On well-posedness for a multi-particle-fluid model

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Abstract In this paper we study a one dimensional fluid modeled by the Burgers equation influenced by an arbitrary but finite number of particles $N(t)$ moving inside the fluid, each one acting as a point-wise drag force with a particle related friction constant λ. For given particle paths $h_i(t)$ we only assume finite speed of particles, allowing for crossing, merging and splitting of particles. This model is an extension of existing models for fluid interactions with a single particle, compare [3] and [10]:

$$\partial_t u(x,t) + \partial_x \left(\frac{u^2}{2}\right) = \sum_{i=1}^{N} \lambda (h_i'(t) - u(t, h_i(t)))\delta(x - h_i(t))$$

Well-posedness for the Cauchy problem, as well as an L^∞ bound is proven under the weak assumption that particle paths are Lipschitz continuous. In this context, an entropy admissibility criteria is shown, using the theory of L^1-dissipative Germs, compare [2], to deal with the moving interfaces resulting from the point-wise particles and the shockwaves from the fluid equation interacting with them.

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1 Introduction

We consider an inviscid fluid with velocity $u(t,x)$ and a finite number of particles moving inside. The fluid is modeled by the inviscid Burgers equation and the particles act as a point-wise drag force on the fluid, namely $\lambda(h_i'(t) - u(t,h_i(t))$, where λ is the friction constant related to the particle and $h_i(t)$ the given path of the i-th particle. The Cauchy problem writes

$$\begin{aligned} \partial_t u + \partial_x(u^2/2) &= \sum_{i=1}^{N} \lambda(h_i'(t) - u(t,h_i(t))\delta(x-h_i(t)), \\ u(0,x) &= u_0(x) \end{aligned} \tag{1}$$

with

$u(x,t)$	velocity of the one-dimensional fluid
$h_i(t)$	the given position of the i-th particle at time t
λ	the friction constant corresponding to a particle
$N(t)$	arbitrary but finite number of particles at time t
$u_0 \in L^\infty(\mathbb{R})$	the given L^∞ initial data for the fluid

Note that this model also bears the difficulty of interpretating the non-conservative product $u(t,h_i(t))\delta(x-h_i(t))$. This problem was tackled in [3] by a regularization of the particle, using sequences of non-negative, compactly supported density functions (see also [7] for a similar approach). However, an analysis of the behaviour of the fluid at the position of the particle allows for a well-posedness proof considering the influence of the particle as a condition on the behaviour of the fluid at a moving interface located at the particle position. The theory extends the analysis of the fluid-solid interaction of [10], [3], where the original model also includes coupling to an ordinary differential equation, to the case of multiple particles. Models of this kind are of increasing interest theoretically, cf. [4], aswell as in applications like trajectory tracking in traffic flow, cf. [5], [6].

We proceed in the following way. In section 2, we give an admissibility condition for the selection of physical shockwaves and therefore a definition of entropy solutions to the problem. At the end of section 2, we will state the main theorem, which is the well-posedness result for problem (1) and a L^∞ bound. Section 3 and 4 give the proof to this theorem, where section 3 contains the existence proof aswell as the L^∞ bound and section 4 is devoted to the uniqueness proof using almost classical Kruzkov-type arguments combinded with the notion of Germs, i.e. sets of admissible states connected by shockwaves, first introduced in [2].

2 Definition of entropy solutions

The behaviour of a solution across one particle is dictated by the drag of the particle. However, there might also be shockwaves originating from the fluid equation. A traveling-wave study with respect to the particle speed was done in [10] regarding a single particle and acts as a building block for the analysis of the behaviour in the case of multiple particles. It is proven in [10] that the following definition of sets describes the admissible jumps across the interface of a single particle.

