \Omega. \tag{1.45}$$

It is noted that the initial data of $\mathbf{q}^0$ can be determined by $\theta_0^0(x)$ through the equation (1.44).

We first state a result on the local existence of smooth solutions to the problem (1.40)–(1.45), one can refer to [7, 8] for a similar proof in details.

Proposition 1.1. *Let $s > 7/2$ be an integer and assume that the initial data $(\rho_0^0, \mathbf{u}_0^0, H_0^0, \theta_0^0)$ satisfy*

$$\rho_0^0, \mathbf{u}_0^0, H_0^0, \theta_0^0 \in H^{s+2}(\Omega),$$

$$0 < \bar{\rho} = \inf_{x \in \Omega} \rho_0^0(x) \le \rho_0^0(x) \le \bar{\bar{\rho}} = \sup_{x \in \Omega} \rho_0^0(x) < +\infty,$$

$$0 < \bar{\theta} = \inf_{x \in \Omega} \theta_0^0(x) \le \theta_0^0(x) \le \bar{\bar{\theta}} = \sup_{x \in \Omega} \theta_0^0(x) < +\infty$$

for some positive constants $\bar{\rho}, \bar{\bar{\rho}}, \bar{\theta}$, *and* $\bar{\bar{\theta}}$. *Then there exist positive constants* T_*(*the maximal time interval*, $0 < T_* \le +\infty$), *and* $\hat{\rho}, \tilde{\rho}, \hat{\theta}, \tilde{\theta}$, *such that the problem* (1.40)–(1.45) *has a unique classical solution* $(\rho^0, \mathbf{u}^0, H^0, \theta^0, \mathbf{q}^0)$ *satisfying*

$$\rho^0, \mathbf{u}^0, H^0, \theta^0 \in C^l([0, T_*), H^{s+2-l}(\Omega)), \quad l = 0, 1;$$

$$\mathbf{q}^0 \in C^0([0, T_*), H^{s+3}(\Omega));$$

$$0 < \hat{\rho} = \inf_{(x,t)\in\Omega\times[0,T_*)} \rho^0(x,t) \le \rho^0(x,t) \le \tilde{\rho} = \sup_{(x,t)\in\Omega\times[0,T_*)} \rho^0(x,t) < +\infty,$$

$$0 < \hat{\theta} = \inf_{(x,t)\in\Omega\times[0,T_*)} \theta^0(x,t) \le \theta^0(x,t) \le \tilde{\theta} = \sup_{(x,t)\in\Omega\times[0,T_*)} \theta^0(x,t) < +\infty.$$

Our convergence results can be stated as follows.

Theorem 1.2. *Let* $s > 7/2$ *be an integer and* $(\rho^0, \mathbf{u}^0, H^0, \theta^0, \mathbf{q}^0)$ *be the unique classical solution to the problem* (1.40)–(1.45) *given in Proposition* 1.1. *Suppose that the initial data* $(\rho_0^\epsilon, \mathbf{u}_0^\epsilon, H_0^\epsilon, \theta_0^\epsilon, I_{00}^\epsilon, \mathbf{I}_{10}^\epsilon)$ *satisfies*

$$\rho_0^\epsilon, \mathbf{u}_0^\epsilon, H_0^\epsilon, \theta_0^\epsilon, I_{00}^\epsilon, \mathbf{I}_{10}^\epsilon \in H^s(\Omega), \quad \inf_{x\in\Omega} \rho_0^\epsilon(x) > 0, \ \inf_{x\in\Omega} \theta_0^\epsilon(x) > 0,$$

and

$$\begin{aligned} &\|(\rho_0^\epsilon - \rho_0^0, \mathbf{u}_0^\epsilon - \mathbf{u}_0^0, H_0^\epsilon - H_0^0, \theta_0^\epsilon - \theta_0^0)\|_s \\ &\quad + \sqrt{\epsilon}\|(I_{00}^\epsilon - (-\Delta)^{-1}\mathrm{div}\, \mathbf{q}_0^0, \mathbf{I}_{10}^\epsilon - \mathbf{q}_0^0)\|_s \le L_0\epsilon \end{aligned} \tag{1.46}$$

for some constant $L_0 > 0$. *Then, for any* $T_0 \in (0, T_*)$, *there exist a constant* $L > 0$, *and a sufficient small constant* $\epsilon_0 > 0$, *such that for any* $\epsilon \in (0, \epsilon_0]$, *the problem* (1.32)–(1.39) *has a unique smooth solution* $(\rho^\epsilon, \mathbf{u}^\epsilon, H^\epsilon, \theta^\epsilon, I_0^\epsilon, \mathbf{I}_1^\epsilon)$ *on* $[0, T_0]$ *satisfying*

$$\begin{aligned} &\|(\rho^\epsilon - \rho^0, \mathbf{u}^\epsilon - \mathbf{u}^0, H^\epsilon - H^0, \theta^\epsilon - \theta^0)\|_s \\ &\quad + \sqrt{\epsilon}\|(I_0^\epsilon - (-\Delta)^{-1}\mathrm{div}\, \mathbf{q}^0, \mathbf{I}_1^\epsilon - \mathbf{q}^0)\|_s \le L\epsilon, \quad t \in [0, T_0]. \end{aligned} \tag{1.47}$$

Here $\mathbf{q}_0^0$ *is defined via the initial datum* θ_0^0 *in the following way*:

$$\mathbf{q}_0^0 = \left(\frac{-\Delta}{I - \Delta} - I\right) \nabla(\theta_0^0)^4$$

and $\|\cdot\|_s$ *denotes the norm of Sobolev space* $H^s(\Omega)$.

