On Uniqueness of Solutions to the two-dimensional Compressible Euler Equations

Master Thesis

by

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Zusammenfassung

In dieser Masterarbeit werden die isentropen kompressiblen Euler Gleichungen in zwei Raumdimensionen studiert. Für spezielle Anfangsdaten, zu denen man eine Lösung explizit angeben kann, wird untersucht, ob es auch noch weitere Lösungen gibt. Dabei kommt die Methode der konvexen Integration zum Einsatz. Nach Klärung der Grundbegriffe im ersten Abschnitt wird in Abschnitt 2 eine kurze Einführung in die Methode der konvexen Integration gegeben. Der dritte Abschnitt ist der Standardlösung gewidmet, d.h. der einen Lösung, die explizit angegeben werden kann. In Abschnitt 4 wird die Methode der konvexen Integration durchgeführt. Mit deren Hilfe wird ein Kriterium für die Existenz von unendlich vielen Lösungen bewiesen. Dieses Kriterium wird im fünften Abschnitt angewandt und schließlich in Abschnitt 6 ein Ausblick auf offene Probleme gegeben.

Abstract

In this master thesis we study the isentropic compressible Euler equations in two space dimensions. For special initial data, for which one can explicitly tell a solution, we investigate if there exist other solutions. In order to do this we use the convex integration method. After an introduction to the basic notions in the first section we present the main ideas of the convex integration method in section 2. The third section deals with the standard solution, i.e. the solution which can be explicitly named. In section 4 the convex integration is carried out. With the help of this method we prove a criterion for the existence of infinitely many solutions. This criterion is applied in the fifth section. Finally in section 6 we give an outlook on open problems.

Notation

Basic sets

- $\mathbb{N} := \{1, 2, \ldots\}$, in particular $0 \notin \mathbb{N}$
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
- $\mathbb{R}^+ := (0, \infty)$, in particular $0 \notin \mathbb{R}^+$
- $\mathbb{R}_0^+ := \mathbb{R} \cup \{0\} = [0, \infty)$
- Let A be a set. We write $B \subset A$ for a subset B, which can also be equal A. For strict subsets C we write $C \subsetneq A$.
- Let $X \subset \mathbb{R}^d$, $d \in \mathbb{N}$. Then
  - X° denotes the interior of X,
  - $\overline{X}$ is the closure of X and
  - X^{co} denotes the convex hull of X, i.e. the smallest convex set which contains X.
- Let $d \in \mathbb{N}$. If we don't say anything different, we endow $\mathbb{R}^d$ with the 2-norm and write $|\cdot|$ for it:
$$|v| = \sqrt{\sum_{i=1}^{d} v_i^2} \quad \text{for all } v \in \mathbb{R}^d.$$
- For $r > 0$ and $v \in \mathbb{R}^d$ we write $B_r^d(v)$ for the d-dimensional ball with radius r and center v:
$$B_r^d(v) := \{w \in \mathbb{R}^d : |v - w| < r\}.$$
If the dimension is clear, we will also omit it and write $B_r(v)$.

Matrices

- $\mathcal{M}^{n\times m}$:= the set of all real $n \times m$ matrices
- Id_n := the $n \times n$ identity matrix
- $\mathrm{Id} := \mathrm{Id}_2$
- $\mathcal{S}^{2\times 2} := \{A \in \mathcal{M}^{2\times 2} | A \text{ symmetric}\}$
- $\mathcal{S}_0^{2\times 2} := \{A \in \mathcal{M}^{2\times 2} | A \text{ symmetric and } \mathrm{tr}(A) = 0\}$
- For $A, B \in \mathcal{S}^{2\times 2}$ we define
  - $A < B :\Leftrightarrow B - A$ positive definite,
  - $A \leq B :\Leftrightarrow B - A$ positive semi-definite.
- For $A, B \in \mathcal{M}^{n\times n}$ we write $:$ for the Frobenius product: $A : B = \sum_{i,j=1}^{n} A_{ij}\, B_{ij}$.

Functions

- $\text{supp}(f) := \overline{\{x \in \mathbb{R}^n | f(x) \neq 0\}}$, the support of a function $f : \mathbb{R}^n \to \mathbb{R}^m$
- $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m) := \{f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \,|\, \text{supp}(f) \text{ is compact}\}$, the set of the functions with compact support
- Let X be a set and $P \subset X$. Then $\mathbf{1}_P : X \to \mathbb{R}$ denotes the indicator function of P, i.e.:
$$x \mapsto \mathbf{1}_P(x) := \begin{cases} 0 & \text{if } x \notin P \\ 1 & \text{if } x \in P \end{cases}$$

Abbreviations

$\partial_i := \partial_{x_i} := \frac{\partial}{\partial x_i}$

Components

Let $v \in \mathbb{R}^2$ be a vector. Then we will denote its components as $v = (v_1, v_2)^T$. If the vector is called $v_i \in \mathbb{R}^2$, where $i \in \{-, +\} \cup \mathbb{N}$, then we consequently name the components $v_i = (v_{i\,1}, v_{i\,2})^T$.

We treat matrices similar, e.g. $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \mathcal{M}^{2\times 2}$ or $u_i = \begin{pmatrix} u_{i\,11} & u_{i\,12} \\ u_{i\,21} & u_{i\,22} \end{pmatrix} \in \mathcal{M}^{2\times 2}$.

Contents

1. Introduction

Consider the 2-dimensional isentropic compressible Euler equations

$$\begin{aligned}\partial_t \rho + \operatorname{div}_x(\rho\, v) &= 0,\\ \partial_t(\rho\, v) + \operatorname{div}_x(\rho\, v \otimes v) + \nabla_x\,[p(\rho)] &= 0,\end{aligned} \tag{1.1}$$

where the density $\rho = \rho(t,x) \in \mathbb{R}^+$ and the velocity $v = v(t,x) \in \mathbb{R}^2$ are functions of the time $t \in [0,\infty)$ and the position $x = (x_1, x_2) \in \mathbb{R}^2$. We will use ∂_i as a short form for the partial derivative with respect to x_i, i.e. $\partial_i := \partial_{x_i}$.
In particular the first equation of (1.1) represents the conservation of mass and the second the conservation of momentum. More precisely the momentum equation is a system of two equations since the momentum $\rho\, v$ is 2-dimensional. By $\rho\, v \otimes v$ we mean the matrix with entries $(\rho\, v \otimes v)_{ij} = \rho\, v_i\, v_j$ and the divergence of this matrix is meant row-wise. So $\operatorname{div}_x(\rho\, v \otimes v)$ is a 2-dimensional vector with components

$$\big(\operatorname{div}_x(\rho\, v \otimes v)\big)_i = \sum_{j=1}^{2} \partial_j(\rho\, v_i\, v_j).$$

Hence (1.1) is a system of $2 + 1 = 3$ conservation laws. Furthermore the pressure $p = p(\rho)$ is a given function of the density ρ. It turns out, that the system (1.1) is strictly hyperbolic if $p' > 0$. We choose the polytropic pressure law $p(\rho) = K\,\rho^\gamma$ with a constant $K \in \mathbb{R}^+$ and the adiabatic coefficient $\gamma = 1 + \frac{2}{f}$, where f is the number of degrees of freedom. It is easy to check that in the case of polytropic pressure law $p'(\rho) > 0$ for all $\rho \in \mathbb{R}^+$. In other words, the system (1.1) equipped with the polytropic pressure law is strictly hyperbolic.
We are interested in solutions to the Cauchy problem consisting of the Euler system (1.1) and the initial data

$$\begin{aligned}\rho(0,x) &= \rho_0(x),\\ v(0,x) &= v_0(x).\end{aligned} \tag{1.2}$$

Hopefully the reader knows that the notion of a classical (i.e. differentiable) solution is not satisfactory in the theory of conservation laws. Even if the initial data is differentiable there might be no classical solution for all times. Thus one introduced weak solutions:

Definition 1.1. *(weak solution)* A weak solution to the Cauchy problem (1.1), (1.2) is a pair of functions $(\rho, v) \in L^\infty([0,\infty)\times\mathbb{R}^2, \mathbb{R}^+\times\mathbb{R}^2)$ such that for all test functions $(\psi, \phi) \in C_c^\infty([0,\infty)\times\mathbb{R}^2, \mathbb{R}\times\mathbb{R}^2)$ the following identities hold:

$$\int_0^\infty \int_{\mathbb{R}^2} \big(\rho\,\partial_t\psi + \rho\, v\cdot\nabla_x\psi\big)dx\,dt + \int_{\mathbb{R}^2} \rho_0(x)\,\psi(0,x)\,dx = 0, \tag{1.3}$$

$$\int_0^\infty \int_{\mathbb{R}^2} \big(\rho\, v\cdot\partial_t\phi + \rho\, v\otimes v : D_x\phi + p(\rho)\operatorname{div}_x\phi\big)dx\,dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x)\cdot\phi(0,x)\,dx = 0. \tag{1.4}$$

Here : denotes the Frobenius product, i.e.

$$\rho\, v\otimes v : D_x\phi = \sum_{i,j=1}^{2} \rho\, v_i\, v_j\, \partial_j\phi_i.$$

We assume that the reader also knows that weak solutions might be non-unique. Hence one introduced the so-called entropy admissibility criterion which states that in addition an inequality, the *entropy inequality*, must hold:

Definition 1.2. *(admissible weak solution)* A weak solution is admissible if in addition for every non-negative test function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R}_0^+)$ the following inequality is fulfilled:

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\left(\rho\, \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi + \left(\rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right) dx\, dt \\ + \int_{\mathbb{R}^2} \left(\rho_0(x)\, \varepsilon(\rho_0(x)) + \rho_0(x)\, \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \;\geq\; 0, \tag{1.5}$$

where the internal energy $\varepsilon = \varepsilon(\rho)$ is given by $p(\rho) = \rho^2\, \varepsilon'(\rho)$. In the case of polytropic pressure law one gets $\varepsilon(\rho) = \frac{K\,\rho^{\gamma-1}}{\gamma-1}$.

Remark. In this thesis we will always be interested in admissible weak solutions. For convenience we will sometimes just use the word "solution" to denote an admissible weak solution.