Definition 1. Let $\mathscr{G}_\lambda$ be the set of possible states left and right of a particle with friction λ. A case-by-case study with respect to u_l, u_R, h gives the characterization

$$(u_L, u_R) \in \mathscr{G}_\lambda$$
$$\Leftrightarrow u_R \in \begin{cases} \{u_L - \lambda\} & \text{if } u_L < h', \\ [2h' - u_L - \lambda, h'] & \text{if } h' \le u_L \le h' + \lambda, \\ \{u_L - \lambda\} \cup [2h' - u_L - \lambda, 2h' - u_L + \lambda] & \text{if } u_L > h' + \lambda. \end{cases}$$

Increasing the number of particles means that the behaviour of the fluid at each particle is governed by an interface admissibility condition $\mathscr{G}_i = \mathscr{G}_{\lambda_i}$, which impose to traces of the solution at the left and right of each particle lie in $\mathscr{G}_\lambda$. Thus we are able to define entropy admissible solutions to the problem as long as the particle paths do not intersect using the notion of admissible particle-related jumps and the notion of Kruzkov entropy η, entropy flux Φ, defined by

$$\eta(a,c) = |a - c|$$
$$\Phi(a,c) = \operatorname{sgn}(a-c)(f(a) - f(c)),$$

which enable comparison to any constant $c \in \mathbb{R}$.

Definition 2. Given $u_0 \in L^\infty, N > 0, h_i(t) \in W^{1,\infty}([0,T]), h_i(t) \neq h_j(t) \forall t \in [0,T], i \neq j$. We call $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ weak entropy solution to the Cauchy Problem, if for $N \in \mathbb{N}$ the finite number of particles, $h_i(t)$ the position and $h_i'(t)$ the velocity of particle i, with $i = 1, \dots, N$, u satisfies for all $c \in \mathbb{R}$ and almost every time t

$$\int_0^T \int_{\mathbb{R}} |u - c| \partial_t \phi + \Phi(u,c) \partial_x \phi)\, dxdt + \int_{\mathbb{R}} |u_0 - c| \phi(0,x) dx \ \ge\ 0 \tag{2}$$

with $\phi \in C^\infty([0,T] \times \mathbb{R}, \mathbb{R}^+), \phi(t, h_i(t)) = 0$, and additionally

$$(\gamma_u^-(t, h_i(t)), \gamma_u^+(t, h_i(t))) \in \mathscr{G}_\lambda(t), \quad \text{for a.e. } t \in (0,T)$$

where we denoted the left an right traces of $u(t,x)$ at the position of the particles by $\gamma_u^-(t,h_i(t)), \gamma_u^+(t,h_i(t))$ respectively. Due to the nature of the Burgers equation, these traces exist a priori, even for L^∞ initial data, cf [11].
Note that whenever two particles are located at the same position, a careful new definition of the particle related Germs has to be taken into account. This is not a

problem for crossing, as the condition is enforced only almost everywhere in time, however if two or more particles merge, the corresponding Germ changes, resulting in the following definition of time-dependent interface-condition

$$\mathscr{G}_\lambda(t) = \mathscr{G}_{n_i(t)\times\lambda}(h_i'(t)), \quad \text{with } n_i(t) := \#\{j \in [0,N], h_i(t) = h_j(t)\}.$$

This definition makes sure that the interface condition really applies the drag of both particles, and does not impose two (maybe contradictory) conditions at the same position. The fact that the influence of the particles adds up like that uses the specific form of the germ $\mathscr{G}_\lambda$, was proven by the authors and can be found in the upcoming publication [8].

Remark 1. The definition of entropy solution is done using the notion of Germs, introduced in [2]. Furthermore the entropy condition can not be distinguished from an entropy condition for a discontinuous flux problem with interfaces located at the particle positions $h_i(t)$, emphasizing the pointwise influence of the particles.

At this point we state our main theorem.

Theorem 1. *Given any finite time T, initial data $u_0(x) \in L^\infty(\mathbb{R})$ and Lipschitz continuous in time particles paths $h_i(t), i \in [1,N]$, then there exists a unique solution $u(t,x) \in L^\infty([0,T]\times\mathbb{R})$, entropy admissble in the sense of Definition 2. Additionally, $u(t,x)$ satisfies for all $t \in [0,T]$*

$$\|u(t,\cdot)\|_{L^\infty} \leq \|u_0(\cdot)\|_{L^\infty} + N\lambda. \tag{3}$$

The proof of this theorem is distributed between the next two sections.