Remark 1.3. The purpose of this paper is to give a rigorous derivation of the system (1.26)–(1.30) from the general radiation magnetichydrodynamics models (1.1)–(1.4) and (1.6), based on two physical approximations: "gray" approximation and P1 (Eddington or Diffusion) approximation.

Remark 1.4. If the domain Ω in the Proposition 1.1 is the whole space $\mathbb{R}^3$, then the conditions $\rho_0^0, \mathbf{u}_0^0, H_0^0, \theta_0^0 \in H^{s+2}(\Omega)$ should be replaced by $\rho_0^0 - \check{\rho}, \mathbf{u}_0^0, H_0^0 - \check{H}, \theta_0^0 - \check{\theta} \in H^{s+2}(\Omega)$ for some positive constant states $\check{\rho}$, $\check{H}$ and $\check{\theta}$. At the same time, the conditions $\rho_0^\epsilon, \mathbf{u}_0^\epsilon, H_0^\epsilon, \theta_0^\epsilon, I_{00}^\epsilon, \mathbf{I}_{10}^\epsilon \in H^s(\Omega)$ in Theorem 1.2 are required to be changed into $\rho_0^\epsilon - \check{\rho}, \mathbf{u}_0^\epsilon, H_0^\epsilon - \check{H}, \theta_0^\epsilon - \check{\theta}, I_{00}^\epsilon - \check{\theta}^4, \mathbf{I}_{10}^\epsilon \in H^s(\Omega)$ accordingly. The corresponding proof is essentially unchanged and can be modified in a direct way.

Remark 1.5. As a consequence of our result, we obtain the local existence of solutions to the primitive system (1.32)–(1.37), and the life-span of which is independent of ϵ. Furthermore, the inequality (1.47) implies that the sequences $(\rho^\epsilon, \mathbf{u}^\epsilon, H^\epsilon, \theta^\epsilon)$ converge strongly to $(\rho^0, \mathbf{u}^0, H^0, \theta^0)$ in $L^\infty(0, T; H^s(\Omega))$ and $(I_0^\epsilon, \mathbf{I}_1^\epsilon)$ converge strongly to $((-\Delta)^{-1}\mathrm{div}\,\mathbf{q}^0, \mathbf{q}^0)$ in $L^\infty(0, T; H^s(\Omega))$ but with different convergence rates.

Remark 1.6. In the local existence for the problem (1.40)–(1.45), the regularity requirement on initial data $(\rho_0^0, \mathbf{u}_0^0, H_0^0, \theta_0^0, \mathbf{q}_0^0) \in H^s(\Omega)$ for $s > 7/2$ is in fact sufficient. Here we have added more regularity assumption in Proposition 1.1 in order to obtain more regular solutions which are needed in the proof of Theorem 1.2.

Remark 1.7. Proposition 1.1 and Theorem 1.2 also hold true for n-dimensional cases ($n \geq 2$). We only deal with three-dimensional case for simplicity clarity of presentation in this paper.

Remark 1.8. The divergence free conditions in (1.34) and (1.42) can be guaranteed automatically by the equations and the divergence free conditions on the corresponding initial datum. Consequently, these two divergence free conditions will not be specified below.

It is necessary for us to give some comments on the proof of Theorem 1.2 and some known results for the related topics. The main difficulty in dealing with our non-relativistic type limit is to control the oscillatory behavior of I_0^ϵ and $\mathbf{I}_1^\epsilon$. The time derivatives of I_0^ϵ and $\mathbf{I}_1^\epsilon$ in (1.36)–(1.37) are multiplied by a small parameter, hence the uniform energy estimates are obtained from the relaxation terms rather than from the time-derivative terms. Besides the singularity in (1.36)–(1.37), there exists an extra singularity caused by the coupling of I_0^ϵ and $\mathbf{I}_1^\epsilon$ in the momentum and temperature equations. In this paper, we shall overcome all these difficulties by adopting and modifying the elaborate nonlinear energy method developed in [4–6]. First, we derive the error system (2.1)–(2.6) by utilizing the original system (1.32)–(1.37) and the limit system (1.40)–(1.44). In this step, we need to find the suitable quantities from the limit system, which are related to I_0^ϵ and $\mathbf{I}_1^\epsilon$. Next, we study the estimates of H^s-norm to the error system. To do so, we shall make full use of the special structure of the error system, the Sobolev imbedding and the Moser-type inequalities, and the regularity of the limit equations. In particular, a very refined analysis is carried out to deal with the higher order nonlinear terms in the system (2.1)–(2.6). It is noted that the damping terms in Eqs. (2.5)–(2.6) also play a crucial role in controlling the nonlinear coupled terms, which are so-called *good* properties from radiation fields. However, compared with the limit problem considered in [6], there is no any diffusion effect in the systems (1.32)–(1.39) and (1.40)–(1.45). Consequently, the symmetrizers of the systems (1.32)–(1.39) and (1.40)–(1.45) are essentially used to overcome the difficulties caused by the flux terms. Finally, we combine these obtained estimates and apply the Gronwall inequality to get the desired results. In addition, we should remark that for fixed ϵ, the global-in-time existence of solutions to the barotropic case without magnetic fields of Eqs. (1.32)–(1.37) is achieved in the critical Besov spaces by Danchin and Ducomet recently in [2]. As is pointed out in [16, 19] that the energy exchange between the hydrodynamics and the radiation field sometime plays a leading role. This is the key reason why we include the energy equation into the system (1.32)–(1.37). Moreover, (1.26)–(1.30) without radiation hydrodynamics has been successfully calculated numerically in [10] with applications in astrophysics [23]. Before extending this numerical method to include radiation, studying the validity of this system of equations is achieved in this paper. In addition, in the case of high temperature hydrodynamics, the magnetic fields effect is an important factor, which cannot ignored in such physical situations. An example of this is the magnetic confinement fusion process. Here one needs to consider both radiation and magnetic effects in the motion of fluids at the same time. This case will greatly increase difficulties of mathematical analysis. By assuming that there is no any interaction between the radiation field and the magnetic field, we obtain the case shown in the system (1.1)–(1.4). To our knowledge, for fixed ϵ the global existence of strong solutions or blowup phenomena to Eqs. (1.32)–(1.37) is still open, which is left for our future study.

