A structure-preserving staggered semi-implicit scheme for continuum mechanics

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Hyperbolic formulation of Newtonian continuum mechanics

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0,$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \left(\rho v_i v_k + p\, \delta_{ik} + \sigma_{ik} \right)}{\partial x_k} = 0,$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial (A_{im} v_m)}{\partial x_k} + v_m \left(\frac{\partial A_{ik}}{\partial x_m} - \frac{\partial A_{im}}{\partial x_k} \right) = -\frac{\psi_{ik}}{\theta_1(\tau_1)},$$

$$\frac{\partial J_k}{\partial t} + \frac{\partial \left(J_m v_m + T \right)}{\partial x_k} + v_m \left(\frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k} \right) = -\frac{H_k}{\theta_2(\tau_2)},$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \left(v_k \rho E + v_i (p\, \delta_{ik} + \sigma_{ik}) + q_k \right)}{\partial x_k} = 0.$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \left(\rho S v_k + \rho H_k \right)}{\partial x_k} = \frac{\rho}{\theta_1(\tau_1) T} \psi_{ik} \psi_{ik} + \frac{\rho}{\theta_2(\tau_2) T} H_i H_i \geq 0$$

Godunov (1961)
Godunov & Romenski (1972)
Romenski (1998)
Peshkov & Romenski (2016)
Dumbser et al. (2016)
Dumbser et al. (2017)

Conservation of mass, momentum and total energy. Entropy inequality.
Evolution of the distortion field *A* and of the thermal impulse vector *J*.

Hyperbolic formulation of Newtonian continuum mechanics

Choice of the total energy potential:

$$E(\rho, s, \mathbf{v}, \mathbf{A}, \mathbf{J}) = E_1(\rho, s) + E_2(\mathbf{A}, \mathbf{J}) + E_3(\mathbf{v}).$$

Classical equation of state, e.g. ideal gas EOS (micro-scale):

$$E_1(\rho, s) = \frac{c_0^2}{\gamma(\gamma-1)}, \; c_0^2 = \gamma \rho^{\gamma-1} e^{s/c_V}$$

Classical kinetic energy (macro-scale):

$$E_3(v_k) = \frac{1}{2} v_i v_i$$

Energy stored in the <u>meso-scale</u>, due to <u>deformations</u> and heat-flux, where c_s is the shear sound speed and c_h is the so-called *second sound*:

$$E_2(A_{ik}, J_k) = \frac{1}{4} c_s^2 \mathring{G}_{ij} \mathring{G}_{ij} + \frac{1}{2} c_h^2 J_k J_k$$

$$\mathring{G}_{ij} = G_{ij} - \frac{1}{3} G_{kk} \, \delta_{ij}$$

Hyperbolic formulation of Newtonian continuum mechanics

The system is symmetrizable and thermodynamically compatible (SHTC).
All source terms and constitutive fluxes as well as the hydrodynamic pressure and the temperature are defined by partial derivatives of the scalar energy potential:

$$\sigma_{ik} = \rho A_{ji} E_{A_{jk}} + \rho J_i E_{J_k} \qquad \psi_{ik} := E_{A_{ik}}$$

$$q_k = \rho E_S E_{J_k} \qquad H_k := E_{J_k}$$

$$p = \rho^2 E_\rho \qquad T = E_S$$

For the previously chosen energy potential we obtain:

$$\sigma_{ik} = \rho c_s^2 G_{ij} \mathring{G}_{jk} + \rho c_h^2 J_i J_k, \qquad q_k = \rho T c_h^2 J_k$$

$$\psi_{ik} = c_s^2 A_{ij} \mathring{G}_{jk}, \qquad H_k = c_h^2 J_k$$

The functions θ in the source terms are chosen as

$$\theta_1(\tau_1) = \frac{1}{3}\tau_1 c_s^2 |\mathbf{A}|^{\frac{5}{3}}, \qquad \theta_2(\tau_2) = \tau_2 \frac{c_h^2}{\rho T}$$

Hyperbolic formulation of Newtonian continuum mechanics

Asymptotic (stiff) relaxation limit of the stress tensor:

$$\frac{\partial \mathbf{G}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{G} + \left(\mathbf{G} \nabla \mathbf{v} + \nabla \mathbf{v}^T \mathbf{G} \right) = \frac{2}{\rho \theta_1} \boldsymbol{\sigma}. \qquad \mathbf{G} = \mathbf{A}^\mathsf{T} \mathbf{A}$$

Chapman-Enskog expansion of G...

$$\mathbf{G} = \mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \tau_1^2 \mathbf{G}_2 + \ldots$$

... and some calculations ...

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots) = -\Big((\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots) \nabla \mathbf{v} + \nabla \mathbf{v}^\mathsf{T} (\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots) \Big) - \frac{6}{\tau_1} |\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots|^{\frac{5}{6}} (\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots) \mathrm{dev}(\mathbf{G}_0 + \tau_1 \mathbf{G}_1 + \ldots),$$

Hyperbolic formulation of Newtonian continuum mechanics

Via formal asymptotic analysis it can be shown that in the stiff relaxation limit of relaxation times τ that tend to zero, the stress tensor and the heat flux reduce to

$$\boldsymbol{\sigma} = -\frac{1}{6}\rho_0 c_s^2 \tau_1 \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} \nabla \cdot \mathbf{v} \mathbf{I} \right) - \frac{c_h^2}{\rho T^2} \tau_2^2 \nabla T \otimes \nabla T.$$

$$\boldsymbol{q} = -c_s^2 \tau_2 \nabla T$$

In the absence of source terms, i.e for relaxation times tending to infinity, the fields A and J remain curl-free for all times if they were initially curl-free. Thse two stationary, linear differential constraints (involutions) are similar to the well-known divergence-free condition of the magnetic field B in the Maxwell and MHD equations.

$$\tau_1 \rightarrow \infty \qquad \frac{\partial A_{ik}}{\partial x_m} - \frac{\partial A_{im}}{\partial x_k} = 0$$

$$\tau_2 \rightarrow \infty \qquad \frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k} = 0$$

Hyperbolic formulation of Newtonian continuum mechanics

The aim of this talk is to construct a new staggered semi-implicit FV scheme with the following properties:

1. **Exactly curl-free on the discrete level for the homogeneous PDE system without source terms. This is achieved via a set of compatible discrete divergence, gradient and curl operators on staggered grids (S2)**
2. **Asymptotic preserving property in the stiff relaxation limit of the source terms, so that a consistent scheme for the Navier-Stokes equations is obtained (S4)**
3. **Conservative pressure-based staggered semi-implicit all Mach number flow solver, valid for both, supersonic flows with shock waves and incompressible flows (S5)**
4. **So far, we are <u>not</u> yet able to combine all the above properties with (S1), i.e. the discrete SHTC property.**

Splitting of the PDE system

The system is then split into a nonlinear convective flux and a pressure flux (see Toro & Vázquez splitting 2012), the terms in the PDE for A and J that need a compatible, curl-free discretization and appropriate vertex fluxes for the stress tensor and the heat flux

$$\partial_t \mathbf{Q} + \nabla \cdot (\mathbf{F}_c(\mathbf{Q}) + \mathbf{F}_p(\mathbf{Q}) + \mathbf{F}_v(\mathbf{Q})) + \nabla \mathbf{G}_v(\mathbf{Q}) + \mathbf{B}_v(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}_v(\mathbf{Q})$$

$$\mathbf{F}_c = \begin{pmatrix} \rho v_k \\ \rho v_i v_k \\ 0 \\ 0 \\ \rho v_k (E_2 + E_3) \end{pmatrix} \quad \mathbf{F}_p = \begin{pmatrix} 0 \\ p\delta_{ik} \\ 0 \\ 0 \\ h\rho v_k \end{pmatrix} \quad \mathbf{F}_v = \begin{pmatrix} 0 \\ -\sigma_{ik} \\ 0 \\ 0 \\ -v_i \sigma_{ik} + q_k \end{pmatrix}$$

$$\mathbf{G}_v(\mathbf{Q}) = \begin{pmatrix} 0 \\ 0 \\ A_{im} v_m \\ J_m v_m + T \\ 0 \end{pmatrix}, \qquad \mathbf{B}_v(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \\ v_m \left(\frac{\partial A_{ik}}{\partial x_m} - \frac{\partial A_{im}}{\partial x_k} \right) \\ v_m \left(\frac{\partial J_k}{\partial x_m} - \frac{\partial J_m}{\partial x_k} \right) \\ 0 \end{pmatrix}$$

Splitting of the PDE system

The original PDE system (Galilean invariant) reads in compact matrix-vector notation

$$\partial_t \mathbf{Q} + \nabla \cdot \mathbf{F}(\mathbf{Q}) + \mathbf{B}(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}(\mathbf{Q})$$

For the case A=I and J=0, the eigenvalues of the original system are obtained from the following characteristic polynomial of the system matrix in x direction:

$$\tilde{\lambda}^9 (c_s^2 - \tilde{\lambda}^2)^2 (a_0 + a_2 \tilde{\lambda}^2 - a_4 \tilde{\lambda}^4) = 0, \qquad \tilde{\lambda} = \lambda - u$$

$$a_0 = c_h^2 T (4 c_s^2 + 3 c_v (\gamma - 1) T),$$

$$a_2 = 4 c_s^2 c_v + 3 (c_h^2 + c_v^2 (\gamma - 1) \gamma) T,$$

$$a_4 = 3 c_v,$$

$$\lambda_{ta} = u \pm \sqrt{\frac{a_2 \mp \sqrt{a_2^2 + 4 a_0 a_4}}{2 a_4}}$$

Slow and fast thermo-acoustic waves (c_h is the second sound)

$$\lambda_s = u \pm c_s$$

xy and xz-shear waves

$$\lambda_a = u$$

transport

Splitting of the PDE system

The part of the system that is discretized explicitly, in the following also called the “convective subsystem”, which is in general not Galilean invariant, reads:

$$\partial_t \mathbf{Q} + \nabla \cdot (\mathbf{F}_c(\mathbf{Q}) + \mathbf{F}_v(\mathbf{Q})) + \nabla \mathbf{G}_v(\mathbf{Q}) + \mathbf{B}_v(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}_v(\mathbf{Q}),$$

Its eigenvalues are

$$\lambda^c_{1,2} = \frac{1}{2}u \pm \frac{1}{2}\sqrt{\frac{4T}{c_v}c_h^2 + u^2}, \qquad \lambda^c_{3,4} = u \pm \frac{2}{3}\sqrt{3}c_s, \qquad \lambda^c_{5,6,7,8} = u \pm c_s, \qquad \lambda^c_{9,10,\cdots,17} = u.$$

The remaining pressure sub-system is formally identical to the Toro-Vázquez system

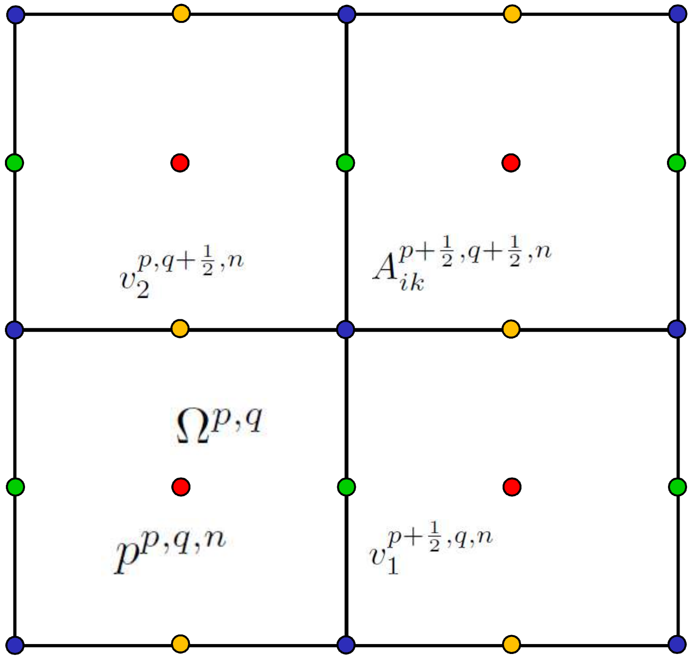
$$\partial_t \mathbf{Q} + \nabla \cdot \mathbf{F}_p(\mathbf{Q}) = 0.$$

The eigenvalues of the pressure system (not Galilean invariant) are

$$\lambda^p_{1,2} = \frac{1}{2}\left(u \pm \sqrt{u^2 + 4c_0^2}\right), \qquad \lambda^p_{3,4,5,\cdots,17} = 0$$

Staggered Cartesian mesh

In our structure-preserving staggered semi-implicit finite volume scheme, we define the quantities on four different staggered mesh locations: in the barycenters, in the vertices, on the *x*-edges and on the *y*-edges of the main grid.



$$p^{p,q,n} = p(x^p, y^q, t^n)$$

$$v_1^{p+\frac{1}{2},q,n} := u^{p+\frac{1}{2},q,n} = v_1(x^{p+\frac{1}{2}}, y^q, t^n)$$

$$v_2^{p,q+\frac{1}{2},n} := v^{p,q+\frac{1}{2},n} = v_2(x^p, y^{q+\frac{1}{2}}, t^n)$$

$$A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n} = A_{ik}(x^{p+\frac{1}{2}}, y^{q+\frac{1}{2}}, t^n)$$

$$J_k^{p+\frac{1}{2},q+\frac{1}{2},n} = J_k(x^{p+\frac{1}{2}}, y^{q+\frac{1}{2}}, t^n)$$