3 Existence

We will prove existence of entropy admissible solutions in the following way. Given initial data $u_0 \in L^\infty(\mathbb{R})$ and any finite time interval $[0,T]$, we divide the problem into several local problems and use the following existence result for the problem with a single particle, which is proven in [3].

Lemma 1. *Given $h \in W^{1,\infty}([0,T])$ and $u_0 \in L^\infty(\mathbb{R})$, then there exists a unique entropy admissible solution u of (1) with $N(t) = 1$.*

Several difficulties arise. Even though the behaviour of the fluid in the presence of a single particle is known, each particle generates waves interfering with the other particles, creating domains of unknown behaviour. Additionally, the possibility of crossing, merging and splitting of particles seem to complicate some of the nice properties that were holding as long as only one particle was present, e.g. the global in time bound on the total variation.
The proof is done using an explicit construction algorithm based on the existence

result in the presence of a single particle, which we will present here for the case of two particles. Note however, that this can be easily extended to any finite number of particles by simply choosing a good timestepping, creating domains where the following analysis applies locally.
At the same time, we will prove the L^∞ bound (3), justifying the existence of a maximum speed of propagation, denoted L from here on, which, though a very natural property of hyperbolic equations, needs to be checked in the presence of source terms. Both the L^∞ bound as well as the existence are constructed using a time-stepping, which ensures that the cones of influence of two particles don't intersect.

Lemma 2. *Given any time $t_i \in [0,T]$, there exists a time $t_{i+1} > t_i$, such that given problem (1) with two particles with particle paths $h_1, h_2 \in Lip([t_i,t_{i+1}])$ with $h_1(t) \neq h_2(t) \in [t_i,t_{i+1}]$ and initial data $u(t_i) \in L^\infty(\mathbb{R})$, then there exists a solution $u(t,x) \in L^\infty([t_i,t_{i+1}] \times \mathbb{R})$, entropy admissible in the sense of (2).*
Additionally, if $u(t_i,x)$ satisfies for $x \in \mathbb{R}$

$$c_{min}(t_i,x) \le u(t_i,x) \le c_{max}(t_i,x),$$

then $u(t,x)$ satisfies for almost every $t \in [t_i,t_{i+1}]$, $x \in \mathbb{R}$

$$c_{min}(t,x) \le u(t,x) \le c_{max}(t,x), \tag{4}$$

with piecewise constant functions

$$c_{min,\,max}(t,x) = \begin{cases} c^1_{min,\,max} & \text{for } x \in \Omega_1(t), t \in [t_i,t_{i+1}] \\ c^2_{min,\,max} & \text{for } x \in \Omega_2(t), t \in [t_i,t_{i+1}] \\ c^3_{min,\,max} & \text{for } x \in \Omega_3(t), t \in [t_i,t_{i+1}] \end{cases}$$

with

$$\begin{aligned} \Omega_1(t) &:= (-\infty, h_1(t)) \\ \Omega_2(t) &:= (h_1(t), h_2(t)) \\ \Omega_3(t) &:= (h_2(t), \infty) \end{aligned}$$

such that for $j = 1,2$

$$c^j_{min,\,max} = c^{j+1}_{min,\,max} + \lambda$$

and $c^{k_1}_{min} = \inf\limits_{\Omega_{k_1}(t_i)} u(t_i,x)$, $c^{k_2}_{max}(t,x) = \sup\limits_{\Omega_{k_2}(t_i)} u(t_i,x)$ with

$$\begin{aligned} k_1 &= \arg\min_{j=1,2,3} \Big\{ \operatorname*{ess\,inf}_{x\in\Omega_1} u(t_i,x), \operatorname*{ess\,inf}_{x\in\Omega_2} \big(u(t_i,x)-\lambda\big), \operatorname*{ess\,inf}_{x\in\Omega_3} \big(u(t_i,x)-2\lambda\big) \Big\} \\ k_2 &= \arg\max_{j=1,2,3} \Big\{ \operatorname*{ess\,sup}_{x\in\Omega_1} u(t_i,x), \operatorname*{ess\,sup}_{x\in\Omega_2} \big(u(t_i,x)-\lambda\big), \operatorname*{ess\,sup}_{x\in\Omega_3} \big(u(t_i,x)-2\lambda\big) \Big\}. \end{aligned}$$