This admissibility criterion can be motivated in two ways. Firstly, it is just the formulation for systems of the well-known entropy admissibility criterion for scalar conservation laws. This criterion states that the entropy inequality has to hold for all entropy-entropy flux-pairs. One can show that for our Cauchy problem (1.1), (1.2) the only non trivial entropy is the total energy $\eta = \rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2}$ (see [CK14, Section 1]) and the corresponding entropy flux is $\Psi = \big(\rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2} + p(\rho)\big)\, v$. This is the reason why (1.5) is sometimes called *energy inequality*. For scalar conservation laws Kružkov showed in [Kru70] that there is exactly one weak solution that fulfills the entropy criterion. In other words this criterion yields a unique solution to scalar conservation laws. The question is, if this is also true for systems. A second way to motivate (1.5) is more physical. The inequality (1.5) is exactly the weak formulation of the local energy criterion

$$\partial_t \left(\rho\, \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) + \operatorname{div}_x \left(\left(\rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2} + p(\rho) \right) v \right) \leq 0.$$

If there was an equality sign in the above inequality, this would be conservation of energy. The $\leq$ sign is motivated as we allow energy to dissipate.
The topic we deal with in this thesis is the uniqueness of admissible weak solution to (1.1), (1.2). We will prove that for special initial data ρ_0, v_0 there are infinitely many admissible weak solution.
In spite of this non-uniqueness result one expects the existence of a unique physically relevant solution. We conjecture that there is another admissibility condition which is able to single out this unique physically relevant solution. Unfortunately such a condition has not been found so far.
The proof of non-uniqueness of admissible weak solution uses the so-called convex integration method developed by C. De Lellis and L. Székelyhidi in [LS09] and [LS10]. Weak solutions which are constructed with the convex integration method are called *wild solutions*, since they have "wild" oscillations. The existence of such

wild solutions depends on the initial data and we denote initial data which yield wild solutions as *wild initial data.* We will present the basic ideas of the convex integration method in section 2.
In this thesis we consider a special Riemann-like initial data as in [Chi12, Chapter 3], [CK14] and [CLK15], namely

$$\begin{aligned} \rho(0,x) = \rho_0(x) &:= \begin{cases} \rho_- & \text{if } x_2 < 0 \\ \rho_+ & \text{if } x_2 > 0 \end{cases}, \\ v(0,x) = v_0(x) &:= \begin{cases} v_- & \text{if } x_2 < 0 \\ v_+ & \text{if } x_2 > 0 \end{cases}, \end{aligned} \tag{1.6}$$

where $\rho_\pm \in \mathbb{R}^+$ and $v_\pm \in \mathbb{R}^2$ are constants (see figure 1). We denote the compo-

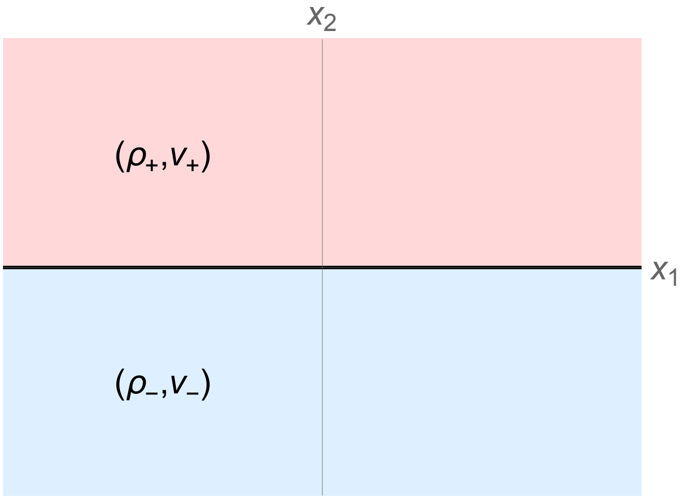

Figure 1: The initial data considered in this thesis

nents of the velocities as $v_- = (v_{-1}, v_{-2})^T$ resp. $v_+ = (v_{+1}, v_{+2})^T$. Furthermore we suppose that $v_{-1} = v_{+1}$, which means that the component of the velocity which is parallel to the discontinuity is equal on both sides of the discontinuity.
The reasons why we look at initial data of the form (1.6) are the following: Firstly, we will see that even for this easy initial condition there might be infinitely many admissible weak solutions. In other words there exist wild initial data of the form (1.6). Hence wild initial data do not need to have “wild” oscillations itself. Furthermore with help of the considered initial condition it is even possible to find wild initial data which are Lipschitz (see [CLK15, Corollary 1.2]). The idea to prove this is to find Lipschitz initial data which yield a unique continuous solution up to a time $T > 0$ and at time $t = T$ this continuous solution collapses to a weak solution which has the form of our initial condition (1.6). Then for $t > T$ the results on the existence of wild solutions to the considered problem (1.1), (1.6) yield that there are infinitely many admissible weak solutions for $t > T$.
In addition to that we have an example for an initial condition for which we can explicitly find a solution (the *standard solution*, see section 3) and likewise we know that there are infinitely many other admissible weak solutions. One could expect that the standard solution is the physically relevant one but this is just a conjecture

based on intuition. If one has an idea for an additional admissibility criterion that could dismiss all the non physical solutions, one can check if this criterion singles out the standard solution. For example in [CK14, Theorem 2] E. Chiodaroli and O. Kreml showed, that there are initial data of the form (1.6) where the entropy rate admissibility criterion does not favor the standard solution. The entropy rate admissibility criterion states that the solution which dissipates the largest amount of energy should be the physically relevant one, provided such a maximum exists. But because of Chiodaroli's and Kreml's results the entropy rate admissibility criterion also seems to be not the desired criterion.
Another reason why the consideration of problems with an initial data like (1.6) is interesting is that there are cases, i.e. for special values of $\rho_\pm$, $v_\pm$, where there are infinitely many solutions and other cases where the standard solution is unique. In view of finding an admissibility condition, that yields a unique physically relevant solution, it could be useful to study the differences of these cases.

2. Main ideas of the convex integration method

In this section we want to present the basic ideas of the convex integration method, which we will use to prove non-uniqueness of admissible weak solutions to the system (1.1), (1.6) for certain values $\rho_\pm, v_\pm$. The first non-uniqueness results for weak solutions to the incompressible Euler equations

$$\begin{aligned} \mathrm{div}_x v &= 0, \\ \partial_t v + \mathrm{div}_x(v \otimes v) + \nabla_x p &= 0 \end{aligned}$$

go back to V. Scheffer in 1993, [Sch93]. Scheffer proved that there is a non-trivial weak solution to the incompressible Euler equations which has a compact support in space and time. So the Cauchy problem consisting of the incompressible Euler equations and a zero initial condition has at least two weak solutions: the trivial one, i.e. $(v, p) \equiv 0$, and another one which is constructed in [Sch93]. An easier proof of the existence of such a non-trivial solution is given by A. Shnirelman in [Shn97]. Later on C. De Lellis and L. Székelyhidi described in [LS09] and [LS10] a new method to prove non-uniqueness, namely the so-called convex integration. The solutions they produced, the *wild solutions*, are essentially bounded, whereas Scheffer's an Shnirelman's solutions are unbounded and in L^2. Another difference of the Scheffer-Shnirelman-solutions and the wild solutions is, that the former obviously violate energy criteria in the sense that the total energy increases over some time interval. In [LS10] it is proved that there exist initial data for which there are infinitely many solutions to the incompressible Euler equations which fulfill several energy criteria, [LS10, Theorem 1].
The next step was to adapt the results described above to the compressible Euler system. De Lellis and Székelyhidi showed in [LS10, Theorem 2] that there is an initial density and velocity ρ_0, v_0, which leads to infinitely many admissible weak solutions to the compressible system (1.1), (1.2). In other words they proved that there is wild initial data for the compressible Euler system. This result was improved by E. Chiodaroli, who proved that for all periodic initial densities $\rho_0 \in C^1$ there is an initial velocity $v_0 \in L^\infty$ such that the initial data ρ_0, v_0 is wild (see [Chi12, Theorem 0.2.1]). The results on wild solutions to the system (1.1) equipped with the

special initial condition (1.6) will be described later in this thesis. Now we're going to turn our attention to the convex integration method, which makes the above results possible.
The basic idea of the convex integration consists of several steps (see also [Chi12, Section 2.4]): We start with a system of non-linear partial differential equations $\mathcal{N}(y) = 0$, where $\mathcal{N}$ is a non-linear differential operator and y are the unknowns.

Example. Let $y = v \in \mathbb{R}^2$ be the unknown, which is a function of $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Consider the pressureless incompressible Euler equations

$$\begin{aligned} \operatorname{div}_x v &= 0, \\ \partial_t v + \operatorname{div}_x(v \otimes v) &= 0, \end{aligned} \tag{2.1}$$

which is a system of non-linear partial differential equations.

1. The first step is to rewrite the system $\mathcal{N}(y) = 0$ as a linear system $\mathcal{L}(z) = 0$ with a non-linear constraint $z \in X$ by introducing new unknowns (z instead of y).
   Example. We introduce the new unknown $u \in \mathcal{S}_0^{2\times 2}$ such that $z = (v, u)$, and rewrite the above system (2.1) as a linear one

$$\begin{aligned} \operatorname{div}_x v &= 0, \\ \partial_t v + \operatorname{div}_x u &= 0, \end{aligned} \tag{2.2}$$

   with the non-linear constraint

$$\begin{aligned} z \in X := \Big\{ (v,u) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\big|\, \big(v(t,x), u(t,x)\big) \in K \\ \text{for all}^1 (t,x) \in \mathbb{R} \times \mathbb{R}^2 \Big\}, \end{aligned}$$

   and

$$K := \Big\{ (v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\big|\, u = v \otimes v - \frac{C}{2}\,\mathrm{Id} \Big\},$$

   where $C > 0$ is a fixed constant.
   Then if $z = (v, u) \in X$ is a solution to (2.2), $y = v$ is a solution to (2.1), what can be checked easily.
   Remark. Another possibility in this step would be to introduce $u \in \mathcal{S}^{2\times 2}$ and use $u = v \otimes v$ as the constraint. But it is easy to realize that a symmetric 2×2 matrix can be uniquely written as a sum of a symmetric tracefree 2×2 matrix and a multiple of the identity matrix. One chose to use the latter formulation with $u \in \mathcal{S}_0^{2\times 2}$ and a constant $C \in \mathbb{R}$. Since C represents the trace of $v \otimes v$, which is simply $|v|^2$, we can assume $C > 0$. However it is also possible to consider C fixed and obtain solutions with $|v|^2 = C$.

2. Relax the constraint: $X \mapsto \widehat{X}$.
   Weak solutions to the linearized system are called subsolutions if they fulfill the relaxed constraint[2], i.e. z_0 is a subsolution provided $\mathcal{L}(z_0) = 0$ and $z_0 \in \widehat{X}$.

[1] Since we are interested in weak solutions, it is enough to require "for almost all".

[2] In some other areas of mathematics, e.g. with respect to elliptic partial differential equations, one uses the term *subsolution* for a totally different object. More precisely, given a scalar differential equation $\mathcal{D}(z) = 0$, one denotes z_0 as a subsolution if $\mathcal{D}(z_0) \leq 0$. Notice that this usage is totally different to ours.

In the following sections of this thesis we will use a slightly different notion of so-called *fan subsolutions*.

3. Find a subsolution z_0.
4. The final step is to construct a sequence of subsolutions $(z_k)_{k\in\mathbb{N}_0} \subset \widehat{X}$ starting with the subsolution z_0 from above step and approaching X. To get z_{k+1} we add a solution of $\mathcal{L}(z) = 0$ to z_k in a way such that $z_{k+1} \in \widehat{X}$ and such that $(z_k)_{k\in\mathbb{N}}$ converges to a $z \in X$. Since we deal with a linear system $\mathcal{L}(z) = 0$, the sum of two solutions to this system is again a solution, i.e. z_{k+1} solves $\mathcal{L}(z_{k+1}) = 0$ and hence each z_k fulfills $\mathcal{L}(z_k) = 0$.