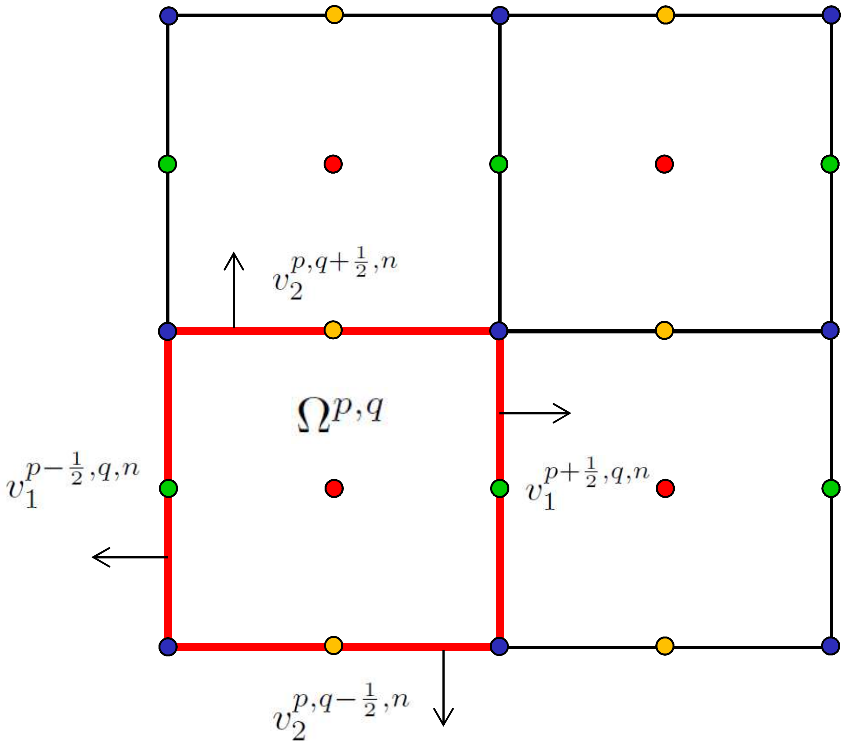
We also use simple central interpolations of all quantities from one mesh to another.

The staggered mesh is a fundamental ingredient for the definition of compatible discrete curl and grad operators needed for our exactly curl-free scheme. It is also important to get a stable and symmetric positive definite discretization of the pressure system

Compatible *div*, *grad* and *curl* operators

The first discrete divergence on a control volume is naturally defined using the edge-based staggered quantities (discrete form of the Gauss theorem):

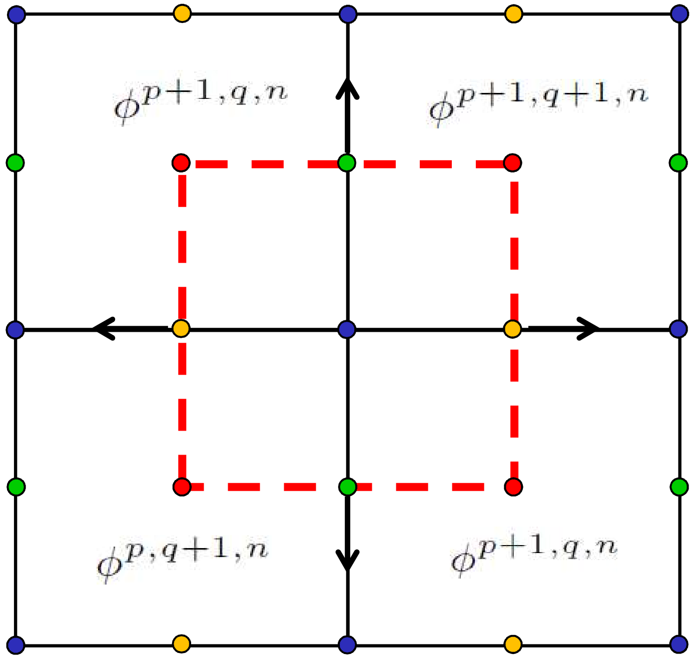
$$\nabla^{p,q} \cdot \mathbf{v}^{h,n} = \partial_k^{p,q} v_k^{h,n} = \frac{v_1^{p+\frac{1}{2},q,n} - v_1^{p-\frac{1}{2},q,n}}{\Delta x} + \frac{v_2^{p,q+\frac{1}{2},n} - v_2^{p,q-\frac{1}{2},n}}{\Delta y}$$



Compatible *div*, *grad* and *curl* operators

The discrete gradient on a vertex is naturally defined using the barycenter quantities (discrete form of the Gauss theorem for tensors):

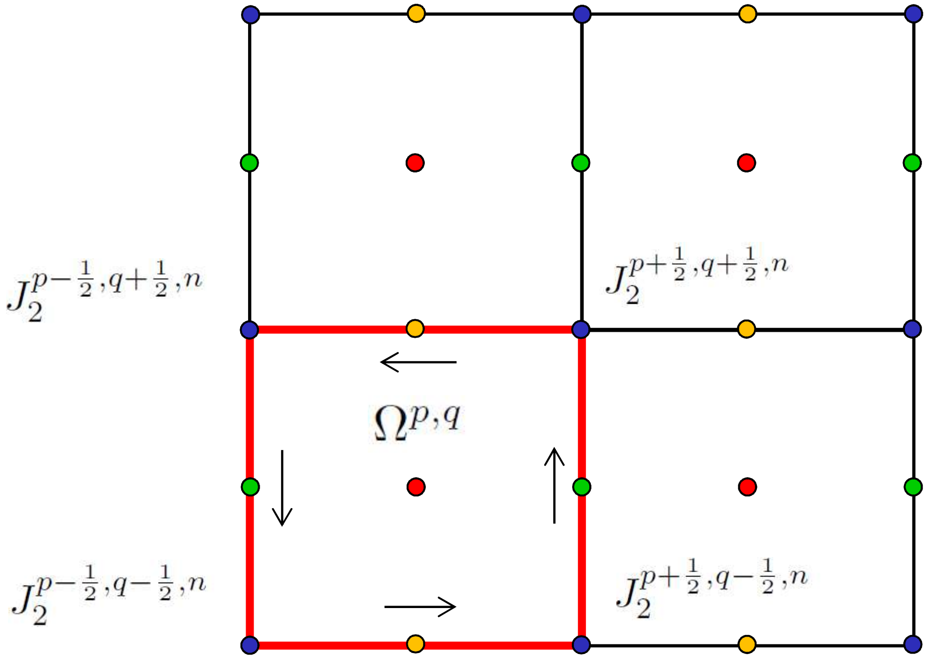
$$\nabla^{p+\frac{1}{2},q+\frac{1}{2}}\phi^{h,n} = \partial_k^{p+\frac{1}{2},q+\frac{1}{2}}\phi^{h,n} = \begin{pmatrix} \frac{1}{2}\frac{\phi^{p+1,q+1,n}+\phi^{p+1,q,n}-\phi^{p,q+1,n}-\phi^{p,q,n}}{\Delta x} \\ \frac{1}{2}\frac{\phi^{p+1,q+1,n}+\phi^{p,q+1,n}-\phi^{p+1,q,n}-\phi^{p,q,n}}{\Delta y} \\ 0 \end{pmatrix}$$



Compatible *div*, *grad* and *curl* operators

The discrete curl on a control volume is naturally defined using the vertex-based staggered quantities (discrete form of the Stokes theorem):

$$
\begin{aligned}
\left(\nabla^{p,q} \times \mathbf{J}^{h,n}\right) \cdot \mathbf{e}_z &= \epsilon_{3jk} \partial_j^{p,q} J_k^{h,n} \\
&= \frac{1}{2} \frac{J_2^{p+\frac{1}{2},q+\frac{1}{2},n} + J_2^{p+\frac{1}{2},q-\frac{1}{2},n} - J_2^{p-\frac{1}{2},q+\frac{1}{2},n} - J_2^{p-\frac{1}{2},q-\frac{1}{2},n}}{\Delta x} - \\
&\quad \frac{1}{2} \frac{J_1^{p+\frac{1}{2},q+\frac{1}{2},n} + J_1^{p-\frac{1}{2},q+\frac{1}{2},n} - J_1^{p+\frac{1}{2},q-\frac{1}{2},n} - J_2^{p-\frac{1}{2},q-\frac{1}{2},n}}{\Delta y},
\end{aligned}
$$



Discrete curl - grad

Combining the previous discrete grad and curl operators, it is then easy to prove that for an arbitrary discrete scalar field defined in the barycenter of the main grid one has

$$\nabla^h \times \nabla^h \phi^{h,n} = 0$$

which is the discrete curl-grad compatibility. Using the previous curl and grad operators, the discretized J equation then reads

$$\begin{aligned} J_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} &= J_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t\, \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} \left(J_m^{h,n} v_m^{h,n} + T^{h,n} \right) \\ &\quad - \Delta t \frac{1}{4} \sum_{r=0}^{1} \sum_{s=0}^{1} v_m^{p+r,q+s,n} \left(\partial_m^{p+r,q+s} J_k^{h,n} - \partial_k^{p+r,q+s} J_m^{h,n} \right) \\ &\quad - \Delta t \frac{\rho^{p+\frac{1}{2},q+\frac{1}{2},n}\, T^{p+\frac{1}{2},q+\frac{1}{2},n}}{\tau_2} J_k^{p+\frac{1}{2},q+\frac{1}{2},n+1}. \end{aligned}$$

If the field J was initially curl-free, it will remain curl-free for all times, if the source term vanishes ($\tau_2 = \infty$).

Discrete curl - grad

Combining the previous discrete grad and curl operators, it is then easy to prove that for an arbitrary discrete scalar field defined in the barycenter of the main grid one has

$$\nabla^h \times \nabla^h \phi^{h,n} = 0$$

which is the discrete curl-grad compatibility. Using the previous curl and grad operators, the discretized A equation then reads

$$\begin{aligned} A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n+1} \quad = \quad & A_{ik}^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t \, \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} \left(A_{im}^{h,n} v_m^{h,n} \right) \\ & - \Delta t \frac{1}{4} \sum_{r=0}^{1} \sum_{s=0}^{1} v_m^{p+r,q+s,n} \left(\partial_m^{p+r,q+s} A_{ik}^{h,n} - \partial_k^{p+r,q+s} A_{im}^{h,n} \right) \\ & - \Delta t \frac{\left| A_{ij}^{p+\frac{1}{2},q+\frac{1}{2},n+1} \right|^{\frac{5}{3}}}{\tau_1} A_{ij}^{p+\frac{1}{2},q+\frac{1}{2},n+1} \mathring{G}_{jk}^{p+\frac{1}{2},q+\frac{1}{2},n+1}. \end{aligned}$$

If the field A was initially curl-free, it will remain curl-free for all times, if the source term vanishes ($\tau_1 = \infty$).

Explicit discretization of the convective fluxes

For the classical convective flux, combined with the fluxes contained in F_v, we employ a standard second order Godunov-type TVD scheme (MUSCL-Hancock method), which reads as follows:

$$\mathbf{Q}^{p,q,*} = \mathbf{Q}^{p,q,n} - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{c,v}^{p+\frac{1}{2},q} - \mathbf{f}_{c,v}^{p-\frac{1}{2},q}\right) - \frac{\Delta t}{\Delta y}\left(\mathbf{g}_{c,v}^{p,q+\frac{1}{2}} - \mathbf{g}_{c,v}^{p,q-\frac{1}{2}}\right)$$

with the numerical fluxes (Rusanov)

$$\begin{aligned}
\mathbf{f}_{c,v}^{p+\frac{1}{2},q} \;=\; & \frac{1}{2}\left(\mathbf{f}_c\left(\mathbf{Q}_-^{p+\frac{1}{2},q,n+\frac{1}{2}}\right) + \mathbf{f}_c\left(\mathbf{Q}_+^{p+\frac{1}{2},q,n+\frac{1}{2}}\right)\right) - \frac{1}{2}s_{\max}^x\left(\mathbf{Q}_+^{p+\frac{1}{2},q,n+\frac{1}{2}} - \mathbf{Q}_-^{p+\frac{1}{2},q,n+\frac{1}{2}}\right) \\
& +\frac{1}{2}\left(\mathbf{f}_v\left(\mathbf{Q}^{p+\frac{1}{2},q+\frac{1}{2},n}\right) + \mathbf{f}_v\left(\mathbf{Q}^{p+\frac{1}{2},q-\frac{1}{2},n}\right)\right), \\
\mathbf{g}_{c,v}^{p,q+\frac{1}{2}} \;=\; & \frac{1}{2}\left(\mathbf{g}_c\left(\mathbf{Q}_-^{p,q+\frac{1}{2},n+\frac{1}{2}}\right) + \mathbf{g}_c\left(\mathbf{Q}_+^{p,q+\frac{1}{2},n+\frac{1}{2}}\right)\right) - \frac{1}{2}s_{\max}^y\left(\mathbf{Q}_+^{p,q+\frac{1}{2},n+\frac{1}{2}} - \mathbf{Q}_-^{p,q+\frac{1}{2},n+\frac{1}{2}}\right) \\
& +\frac{1}{2}\left(\mathbf{g}_v\left(\mathbf{Q}^{p+\frac{1}{2},q+\frac{1}{2},n}\right) + \mathbf{g}_v\left(\mathbf{Q}^{p-\frac{1}{2},q+\frac{1}{2},n}\right)\right).
\end{aligned}$$

The fluxes f_v and g_v are directly evaluated on the vertices, where A and J are defined (corner fluxes).