The last statement (4) is actually a stronger result than the L^∞ bound, as (3) follows directly from (4) as soon as it is established for all times $t \in [0,T]$. To see this, it is very important to note that the time dependence of c_{min}, c_{max} is only due to the position of the particles and does not change the values of the two functions, cf. Figure 1.

Proof. To be able to make use of the existing results for the case of a single particle, i.e. Lemma 1, we choose t_{i+1} such that the waves propagating from the two particles can not intersect in $[t_i, t_{i+1}] \times \mathbb{R}$. This is achieved by defining

$$t_{i+1} = t_i + \frac{h_2(t_i) - h_1(t_i) - 2\varepsilon}{2L}.$$

where $L = L(\|u\|_{L^\infty}, h_1', h_2') = \max_{x \in \Omega} (c_{\max}(0,x), -c_{\min}(0,x))$ denotes the finite speed of propagation and $\varepsilon > 0$ can be chosen arbitrarily small. We define the superposition of $[t_i, t_{i+1}] \times \mathbb{R} = B_1 \cup P_1 \cup B_2 \cup P_2 \cup B_3$ such that P_1, P_2 contain the particles and all waves emanating from them.

$$\begin{aligned} P_{1,2}(t) &:= [h_{1,2}(t_{i+1}) - L(t_{i+1} - t), h_{1,2}(t_{i+1}) + L(t_{i+1} - t)] \\ B_1(t) &:= (-\infty, h_1(t_{i+1}) - L(t_{i+1} - t)] \\ B_2(t) &:= [h_1(t_{i+1}) + L(t_{i+1} - t), h_2(t_{i+1}) - L(t_{i+1} - t)] \\ B_3(t) &:= [h_2(t_{i+1}) + L(t_{i+1} - t), \infty). \end{aligned}$$

From the analysis done for a single particle, we know that given $u(t_i, \cdot) \in L^\infty(P_1)$ and given that the solution $u(t,x)$ with $x \in \mathbb{R} \setminus P_1$ in the adjacent regions to P_1 satisfies $c_{\min}(t,x) \le u(t,x) \le c_{\max}(t,x)$, the bounds are also true in P_1 [1], namely

$$c_{\min}(t,x) \le u(t,x) \le c_{\max}(t,x) \quad \text{for } x \in P_1$$

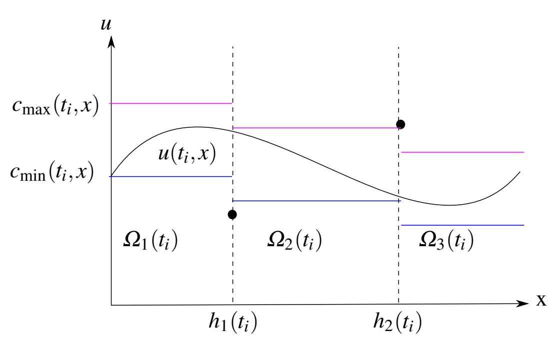

Fig. 1 The boundaries $c_{\min}, c_{\max}$ on the solution in the regions between the particles at time t_i. As time passes, the particles will change their position and the respective bounds will shift along the x-Axis.