Now the questions are how to relax the constraint and what kind of solutions to the linearized system we should add in each step to achieve remaining in $\widehat{X}$ and convergence of the arising sequence to an element of X. Of course these two questions are connected: We should relax the constraint in a way such that adding a special type of a solution to the linear system automatically yield a solution which still fulfills the relaxed constraint. Furthermore we should guarantee that the set of functions fulfilling the relaxed constraint $\widehat{X}$ is sufficiently large, such that we have much freedom to construct the sequence.

Example. However one chooses localized plane wave solutions of (2.2) to add them in order to construct the desired sequence. Plane waves are solutions of (2.2) of the form $a\,h\big((x,t)\cdot\eta\big)$ where $h \in C^1\big(\mathbb{R},[-1,1]\big)$ is a function, $a \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ a constant and $\eta \in \mathbb{R}^3$ a direction in space-time. By $\cdot$ we mean the scalar product in $\mathbb{R}^3$, in other words $(x,t)\cdot\eta = x_1\,\eta_1 + x_2\,\eta_2 + t\,\eta_3$. Then the plane wave $a\,h\big((x,t)\cdot\eta\big)$ takes values in the segment $[-a,a] \subset \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$. We want to add localized versions of plane waves. That means solutions which have compact support and take values in a neighborhood of a segment. We will prove that such solutions exist. The question is: How do we relax the constraint to achieve that adding a localized plane wave yields an element, which also fulfills the relaxed constraint?
The answer is to change over to the interior of the convex hull $(K^{\mathrm{co}})^\circ$ of K and define

$$\widehat{X} := \Big\{(v,u) : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\big|\, \big(v(t,x),u(t,x)\big) \in (K^{\mathrm{co}})^\circ \text{ for almost all } (t,x) \in \mathbb{R}\times\mathbb{R}^2\Big\}.$$

If we let $(v,u) \in \widehat{X}$ and fix a point $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^2$ we will find an a such that $\big(v(t_0,x_0),u(t_0,x_0)\big) + [-a,a] \subset (K^{\mathrm{co}})^\circ$. Since we find a localized plane wave which takes values near the segment $[-a,a]$ and if we require (v,u) to be continuous we can achieve that the sum of (v,u) and the localized plane wave takes values in $(K^{\mathrm{co}})^\circ$ and hence lies in $\widehat{X}$.

The reader might ask why we chose the pressureless incompressible Euler equations as an example. This is because we won't apply the convex integration method directly to the compressible Euler equations. We will do it essentially for the system (2.1) and show that infinitely many solutions to this system may also lead to infinitely many solutions to the compressible Euler system (1.1).
So far we explained the basic ideas of the convex integration. The way how we really prove non-uniqueness slightly differs from this ideas. More precisely we won't construct the mentioned sequence explicitly. Instead we will use some Baire arguments.

3. The standard solution

Observe that the initial functions ρ_0, v_0 in (1.6) do not depend on x_1. Additionally $v_{-1} = v_{+1}$ and hence the initial condition is truly 1-dimensional. So a reasonable approach is now to solve the 1-dimensional Riemann problem

$$\begin{aligned} \partial_t \rho + \partial_2(\rho\, v_2) &= 0, \\ \partial_t(\rho\, v_1) + \partial_2(\rho\, v_1\, v_2) &= 0, \\ \partial_t(\rho\, v_2) + \partial_2\big(\rho\, v_2^2 + p(\rho)\big) &= 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \rho(0,x_2) &= \begin{cases} \rho_- & \text{if } x_2 < 0 \\ \rho_+ & \text{if } x_2 > 0 \end{cases}, \\ v(0,x_2) &= \begin{cases} v_- & \text{if } x_2 < 0 \\ v_+ & \text{if } x_2 > 0 \end{cases}, \end{aligned} \tag{3.2}$$

where the unknowns $\rho = \rho(t,x_2) \in \mathbb{R}^+$ and $v = v(t,x_2) \in \mathbb{R}^2$ are now functions of the time $t \in [0,\infty)$ and the position $x_2 \in \mathbb{R}$. We additionally demand the validity of the entropy inequality

$$\partial_t\left(\rho\,\varepsilon(\rho) + \rho\,\frac{|v|^2}{2}\right) + \partial_2\left(\left(\rho\,\varepsilon(\rho) + \rho\,\frac{|v|^2}{2} + p(\rho)\right)v_2\right) \leq 0. \tag{3.3}$$

It is not difficult to prove that solutions to (3.1), (3.2), which fulfill (3.3), are also x_1-independent admissible solutions[3] to (1.1), (1.6).
We denote $m = \rho\, v$, whereby (3.1), (3.2) and (3.3) turn into

$$\begin{aligned} \partial_t \rho + \partial_2 m_2 &= 0, \\ \partial_t m_1 + \partial_2\left(\frac{m_1\, m_2}{\rho}\right) &= 0, \\ \partial_t m_2 + \partial_2\left(\frac{m_2^2}{\rho} + p(\rho)\right) &= 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \rho(0,x_2) &= \begin{cases} \rho_- & \text{if } x_2 < 0 \\ \rho_+ & \text{if } x_2 > 0 \end{cases}, \\ m(0,x_2) &= \begin{cases} \rho_-\, v_- & \text{if } x_2 < 0 \\ \rho_+\, v_+ & \text{if } x_2 > 0 \end{cases}, \end{aligned} \tag{3.5}$$

$$\partial_t\left(\rho\,\varepsilon(\rho) + \frac{|m|^2}{2\,\rho}\right) + \partial_2\left(\left(\varepsilon(\rho) + \frac{|m|^2}{2\,\rho^2} + \frac{p(\rho)}{\rho}\right)m_2\right) \leq 0. \tag{3.6}$$

We are going to solve the above 1-dimensional Riemann problem (3.4), (3.5) with admissibility condition (3.6). It is well-known that the 1-dimensional Riemann problem has a self similar solution, which consists of shocks, contact discontinuities and rarefactions. The strategy to find this self similar solution is also well-known and the reader should have some basic knowledge about it. If not, we refer to textbooks like

[3] More precisely we are interested in weak solutions to (3.1), (3.2), but we leave a weak formulation of (3.1), (3.2) and (3.3) up to the reader. We then get that weak solutions to (3.1), (3.2) and (3.3) are x_1-independent admissible weak solution to the 2-dimensional problem in the sense of definitions 1.1 and 1.2.

[Daf16, chapters 7-9] or [LeV04, chapters 13,14]. We will follow E. Chiodaroli and O. Kreml [CK14, section 2], [Chi12, chapter 4], who computed the wanted solution, too.
First of all set $U := (\rho, m_1, m_2)$, which is called state vector. Via defining

$$F(U) := \begin{pmatrix} m_2 \\ \frac{m_1 m_2}{\rho} \\ \frac{m_2^2}{\rho} + p(\rho) \end{pmatrix},$$

we can write (3.4) as

$$\partial_t U + \partial_2 F(U) = 0.$$

It is easy to check that the Jacobian of F reads

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{m_1 m_2}{\rho^2} & \frac{m_2}{\rho} & \frac{m_1}{\rho} \\ -\frac{m_2^2}{\rho^2} + p'(\rho) & 0 & \frac{2 m_2}{\rho} \end{pmatrix}.$$

In the case $p'(\rho) > 0$, which is true for our choice of pressure law $p(\rho) = K \rho^\gamma$ ($K > 0$, $\gamma > 1$, see above) and $\rho > 0$, the eigenvalues of $DF(U)$ are

$$\lambda_1 = \frac{m_2}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m_2}{\rho}, \qquad \lambda_3 = \frac{m_2}{\rho} + \sqrt{p'(\rho)},$$

and the corresponding eigenvectors

$$R_1 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} + \sqrt{p'(\rho)} \end{pmatrix}.$$

Since $p'(\rho) > 0$, there are three real and pairwise different eigenvalues and the corresponding eigenvectors are linearly independent, what means that the system (3.4) is strictly hyperbolic.
Furthermore it is not difficult to verify that the functions

$$\omega_1 = \frac{m_2}{\rho} - \int_0^\rho \frac{\sqrt{p'(r)}}{r}\, dr, \qquad \omega_2 = \frac{m_1}{\rho}, \qquad \omega_3 = \frac{m_2}{\rho} + \int_0^\rho \frac{\sqrt{p'(r)}}{r}\, dr$$

are 2- and 3-, 1- and 3-, 1- and 2-Riemann invariants of the system (3.4).
Next we check if the i-th characteristic families are genuinely non-linear or linearly degenerate ($i \in \{1, 2, 3\}$). With the easy computation

$$\rho\, p''(\rho) + 2\, p'(\rho) = \rho\ K\, \gamma\, (\gamma - 1)\, \rho^{\gamma-2} + 2\ K\, \gamma\, \rho^{\gamma-1} = K\, \gamma\, (\gamma + 1)\, \rho^{\gamma-1} > 0$$

we obtain

$$\begin{aligned} D\lambda_1 \cdot R_1 &= -\frac{1}{2\,\rho\,\sqrt{p'(\rho)}} \big(\rho\, p''(\rho) + 2\, p'(\rho)\big) < 0, \\ D\lambda_2 \cdot R_2 &= 0, \\ D\lambda_3 \cdot R_3 &= \frac{1}{2\,\rho\,\sqrt{p'(\rho)}} \big(\rho\, p''(\rho) + 2\, p'(\rho)\big) > 0. \end{aligned}$$

Hence the 1st and the 3rd characteristic families are genuinely non-linear while the 2nd characteristic family is linearly degenerate. The former implies that the 1- and 3-wave is either a shock or a rarefaction and the latter means that the 2-wave is a contact discontinuity.