Before ending this introduction, we give some notations and recall some basic facts which will be frequently used throughout this paper.

(1) We denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $L^2(\Omega)$ with $\langle f,f\rangle = \|f\|^2$, by H^k the standard Sobolev space $W^{k,2}$ with norm $\|\cdot\|_k$. The notation $\|(A_1, A_2, \ldots, A_l)\|_k$ means the summation of $\|A_i\|_k$ from $i = 1$ to $i = l$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$. For an integer m, the symbol D_x^m denotes the summation of all terms ∂_x^α with the multi-index α satisfying $|\alpha| = m$. We use C_i, δ_i, K_i, and K to denote the constants which are independent of ϵ and may change from line to line. We also omit the spatial domain Ω in integrals for convenience.

(2) We shall frequently use the following Moser-type calculus inequalities (see [9]):

(i) For $f, g \in H^s(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| \le s$, $s > n/2$, it holds that

$$\|\partial_x^\alpha(fg)\| \le C_s(\|f\|_{L^\infty}\|D_x^s g\| + \|g\|_{L^\infty}\|D_x^s f\|). \tag{1.48}$$

(ii) For $f \in H^s(\Omega), D_x^1 f \in L^\infty(\Omega), g \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| \le s$, $s > n/2 + 1$, it holds that

$$\|\partial_x^\alpha(fg) - f\partial_x^\alpha g\| \le C_s(\|D_x^1 f\|_{L^\infty}\|D_x^{s-1} g\| + \|g\|_{L^\infty}\|D_x^s f\|). \tag{1.49}$$

(3) Let $s > n/2$, $f \in C^s(\Omega)$, and $u \in H^s(\Omega)$; then for each multi-index α, $1 \le |\alpha| \le s$, we have (see [9, 17]):

$$\|\partial_x^\alpha(f(u))\| \le C(1 + \|u\|_{L^\infty}^{|\alpha|-1})\|u\|_{|\alpha|}. \tag{1.50}$$

Moreover, if $f(0) = 0$, then (see [3])

$$\|\partial_x^\alpha(f(u))\| \le C(\|u\|_s)\|u\|_s. \tag{1.51}$$

This paper is organized as follows. In Sec. 2, we utilize the primitive system (1.32)–(1.37) and the target system (1.40)–(1.44) to derive the error system and state the local existence of the solution. In Sec. 3 we give the *a priori* energy estimates of the error system and present the proof of Theorem 1.2.

2. Derivation of the Error System

In this section we first derive the error system from the original system (1.32)–(1.37) and the limiting equations (1.40)–(1.44), then we state the local existence of solution to this error system.

Setting $N^\epsilon = \rho^\epsilon - \rho^0, \mathbf{U}^\epsilon = \mathbf{u}^\epsilon - \mathbf{u}^0, B^\epsilon = H^\epsilon - H^0, \Theta^\epsilon = \theta^\epsilon - \theta^0, J_0^\epsilon = I_0^\epsilon - (-\Delta)^{-1}\mathrm{div}\,\mathbf{q}^0$, and $\mathbf{J}_1^\epsilon = \mathbf{I}_1^\epsilon - \mathbf{q}^0$, and utilizing the system (1.32)–(1.37) and the system (1.40)–(1.44), we obtain that

$$\partial_t N^\epsilon + (N^\epsilon + \rho^0)\mathrm{div}\,\mathbf{U}^\epsilon + (\mathbf{U}^\epsilon + \mathbf{u}^0)\cdot\nabla N^\epsilon = -N^\epsilon \mathrm{div}\,\mathbf{u}^0 - \nabla\rho^0\cdot\mathbf{U}^\epsilon, \tag{2.1}$$