Implicit discretization of the pressure terms

Now all terms apart from the pressure fluxes have been discretized. The discrete momentum equations including the pressure gradient read

$$
\begin{aligned}
(\rho v)_1^{p+\frac{1}{2},q,n+1} &= (\rho v)_1^{p+\frac{1}{2},q,*} - \frac{\Delta t}{\Delta x}\left(p^{p+1,q,n+1} - p^{p,q,n+1}\right) \\
(\rho v)_2^{p,q+\frac{1}{2},n+1} &= (\rho v)_2^{p,q+\frac{1}{2},*} - \frac{\Delta t}{\Delta y}\left(p^{p,q+1,n+1} - p^{p,q,n+1}\right)
\end{aligned}
$$

and the discrete total energy conservation law is written as follows:

$$
\begin{aligned}
\rho E_1\left(p^{p,q,n+1}\right) + \rho E_2^{p,q,n+1} + \rho \tilde{E}_3^{p,q,n+1} &= \rho E^{p,q,*} \\
-\frac{\Delta t}{\Delta x}\left(\tilde{h}^{p+\frac{1}{2},q,n+1}(\rho v)_1^{p+\frac{1}{2},q,n+1} - \tilde{h}^{p-\frac{1}{2},q,n+1}(\rho v)_1^{p-\frac{1}{2},q,n+1}\right) \\
-\frac{\Delta t}{\Delta y}\left(\tilde{h}^{p,q+\frac{1}{2},n+1}(\rho v)_2^{p,q+\frac{1}{2},n+1} - \tilde{h}^{p,q-\frac{1}{2},n+1}(\rho v)_2^{p,q-\frac{1}{2},n+1}\right).
\end{aligned}
$$

Implicit discretization of the pressure terms

Inserting the discrete momentum equations into the discrete energy equation leads to the following mildly nonlinear (or linear for ideal gas EOS) pressure system, where we use a simple Picard iteration to update the kinetic energy at time t^{n+1}:

$$\begin{aligned}
&\rho^{p,q,n+1} E_1\left(p_{r+1}^{p,q,n+1}, \rho^{p,q,n+1}\right) \\
-\frac{\Delta t^2}{\Delta x^2}&\left(h_r^{p+\frac{1}{2},q,n+1}\left(p_{r+1}^{p+1,j,n+1} - p_{r+1}^{p,q,n+1}\right) - h_r^{p-\frac{1}{2},q,n+1}\left(p_{r+1}^{p,q,n+1} - p_{r+1}^{p-1,q,n+1}\right)\right) \\
-\frac{\Delta t^2}{\Delta y^2}&\left(h_r^{p,q+\frac{1}{2},n+1}\left(p_{r+1}^{p,q+1,n+1} - p_{r+1}^{p,q,n+1}\right) - h_r^{p,q-\frac{1}{2},n+1}\left(p_{r+1}^{p,q,n+1} - p_{r+1}^{p,q-1,n+1}\right)\right) \quad = \quad b_r^{p,q,n},
\end{aligned}$$

with the known right hand side

$$\begin{aligned}
b_{i,j}^r = \rho E^{p,q,*} &- \rho E_2^{p,q,n+1} - \rho E_{3,r}^{p,q,n+1} \\
-\frac{\Delta t}{\Delta x}&\left(h_r^{p+\frac{1}{2},q,n+1}(\rho v)_1^{p+\frac{1}{2},q,*} - h_r^{p-\frac{1}{2},q,n+1}(\rho v)_1^{p-\frac{1}{2},q,*}\right) \\
-\frac{\Delta t}{\Delta y}&\left(h_r^{p,q+\frac{1}{2},n+1}(\rho v)_2^{p,q+\frac{1}{2},*} - h_r^{p,q-\frac{1}{2},n+1}(\rho v)_2^{p,q-\frac{1}{2},*}\right).
\end{aligned}$$

Implicit discretization of the pressure terms

For general EOS, the mildly nonlinear pressure system

$$\rho \mathbf{E}_1 \left(\mathbf{p}_{r+1}^{n+1} \right) + \mathbf{M}_r \cdot \mathbf{p}_{r+1}^{n+1} = \mathbf{b}_r^n$$

can be easily solved via the Newton method of Brugnano and Casulli (2007,2009), or the nested Newton method of Casulli and Zanolli (2012). Once the new pressure has been obtained, the momentum for the next Picard iteration is computed by

$$\begin{aligned}
(\rho v)_{1,r+1}^{p+\frac{1}{2},q,n+1} &= (\rho v)_1^{p+\frac{1}{2},q,*} - \frac{\Delta t}{\Delta x} \left(p_{r+1}^{p+1,q,n+1} - p_{r+1}^{p,q,n+1} \right) \\
(\rho v)_{2,r+1}^{p,q+\frac{1}{2},n+1} &= (\rho v)_2^{p,q+\frac{1}{2},*} - \frac{\Delta t}{\Delta y} \left(p_{r+1}^{p,q+1,n+1} - p_{r+1}^{p,q,n+1} \right)
\end{aligned}$$

The total energy is finally updated as

$$\begin{aligned}
(\rho E)^{p,q,n+1} &= (\rho E)^{p,q,*} - \frac{\Delta t}{\Delta x} \left(h^{p+\frac{1}{2},q,n+1} (\rho v)_1^{p+\frac{1}{2},q,n+1} - h^{p-\frac{1}{2},q,n+1} (\rho v)_1^{p-\frac{1}{2},q,n+1} \right) \\
&\quad - \frac{\Delta t}{\Delta y} \left(h^{p,q+\frac{1}{2},n+1} (\rho v)_2^{p,q+\frac{1}{2},n+1} - h^{p,q-\frac{1}{2},n+1} (\rho v)_2^{p,q-\frac{1}{2},n+1} \right)
\end{aligned}$$

Asymptotic preserving property

We start with the analysis of the semi-discrete equation for the specific thermal impulse

$$\partial_t J_k^{p+\frac{1}{2},q+\frac{1}{2}} + \partial_k^{p+\frac{1}{2},q+\frac{1}{2}}\left(J_m^h v_m^h\right) + \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} T^h$$
$$+\frac{1}{4}\sum_{r=0}^{1}\sum_{s=0}^{1} v_m^{p+r,q+s,n}\left(\partial_m^{p+r,q+s} J_k^h - \partial_k^{p+r,q+s} J_m^h\right) = -\frac{\rho^{p+\frac{1}{2},q+\frac{1}{2}}\, T^{p+\frac{1}{2},q+\frac{1}{2}}}{\tau_2} J_k^{p+\frac{1}{2},q+\frac{1}{2}}.$$

Chapman-Enskog expansion of the discrete J:

$$J_k^{p+\frac{1}{2},q+\frac{1}{2}} = J_{k,(0)}^{p+\frac{1}{2},q+\frac{1}{2}} + \tau_2 J_{k,(1)}^{p+\frac{1}{2},q+\frac{1}{2}} + \cdots$$

Inserting the expansion into the semi-discrete equation for J leads to

$$J_{k,(0)}^{p+\frac{1}{2},q+\frac{1}{2}} = 0 \qquad\qquad J_k^{p+\frac{1}{2},q+\frac{1}{2}} = -\frac{\tau_2}{\rho^{p+\frac{1}{2},q+\frac{1}{2}}}\,\partial_k^{p+\frac{1}{2},q+\frac{1}{2}} T^h$$

One finally obtains the following discrete AP heat flux based on the *corner gradient* of T:

$$q_k^{p+\frac{1}{2},q+\frac{1}{2}} = -\tau_2 c_h^2\, \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} T^h$$

Asymptotic preserving property

After some algebra, the semi-discrete equation for A can be rewritten as

$$\partial_t A_{ik}^{p+\frac{1}{2},q+\frac{1}{2}} + \bar{v}_m^{p+\frac{1}{2},q+\frac{1}{2}} \partial_m^{p+\frac{1}{2},q+\frac{1}{2}} A_{ik}^h + A_{im}^{p+\frac{1}{2},q+\frac{1}{2}} \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} v_m^h + \mathcal{O}(\Delta x^2, \Delta y^2) = -\frac{\left|A_{ij}^{p+\frac{1}{2},q+\frac{1}{2}}\right|^{\frac{5}{3}}}{\tau_1} A_{ij}^{p+\frac{1}{2},q+\frac{1}{2}} \mathring{G}_{jk}^{p+\frac{1}{2},q+\frac{1}{2}}.$$

The discrete convective term is of second order of accuracy and can therefore be absorbed in the material derivative

$$\frac{D}{Dt} A_{ik}^{p+\frac{1}{2},q+\frac{1}{2}} + A_{im}^{p+\frac{1}{2},q+\frac{1}{2}} \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} v_m^h + \mathcal{O}(\Delta x^2, \Delta y^2) = -\frac{\left|A_{ij}^{p+\frac{1}{2},q+\frac{1}{2}}\right|^{\frac{5}{3}}}{\tau_1} A_{ij}^{p+\frac{1}{2},q+\frac{1}{2}} \mathring{G}_{jk}^{p+\frac{1}{2},q+\frac{1}{2}}.$$

Asymptotic preserving property

Multiplication with the discrete A^T from the left, and summing the transposed equation multiplied with the discrete A from the right yields

$$\frac{D}{Dt} G_{ik}^{p+\frac{1}{2},q+\frac{1}{2}} + G_{im}^{p+\frac{1}{2},q+\frac{1}{2}} \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} v_m^h + \mathcal{O}(\Delta x^2, \Delta y^2) = -2\frac{\left|A_{ij}^{p+\frac{1}{2},q+\frac{1}{2}}\right|^{\frac{5}{3}}}{\tau_1} G_{ij}^{p+\frac{1}{2},q+\frac{1}{2}} \mathring{G}_{jk}^{p+\frac{1}{2},q+\frac{1}{2}}.$$

Chapman-Enskog expansion of the discrete solution for the metric tensor G:

$$G_{ij}^{p+\frac{1}{2},q+\frac{1}{2}} = G_{ij,(0)}^{p+\frac{1}{2},q+\frac{1}{2}} + \tau_1 G_{ij,(1)}^{p+\frac{1}{2},q+\frac{1}{2}} + \cdots$$

and some algebra yield the desired quasi AP result, which is the compressible Navier-Stokes stress tensor computed using discrete corner gradients:

$$\sigma_{ik}^{p+\frac{1}{2},q+\frac{1}{2}} = \frac{1}{6}\rho_0 c_s^2 \tau_1 \left(\partial_i^{p+\frac{1}{2},q+\frac{1}{2}} v_k^h + \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} v_i^h - \frac{2}{3}\delta_{ik}\, \partial_m^{p+\frac{1}{2},q+\frac{1}{2}} v_m^h \right) + \mathcal{O}(\Delta x^2, \Delta y^2)$$

What about numerical viscosity?

In order to introduce a *compatible numerical viscosity* into our scheme, we need to recall the definition of the vector Laplacian

$$\nabla^2 \mathbf{J} = \nabla \left(\nabla \cdot \mathbf{J} \right) - \nabla \times \nabla \times \mathbf{J}$$

On the discrete level, we can define a *second* discrete divergence as

$$\begin{aligned} \nabla^{p,q} \cdot \mathbf{J}^{h,n} &= \partial_k^{p,q} J_k^{h,n} = \\ & \frac{1}{2} \frac{J_1^{p+\frac{1}{2},q+\frac{1}{2},n} + J_1^{p+\frac{1}{2},q-\frac{1}{2},n} - J_1^{p-\frac{1}{2},q+\frac{1}{2},n} - J_1^{p-\frac{1}{2},q-\frac{1}{2},n}}{\Delta x} + \\ & \frac{1}{2} \frac{J_2^{p+\frac{1}{2},q+\frac{1}{2},n} + J_2^{p-\frac{1}{2},q+\frac{1}{2},n} - J_2^{p+\frac{1}{2},q-\frac{1}{2},n} - J_2^{p-\frac{1}{2},q-\frac{1}{2},n}}{\Delta y} \end{aligned}$$

Then the above definition of the vector Laplacian also holds at the discrete level:

$$\nabla^{p+\frac{1}{2},q+\frac{1}{2}} \cdot \nabla^h \mathbf{J}^{h,n} = \nabla^{p+\frac{1}{2},q+\frac{1}{2}} \left(\nabla^h \cdot \mathbf{J}^{h,n} \right) - \nabla^{p+\frac{1}{2},q+\frac{1}{2}} \times \nabla^h \times \mathbf{J}^{h,n}$$

What about numerical viscosity?