3.1. 2-Contact discontinuity

At the 2-contact discontinuity the corresponding eigenvalue $\lambda_2 = \frac{m_2}{\rho}$ does not jump. The Rankine Hugoniot conditions at the contact discontinuity which connects a left state $U_L = (\rho_L, m_{L\,1}, m_{L\,2})$ to a right state $U_R = (\rho_R, m_{R\,1}, m_{R\,2})$ read

$$\lambda_2\,(\rho_L - \rho_R) = m_{L\,2} - m_{R\,2}, \tag{3.7}$$

$$\lambda_2\,(m_{L\,1} - m_{R\,1}) = \frac{m_{L\,1}\,m_{L\,2}}{\rho_L} - \frac{m_{R\,1}\,m_{R\,2}}{\rho_R}, \tag{3.8}$$

$$\lambda_2\,(m_{L\,2} - m_{R\,2}) = \frac{m_{L\,2}^2}{\rho_L} - \frac{m_{R\,2}^2}{\rho_R} + p(\rho_L) - p(\rho_R). \tag{3.9}$$

The second condition (3.8) is fulfilled since $\lambda_2 = \frac{m_2}{\rho} = \frac{m_{L\,2}}{\rho_L} = \frac{m_{R\,2}}{\rho_R}$. The third (3.9) yields $p(\rho_L) = p(\rho_R)$ and because p is strictly monotone ($p' > 0$) and hence injective, this implies $\rho_L = \rho_R$. Then by (3.7) we obtain $m_{L\,2} = m_{R\,2}$. The contact discontinuity is admissible in the sense of (3.6) because

$$\begin{aligned}
&\lambda_2 \left(\rho_L\,\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L} - \rho_R\,\varepsilon(\rho_R) - \frac{|m_R|^2}{2\,\rho_R}\right) \\
&= \left(\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L^2} + \frac{p(\rho_L)}{\rho_L}\right) m_{L\,2} - \left(\varepsilon(\rho_R) + \frac{|m_R|^2}{2\,\rho_R^2} + \frac{p(\rho_R)}{\rho_R}\right) m_{R\,2}
\end{aligned}$$

holds, what is easy to check. To summarize, at the 2-contact discontinuity, which is admissible, ρ and m_2 are continuous whereas m_1 may have a jump. These results can also be achieved by looking at the 2-Riemann invariants ω_1 and ω_3.

3.2. Admissible Shocks

Let us now investigate the 1- and 3-wave. First we suppose that the 1- or 3-wave is a shock. The Rankine Hugoniot conditions which belong to a shock that connects a left state $U_L = (\rho_L, m_{L\,1}, m_{L\,2})$ to a right state $U_R = (\rho_R, m_{R\,1}, m_{R\,2})$ are

$$\sigma\,(\rho_L - \rho_R) = m_{L\,2} - m_{R\,2}, \tag{3.10}$$

$$\sigma\,(m_{L\,1} - m_{R\,1}) = \frac{m_{L\,1}\,m_{L\,2}}{\rho_L} - \frac{m_{R\,1}\,m_{R\,2}}{\rho_R}, \tag{3.11}$$

$$\sigma\,(m_{L\,2} - m_{R\,2}) = \frac{m_{L\,2}^2}{\rho_L} - \frac{m_{R\,2}^2}{\rho_R} + p(\rho_L) - p(\rho_R), \tag{3.12}$$

where σ is the corresponding shock speed. A shock is admissible in the sense of (3.6) if

$$\begin{aligned}
&\sigma \left(\rho_L\,\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L} - \rho_R\,\varepsilon(\rho_R) - \frac{|m_R|^2}{2\,\rho_R}\right) \\
&\leq \left(\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L^2} + \frac{p(\rho_L)}{\rho_L}\right) m_{L\,2} - \left(\varepsilon(\rho_R) + \frac{|m_R|^2}{2\,\rho_R^2} + \frac{p(\rho_R)}{\rho_R}\right) m_{R\,2}.
\end{aligned} \tag{3.13}$$

From (3.11) we get

$$\frac{m_{L\,1}}{\rho_L}\,(m_{L\,2} - \sigma\,\rho_L) = \frac{m_{R\,1}}{\rho_R}\,(m_{R\,2} - \sigma\,\rho_R),$$

and from (3.10)

$$m_{L\,2} - \sigma\,\rho_L = m_{R\,2} - \sigma\,\rho_R.$$

This yields[4]

$$\frac{m_{L\,1}}{\rho_L} = \frac{m_{R\,1}}{\rho_R}. \tag{3.14}$$

Multiplying (3.12) with $(\rho_L - \rho_R)$ and substituting (3.10) we obtain

$$\big(m_{L\,2} - m_{R\,2}\big)^2 = \big(\rho_L - \rho_R\big)\left(\frac{m_{L\,2}^2}{\rho_L} - \frac{m_{R\,2}^2}{\rho_R} + p(\rho_L) - p(\rho_R)\right),$$

which is equivalent to

$$\rho_L\,\rho_R\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)^2 = \big(\rho_L - \rho_R\big)\,\big(p(\rho_L) - p(\rho_R)\big). \tag{3.15}$$

Now we discuss if a shock is admissible. To do this we first use (3.10) to replace σ in (3.13)[5] to obtain

$$\begin{aligned}
&\frac{m_{L\,2} - m_{R\,2}}{\rho_L - \rho_R}\left(\rho_L\,\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L} - \rho_R\,\varepsilon(\rho_R) - \frac{|m_R|^2}{2\,\rho_R}\right)\\
&\quad\le \left(\varepsilon(\rho_L) + \frac{|m_L|^2}{2\,\rho_L^2} + \frac{p(\rho_L)}{\rho_L}\right) m_{L\,2} - \left(\varepsilon(\rho_R) + \frac{|m_R|^2}{2\,\rho_R^2} + \frac{p(\rho_R)}{\rho_R}\right) m_{R\,2}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\frac{\rho_L\,\rho_R}{\rho_L - \rho_R}\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)\left(\frac{|m_L|^2}{2\,\rho_L^2} - \frac{|m_R|^2}{2\,\rho_R^2}\right)\\
&\le p(\rho_L)\,\frac{m_{L\,2}}{\rho_L} - p(\rho_R)\,\frac{m_{R\,2}}{\rho_R} + \frac{\rho_L\,\rho_R}{\rho_L - \rho_R}\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)\big(\varepsilon(\rho_L) - \varepsilon(\rho_R)\big).
\end{aligned} \tag{3.16}$$

Let's use (3.14) and (3.15) to simplify the left-hand side further:

$$\begin{aligned}
&\frac{\rho_L\,\rho_R}{\rho_L - \rho_R}\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)\left(\frac{|m_L|^2}{2\,\rho_L^2} - \frac{|m_R|^2}{2\,\rho_R^2}\right)\\
&\quad= \frac{\rho_L\,\rho_R}{\rho_L - \rho_R}\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)\left(\frac{m_{L\,1}^2}{2\,\rho_L^2} + \frac{m_{L\,2}^2}{2\,\rho_L^2} - \frac{m_{R\,1}^2}{2\,\rho_R^2} - \frac{m_{R\,2}^2}{2\,\rho_R^2}\right)\\
&\quad= \frac{\rho_L\,\rho_R}{2\,(\rho_L - \rho_R)}\left(\frac{m_{L\,2}}{\rho_L} - \frac{m_{R\,2}}{\rho_R}\right)^2\left(\frac{m_{L\,2}}{\rho_L} + \frac{m_{R\,2}}{\rho_R}\right)\\
&\quad= \frac{1}{2}\,\big(p(\rho_L) - p(\rho_R)\big)\left(\frac{m_{L\,2}}{\rho_L} + \frac{m_{R\,2}}{\rho_R}\right).
\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) we get

$$\left(\frac{m_{R\,2}}{\rho_R} - \frac{m_{L\,2}}{\rho_L}\right)\left(p(\rho_L) + p(\rho_R) - 2\,\rho_L\,\rho_R\,\frac{\varepsilon(\rho_L) - \varepsilon(\rho_R)}{\rho_L - \rho_R}\right) \le 0. \tag{3.18}$$

[4]Note that $m_{L\,2} - \sigma\,\rho_L = m_{R\,2} - \sigma\,\rho_R \neq 0$. If not, we had $\sigma = \frac{m_{L\,2}}{\rho_L} = \frac{m_{R\,2}}{\rho_R}$. With $\lambda_2 = \frac{m_2}{\rho}$ in mind this would imply that the shock coincides with the 2-contact discontinuity, which is a contradiction.

[5]We can assume that $\rho_L \neq \rho_R$. If not, (3.10) would yield that $m_{L\,2} = m_{R\,2}$. Then using (3.11) we observe that $\sigma\,(m_{L\,1} - m_{R\,1}) = \frac{m_{L\,2}}{\rho_L}\,(m_{L\,1} - m_{R\,1})$, which means that $m_{L\,1} = m_{R\,1}$ or $\sigma = \frac{m_{L\,2}}{\rho_L}$. The former implies that $U_L = U_R$, i.e. there is no jump and therefore no shock, and the latter that the shock coincides with the 2-contact discontinuity, what is a contradiction.

We're going to show now, that

$$p(\rho_L) + p(\rho_R) - 2\,\rho_L\,\rho_R\,\frac{\varepsilon(\rho_L) - \varepsilon(\rho_R)}{\rho_L - \rho_R} > 0$$

for all $\rho_L, \rho_R \in \mathbb{R}^+$ with $\rho_L \neq \rho_R$. Without loss of generality we can assume that $\rho_L < \rho_R$. For the adiabatic pressure law the above equation is equivalent to

$$K\left(\rho_L^\gamma + \rho_R^\gamma - 2\,\rho_L\,\rho_R\,\frac{\rho_L^{\gamma-1} - \rho_R^{\gamma-1}}{(\gamma-1)\,(\rho_L - \rho_R)}\right) > 0.$$

Multiply by $(\gamma - 1)$ (note that $\gamma > 1$) and $(\rho_L - \rho_R)$, divide by $\rho_L^{\gamma+1}$ and K (note that $K > 0$), and denote $z = \frac{\rho_R}{\rho_L}$ to obtain

$$(\gamma-1)\left(z^{\gamma+1} - 1\right) - (\gamma+1)\left(z^\gamma - z\right) > 0,$$

since $\rho_L - \rho_R < 0$ by assumption. Introduce the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(z) := (\gamma-1)\left(z^{\gamma+1} - 1\right) - (\gamma+1)\left(z^\gamma - z\right).$$

It remains to show that $f(z) > 0$ for $z > 1$. We get

$$\begin{aligned} f'(z) &= (\gamma-1)\,(\gamma+1)\,z^\gamma - (\gamma+1)\,(\gamma\,z^{\gamma-1} - 1), \\ f''(z) &= (\gamma-1)\,(\gamma+1)\,\gamma\,z^{\gamma-2}\,(z-1), \end{aligned}$$

and therefore $f''(z) > 0$ for $z > 1$. Using $f(1) = f'(1) = 0$ the recently shown convexity of f on $z > 1$ implies that f is increasing on $z > 1$ and hence $f(z) > 0$ for $z > 1$.

Finally (3.18) implies with the just proved estimate that a shock, which connects U_L and U_R, is admissible if

$$\frac{m_{L\,2}}{\rho_L} \geq \frac{m_{R\,2}}{\rho_R}. \tag{3.19}$$

Let us summarize this subsection.

Lemma 3.1.