$$\begin{aligned}
&\partial_t \mathbf{U}^\epsilon + [(\mathbf{U}^\epsilon + \mathbf{u}^0)\cdot\nabla]\mathbf{U}^\epsilon + \nabla\Theta^\epsilon + \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0}\nabla N^\epsilon \\
&\quad - \frac{1}{4\pi(N^\epsilon + \rho^0)}(\nabla\times B^\epsilon)\times(B^\epsilon + H^0) \\
&\quad = -(\mathbf{U}^\epsilon\cdot\nabla)\mathbf{u}^0 - \left[\frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} - \frac{\theta^0}{\rho^0}\right]\nabla\rho^0 + \frac{1}{4\pi(N^\epsilon + \rho^0)}(\nabla\times H^0)\times B^\epsilon \\
&\qquad + \frac{1}{4\pi}\left[\frac{1}{N^\epsilon + \rho^0} - \frac{1}{\rho^0}\right](\nabla\times H^0)\times H^0 + \frac{\epsilon^2}{N^\epsilon + \rho^0}(\mathbf{J}_1^\epsilon + \mathbf{q}^0),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\partial_t B^\epsilon + (\mathbf{U}^\epsilon + \mathbf{u}^0)\nabla B^\epsilon + (B^\epsilon + H^0)\mathrm{div}\,\mathbf{U}^\epsilon - (B^\epsilon + H^0)\nabla\mathbf{U}^\epsilon \\
&\quad = -B^\epsilon \mathrm{div}\,\mathbf{u}^0 + B^\epsilon\nabla\mathbf{u}^0 - \mathbf{U}^\epsilon\nabla H^0,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\partial_t\Theta^\epsilon + [(\mathbf{U}^\epsilon + \mathbf{u}^0)\cdot\nabla]\Theta^\epsilon + (\Theta^\epsilon + \theta^0)\,\mathrm{div}\,\mathbf{U}^\epsilon \\
&\quad = -(\mathbf{U}^\epsilon\cdot\nabla)\theta^0 - \Theta^\epsilon\mathrm{div}\,\mathbf{u}^0 - \left[\frac{1}{N^\epsilon + \rho^0} - \frac{1}{\rho^0}\right]\mathrm{div}\,\mathbf{q}^0 \\
&\qquad - \frac{\epsilon(\mathbf{q}^0 + \mathbf{J}_1^\epsilon)(u^0 + \mathbf{U}^\epsilon)}{N^\epsilon + \rho^0} + \frac{1}{N^\epsilon + \rho^0} \\
&\qquad \times\left\{J_0^\epsilon - (\Theta^\epsilon)^4 - 4(\Theta^\epsilon)^3\theta^0 - 6(\Theta^\epsilon)^2(\theta^0)^2 - 4\Theta^\epsilon(\theta^0)^3\right\},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\epsilon\partial_t J_0^\epsilon + \mathrm{div}\,\mathbf{J}_1^\epsilon + J_0^\epsilon &= (\Theta^\epsilon)^4 + 4(\Theta^\epsilon)^3\theta^0 + 6(\Theta^\epsilon)^2(\theta^0)^2 + 4\Theta^\epsilon(\theta^0)^3 \\
&\quad - \epsilon\partial_t(-\Delta)^{-1}\mathrm{div}\,\mathbf{q}^0,
\end{aligned} \tag{2.5}$$

$$\epsilon\partial_t\mathbf{J}_1^\epsilon + \nabla J_0^\epsilon + \mathbf{J}_1^\epsilon = -\epsilon\partial_t\mathbf{q}^0; \tag{2.6}$$

here we used the following identity in (2.3).

$$\nabla \times (\mathbf{u} \times H) = \mathbf{u}(\operatorname{div} H) - H(\operatorname{div} \mathbf{u}) + H \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla H,$$

and the related initial data can be written as

$$\begin{aligned}
&(N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon)|_{t=0} = (N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon, \Theta_0^\epsilon, J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon)\\
&\quad = \big(\rho_0^\epsilon - \rho_0^0, \mathbf{u}_0^\epsilon - \mathbf{u}_0^0, H_0^\epsilon - H_0^0, \theta_0^\epsilon - \theta_0^0, I_0^\epsilon\\
&\qquad - (-\Delta)^{-1}\operatorname{div} \mathbf{q}_0^0, \mathbf{I}_{10}^\epsilon - \mathbf{q}_0^0\big).
\end{aligned} \tag{2.7}$$

We remark that in (2.6) we have used the fact that

$$\mathbf{q}^0 = \nabla \operatorname{div} \mathbf{q}^0 - \nabla(\theta^0)^4 = \left(\frac{-\Delta}{I - \Delta} - I\right) \nabla(\theta^0)^4 = \frac{-\nabla}{I - \Delta}(\theta^0)^4$$

is a gradient.