[1] This was a byproduct of constructing the L^∞ bound in [3] and can be found in the proof of the corresponding Lemma.

and the same holds equivalently for P_2. Also, we know for the regions B_j, $j = 1,2,3$, given $u(t_i,\cdot) \in L^\infty(B_j)$, $u(t,x)$ on the boundaries of B_j and given that the solution $u(t,x)$ with $x \in \mathbb{R} \setminus B_j$ in the adjacent region to B_j satisfies $c_{\min}(t_{in},x) \leq u(t,x) \leq c_{\max}(t_{in},x)$, the bounds are also true in B_j

$$c_{\min}(t,x) \leq u(t,x) \leq c_{\max}(t,x). \quad \text{for } x \in B_j,$$

as the Burgers equation with L^∞ boundary data satisfies an L^∞ bound for any finite time. Piecing together the different regions, given $c_{\min}(t_i,x) \leq u(t_i,x) \leq c_{\max}(t_i,x)$, we obtain (4).
Therefore, defining the new superposition of $[t_i,t_{i+1}] \times \mathbb{R} = \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_1(t) = (-\infty, h_2(t_i) - L(t - t_i)]$$
$$\Sigma_2(t) = (h_1(t_i) + L(t - t_i), \infty)$$

Each of those regions contains only one particle, and therefore, applying Lemma 2 twice, we obtain existence of an entropy solution in $[t_i,t_{i+1}] \times \mathbb{R}$. □

Iterating this by using $t_i = t_{i+1}$ as new starting time for Lemma 3 until reaching time T gives the existence result of theorem 1 and the L^∞ bound follows from property (4) as long as the particle paths do not intersect.
It remains to investigate the case of particles being located at the same position at some time $t \leq T$. We choose to stop the current timestep whenever two particles are located at the same position, thus from the three considered cases of particle interactions, i.e. crossing, merging and splitting of particles, only the following two cases need to be dealt with. Again, we restrict us to two particles for simplicity, as the case of more particles follows using the same mechanism, see Figure 3.

1. Two particles are located at the same position at the end of a given time Interval $[t_0,T]$. (Merging)
2. Two particles are located at the same position at the initial time of a given time Interval $[t_0,T]$. (Splitting)

Case 1: $h_1(T) = h_2(T)$. The difficulty of this case lies in the timestepping, as at first glimpse, it is unclear wether or not the proposed method of construction used in the proof of Lemma 3 can actually reach time T. The reason is, that

$$t_{i+1} - t_i = \frac{h_2(t_i) - h_1(t_i) - 2\varepsilon}{2L},$$

meaning the length of each timestep depends on the distance between the particles $h_2(t_i) - h_1(t_i)$ which goes to zero as t goes to T.
However, the method of construction is equivalent to finding the root of the distance between the particles, denoted $d(t)$, by means of a simplified Newton method, which can be seen by measuring the distance against time and including the method in the picture, cf. Figure 2. Therefore, Lemma 3 holds for all timesteps where $t_i \in [t_0,T)$,

reaching time T either in a finite number of timesteps or as the limit of $n \to \infty$ if the particles have zero contact angle.

Case 2: $h_1(t_0) = h_2(t_0)$. This case is more delicate, as the method of construction fails to construct any solution between the two particles because all information about this region emanates from the two particles. There is no first timestep, as the choice of each timestep depends on finding a superposition suitable in the sense that each domain contains only waves coming from one particle (or none).
We solve this problem by shifting the particles apart, defining the particle paths of the approximated problem by

$$h_1^\varepsilon(t) = h_1(t),$$
$$h_2^\varepsilon(t) = h_2(t) + \varepsilon.$$

Therefore the particle paths do not intersect anymore and we meet the conditions of Lemma 3. Using the method of construction we obtain existence and the L^∞ bound for u^ε and any given, finite time T. It remains to show convergence of the approximate solution u^ε to the solution of the original problem, which is done using Helly's theorem. In order to be able to apply the latter, a bound in the total variation