- *The 1-wave is an admissible shock, which connects the state $U_M = (\rho_M, m_{M\,1}, m_{M\,2})$ on the right to the state $U_- = (\rho_-, m_{-\,1}, m_{-\,2})$ on the left if*

$$\begin{aligned} \rho_M &> \rho_-, \\ \frac{m_{M\,1}}{\rho_M} &= \frac{m_{-\,1}}{\rho_-}, \\ \frac{m_{M\,2}}{\rho_M} &= \frac{m_{-\,2}}{\rho_-} - \sqrt{\frac{(\rho_M - \rho_-)\,\big(p(\rho_M) - p(\rho_-)\big)}{\rho_M\,\rho_-}}. \end{aligned}$$

- *The 3-wave is an admissible shock, which connects the state $U_M = (\rho_M, m_{M\,1}, m_{M\,2})$ on the left to the state $U_+ = (\rho_+, m_{+\,1}, m_{+\,2})$ on the right if*

$$\begin{aligned} \rho_M &> \rho_+, \\ \frac{m_{M\,1}}{\rho_M} &= \frac{m_{+\,1}}{\rho_+}, \\ \frac{m_{M\,2}}{\rho_M} &= \frac{m_{+\,2}}{\rho_+} + \sqrt{\frac{(\rho_M - \rho_+)\,\big(p(\rho_M) - p(\rho_+)\big)}{\rho_M\,\rho_+}}. \end{aligned}$$

Proof. The claims are simple consequences of (3.14), (3.15) and (3.19). We're going to find out the proper sign when solving (3.15) for $\frac{m_{M\,2}}{\rho_M}$.
We start with a 1-shock. The equation (3.15) leads to[6]

$$\frac{m_{M\,2}}{\rho_M} = \frac{m_{-\,2}}{\rho_-} \pm \sqrt{\frac{\left(\rho_M - \rho_-\right)\left(p(\rho_M) - p(\rho_-)\right)}{\rho_M\,\rho_-}}. \tag{3.20}$$

One possibility to find the proper sign is the following (see [LeV04, Chapter 13.7]). For small shocks, i.e. if $\rho_M \approx \rho_-$ and $m_{M\,2} \approx m_{-\,2}$, we expect the linearized theory to be true. That means that the Hugoniot locus in U_- should be parallel to the eigenvector R_1. Hence for $\rho_M = \rho_- + \alpha$ with $\alpha \approx 0$ we expect

$$m_{M\,2} \approx m_{-\,2} + \alpha\left(\frac{m_{-\,2}}{\rho_-} - \sqrt{p'(\rho_-)}\right). \tag{3.21}$$

Let us now compute $m_{M\,2}$ for small α using (3.20) and compare the result to (3.21).

$$\begin{aligned}
m_{M\,2} &= \rho_M\,\frac{m_{-\,2}}{\rho_-} \pm \rho_M\sqrt{\frac{\left(\rho_M - \rho_-\right)\left(p(\rho_M) - p(\rho_-)\right)}{\rho_M\,\rho_-}} \\
&= (\rho_- + \alpha)\,\frac{m_{-\,2}}{\rho_-} \pm (\rho_- + \alpha)\sqrt{\frac{\alpha\left(p(\rho_- + \alpha) - p(\rho_-)\right)}{(\rho_- + \alpha)\,\rho_-}} \\
&= m_{-\,2} + \alpha\,\frac{m_{-\,2}}{\rho_-} \pm |\alpha|\sqrt{\frac{\rho_- + \alpha}{\rho_-}}\sqrt{\frac{p(\rho_- + \alpha) - p(\rho_-)}{\alpha}} \\
&\approx m_{-\,2} + \alpha\left(\frac{m_{-\,2}}{\rho_-} \pm \operatorname{sign}(\alpha)\sqrt{p'(\rho_-)}\right).
\end{aligned}$$

In view of (3.21) the proper sign is "$-$" if $\alpha > 0$, which means that $\rho_M > \rho_-$ and "$+$" if $\alpha < 0$, which means that $\rho_M < \rho_-$.
On the other hand the admissibility condition (3.19) determines the sign to be "$-$". Therefore we conclude that the 1-shock is admissible if $\rho_M > \rho_-$ and the proper sign is a "$-$".
Analogously one proves that for admissible 3-shocks it holds that

$$\frac{m_{M\,2}}{\rho_M} = \frac{m_{+\,2}}{\rho_+} + \sqrt{\frac{\left(\rho_M - \rho_+\right)\left(p(\rho_M) - p(\rho_+)\right)}{\rho_M\,\rho_+}}$$

and $\rho_M > \rho_+$. □

3.3. Rarefactions

We now turn our attention to rarefactions. Suppose that the 1-wave is a rarefaction that connects the state $U_- = (\rho_-, m_{-\,1}, m_{-\,2})$ on the left to the state $U_M = (\rho_M, m_{M\,1}, m_{M\,2})$ on the right. Then the fact that ω_2 and ω_3 from above are 1-Riemann invariants implies that

$$\frac{m_{-\,1}}{\rho_-} = \frac{m_{M\,1}}{\rho_M} \tag{3.22}$$

[6] Note that p is increasing since $p' > 0$. Hence $p(\rho_M) - p(\rho_-)$ and $\rho_M - \rho_-$ have the same sign, which means that the radicand is positive and the root is well-defined.

and

$$\frac{m_{-2}}{\rho_-} + \int_0^{\rho_-} \frac{\sqrt{p'(r)}}{r}\, dr = \frac{m_{M2}}{\rho_M} + \int_0^{\rho_M} \frac{\sqrt{p'(r)}}{r}\, dr. \tag{3.23}$$

Similarly, if there is a 3-rarefaction connecting the state $U_M = (\rho_M, m_{M1}, m_{M2})$ on the left to the state $U_+ = (\rho_+, m_{+1}, m_{+2})$ on the right then the 3-Riemann invariants ω_1 and ω_2 yield

$$\frac{m_{+1}}{\rho_+} = \frac{m_{M1}}{\rho_M} \tag{3.24}$$

and

$$\frac{m_{+2}}{\rho_+} - \int_0^{\rho_+} \frac{\sqrt{p'(r)}}{r}\, dr = \frac{m_{M2}}{\rho_M} - \int_0^{\rho_M} \frac{\sqrt{p'(r)}}{r}\, dr. \tag{3.25}$$

Lemma 3.2.

- *The 1-wave is a rarefaction, which connects the state $U_M = (\rho_M, m_{M1}, m_{M2})$ on the right to the state $U_- = (\rho_-, m_{-1}, m_{-2})$ on the left if*

$$\begin{aligned} \rho_M &< \rho_-, \\ \frac{m_{M1}}{\rho_M} &= \frac{m_{-1}}{\rho_-}, \\ \frac{m_{M2}}{\rho_M} &= \frac{m_{-2}}{\rho_-} + \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\, dr. \end{aligned}$$

- *The 3-wave is a rarefaction, which connects the state $U_M = (\rho_M, m_{M1}, m_{M2})$ on the left to the state $U_+ = (\rho_+, m_{+1}, m_{+2})$ on the right if*

$$\begin{aligned} \rho_M &< \rho_+, \\ \frac{m_{M1}}{\rho_M} &= \frac{m_{+1}}{\rho_+}, \\ \frac{m_{M2}}{\rho_M} &= \frac{m_{+2}}{\rho_+} - \int_{\rho_M}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\, dr. \end{aligned}$$

Proof. The above claims are simple consequences of (3.22) - (3.25). What remains to show are the conditions $\rho_M < \rho_-$ respectively $\rho_M < \rho_+$. Rarefaction waves are always admissible but they are only well-defined if the speeds are ordered properly. For the 1-rarefaction this means that $\lambda_1(U_-) < \lambda_1(U_M)$. Using (3.23) this is equivalent to

$$\sqrt{p'(\rho_M)} - \sqrt{p'(\rho_-)} < \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r} dr.$$

We compute the right-hand side further to

$$\begin{aligned} \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r} dr = \int_{\rho_M}^{\rho_-} \frac{\sqrt{K\,\gamma\, r^{\gamma-1}}}{r} dr = \sqrt{K\gamma} \int_{\rho_M}^{\rho_-} r^{\frac{\gamma-1}{2}-1} dr \\ = \frac{2}{\gamma-1}\Big(\sqrt{p'(\rho_-)} - \sqrt{p'(\rho_M)}\Big). \end{aligned} \tag{3.26}$$

This yields

$$\sqrt{p'(\rho_M)} - \sqrt{p'(\rho_-)} < -\frac{2}{\gamma-1}\Big(\sqrt{p'(\rho_M)} - \sqrt{p'(\rho_-)}\Big).$$

Since $1 > -\frac{2}{\gamma-1}$, this is equivalent to $\sqrt{p'(\rho_M)} < \sqrt{p'(\rho_-)}$. Because the functions $x \mapsto \sqrt{x}$ and $x \mapsto x^2$ are strictly monotone for $x > 0$, we get $p'(\rho_M) < p'(\rho_-)$. Again $\rho \mapsto p'(\rho)$ is strictly monotone since $\gamma > 1$ and $p''(\rho) = K\,\gamma\,(\gamma-1)\,\rho^{\gamma-2} > 0$. Therefore the inverse p'^{-1} is monotone, too, and we obtain that a 1-rarefaction makes sense if and only if $\rho_M < \rho_-$.
The condition $\rho_M < \rho_+$ for the 3-rarefaction wave can be shown analogously. □

3.4. Conclusion

Now we are ready to put all the things together and state the standard solution.

Proposition 3.3. (see [CK14, Lemma 2.4]) *Let $\rho_\pm \in \mathbb{R}^+$ and $v_\pm \in \mathbb{R}^2$ given constants, where $v_{-\,1} = v_{+\,1}$.*

1. *If*

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr \right| < v_{+\,2} - v_{-\,2} < \int_0^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr + \int_0^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr, \qquad (3.27)$$

then there is a unique self similar solution to the 1-dimensional Riemann problem (3.1), (3.2), which is admissible in the sense of (3.3) and consists of a 1-rarefaction and a 3-rarefaction. The intermediate state $(\rho_M, v_{M\,1}, v_{M\,2})$ is given by

$$\begin{aligned} \rho_M &< \min\{\rho_-, \rho_+\}, \\ v_{+\,2} - v_{-\,2} &= \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr + \int_{\rho_M}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr, \\ v_{M\,1} &= v_{-\,1} = v_{+\,1}, \\ v_{M\,2} &= v_{-\,2} + \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr. \end{aligned}$$

2. *If $\rho_- > \rho_+$ and*

$$-\sqrt{\frac{(\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+}} < v_{+\,2} - v_{-\,2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr,$$

then there is a unique self similar solution to the 1-dimensional Riemann problem (3.1), (3.2), which is admissible in the sense of (3.3) and consists of a 1-rarefaction and a 3-shock. The intermediate state $(\rho_M, v_{M\,1}, v_{M\,2})$ is given by

$$\begin{aligned} &\rho_+ < \rho_M < \rho_-, \\ v_{+\,2} - v_{-\,2} &= \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr - \sqrt{\frac{(\rho_M - \rho_+)\,\big(p(\rho_M) - p(\rho_+)\big)}{\rho_M\,\rho_+}}, \\ v_{M\,1} &= v_{-\,1} = v_{+\,1}, \\ v_{M\,2} &= v_{-\,2} + \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr. \end{aligned}$$