Denote

$$\mathbf{W}^\epsilon = \begin{pmatrix} N^\epsilon \\ \mathbf{U}^\epsilon \\ B^\epsilon \\ \Theta^\epsilon \\ J_0^\epsilon \\ \mathbf{J}_1^\epsilon \end{pmatrix}, \quad \mathbf{W}_0^\epsilon = \begin{pmatrix} N_0^\epsilon \\ \mathbf{U}_0^\epsilon \\ B_0^\epsilon \\ \Theta_0^\epsilon \\ J_{00}^\epsilon \\ \mathbf{J}_{00}^\epsilon \end{pmatrix}, \quad \mathbf{D}^\epsilon = \begin{pmatrix} \mathbb{I}_8 & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} \epsilon & \mathbf{0} \\ \mathbf{0} & \epsilon \mathbb{I}_3 \end{pmatrix} \end{pmatrix},$$

$$\mathbf{A}_i^\epsilon = \begin{pmatrix} \hat{\mathbf{A}}_i & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 0 & e_i^{\mathrm{T}} \\ e_i & \mathbf{0} \end{pmatrix} \end{pmatrix}, \quad i = 1, 2, 3, \quad \mathbf{M}^\epsilon = \begin{pmatrix} \mathbf{0}_{8\times 8} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_4 \end{pmatrix},$$

$$\mathbf{S}^\epsilon(\mathbf{W}^\epsilon) = \begin{pmatrix} -N^\epsilon \operatorname{div} \mathbf{u}^0 - \nabla \rho^0 \cdot \mathbf{U}^\epsilon \\ \mathbf{R}_1^\epsilon \\ \mathbf{R}_2^\epsilon \\ R_3^\epsilon \\ R_4^\epsilon \\ -\epsilon \partial_t \mathbf{q}^0 \end{pmatrix},$$

with

$$\hat{A}_1^\epsilon = \begin{pmatrix} \mathbf{u}_1^\epsilon & \rho^\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{\theta^\epsilon}{\rho^\epsilon} & \mathbf{u}_1^\epsilon & 0 & 0 & 0 & \dfrac{H_2^\epsilon}{4\pi\rho^\epsilon} & \dfrac{H_3^\epsilon}{4\pi\rho^\epsilon} & 1 \\ 0 & 0 & \mathbf{u}_1^\epsilon & 0 & 0 & -\dfrac{H_1^\epsilon}{4\pi\rho^\epsilon} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{u}_1^\epsilon & 0 & 0 & -\dfrac{H_1^\epsilon}{4\pi\rho^\epsilon} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{u}_1^\epsilon & 0 & 0 & 0 \\ 0 & H_2^\epsilon & -H_1^\epsilon & 0 & 0 & \mathbf{u}_1^\epsilon & 0 & 0 \\ 0 & H_3^\epsilon & 0 & -H_1^\epsilon & 0 & 0 & \mathbf{u}_1^\epsilon & 0 \\ 0 & \theta^\epsilon & 0 & 0 & 0 & 0 & 0 & \mathbf{u}_1^\epsilon \end{pmatrix},$$

$$\hat{A}_2^\epsilon = \begin{pmatrix} \mathbf{u}_2^\epsilon & 0 & \rho^\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{u}_2^\epsilon & 0 & 0 & -\dfrac{H_2^\epsilon}{4\pi\rho^\epsilon} & 0 & 0 & 0 \\ \dfrac{\theta^\epsilon}{\rho^\epsilon} & 0 & \mathbf{u}_2^\epsilon & 0 & \dfrac{H_1^\epsilon}{4\pi\rho^\epsilon} & 0 & \dfrac{H_3^\epsilon}{4\pi\rho^\epsilon} & 1 \\ 0 & 0 & 0 & \mathbf{u}_2^\epsilon & 0 & 0 & -\dfrac{H_2^\epsilon}{4\pi\rho^\epsilon} & 0 \\ 0 & -H_2^\epsilon & H_1^\epsilon & 0 & \mathbf{u}_2^\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{u}_2^\epsilon & 0 & 0 \\ 0 & 0 & H_3^\epsilon & -H_2^\epsilon & 0 & 0 & \mathbf{u}_2^\epsilon & 0 \\ 0 & 0 & \theta^\epsilon & 0 & 0 & 0 & 0 & \mathbf{u}_2^\epsilon \end{pmatrix},$$

$$\hat{A}_3^\epsilon = \begin{pmatrix} \mathbf{u}_3^\epsilon & 0 & 0 & \rho^\epsilon & 0 & 0 & 0 & 0 \\ 0 & \mathbf{u}_3^\epsilon & 0 & 0 & -\dfrac{H_3^\epsilon}{4\pi\rho^\epsilon} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{u}_3^\epsilon & 0 & 0 & -\dfrac{H_3^\epsilon}{4\pi\rho^\epsilon} & 0 & 0 \\ \dfrac{\theta^\epsilon}{\rho^\epsilon} & 0 & 0 & \mathbf{u}_3^\epsilon & \dfrac{H_1^\epsilon}{4\pi\rho^\epsilon} & \dfrac{H_2^\epsilon}{4\pi\rho^\epsilon} & 0 & 1 \\ 0 & -H_3^\epsilon & 0 & H_1^\epsilon & \mathbf{u}_3^\epsilon & 0 & 0 & 0 \\ 0 & 0 & -H_3^\epsilon & H_2^\epsilon & 0 & \mathbf{u}_3^\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{u}_3^\epsilon & 0 \\ 0 & 0 & 0 & \theta^\epsilon & 0 & 0 & 0 & \mathbf{u}_3^\epsilon \end{pmatrix},$$

where $\mathbf{R}_1^\epsilon, \mathbf{R}_2^\epsilon, R_3^\epsilon$, and R_4^ϵ denote the right-hand side of (2.2), (2.3), (2.4), and (2.5), respectively; (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.