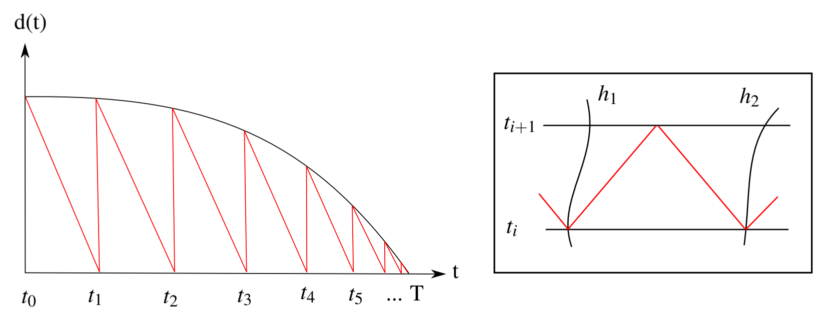

Fig. 2 On the left a visualization of the convergence for the construction method and $t_i \to T$. The method behaves like a simplified Newton method, where the slope corresponding to the maximal speed of propagation remains fixed. On the right the construction of the length of a single timestep, given by the longest possible time, s.t. waves propagating from the two particles don't intersect.

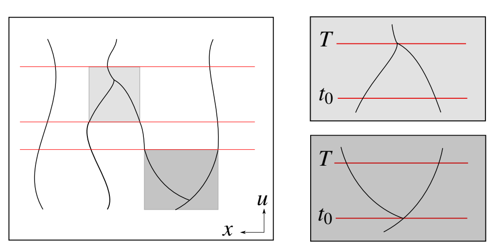

Fig. 3 On the left a random movement of particles, including sections where particles merge, corresponding to Case 1 (upper right), and split, corresponding to Case 2 (lower right). Whenever there is one of the two special cases, the method of construction has to be adapted.

has to be established, which can be proven using a bound on the total variation for the problem with only a single particle, which was proven in [3] using a wave front tracking method, and regularized initial data

$$u_0^\delta(x) = u_0(x) * \rho^\delta$$

with ρ^δ being a regularizing kernel such that $u_0^\delta \in L^\infty \cap BV_{loc}(\mathbb{R})$.
For a more in-depth analysis of the latter cases, which would exceed the purpose of this article, we refer the reader to the upcoming publication [8].

4 Uniqueness of entropy solutions

This section is devoted to proving the uniqueness of solutions to problem (1) whenever the admissiblity condition (2) is satisfied. Following the ideas of Kruzkov, this is done by using the method of doubling of variables and the framework of germs, introduced by Andreianov, Karlsen and Risebro [2]. They key property of an admissibility germ to allow to conclude uniqueness, as proven in their paper, is dissipativity. We state this property of $\mathscr{G}_\lambda$ in the following Lemma

Lemma 3. *The admissibility germ $\mathscr{G}_\lambda$ corresponding to the particle with velocity h' and friction λ is dissipative in the sense that*

$$(c_l, c_r) \in \mathscr{G}_\lambda \;\Leftrightarrow\; \left[\forall (b_l, b_r) \in \mathscr{G}_\lambda : \overline{\Phi}(h'; c_l, b_l) \geq \overline{\Phi}(h'; c_r, b_r)\right] \tag{5}$$

where

$$\overline{\Phi}(h'; c, b) = \Phi(c, b) - h'|c - b|.$$

Let $\Omega = [0, T] \times \mathbb{R}$, $\phi \in C_c^\infty(\Omega)$ be a classical, compactly supported testfunction and

$$w_\varepsilon(x) = \begin{cases} 0, & \text{when } |x| \leq \frac{\varepsilon}{2}, \\ 1, & \text{when } |x| \geq \varepsilon. \end{cases}$$

a continuous function with $w'_\varepsilon(x) = \mathrm{sgn}(x)\,\frac{2}{\varepsilon}$ for $\frac{\varepsilon}{2} \leq |x| \leq \varepsilon$.