Since the discrete vector Laplacian satisfies

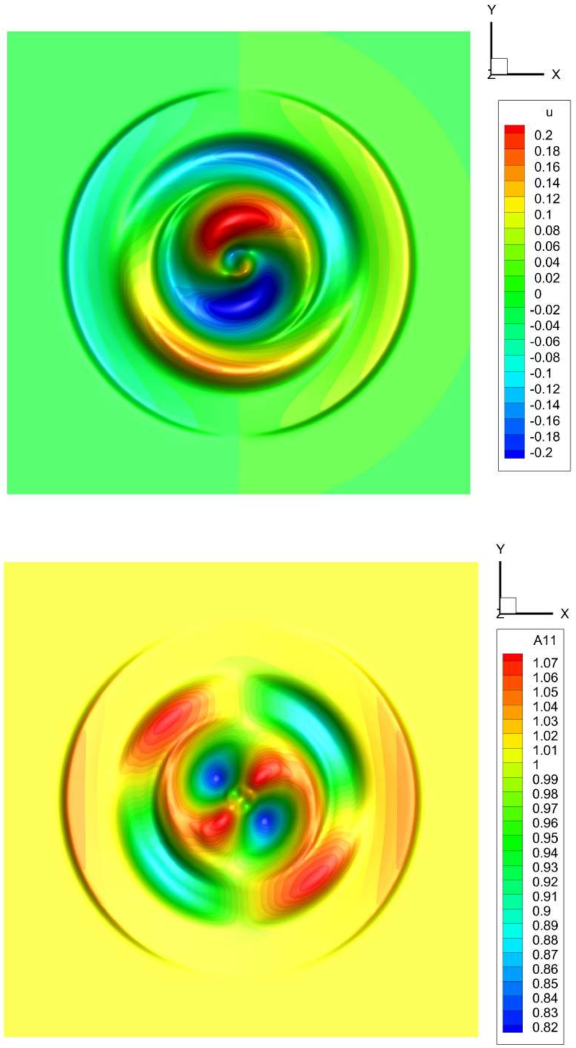
$$\nabla^{p+\frac{1}{2},q+\frac{1}{2}} \cdot \nabla^h \mathbf{J}^{h,n} = \nabla^{p+\frac{1}{2},q+\frac{1}{2}} \left(\nabla^h \cdot \mathbf{J}^{h,n} \right) - \nabla^{p+\frac{1}{2},q+\frac{1}{2}} \times \nabla^h \times \mathbf{J}^{h,n}$$

and in order to keep the discretely curl-free structure, we need to introduce the numerical viscosity via a discrete analogue of the above definition as follows:

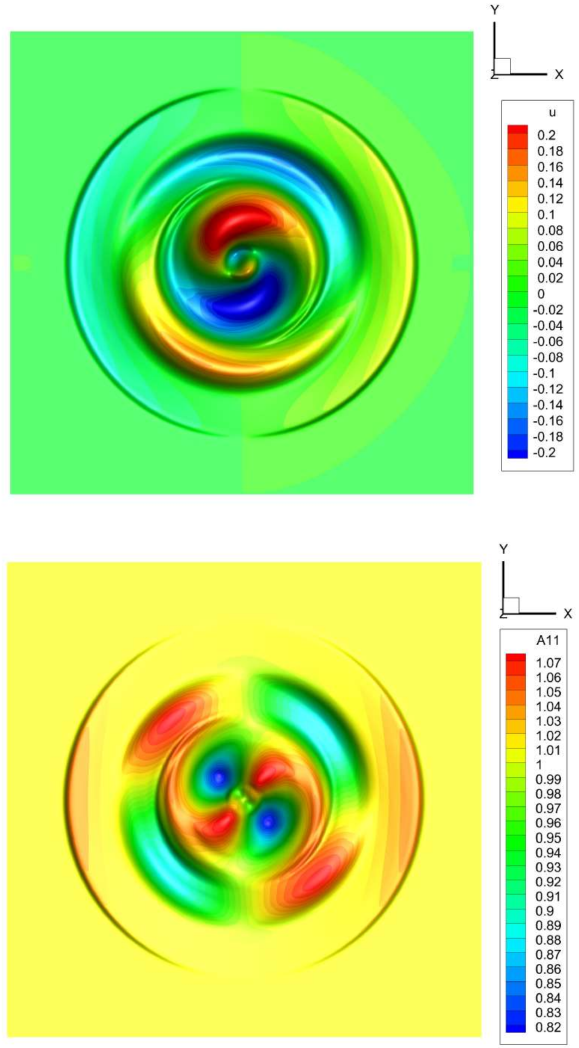
$$\begin{aligned} J_k^{p+\frac{1}{2},q+\frac{1}{2},n+1} \quad = \quad & J_k^{p+\frac{1}{2},q+\frac{1}{2},n} - \Delta t \, \partial_k^{p+\frac{1}{2},q+\frac{1}{2}} \left(J_m^{h,n} v_m^{h,n} + T^{h,n} - h c_h \partial_k^h J_k^{h,n} \right) \\ & + \Delta t h c_h \epsilon_{ki3} \partial_i^{p+\frac{1}{2},q+\frac{1}{2}} \epsilon_{3lm} \partial_l^h J_m^{h,n} \\ & - \Delta t \frac{1}{4} \sum_{r=0}^{1} \sum_{s=0}^{1} v_m^{p+r,q+s,n} \left(\partial_m^{p+r,q+s} J_k^{h,n} - \partial_k^{p+r,q+s} J_m^{h,n} \right) \\ & - \Delta t \frac{\rho^{p+\frac{1}{2},q+\frac{1}{2},n} T^{p+\frac{1}{2},q+\frac{1}{2},n}}{\tau_2} J_k^{p+\frac{1}{2},q+\frac{1}{2},n+1}. \end{aligned}$$

If the field J was initially curl-free, even with the dissipative terms it will still remain curl-free for all times, if the source term vanishes ($\tau_2 = \infty$).

Solid rotor problem

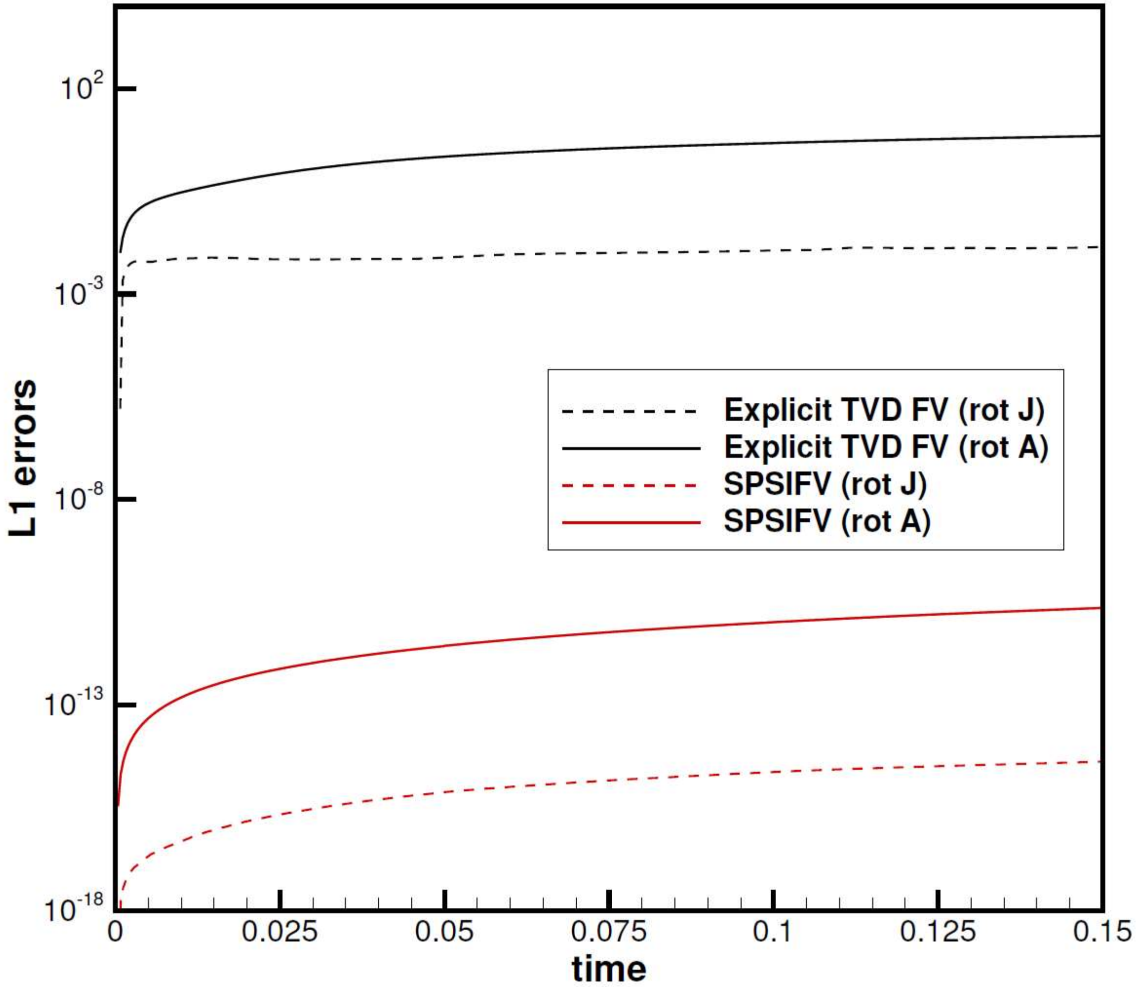


SPSIFV



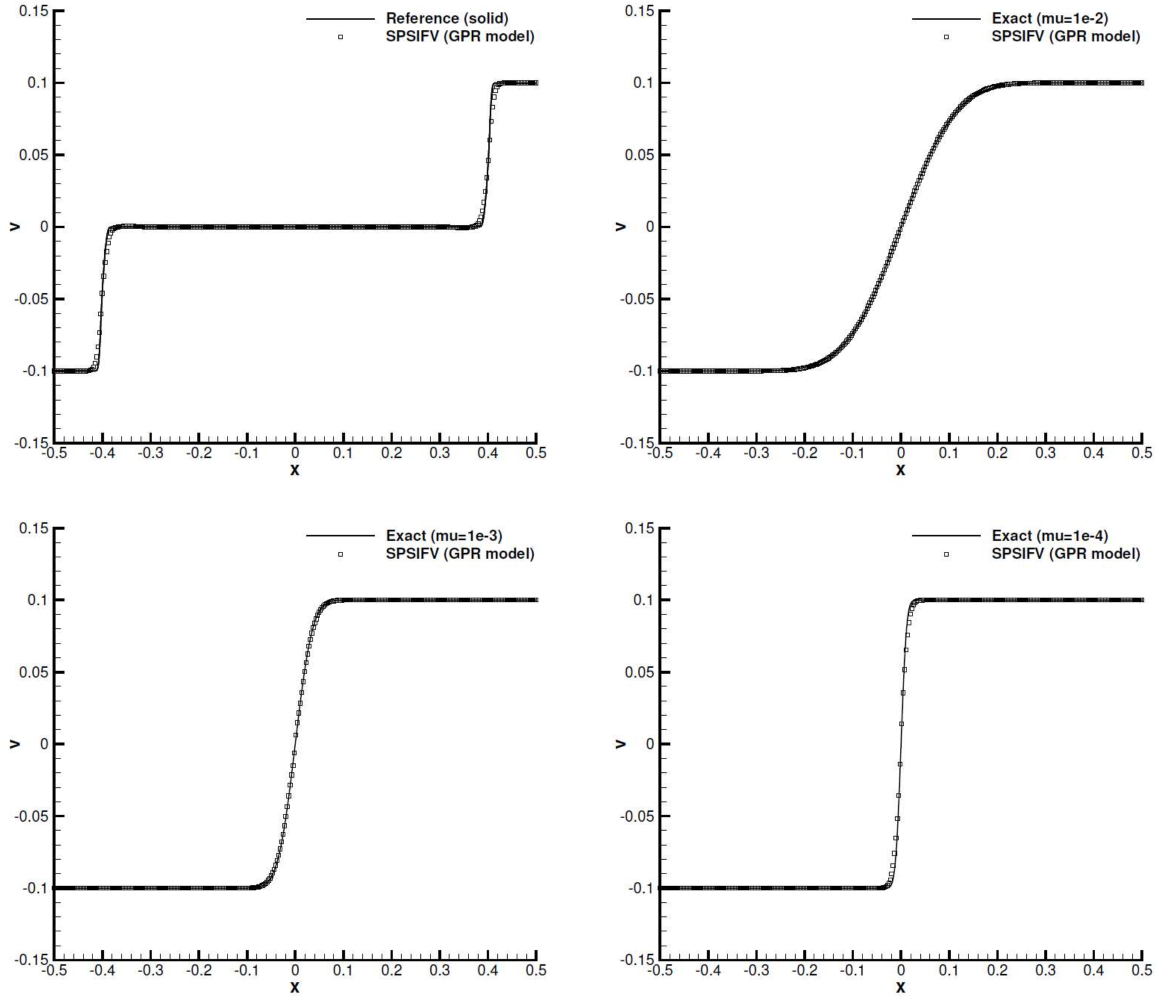
reference solution (MUSCL TVD)