Using these notations we can rewrite the problem (2.1)–(2.7) in the form:

$$\begin{cases} \mathbf{D}^\epsilon \partial_t \mathbf{W}^\epsilon + \sum_{i=1}^3 \mathbf{A}_i^\epsilon \partial_{x_i} \mathbf{W}^\epsilon + \mathbf{M}\mathbf{W}^\epsilon = \mathbf{S}^\epsilon(\mathbf{W}^\epsilon), \\ \mathbf{W}^\epsilon|_{t=0} = \mathbf{W}_0^\epsilon. \end{cases} \tag{2.8}$$

It is not difficult to see that the system for $\mathbf{W}^\epsilon$ in (2.8) can be reduced to a quasilinear symmetric hyperbolic–parabolic one. In fact, if we introduce

$$\mathbf{A}^\epsilon = \begin{pmatrix} \begin{pmatrix} \dfrac{\Theta^\epsilon + \theta^0}{(N^\epsilon + \rho^0)^2} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbb{D}_6 & \mathbf{0} \\ 0 & \mathbf{0} & \dfrac{1}{\Theta^\epsilon + \theta^0} \end{pmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_4 \end{pmatrix} \tag{2.9}$$

with

$$\mathbb{D}_6 = \mathrm{diag}(1, 1, 1, 1/4\pi\rho^\epsilon, 1/4\pi\rho^\epsilon, 1/4\pi\rho^\epsilon).$$

$\mathbf{A}^\epsilon$ is positively definite when $\|N^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\rho}/2$ and $\|\Theta^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\theta}/2$, then $\tilde{\mathbf{A}}_0^\epsilon = \mathbf{A}^\epsilon \mathbf{D}^\epsilon$ is positive symmetric and $\tilde{\mathbf{A}}_i^\epsilon = \mathbf{A}^\epsilon \mathbf{A}_i^\epsilon$ are symmetric on $[0, T]$ for all $1 \le i \le 3$. Thus, for fixed $\epsilon > 0$, we can apply the result of multi-dimensional symmetric quasi-linear hyperbolic conservation laws [15, 22] to obtain the following local existence for the problem (2.8).

Proposition 2.1. *Let $s > 7/2$ be an integer and $(\rho^0, \mathbf{u}^0, H^0, \theta^0, \mathbf{q}^0)$ satisfy the conditions in Proposition* 1.1. *Assume that the initial data $(N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon \Theta_0^\epsilon, J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon)$ satisfy*

$$N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon, \Theta_0^\epsilon, J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon \in H^s(\Omega) \text{ and } \|(N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon, \Theta_0^\epsilon, J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon)\|_s \le \delta$$

for some small constant $\delta > 0$. Then there exist positive constants $T^\epsilon (0 < T^\epsilon \le +\infty)$ and K, such that the Cauchy problem (2.8) *has a unique classical solution $(N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon)$ satisfying*

$$N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon \in C^l([0, T^\epsilon), H^{s-l}), \ l = 0, 1;$$

$$\|(N^\epsilon(t), U^\epsilon(t), B^\epsilon, \Theta^\epsilon(t), J_0^\epsilon(t), \mathbf{J}_1^\epsilon(t))\|_s \le K\delta, \quad t \in [0, T^\epsilon).$$

Note that if $\|N^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\rho}/2$ and $\|\Theta^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\theta}/2$, then for smooth solutions, the system (1.32)–(1.37) with initial data (1.39) are equivalent to (2.1)–(2.7) or (2.8) on $[0, T]$, $T < \min\{T^\epsilon, T_*\}$. Usually, the life-span T^ϵ depends on ϵ and may shrink to zero as $\epsilon \to 0$. Therefore, in order to avoid this situation and to obtain the convergence of system (1.32)–(1.37) to the system (1.40)–(1.44), we only need to establish the uniform decay estimates with respect to the parameter ϵ of the solution to the error system (2.8). This will be achieved by the elaborate energy method presented in next section.

3. Uniform Energy Estimates and Proof of Theorem 1.2

In this section we derive the uniform *a priori* energy estimates with respect to the parameter ϵ of the solution to the problem (2.8) and justify rigorously the convergence of the system (1.32)–(1.37) to the system (1.40)–(1.44). Here we adopt and modify some techniques developed in [4–6] and put main efforts on the estimates of higher order nonlinear terms.

We first establish the convergence rate of the error system by establishing the *a priori* estimates uniformly in ϵ. For conciseness of presentation, we define

$$\begin{aligned}
\|\mathcal{E}^\epsilon(t)\|_s^2 &:= \|(N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon)(t)\|_s^2, \\
|||\mathcal{E}^\epsilon(t)|||_s^2 &:= \|\mathcal{E}^\epsilon(t)\|_s^2 + \epsilon\|(J_0^\epsilon, \mathbf{J}_1^\epsilon)(t)\|_s^2, \\
|||\mathcal{E}^\epsilon|||_{s,T} &:= \sup_{0<t<T} |||\mathcal{E}^\epsilon(t)\,|||_s\,.
\end{aligned}$$

The crucial estimate of this paper is the following decay result on the error system (2.1)–(2.6).