3. If $\rho_- < \rho_+$ *and*

$$-\sqrt{\frac{(\rho_- - \rho_+)\,(p(\rho_-) - p(\rho_+))}{\rho_-\,\rho_+}} < v_{+\,2} - v_{-\,2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr,$$

then there is a unique self similar solution to the 1-dimensional Riemann problem (3.1), (3.2), *which is admissible in the sense of* (3.3) *and consists of a 1-shock and a 3-rarefaction. The intermediate state* $(\rho_M, v_{M\,1}, v_{M\,2})$ *is given by*

$$\begin{aligned}
\rho_- &< \rho_M < \rho_+,\\
v_{+\,2} - v_{-\,2} &= \int_{\rho_M}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr - \sqrt{\frac{(\rho_M - \rho_-)\,(p(\rho_M) - p(\rho_-))}{\rho_M\,\rho_-}},\\
v_{M\,1} &= v_{-\,1} = v_{+\,1},\\
v_{M\,2} &= v_{-\,2} - \sqrt{\frac{(\rho_M - \rho_-)\,(p(\rho_M) - p(\rho_-))}{\rho_M\,\rho_-}}.
\end{aligned}$$

4. If

$$v_{+\,2} - v_{-\,2} < -\sqrt{\frac{(\rho_- - \rho_+)\,(p(\rho_-) - p(\rho_+))}{\rho_-\,\rho_+}},$$

then there is a unique self similar solution to the 1-dimensional Riemann problem (3.1), (3.2), *which is admissible in the sense of* (3.3) *and consists of a 1-shock and a 3-shock. The intermediate state* $(\rho_M, v_{M\,1}, v_{M\,2})$ *is given by*

$$\begin{aligned}
\rho_M &> \max\{\rho_-, \rho_+\},\\
v_{+\,2} - v_{-\,2} &= -\sqrt{\frac{(\rho_M - \rho_+)\,(p(\rho_M) - p(\rho_+))}{\rho_M\,\rho_+}} - \sqrt{\frac{(\rho_M - \rho_-)\,(p(\rho_M) - p(\rho_-))}{\rho_M\,\rho_-}},\\
v_{M\,1} &= v_{-\,1} = v_{+\,1},\\
v_{M\,2} &= v_{-\,2} - \sqrt{\frac{(\rho_M - \rho_-)\,(p(\rho_M) - p(\rho_-))}{\rho_M\,\rho_-}}.
\end{aligned}$$

In each case the unique solution is an admissible weak solution to the 2-dimensional problem (1.1), (1.6), *too, and is called* standard solution*.*

Remark. Note that not every case is covered by the above proposition. For completeness we want to mention the remaining cases. First of all if

$$v_{+\,2} - v_{-\,2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr + \int_0^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr,$$

then there is a unique admissible solution to the 1-dimensional Riemann problem (3.4), (3.5), which consists of a 1-rarefaction and a 3-rarefaction. Here the intermediate state is vacuum, i.e. $\rho_M = 0$ (see [CK14, Lemma 2.4, Case 1]). Since in this thesis we consider $\rho > 0$, we're going to omit this case.
If

$$v_{+\,2} - v_{-\,2} = \left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr \right|,$$

there is a unique self similar solution, which consists of just one rarefaction.
If

$$v_{+2} - v_{-2} = -\sqrt{\frac{\left(\rho_- - \rho_+\right)\left(p(\rho_-) - p(\rho_+)\right)}{\rho_- \, \rho_+}},$$

there is a unique self similar solution, which consists of just one admissible shock.
In this thesis we will only consider the cases covered by proposition 3.3.

Proof. We start with the first component of the velocity. Since this component remains constant at the 1- and 3-wave (see lemma 3.1 and 3.2) it can only jump at the 2-contact discontinuity. But with our assumption that $v_{-1} = v_{+1}$ there actually is no change of v_1 at all. Since ρ and v_2 are continuous at the 2-contact discontinuity, too, nothing happens there. In other words the contact discontinuity is not apparent. So we have that $v_{M\,1} = v_{-1} = v_{+1}$.
Next we try to find out the values of ρ and the second component of the velocity v_2 of the intermediate state between the 1- and the 3-wave. Using lemmas 3.1 and 3.2 we conclude that these values are given by the point of intersection of the curves

$$v_2(\rho) = \begin{cases} v_{-2} - \sqrt{\frac{(\rho-\rho_-)\,(p(\rho)-p(\rho_-))}{\rho\,\rho_-}} & \text{if } \rho > \rho_- \\ v_{-2} + \int_\rho^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr & \text{if } \rho < \rho_- \end{cases} \tag{3.28}$$

and

$$v_2(\rho) = \begin{cases} v_{+2} + \sqrt{\frac{(\rho-\rho_+)\,(p(\rho)-p(\rho_+))}{\rho\,\rho_+}} & \text{if } \rho > \rho_+ \\ v_{+2} - \int_\rho^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr & \text{if } \rho < \rho_+. \end{cases} \tag{3.29}$$

The question, on which part of the curves the point of intersection lies, gives also an answer to the question if the 1-, resp. 3-wave is a shock or a rarefaction.

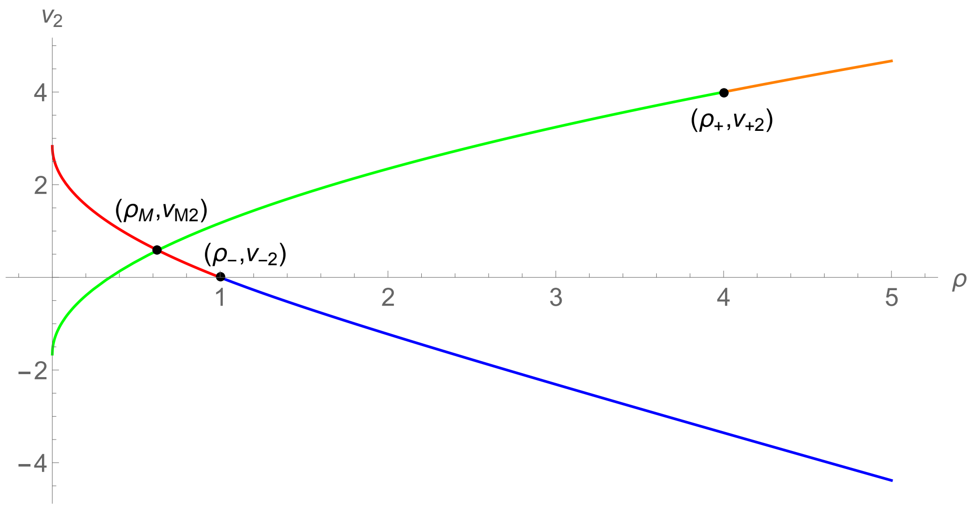

Figure 2: The curves given in (3.28) and (3.29) through the initial values $(\rho_-, v_{-2}) = (1,0)$ and $(\rho_+, v_{+2}) = (4,4)$ and for $p(\rho) = \rho^2$. The rarefaction parts are red, resp. green, whereas the shock parts are blue, resp. orange. The point of intersection lies on the rarefaction parts of the two curves, which means that the 1- and 3-wave both are rarefactions. The intermediate state is given by the point of intersection.

Assume that the initial values $\rho_\pm$ and $v_{\pm 2}$ are such that (3.27) is true. Then one can check that there is a $\rho_M < \min\{\rho_-, \rho_+\}$ such that the curves (3.28) and (3.29) intersect at $(\rho_M, v_{M\,2})$ where ρ_M fulfills

$$v_{+\,2} - v_{-\,2} = \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr + \int_{\rho_M}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr$$

and $v_{M\,2}$ can be computed by

$$v_{M\,2} = v_{-\,2} + \int_{\rho_M}^{\rho_-} \frac{\sqrt{p'(r)}}{r}\,dr.$$

Hence in this case the 1- and 3-wave both are rarefactions. An example is given in figure 2. The other cases can be treated analogously.
For the uniqueness of self similar solutions to the 1-dimensional Riemann problem we refer to textbooks or [CLK15, Proposition 8.1]. □

4. A sufficient condition for non-uniqueness

In this section we want to show that there are infinitely many admissible weak solutions to the Cauchy problem (1.1), (1.6) if a so-called admissible fan subsolution exists. First of all we have to define what such an admissible fan subsolution is.

4.1. Definitions

Definition 4.1. *(fan partition, see [CLK15, Definition 3.3])* Let $N \in \mathbb{N}$ and $\nu_0 < \nu_1 < \ldots < \nu_N$ real numbers. A fan partition of $(0,\infty) \times \mathbb{R}^2$ is a set of open sets $P_-, P_1, \ldots, P_N, P_+$ of the form

$$\begin{aligned} P_- &= \{(t,x) : t > 0 \text{ and } x_2 < \nu_0\, t\}, \\ P_i &= \{(t,x) : t > 0 \text{ and } \nu_{i-1}\, t < x_2 < \nu_i\, t\}, \\ P_+ &= \{(t,x) : t > 0 \text{ and } x_2 > \nu_N\, t\}, \end{aligned}$$

see figure 3.

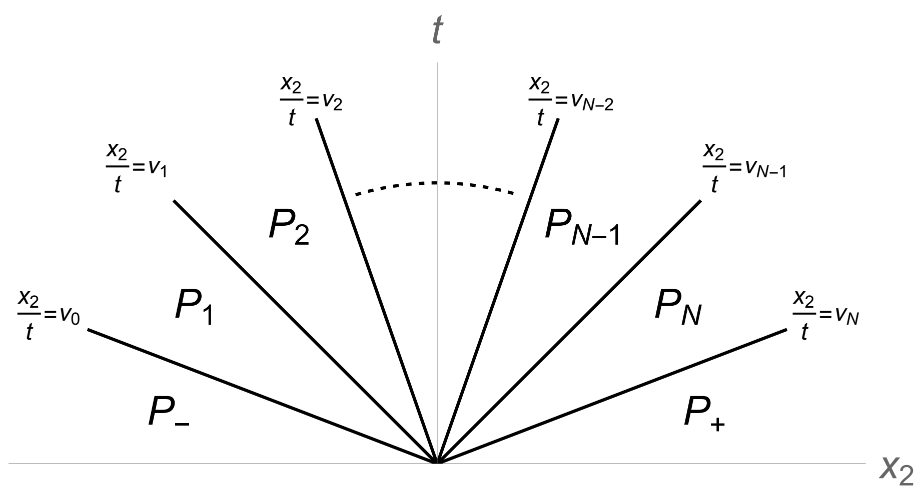

Figure 3: fan partition

Definition 4.2. *(admissible fan subsolution)* A fan subsolution to the Euler system (1.1) with initial condition (1.6) is a triple $(\overline{\rho}, \overline{v}, \overline{u}) : (0,\infty) \times \mathbb{R}^2 \to (\mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ of piecewise constant functions, which satisfies the following properties:

1. There is a fan partition $P_-, P_1, \dots, P_N, P_+$ of $(0,\infty) \times \mathbb{R}^2$ such that

$$(\overline{\rho}, \overline{v}, \overline{u}) = (\rho_-, v_-, u_-)\, \mathbf{1}_{P_-} + \sum_{i=1}^{N} (\rho_i, v_i, u_i)\, \mathbf{1}_{P_i} + (\rho_+, v_+, u_+)\, \mathbf{1}_{P_+},$$

   where $\rho_i \in \mathbb{R}^+$, $v_i \in \mathbb{R}^2$, $u_i \in \mathcal{S}_0^{2\times 2}$ are constants, $u_\pm := v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2 \,\mathrm{Id}$ and $\mathbf{1}_A$ denotes the indicator function on the set A.