Solid rotor problem



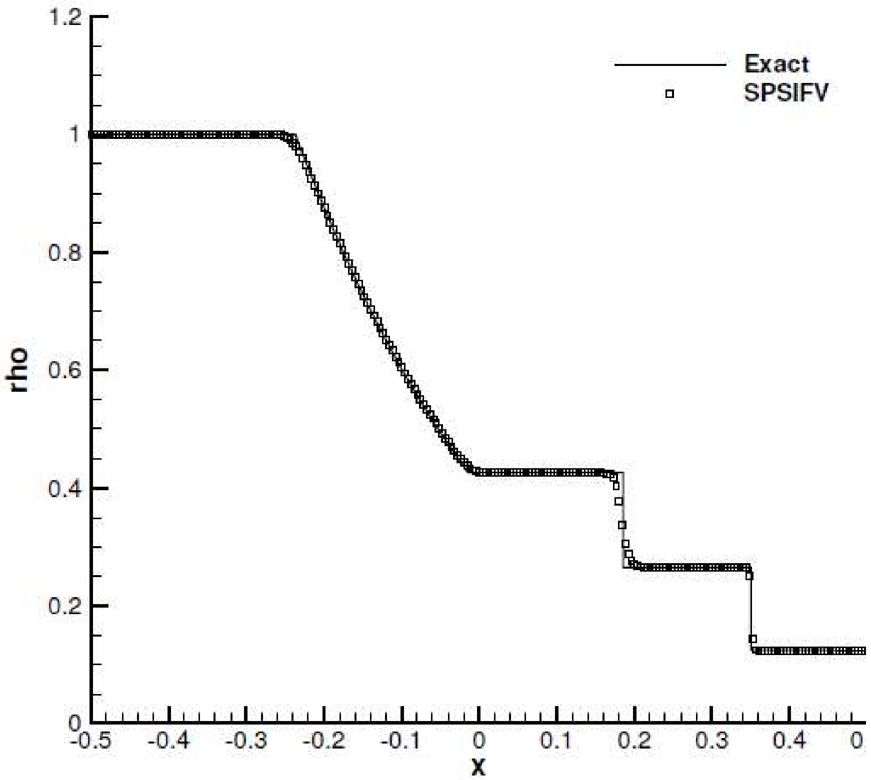
Temporal evolution of the curl errors in J and A
With a standard scheme and the new SPSIFV method