Proposition 3.1. *Let $s > 7/2$ be an integer and assume that the initial data $(N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon, \Theta_0^\epsilon, J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon)$ satisfy*

$$\|(N_0^\epsilon, \mathbf{U}_0^\epsilon, B_0^\epsilon, \Theta_0^\epsilon)\|_s^2 + \epsilon\|(J_{00}^\epsilon, \mathbf{J}_{10}^\epsilon)\|_s^2 = |||\mathcal{E}^\epsilon(t=0)|||_s^2 \le M_0\epsilon^2 \tag{3.1}$$

for sufficiently small ϵ and some constant $M_0 > 0$ independent of ϵ. Then, for any $T_0 \in (0, T_*)$, there exist two constants $M_1 > 0$ and $\epsilon_1 > 0$ depending only on T_0, such that for all $\epsilon \in (0, \epsilon_1]$, it holds that $T^\epsilon \ge T_0$ and the solution $(N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon)$ of the problem (2.1)–(2.7), *well-defined in* $[0, T_0]$, *satisfies that*

$$|||\,\mathcal{E}^\epsilon|||_{s,T_0} \le M_1\epsilon. \tag{3.2}$$

Once this proposition is established, the proof of Theorem 1.2 is a direct procedure. In fact, we have the following proof.

Proof of Theorem 1.2. Suppose that Proposition 3.1 holds. According to the definition of the error functions $(N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon)$ and the regularity of $(\rho^0, \mathbf{u}^0, H^0, \theta^0, \mathbf{q}^0)$, the error system (2.1)–(2.6) and the primitive system (1.32)–(1.37) are equivalent on $[0, T]$ for some $T > 0$. Therefore, the assumption (1.46) in Theorem 1.2 implies the assumption (3.1) in Proposition 3.1, and hence (3.2) gives (1.47). □

Therefore, our main goal next is to prove Proposition 3.1 which can be approached by the following *a priori* estimates. For some given $\hat{T} < 1$ and any $\tilde{T} < \hat{T}$ independent of ϵ, we denote $T \equiv T_\epsilon = \min\{\tilde{T}, T^\epsilon\}$.

Lemma 3.2. *Let the assumptions in Proposition* 3.1 *hold. Then, for all* $0 < t < T$ *and sufficiently small* ϵ, *there exists a generic positive constant* C, *such that*

$$\begin{aligned}
&|||\mathcal{E}^\epsilon(t)\,|||_s^2 + \frac{1}{4}\int_0^t \{\|J_0^\epsilon\|_s^2 + \|\mathbf{J}_1^\epsilon\|_s^2\}(\tau)\mathrm{d}\tau \\
&\quad \le C\,|||\,\mathcal{E}^\epsilon(t=0)\,|||_s^2 + C\int_0^t \left\{(\|\mathcal{E}^\epsilon\|_s^{2(s+1)} + 1)\|\mathcal{E}^\epsilon\|_s^2\right\}(\tau)\mathrm{d}\tau + C\epsilon^2. \qquad (3.3)
\end{aligned}$$

Proof. For the simplicity of presentation, we denote

$$\mathbf{\Phi}^\epsilon = (N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon), \quad \mathbf{\Psi}^\epsilon = (N^\epsilon, \mathbf{U}^\epsilon, B^\epsilon, \Theta^\epsilon), \quad \mathbf{\Pi}^\epsilon = (J_0^\epsilon, \mathbf{J}_1^\epsilon),$$

Let $0 \le |\alpha| \le s$. In the following arguments, the commutators will disappear in the case of $|\alpha| = 0$.

Applying the operator ∂_x^α to the system (2.8), multiplying the resulting equations by $\partial_x^\alpha \mathbf{\Phi}^\epsilon \mathbf{A}^\epsilon$ and integrating over Ω, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \tilde{\mathbf{A}}_0^\epsilon\rangle + \langle \partial_x^\alpha \mathbf{\Pi}^\epsilon, \partial_x^\alpha \mathbf{\Pi}^\epsilon\rangle \\
&\quad = \frac{1}{2}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \partial_t \tilde{\mathbf{A}}_0^\epsilon\rangle - \sum_{i=1}^3 \langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \partial_{x_i} \mathbf{\Phi}^\epsilon \tilde{\mathbf{A}}_i^\epsilon\rangle \\
&\qquad + \langle \partial_x^\alpha \mathbf{S}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \mathbf{A}^\epsilon\rangle + \mathcal{R} \\
&\quad = \frac{1}{2}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \partial_t \mathbf{A}^\epsilon\rangle + \sum_{i=1}^3 \langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \partial_{x_i} \tilde{\mathbf{A}}_i^\epsilon\rangle \\
&\qquad + \langle \partial_x^\alpha \mathbf{S}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \mathbf{A}^\epsilon\rangle + \mathcal{R}. \qquad (3.4)
\end{aligned}$$