2. There exist disjoint sets $I_=, I_C \subset \{1, \dots, N\}$ with $I_C \neq \varnothing$, $I_= \cap I_C = \varnothing$ and $I_= \cup I_C = \{1, \dots, N\}$, such that
   a) for every $i \in I_=$ it holds that

$$v_i \otimes v_i - u_i = \frac{1}{2}\, |v_i|^2 \,\mathrm{Id}, \tag{4.1}$$

   b) for every $i \in I_C$ there is a constant $C_i \in \mathbb{R}^+$ such that[7]

$$v_i \otimes v_i - u_i < \frac{1}{2}\, C_i \,\mathrm{Id}. \tag{4.2}$$

3. For all test functions $(\psi, \phi) \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2)$ the following identities hold:

$$\int_0^\infty \int_{\mathbb{R}^2} \big(\overline{\rho}\, \partial_t \psi + \overline{\rho}\, \overline{v} \cdot \nabla_x \psi\big) dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, \psi(0,x)\, dx \;=\; 0, \tag{4.3}$$

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} \Bigg[\overline{\rho}\, \overline{v} \cdot \partial_t \phi + \overline{\rho}\, \overline{u} : D_x \phi + \Bigg(p(\overline{\rho}) + \frac{1}{2}\overline{\rho} \Bigg(|v_-|^2\, \mathbf{1}_{P_-} \\ + \sum_{i \in I_=} |v_i|^2\, \mathbf{1}_{P_i} + \sum_{i \in I_C} C_i\, \mathbf{1}_{P_i} + |v_+|^2\, \mathbf{1}_{P_+} \Bigg) \Bigg) \mathrm{div}_x \phi \Bigg] dx\, dt \\ + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx \;=\; 0,\end{aligned} \tag{4.4}$$

   where again : is the Frobenius product

$$\overline{\rho}\, \overline{u} : D_x \phi = \sum_{i,j=1}^{2} \overline{\rho}\, \overline{u}_{ij}\, \partial_j \phi_i.$$

A fan subsolution is admissible if in addition for every non-negative test function

[7] Here we have an inequality of matrices, which is meant in the sense of definiteness. That means, that $A < B$ for $A, B \in \mathcal{S}^{2\times 2}$, if $B - A$ is positive definite.

$\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R}_0^+)$ the inequality

$$\int_0^\infty \int_{\mathbb{R}^2} \Bigg[\Bigg(\overline{\rho}\,\varepsilon(\overline{\rho}) + \frac{1}{2}\overline{\rho} \Big(|v_-|^2 \mathbf{1}_{P_-} + \sum_{i \in I_=} |v_i|^2 \mathbf{1}_{P_i} + \sum_{i \in I_C} C_i \mathbf{1}_{P_i} + |v_+|^2 \mathbf{1}_{P_+} \Big) \Bigg) \partial_t \varphi$$
$$+ \Bigg(\overline{\rho}\,\varepsilon(\overline{\rho}) + p(\overline{\rho}) + \frac{1}{2}\overline{\rho} \Big(|v_-|^2 \mathbf{1}_{P_-} + \sum_{i \in I_=} |v_i|^2 \mathbf{1}_{P_i} + \sum_{i \in I_C} C_i \mathbf{1}_{P_i} + |v_+|^2 \mathbf{1}_{P_+} \Big) \Bigg)$$
$$\overline{v} \cdot \nabla_x \varphi \Bigg] \, dx\, dt + \int_{\mathbb{R}^2} \rho_0(x) \left(\varepsilon(\rho_0(x)) + \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \quad \geq \quad 0 \qquad (4.5)$$

is fulfilled.

Remark. We chose to present a definition of an admissible fan subsolution which is as general as possible and it is therefore slightly different from [CK14, Definitions 5, 6] and [CLK15, Definitions 3.4, 3.5]. The latter definitions are special cases of our definition. Later in section 5, where we discuss existence of admissible fan subsolutions, we will be only interested in simple fan subsolutions. More precisely we will look for admissible fan subsolution where the corresponding fan partition consists of just three parts P_-, P_1, P_+ (i.e. $N = 1$, $I_C = \{1\}$ and $I_= = \varnothing$).
Furthermore the notion of a fan subsolution slightly differs from the notion of a subsolution in the actual sense described in section 2. More precisely a fan subsolution is a subsolution in the actual sense in all wedges P_i with $i \in I_C$ and a weak solution in the wedges P_i with $i \in \{-,+\} \cup I_=$.

4.2. The condition

Now we are ready to claim the condition for the existence of infinitely many admissible weak solutions.

Theorem 4.3. ([CLK15, Proposition 3.6] and [CK14, Proposition 3.1]) *Let* $(\rho_\pm, v_\pm)$ *be such that there exists an admissible fan subsolution* $(\overline{\rho}, \overline{v}, \overline{u})$ *to the Cauchy problem* (1.1), (1.6)*. Then there are infinitely many admissible weak solutions* (ρ, v) *to* (1.1), (1.6) *with* $\rho = \overline{\rho}$*.*

So in order to prove that a solution is non-unique we will try to show existence of an admissible fan subsolution. In this case theorem 4.3 will yield infinitely many admissible weak solutions.

4.3. Proof of the condition

The proof of theorem 4.3 is organized as follows. First we prove the actual theorem using another proposition, namely the convex integration proposition 4.4, which deals with the existence of infinitely many weak solutions to the pressureless incompressible Euler system. This proposition is proved by the convex integration method. The heart of the convex integration method, the perturbation property, is summarized in lemma 4.5. Hence we prove proposition 4.4 with help of lemma 4.5 and finally show that this lemma is true.

4.3.1. Proof of the condition (Theorem 4.3)

To prove theorem 4.3 we will use the convex integration method for the pressureless incompressible Euler equations developed by C. De Lellis and L. Székelyhidi and introduced in section 2. This method - worked out in proposition 4.4 - yields infinitely many weak solutions to the pressureless incompressible Euler equations on the sets P_i where $i \in I_C$ which we will add to the subsolution.

Proposition 4.4. (convex integration proposition, [CLK15, Lemma 3.7] and [CK14, Lemma 3.2]) *Let* $(\widetilde{v}, \widetilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ *and* $C > 0$ *such that* $\widetilde{v} \otimes \widetilde{v} - \widetilde{u} < \frac{C}{2} Id$*. Furthermore let* $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ *open. Then there exist infinitely many maps* $(\underline{v}, \underline{u}) \in L^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ *with the following properties.*

1. $\underline{v}$ *and* $\underline{u}$ *vanish outside* Ω.
2. *For all test functions* $(\psi, \phi) \in C_c^\infty(\Omega, \mathbb{R} \times \mathbb{R}^2)$ *it holds that*
$$\iint_\Omega \underline{v} \cdot \nabla_x \psi \, dx \, dt = 0,$$
$$\iint_\Omega (\underline{v} \cdot \partial_t \phi + \underline{u} : D_x \phi) \, dx \, dt = 0.$$
3. $(\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) - (\widetilde{u} + \underline{u}) = \frac{C}{2} Id$ *is fulfilled almost everywhere on* Ω.

Let us now prove theorem 4.3.

Proof. (see [CLK15, Proof of proposition 3.6])
Let $(\overline{\rho}, \overline{v}, \overline{u})$ be an admissible fan subsolution to the initial value problem (1.1), (1.6), which exists by assumption. For every $i \in I_C$ we apply proposition 4.4 to $(\widetilde{v}, \widetilde{u}) = (v_i, u_i)$, $C = C_i$ and $\Omega = P_i$ to obtain infinitely many maps $(\underline{v}_i, \underline{u}_i) \in L^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ that fulfill the three properties given in proposition 4.4. Now we define
$$\rho = \overline{\rho} \qquad \text{and} \qquad v = \overline{v} + \sum_{i \in I_C} \underline{v}_i$$
to obtain infinitely many maps (ρ, v) since for all $i \in I_C$ there are infinitely many $\underline{v}_i$. We will show that the pairs (ρ, v) defined as above are in fact admissible weak solutions to (1.1), (1.6). Note that the ρ_i, v_i, u_i are constants, whereas $\underline{v}_i, \underline{u}_i$ are functions of t and x.
We have to show, that the equations (1.3), (1.4) and the inequality (1.5) hold for all test functions $(\psi, \phi, \varphi) \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_0^+)$, so let them be arbitrary. Let us start with (1.3):
$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^2} \big(\rho \, \partial_t \psi + \rho \, v \cdot \nabla_x \psi\big) dx \, dt + \int_{\mathbb{R}^2} \rho_0(x) \, \psi(0,x) \, dx \\
&= \int_0^\infty \int_{\mathbb{R}^2} \left[\overline{\rho} \, \partial_t \psi + \overline{\rho} \left(\overline{v} + \sum_{i \in I_C} \underline{v}_i\right) \cdot \nabla_x \psi\right] dx \, dt + \int_{\mathbb{R}^2} \rho_0(x) \, \psi(0,x) \, dx \\
&= \underbrace{\int_0^\infty \int_{\mathbb{R}^2} \big(\overline{\rho} \, \partial_t \psi + \overline{\rho} \, \overline{v} \cdot \nabla_x \psi\big) dx \, dt + \int_{\mathbb{R}^2} \rho_0(x) \, \psi(0,x) \, dx}_{=0, \text{ because } (\overline{\rho}, \overline{v}, \overline{u}) \text{ is a fan subsolution}} + \sum_{i \in I_C} \iint_{P_i} \overline{\rho} \, \underline{v}_i \cdot \nabla_x \psi \, dx \, dt \\
&= \sum_{i \in I_C} \rho_i \underbrace{\iint_{P_i} \underline{v}_i \cdot \nabla_x \psi \, dx \, dt}_{=0, \text{ because of proposition 4.4}} = 0.
\end{aligned}$$

Hence (1.3) holds. Now we check (1.4):

$$\int_0^\infty \int_{\mathbb{R}^2} \big(\rho\, v \cdot \partial_t \phi + \rho\, v \otimes v : D_x\phi + p(\rho)\operatorname{div}_x \phi\big) dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx$$
$$= \int_0^\infty \int_{\mathbb{R}^2} \Bigg[\overline{\rho} \left(\overline{v} + \sum_{i\in I_C} \underline{v}_i \right) \cdot \partial_t \phi$$
$$+ \overline{\rho} \left(\sum_{i\in I_= \cup\{-,+\}} v_i \otimes v_i\, \mathbf{1}_{P_i} + \sum_{i\in I_C} (v_i + \underline{v}_i) \otimes (v_i + \underline{v}_i)\, \mathbf{1}_{P_i} \right) : D_x\phi$$
$$+ p(\overline{\rho}) \operatorname{div}_x \phi \Bigg] dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx.$$