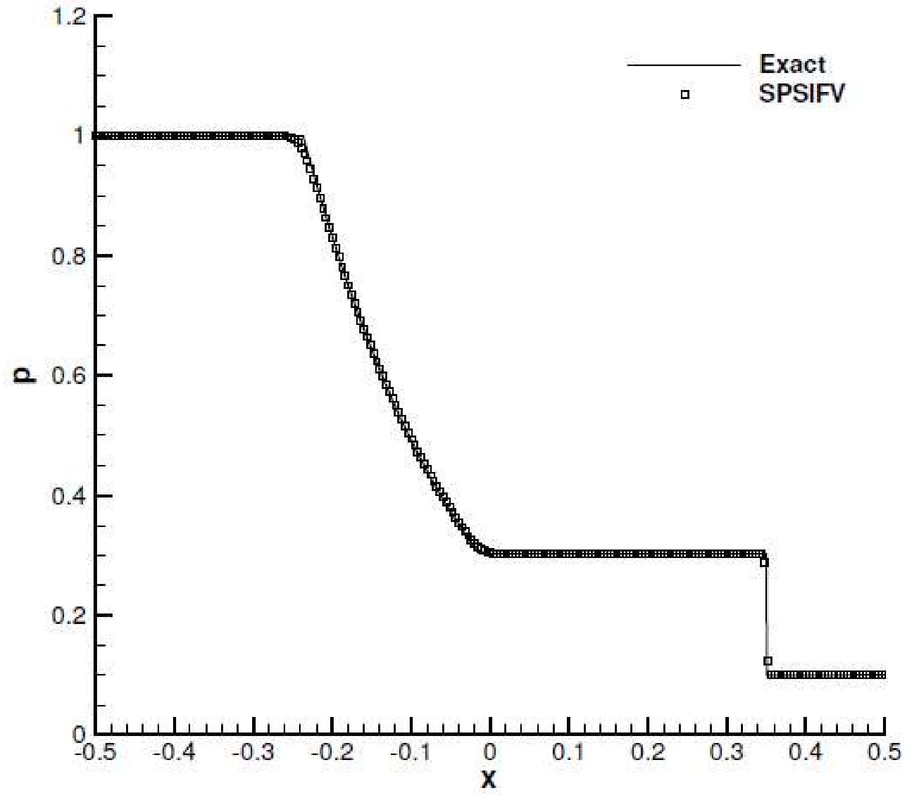
Simple shear flow



Simple shear flow for solid (top left) and fluids with decreasing viscosity

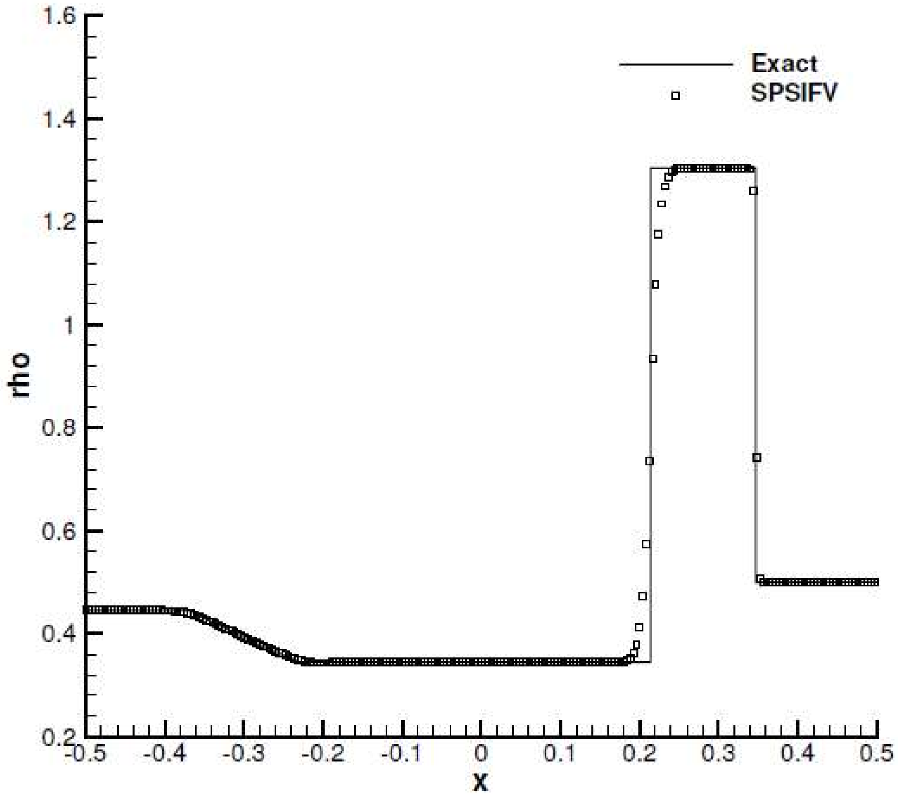
Riemann Problems

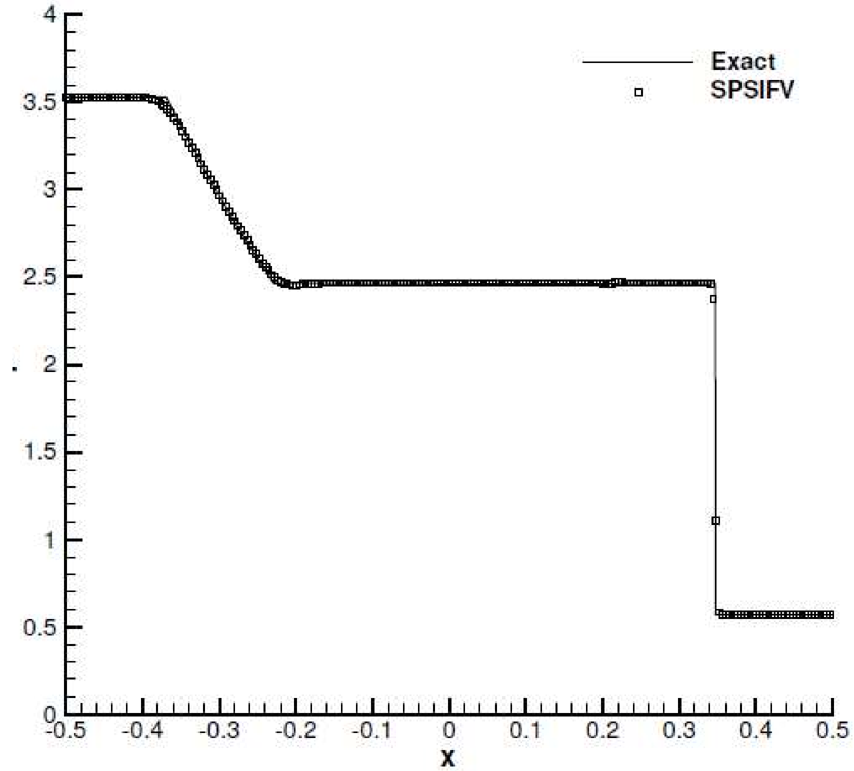




Sod shock tube problem

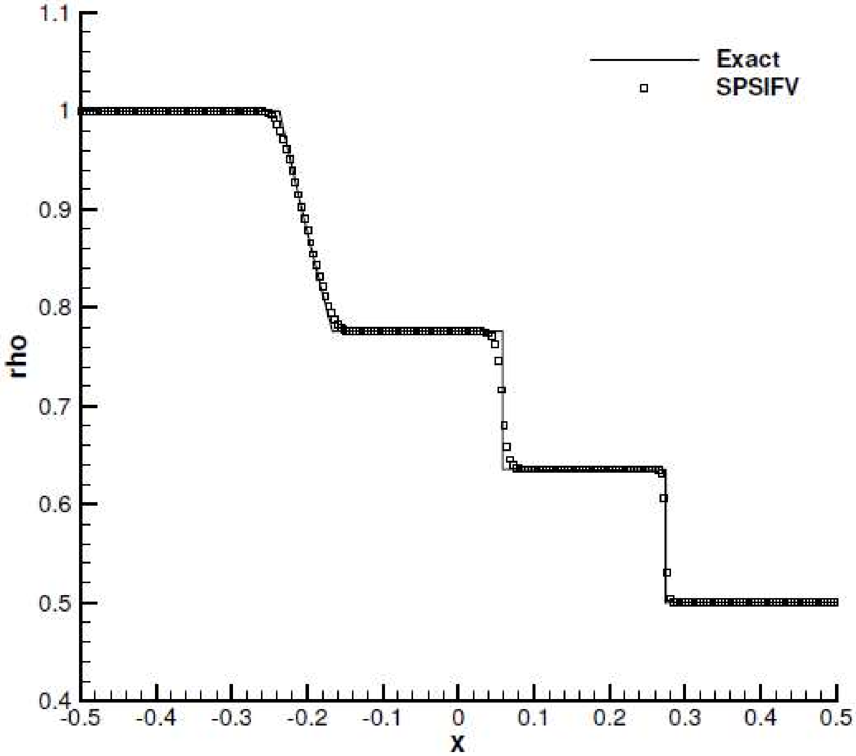
Riemann Problems

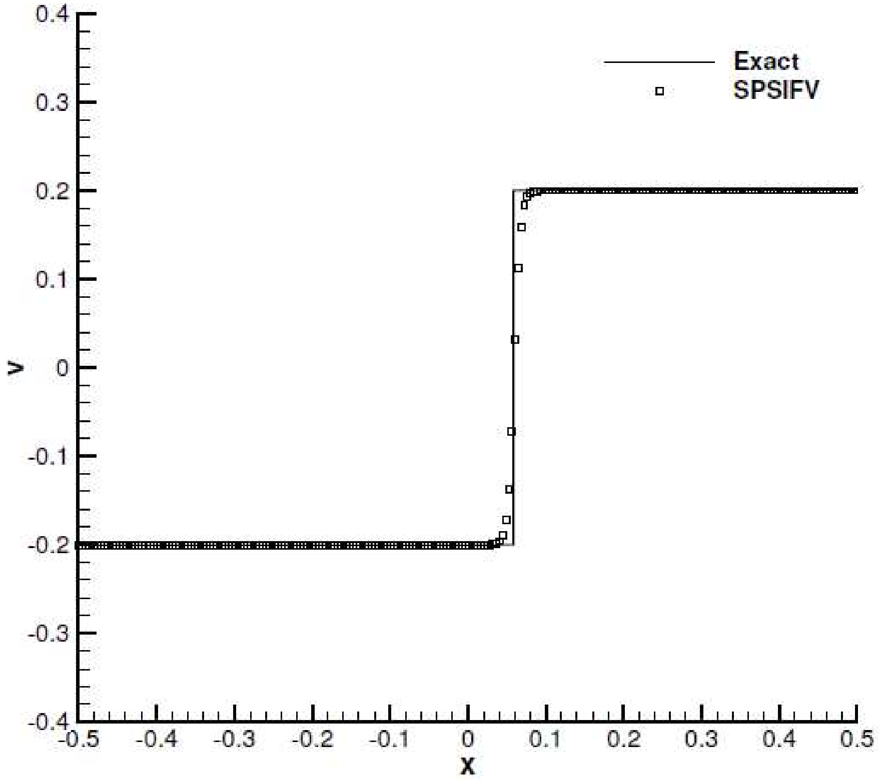




Lax shock tube problem

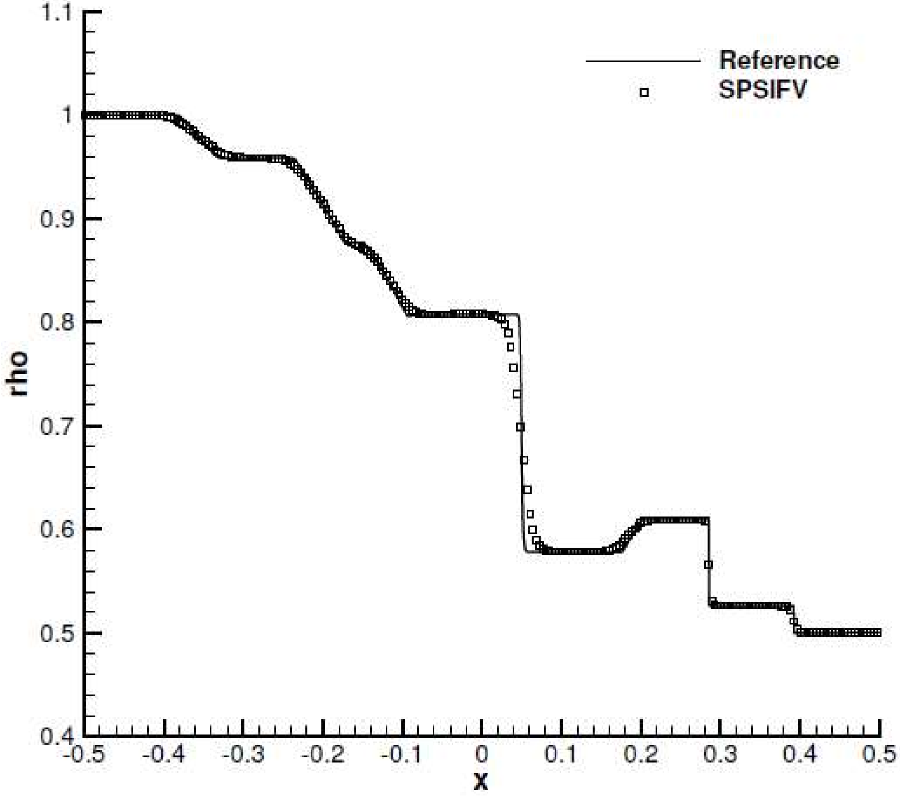
Riemann Problems

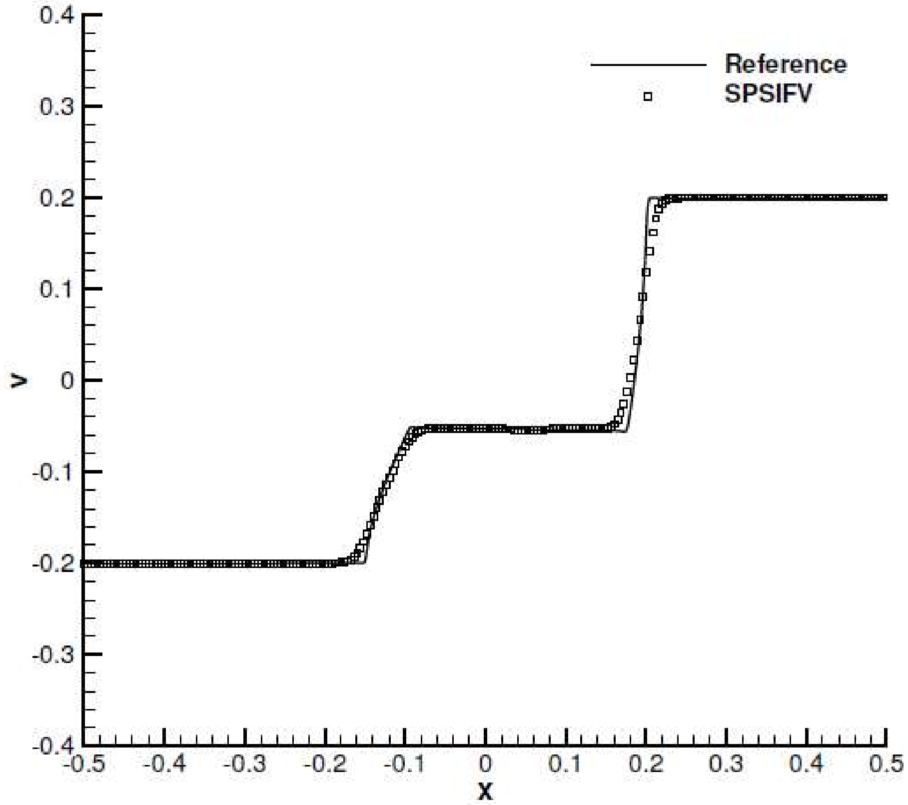




Riemann problem with shear (fluid limit)

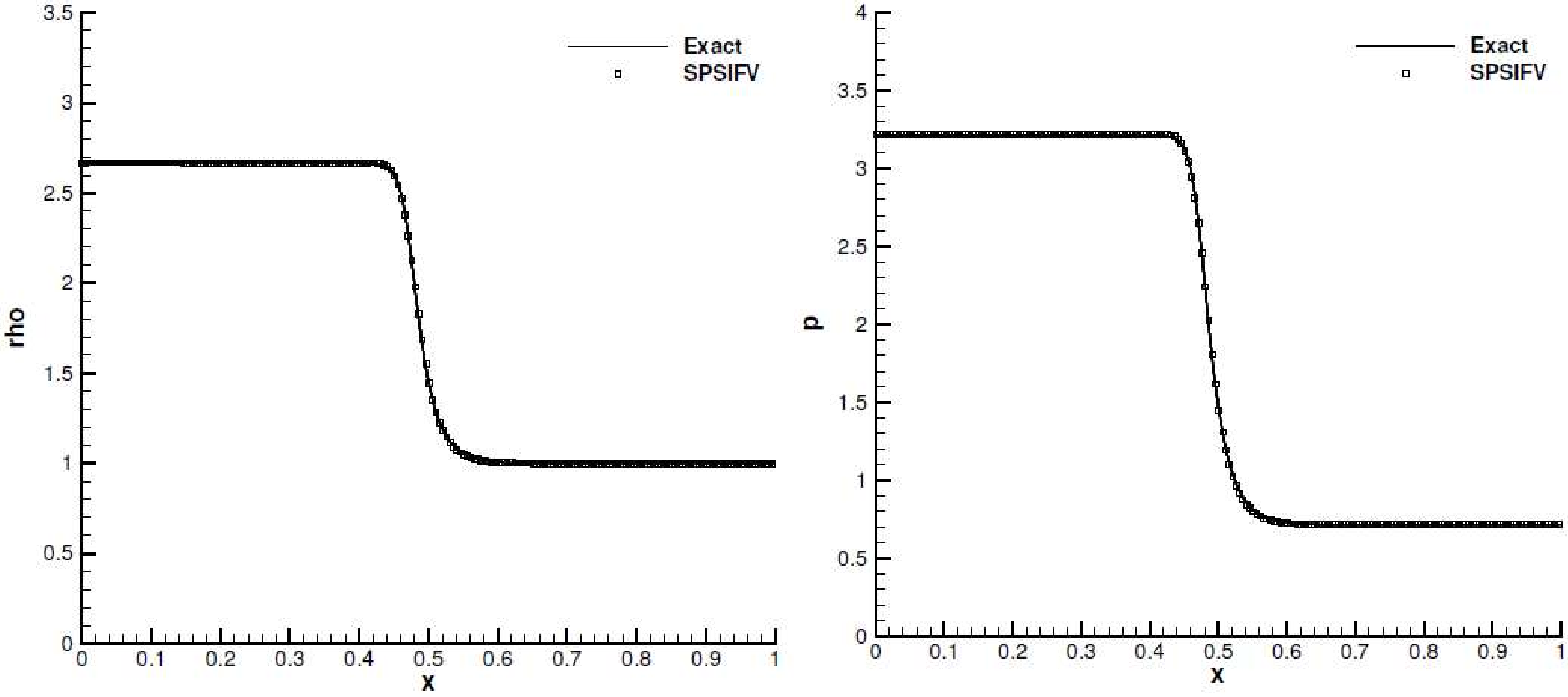
Riemann Problems





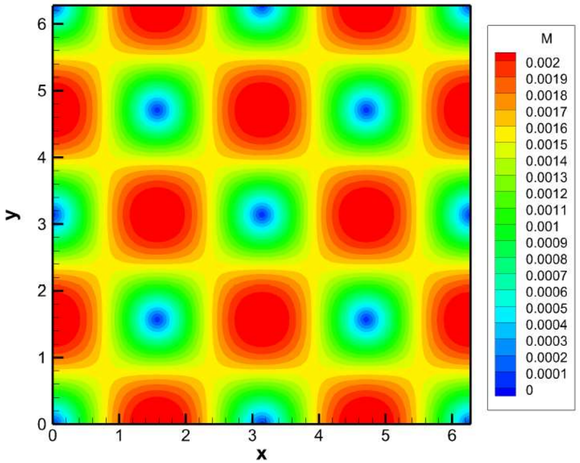
Riemann problem with shear (solid limit)

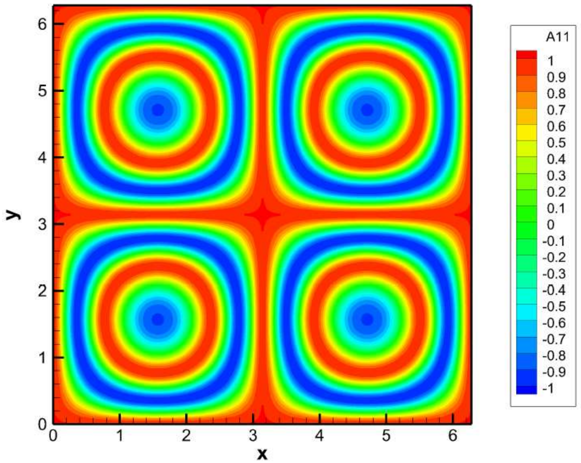
Stationary viscous shock wave

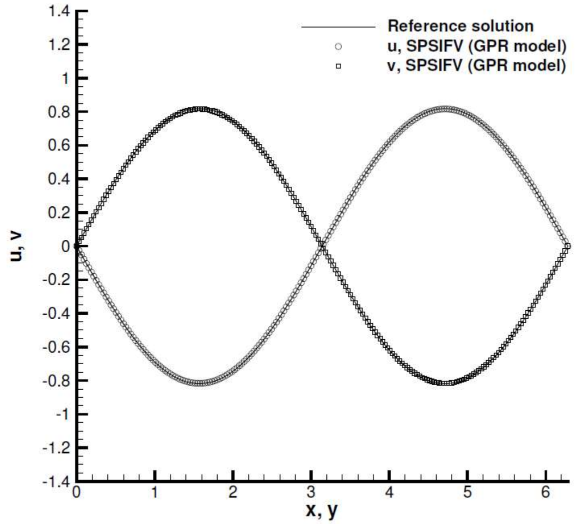


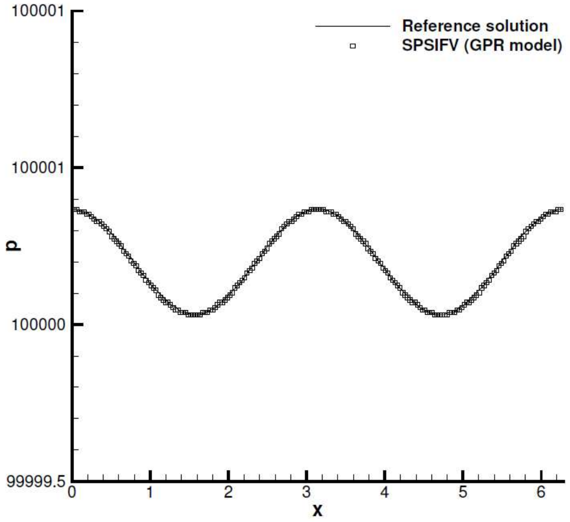
Exact solution of the full compressible Navier-Stokes equations [Becker 1923] for Pr=0.75 taken as reference solution for the GPR model

2D Taylor-Green vortex









2D Taylor-Green vortex

$N_x = N_y$	$L_1(u, 0.2)$	$\mathcal{O}(u)$	$L_2(u, 0.2)$	$\mathcal{O}(u)$	$L_\infty(u, 0.2)$	$\mathcal{O}(u)$
25	9.9081E-03		1.2214E-02		2.4335E-02	
50	2.1079E-03	2.2	2.5905E-03	2.2	5.1097E-03	2.3
100	5.1362E-04	2.0	6.2444E-04	2.1	1.1784E-03	2.1
200	1.3577E-04	1.9	1.6400E-04	1.9	2.9661E-04	2.0