Given two entropy solutions u, v, with the same initial data $u_0 = v_0$, we apply the method of doubling of variables, cf. [9], and choosing as a testfunction $\psi(x,t) = \phi(x,t) \times w_\varepsilon(t, x - h_1(t)) \times \ldots \times w_\varepsilon(t, x - h_N(t))$, we obtain

$$\int_\Omega |u - v| \partial_t \big(\phi \times \prod_{1 \leq i \leq N} w_\varepsilon(x - h_i(t)\big) + \int_{\mathbb{R}} |u_0 - v_0| \big(\phi(0, x) \times \prod_{1 \leq i \leq N} w_\varepsilon(x - h_i(0)\big) dx$$
$$+ \int_\Omega \Phi(u, v) \partial_x \big(\phi \times \prod_{1 \leq i \leq N} w_\varepsilon(x - h_i(t)\big) \geq 0.$$

Remark 2. Note that ψ is not C^∞ and one should regularize w_ε using classical mollifiers to this aim, but this would introduce unnecessary heavy notations that we skip for the sake of brevity.

Due to the choice of testfunction, we can not see the interfaces and the method of Kruzkov works classically. Using chain rule and recognizing that

$$\partial_t w_\varepsilon(x-h_i(t)) = w'_\varepsilon(x-h_i(t))(-h'_i(t))$$
$$\partial_x w_\varepsilon(x-h_i(t)) = w'_\varepsilon(x-h_i(t))$$

gives

$$\begin{aligned}&\int_\Omega |u-v| \left(\partial_t\phi \prod_{1\le i\le N} w_\varepsilon(x-h_i(t)) + \phi \sum_{i=1}^N \prod_{1\le j\neq i\le N} (-h'_i(t)) w'_\varepsilon(x-h_i(t)) w_\varepsilon(x-h_j(t))\right)\\ &+\int_\Omega \Phi(u,v)\left(\partial_x\phi \prod_{1\le i\le N} w_\varepsilon(x-h_i(t)) + \phi \sum_{i=1}^N \prod_{1\le j\neq i\le N} w'_\varepsilon(x-h_i(t)) w_\varepsilon(x-h_j(t))\right)\\ &+\int_{\mathbb{R}} |u_0-v_0|\big(\phi(0,x)\times \prod_{1\le i\le N} w_\varepsilon(x-h_i(0)\big)dx \;\ge\; 0.\end{aligned}$$

Using that we know the form of the derivative of w_ε, namely $w'_\varepsilon(x-h_i(t)) = -\frac{2}{\varepsilon}\mathbf{1}_{[h_i-\varepsilon,h_i-\frac{\varepsilon}{2}]} + \frac{2}{\varepsilon}\mathbf{1}_{[h_i+\frac{\varepsilon}{2},h_i+\varepsilon]}$, we can pass to the limit $\varepsilon \to 0$ and recognizing that in the sense of distributions

$$\lim_{\varepsilon\to 0} w_\varepsilon(x-h_i(t)) = \mathbf{1}$$

reincorporates the interfaces created by the particles and the related terms. Making use of the traces $\gamma_i^\pm(u), \gamma_i^\pm(v)$ respectively at the position of the interfaces $h_i(t)$, we obtain

$$\begin{aligned}&\int_\Omega |u-v|\partial_t\phi + \Phi(u,v)\partial_x\phi\, dx\,dt + \int_{\mathbb{R}} |u_0-v_0|\phi(0,x)\,dx\\ &\quad\ge\; \sum_{i=1}^N \int_0^T \big(\overline{\Phi}(h'_i,\gamma_i^-(u),\gamma_i^-(v))\phi(h_i(s),s) - \overline{\Phi}(h'_i,\gamma_i^+(u),\gamma_i^+(v))\phi(h_i(s),s)\big)\,ds.\end{aligned}$$

Using the dissipativity of the germs for each particle, given by Lemma 3, we get the good signs of the right-side terms of the last inequality, which we then can drop to obtain the Kato inequality,

$$\int_\Omega |u-v|\partial_t\phi + \Phi(u,v)\partial_x\phi\, dx\,dt + \int_{\mathbb{R}} |u_0-v_0|\phi(0,x)\,dx \ge 0,$$

which classically gives uniqueness of entropy solutions. Furthermore, integrating along the cone $C := \{(x,t), |x| = R + L(T-t), t\in[0,T]\}$ gives the L^1-contraction property.

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