$\mathcal{R}$ stands for the commutators, which is defined as follows:

$$\mathcal{R} = -\left\langle \mathbf{A}^\epsilon \partial_x^\alpha \left(\sum_{i=1}^3 \mathbf{A}_i^\epsilon \partial_{x_i} \mathbf{\Phi}^\epsilon\right), \partial_x^\alpha \mathbf{\Phi}^\epsilon \right\rangle + \left\langle \sum_{i=1}^3 \tilde{\mathbf{A}}_i^\epsilon \partial_x^\alpha \partial_{x_i} \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \right\rangle.$$

First, by the explicit formula of $\mathbf{A}^\epsilon$, and Eqs. (1.40), (1.43), (2.1) and (2.4), we obtain

$$\left|\frac{1}{2}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \partial_t \mathbf{A}^\epsilon\rangle\right| \le C\langle \partial_x^\alpha \mathbf{\Psi}^\epsilon, \partial_x^\alpha \mathbf{\Psi}^\epsilon\rangle. \qquad (3.5)$$

Here we used the *a priori* assumptions that

$$\|N^\epsilon, \mathbf{U}^\epsilon, \Theta^\epsilon, J_0^\epsilon, \mathbf{J}_1^\epsilon, \nabla N^\epsilon, \nabla \mathbf{U}^\epsilon, \nabla \Theta^\epsilon\|_{L^\infty} \le C \qquad (3.6)$$

and the facts that

$$\|\rho^0, \mathbf{u}^0, \theta^0, \nabla\rho^0, \nabla\mathbf{u}^0, \nabla\theta^0, \mathbf{q}^0, \operatorname{div}\mathbf{q}^0\|_{L^\infty} \le C.$$

It is not hard to find the *a priori* assumption (3.6) can be closed by proving the inequality (3.2) provided $s > 5/2$. Similarly,

$$\left|\sum_{i=1}^{3}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \partial_{x_i} \tilde{\mathbf{A}}_i^\epsilon\rangle\right| \le C\langle \partial_x^\alpha \mathbf{\Psi}^\epsilon, \partial_x^\alpha \mathbf{\Psi}^\epsilon\rangle. \tag{3.7}$$

The estimates of the source terms $\langle \partial_x^\alpha \mathbf{S}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \mathbf{A}^\epsilon\rangle$ and the commutators $\mathcal{R}$ are the same as those in Sec. 3 in [6], we only write down the related estimates as follows for simplicity:

$$\begin{aligned} &|\langle \partial_x^\alpha \mathbf{S}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \mathbf{A}^\epsilon\rangle, \mathcal{R}| \\ &\quad \le C_{\eta_1,\eta_2}[(\|\mathcal{E}^\epsilon(t)\|_s^{2s+4} + \|\mathcal{E}^\epsilon(t)\|_s^2)] + \eta_1\|J_0^\epsilon\|_s^2 + \eta_2\|\mathbf{J}_1^\epsilon\|_s^2 + C\epsilon^2. \end{aligned} \tag{3.8}$$

Summing (3.4) up α with $0 \le |\alpha| \le s$, using the estimates (3.5)–(3.8) and the fact that $N^\epsilon + \rho^0 \ge \hat{N} + \hat{\rho} > 0$, choosing η_i $(i = 1, 2)$ sufficiently small, and noticing that $s > 7/2$ is an integer.

In addition, noticing the norms $\sum_{|\alpha|\le s}\langle \partial_x^\alpha \mathbf{\Phi}^\epsilon, \partial_x^\alpha \mathbf{\Phi}^\epsilon \tilde{\mathbf{A}}_0^\epsilon\rangle$ and $|||\mathcal{E}^\epsilon(t)|||_s^2$ are equivalent provided that $\|N^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\rho}/2$ and $\|\Theta^\epsilon\|_{L_T^\infty L_x^\infty} \le \hat{\theta}/2$. Consequently, (3.3) holds true. This completes the proof of Lemma 3.2. □

With the estimate (3.3) in hand, we can now prove Proposition 3.1.

Proof of Proposition 3.1. As in [4–6], we introduce an ϵ-weighted energy functional

$$\Gamma^\epsilon(t) = |||\mathcal{E}^\epsilon(t)\,|||_s^2\,.$$

Then, it follows from (3.3) that there exists a constant $\epsilon_1 > 0$ depending only on T, such that for any $\epsilon \in (0, \epsilon_1]$ and any $t \in (0, T]$,

$$\Gamma^\epsilon(t) \le C\Gamma^\epsilon(t=0) + C\int_0^t \{((\Gamma^\epsilon)^{2(s+1)} + 1)\Gamma^\epsilon\}(\tau)\mathrm{d}\tau + C\epsilon^2. \tag{3.9}$$

Thus, applying the Gronwall lemma to (3.9), and keeping in mind that $\Gamma^\epsilon(t = 0) \le C\epsilon^2$ and Proposition 3.1, we find that there exist a $0 < T_1 < 1$ and an $\epsilon > 0$, such that $T^\epsilon \ge T_1$ for all $\epsilon \in (0, \epsilon_1]$ and $\Gamma^\epsilon(t) \le C\epsilon^2$ for all $t \in (0, T_1]$. Therefore, the desired *a priori* estimate (3.2) holds. Moreover, by the standard continuation induction argument, we can extend $T^\epsilon \ge T_0$ to any $T_0 < T_*$. □

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