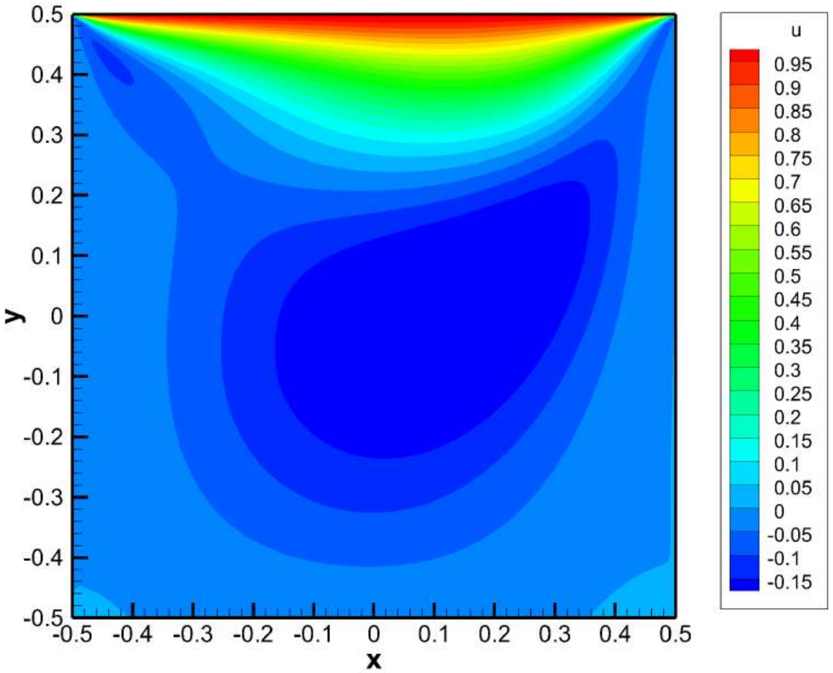
Numerical convergence table for the 2D Taylor-Green vortex

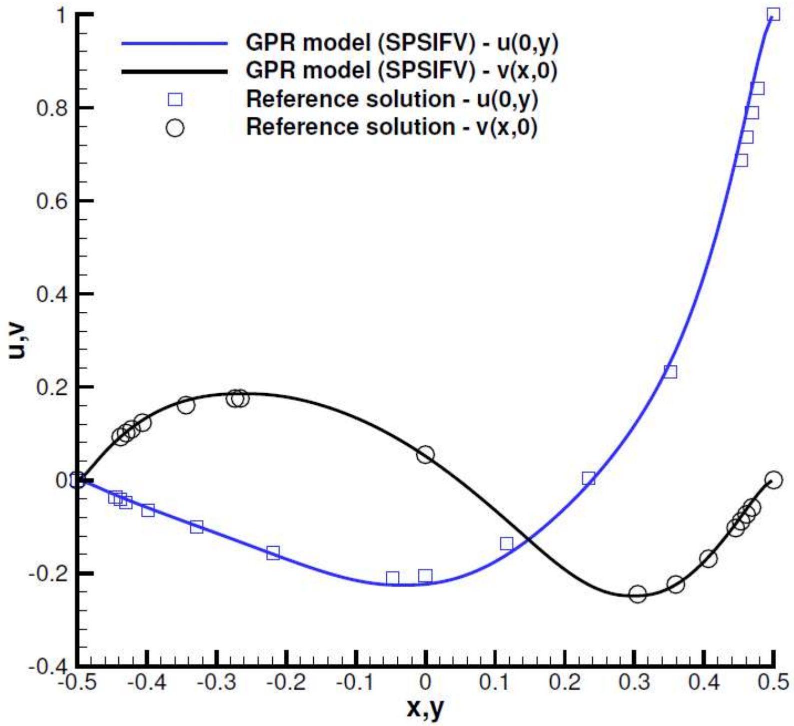
2D Taylor-Green vortex

P_0	M	$L^\infty(\rho)$	$L^\infty(\nabla \cdot \mathbf{v})$	$\mathcal{O}_M(\rho)$	$\mathcal{O}_M(\nabla \cdot \mathbf{v})$
1.00E+03	2.67E-02	5.30E-05	7.55E-06		
1.00E+04	8.45E-03	5.29E-06	7.00E-07	2.0	2.1
1.00E+05	2.67E-03	5.29E-07	7.00E-08	2.0	2.0
1.00E+06	8.45E-04	5.29E-08	7.00E-09	2.0	2.0
1.00E+07	2.67E-04	5.29E-09	7.00E-10	2.0	2.0
1.00E+08	8.45E-05	5.29E-10	7.00E-11	2.0	2.0
1.00E+09	2.67E-05	5.29E-11	7.00E-12	2.0	2.0
1.00E+10	8.45E-06	5.29E-12	7.00E-13	2.0	2.0
1.00E+11	2.67E-06	5.29E-13	7.00E-14	2.0	2.0
1.00E+12	8.45E-07	5.29E-14	7.00E-15	2.0	2.0
1.00E+13	2.67E-07	5.29E-15	7.00E-16	2.0	2.0
1.00E+14	8.45E-08	5.29E-16	7.00E-17	2.0	2.0
1.00E+15	2.67E-08	5.29E-17	7.00E-18	2.0	2.0
1.00E+16	8.45E-09	5.28E-18	6.96E-19	2.0	2.0

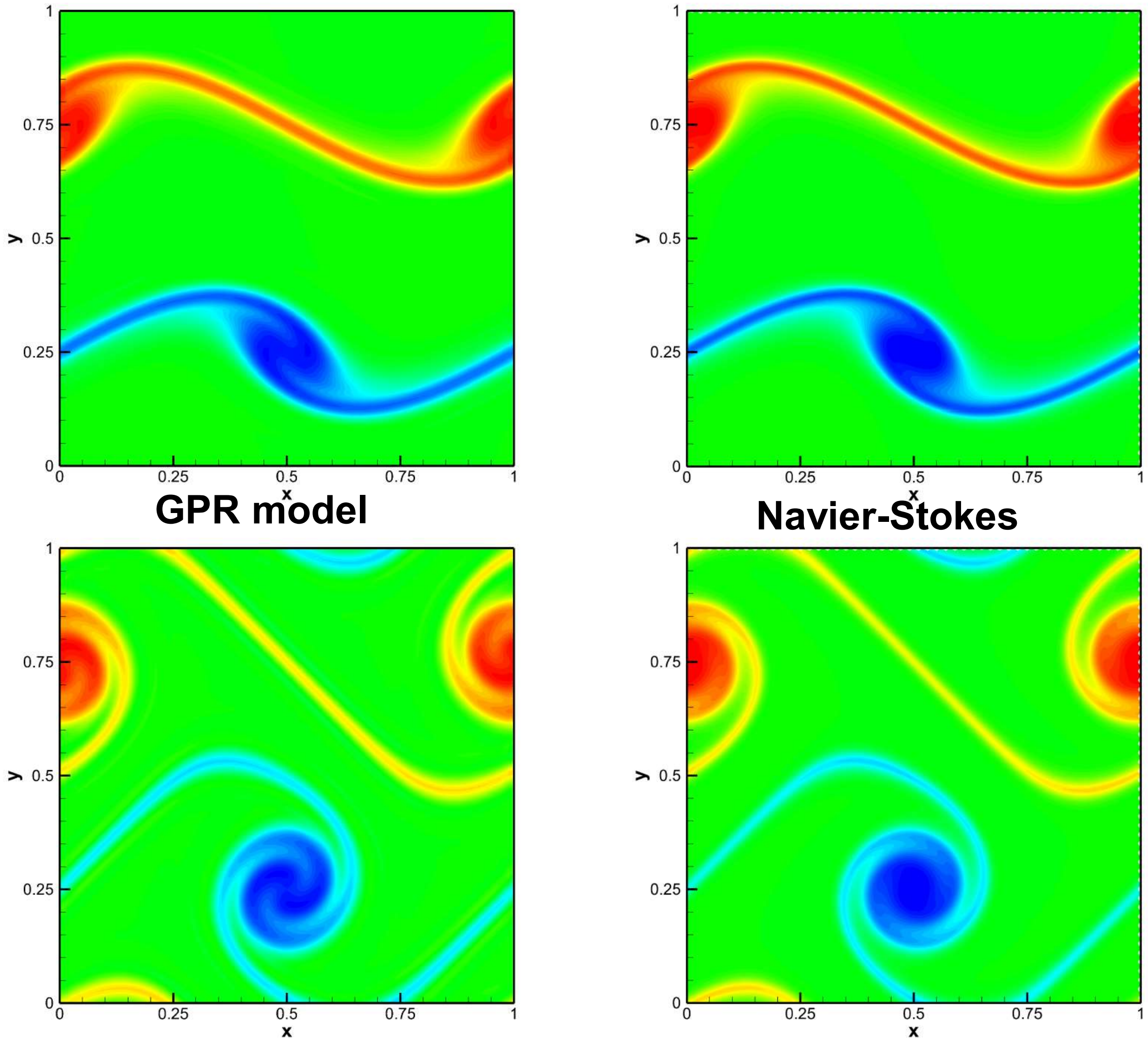
Very low Mach number limit

Lid-driven cavity flow at Re=100 and M = $2\cdot10^{-3}$

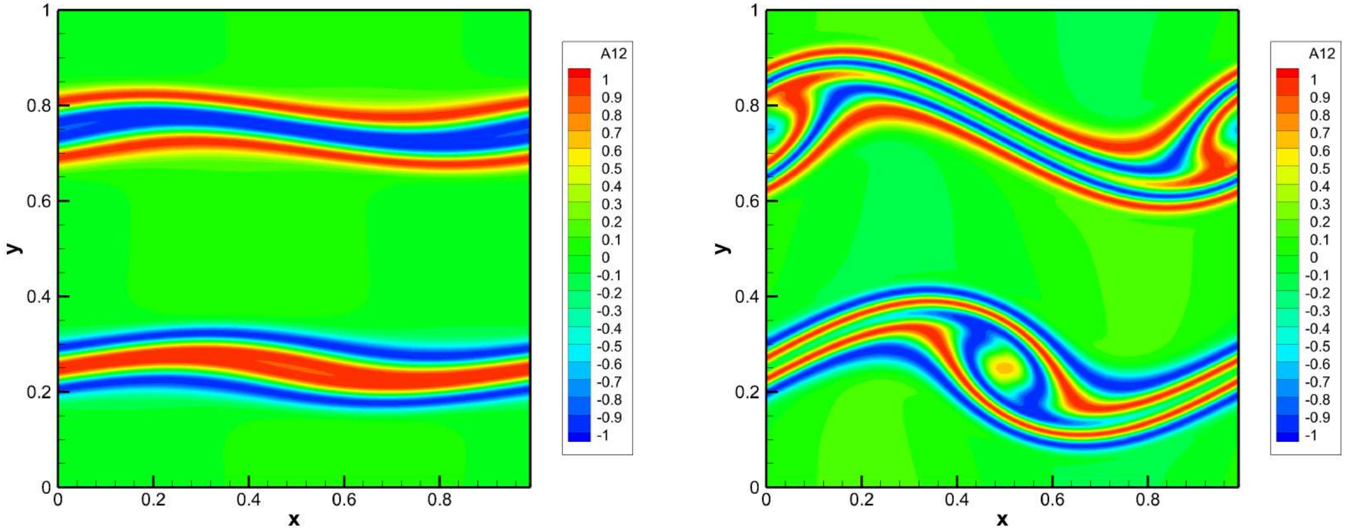




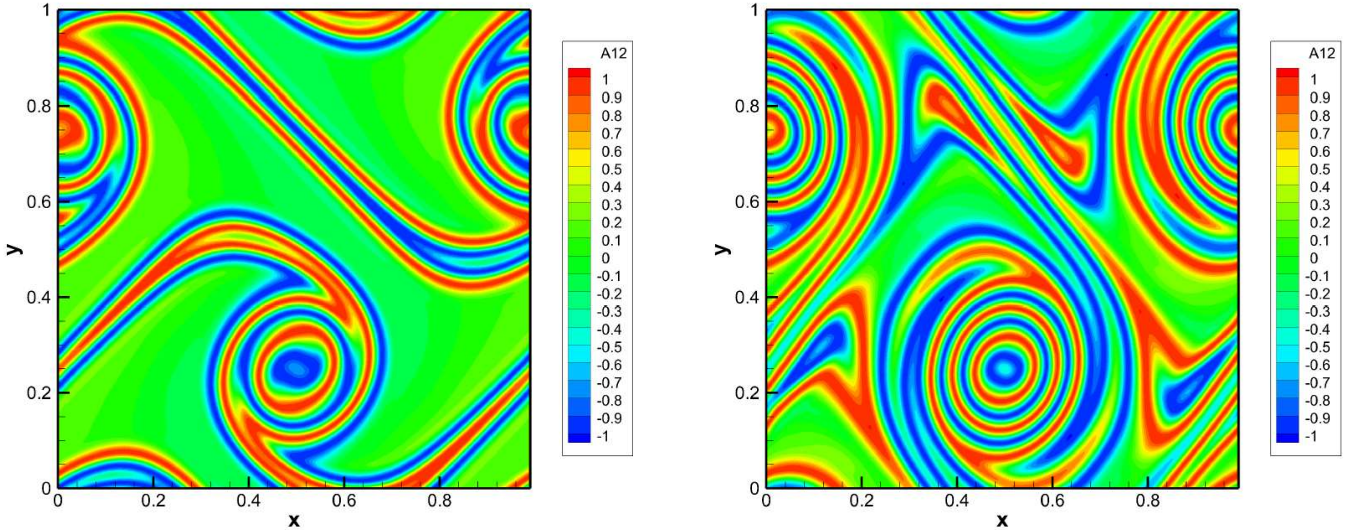
Double Shear-Layer at low Mach number ($M = 2 \cdot 10^{-3}$)



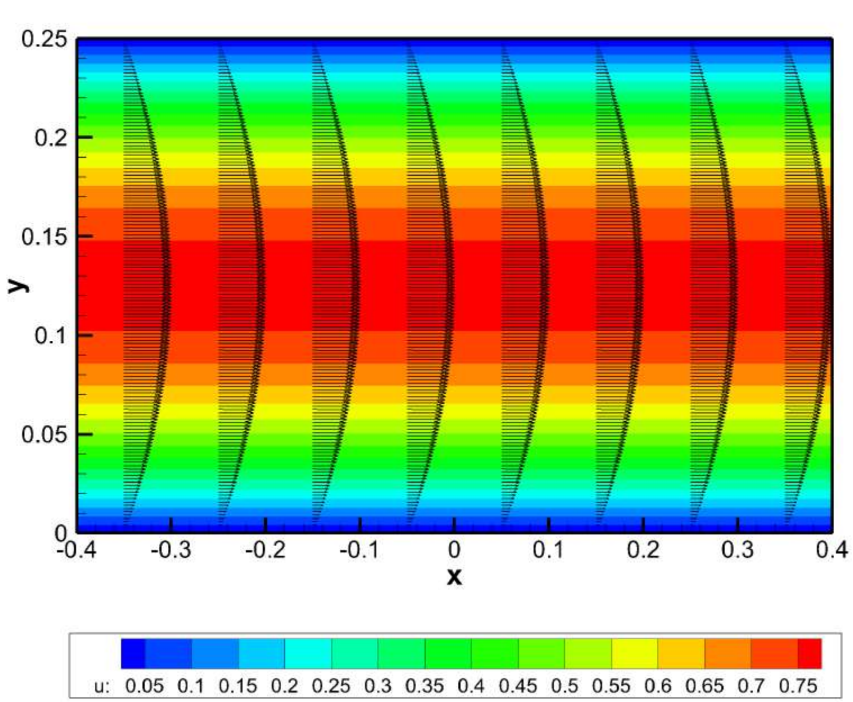
Double Shear-Layer

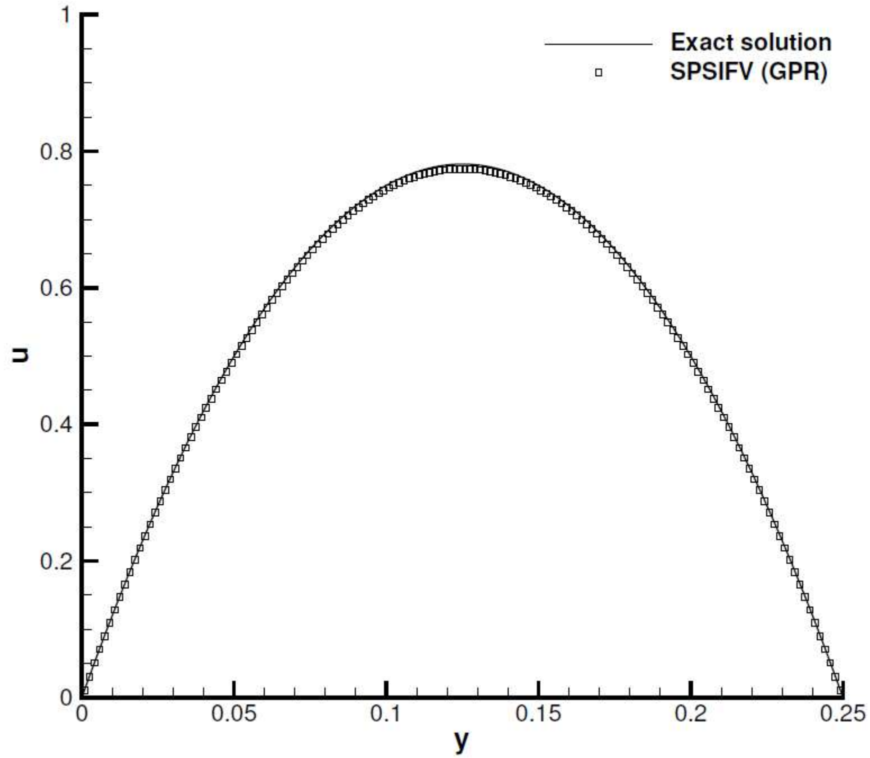


distorsion field component A_{12}



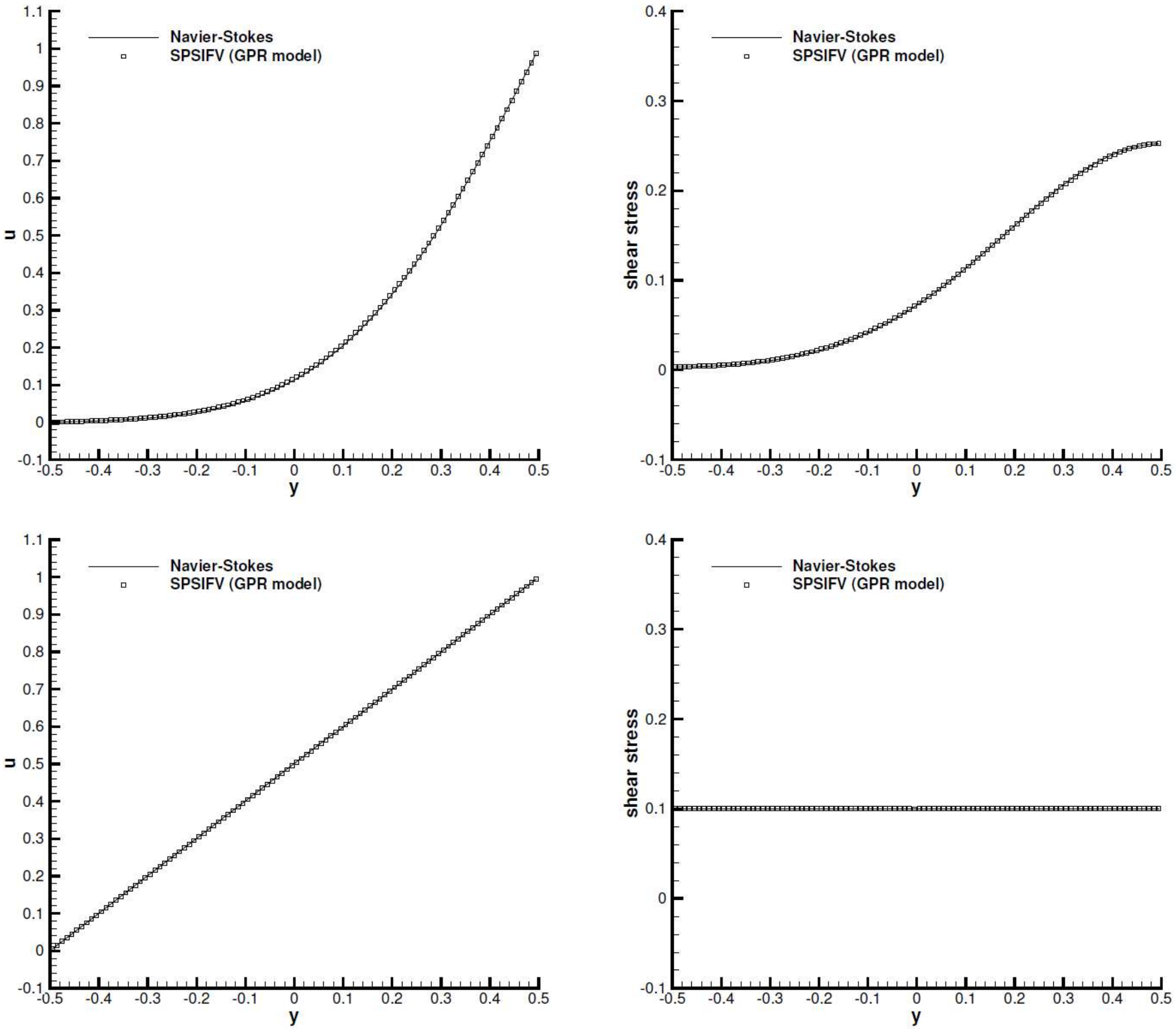
Hagen-Poiseuille flow





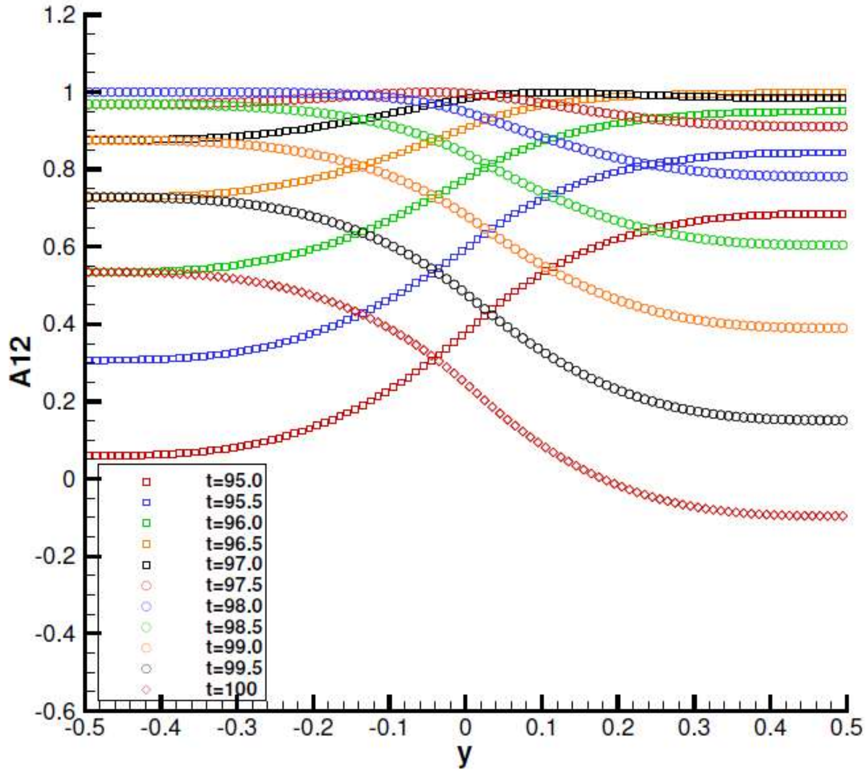
Velocity profile compared with the exact solution of the incompressible Navier-Stokes equations

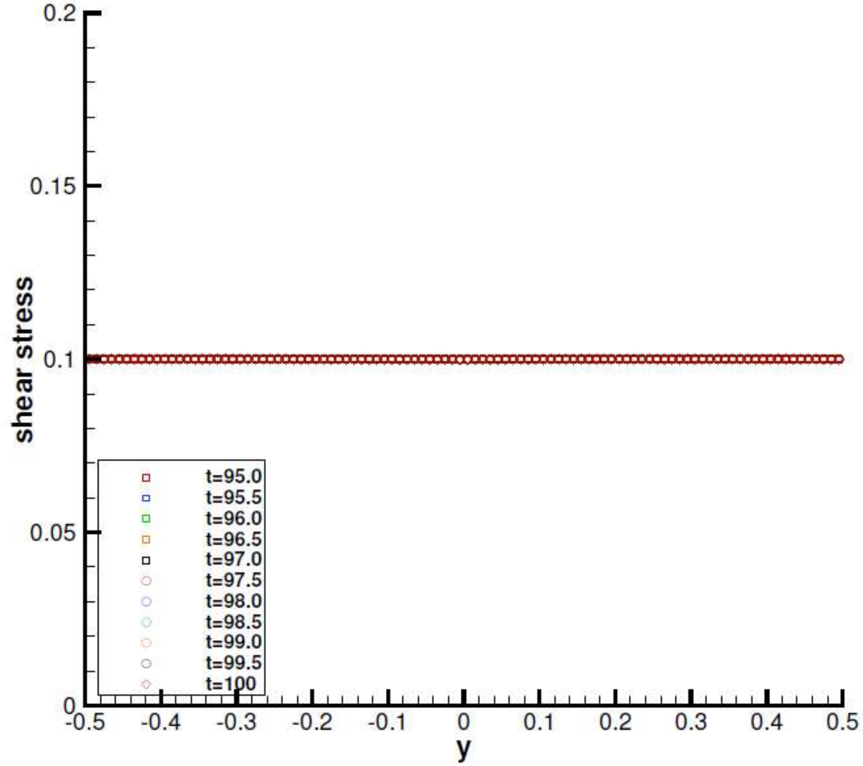
Couette flow Re=10 and M = $1 \cdot 10^{-2}$



velocity **stress**

Couette flow Re=10 and M = $1\cdot10^{-2}$





unsteady distortion field A

steady stress tensor

Conclusions

- First order **hyperbolic** model of Newtonian continuum mechanics (GPR model), including viscous Newtonian fluids as well as elastic and visco-plastic solids in a single, universal formulation with **finite** wave speeds!
- Model formulation is based on the main work of **Godunov** and **Romenski** on **symmetric hyperbolic** & **thermodynamically compatible (SHTC) systems**
- Generalized to viscous fluids for the first time by **Peshkov** & **Romenski** in 2014
- PDE system does not change type whether dissipative terms are present or not
- All dissipative processes have the **same structure** as the **Ohm law**
- **First compatible scheme for the GPR model that is asymptotic preserving for the stiff limit (fluids), curl-free for the homogeneous system (solids) and that is at the same time able to deal with both high and low Mach numbers**
- **Future work: extension to high order DG schemes, extension to non-Newtonian fluids and to continuum mechanics with electro-magnetic fields**

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