From (4.1) and the 3rd property in proposition 4.4 we have that $v_i \otimes v_i = u_i + \frac{|v_i|^2}{2} \operatorname{Id}$ for all $i \in I_= \cup \{-,+\}$ and $(v_i + \underline{v}_i) \otimes (v_i + \underline{v}_i) = u_i + \underline{u}_i + \frac{C_i}{2} \operatorname{Id}$ for all $i \in I_C$ and almost every $(t,x) \in P_i$. Therefore, using that $\operatorname{Id} : D_x\phi = \operatorname{div}_x \phi$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^2} \big(\rho\, v \cdot \partial_t \phi + \rho\, v \otimes v : D_x\phi + p(\rho)\operatorname{div}_x \phi\big) dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx$$
$$= \int_0^\infty \int_{\mathbb{R}^2} \Bigg[\overline{\rho} \left(\overline{v} + \sum_{i\in I_C} \underline{v}_i \right) \cdot \partial_t \phi$$
$$+ \overline{\rho} \left(\overline{u} + \sum_{i\in I_= \cup\{-,+\}} \frac{|v_i|^2}{2} \operatorname{Id} \mathbf{1}_{P_i} + \sum_{i\in I_C} \left(\underline{u}_i + \frac{C_i}{2} \operatorname{Id} \right) \mathbf{1}_{P_i} \right) : D_x\phi$$
$$+ p(\overline{\rho}) \operatorname{div}_x \phi \Bigg] dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx$$
$$= \int_0^\infty \int_{\mathbb{R}^2} \Bigg[\overline{\rho}\, \overline{v} \cdot \partial_t \phi + \overline{\rho}\, \overline{u} : D_x\phi + \Bigg(p(\overline{\rho}) + \overline{\rho} \sum_{i\in I_= \cup\{-,+\}} \frac{|v_i|^2}{2} \mathbf{1}_{P_i}$$
$$+ \overline{\rho} \sum_{i\in I_C} \frac{C_i}{2} \mathbf{1}_{P_i} \Bigg) \operatorname{div}_x \phi \Bigg] dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx$$
$$+ \sum_{i\in I_C} \underbrace{\iint_{P_i} \big(\overline{\rho}\, \underline{v}_i \cdot \partial_t \phi + \overline{\rho}\, \underline{u}_i : D_x \phi \big)\, dx\, dt}_{=\rho_i \iint_{P_i} (\underline{v}_i \cdot \partial_t \phi + \underline{u}_i : D_x\phi)\, dx\, dt = 0} \quad = \quad 0.$$

Here we again used that $(\overline{\rho}, \overline{v}, \overline{u})$ is a fan subsolution and proposition 4.4.

What remains is showing that (1.5) holds:

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \left[\left(\rho\, \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi + \left(\rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dx\, dt \\
&\quad + \int_{\mathbb{R}^d} \left(\rho_0(x)\, \varepsilon(\rho_0(x)) + \rho_0(x)\, \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \\
&= \int_0^\infty \int_{\mathbb{R}^d} \left[\left(\overline{\rho}\, \varepsilon(\overline{\rho}) + \sum_{i \in I_= \cup \{-,+\}} \overline{\rho}\, \frac{|v_i|^2}{2}\, \mathbf{1}_{P_i} + \sum_{i \in I_C} \overline{\rho}\, \frac{|v_i + \underline{v}_i|^2}{2}\, \mathbf{1}_{P_i} \right) \partial_t \varphi \right. \\
&\qquad + \left(\overline{\rho}\, \varepsilon(\overline{\rho}) + \sum_{i \in I_= \cup \{-,+\}} \overline{\rho}\, \frac{|v_i|^2}{2}\, \mathbf{1}_{P_i} + \sum_{i \in I_C} \overline{\rho}\, \frac{|v_i + \underline{v}_i|^2}{2}\, \mathbf{1}_{P_i} + p(\overline{\rho}) \right) \\
&\qquad\qquad \left. \cdot \left(\overline{v} + \sum_{i \in I_C} \underline{v}_i \right) \cdot \nabla_x \varphi \right] dx\, dt \\
&\qquad + \int_{\mathbb{R}^d} \left(\rho_0(x)\, \varepsilon(\rho_0(x)) + \rho_0(x)\, \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx.
\end{aligned}
$$

Again the 3rd property in proposition 4.4 yields for all $i \in I_C$ that

$$
\begin{aligned}
|v_i + \underline{v}_i|^2 &= \mathrm{tr}\big((v_i + \underline{v}_i) \otimes (v_i + \underline{v}_i) \big) \\
&= \mathrm{tr}\Big(u_i + \underline{u}_i + \frac{C_i}{2}\, \mathrm{Id} \Big) \\
&= C_i,
\end{aligned}
$$

because $\mathrm{tr}(u_i) = \mathrm{tr}(\underline{u}_i) = 0$. Using this we get

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \left[\left(\rho\, \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi + \left(\rho\, \varepsilon(\rho) + \rho\, \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dx\, dt \\
&\quad + \int_{\mathbb{R}^d} \left(\rho_0(x)\, \varepsilon(\rho_0(x)) + \rho_0(x)\, \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \\
&= \int_0^\infty \int_{\mathbb{R}^d} \left[\left(\overline{\rho}\, \varepsilon(\overline{\rho}) + \sum_{i \in I_= \cup \{-,+\}} \overline{\rho}\, \frac{|v_i|^2}{2}\, \mathbf{1}_{P_i} + \sum_{i \in I_C} \overline{\rho}\, \frac{C_i}{2}\, \mathbf{1}_{P_i} \right) \partial_t \varphi \right. \\
&\qquad + \left(\overline{\rho}\, \varepsilon(\overline{\rho}) + \sum_{i \in I_= \cup \{-,+\}} \overline{\rho}\, \frac{|v_i|^2}{2}\, \mathbf{1}_{P_i} + \sum_{i \in I_C} \overline{\rho}\, \frac{C_i}{2}\, \mathbf{1}_{P_i} + p(\overline{\rho}) \right) \\
&\qquad\qquad \left. \cdot \left(\overline{v} + \sum_{i \in I_C} \underline{v}_i \right) \cdot \nabla_x \varphi \right] dx\, dt \\
&\qquad + \int_{\mathbb{R}^d} \left(\rho_0(x)\, \varepsilon(\rho_0(x)) + \rho_0(x)\, \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \\
&\geq \sum_{i \in I_C} \iint_{P_i} \left(\overline{\rho}\, \varepsilon(\overline{\rho}) + \sum_{i \in I_= \cup \{-,+\}} \overline{\rho}\, \frac{|v_i|^2}{2}\, \mathbf{1}_{P_i} + \sum_{i \in I_C} \overline{\rho}\, \frac{C_i}{2}\, \mathbf{1}_{P_i} + p(\overline{\rho}) \right) \underline{v}_i \cdot \nabla_x \varphi \; dx\, dt \\
&= \sum_{i \in I_C} \iint_{P_i} \left(\rho_i\, \varepsilon(\rho_i) + \rho_i\, \frac{C_i}{2} + p(\rho_i) \right) \underline{v}_i \cdot \nabla_x \varphi \; dx\, dt \\
&= \sum_{i \in I_C} \left(\rho_i\, \varepsilon(\rho_i) + \rho_i\, \frac{C_i}{2} + p(\rho_i) \right) \iint_{P_i} \underline{v}_i \cdot \nabla_x \varphi \; dx\, dt \quad = \quad 0,
\end{aligned}
$$

where we again applied that the fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ is admissible and proposition 4.4.
Hence (ρ, v) is an admissible weak solution of (1.1), (1.6), i.e. we proved theorem 4.3 with help of proposition 4.4. □

4.3.2. Proof of the convex integration proposition (Proposition 4.4)

In this section we're going to prove proposition 4.4 using the convex integration method introduced in section 2. In this proof we will also make use of some Baire arguments. The results of Baire's theory we need are presented in appendix B. Apart from that one needs some knowledge about the weak* topology on L^∞, what we exhibit in appendix A.

Proof. (see [CLK15, Proof of lemma 3.7])
We start with the set

$$X_0 := \Big\{ (\underline{v}, \underline{u}) \in C_c^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}) \Big| \\ \operatorname{div}_x \underline{v} = 0,\ \partial_t \underline{v} + \operatorname{div}_x \underline{u} = 0 \text{ and } (\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) - (\widetilde{u} + \underline{u}) < \frac{C}{2} \operatorname{Id} \Big\}.$$

First of all we're going to show that this set is bounded in $L^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$, what results from the inequality

$$(\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) - (\widetilde{u} + \underline{u}) < \frac{C}{2} \operatorname{Id}.$$

This inequality means, that the symmetric 2×2 matrix

$$\frac{C}{2} \operatorname{Id} - (\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) + (\widetilde{u} + \underline{u})$$

is positive definite. Hence for all $(x, y)^T \in \mathbb{R}^2 \setminus \{0\}$ the following holds:

$$\begin{aligned} 0 &< (x, y) \cdot \left(\frac{C}{2} \operatorname{Id} - (\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) + (\widetilde{u} + \underline{u}) \right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{C}{2}(x^2 + y^2) - \big((\widetilde{v}_1 + \underline{v}_1)\, x + (\widetilde{v}_2 + \underline{v}_2)\, y \big)^2 \\ &\qquad + (\widetilde{u}_{11} + \underline{u}_{11})\,(x^2 - y^2) + 2\,(\widetilde{u}_{12} + \underline{u}_{12})\, x\, y. \end{aligned}$$

Here we used, that $\widetilde{u} + \underline{u}$ has the form

$$\widetilde{u} + \underline{u} = \begin{pmatrix} \widetilde{u}_{11} + \underline{u}_{11} & \widetilde{u}_{12} + \underline{u}_{12} \\ \widetilde{u}_{12} + \underline{u}_{12} & -\widetilde{u}_{11} - \underline{u}_{11} \end{pmatrix}$$

for $\widetilde{u}, \underline{u} \in \mathcal{S}_0^{2\times 2}$. By choosing $(x, y)^T$ equal $(1, 0)^T$, $(0, 1)^T$, $\big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big)^T$ and $\big(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\big)^T$, we get

$$\begin{aligned} 0 &< \frac{C}{2} - (\widetilde{v}_1 + \underline{v}_1)^2 + (\widetilde{u}_{11} + \underline{u}_{11}), \\ 0 &< \frac{C}{2} - (\widetilde{v}_2 + \underline{v}_2)^2 - (\widetilde{u}_{11} + \underline{u}_{11}), \\ 0 &< \frac{C}{2} - \frac{1}{2}\big((\widetilde{v}_1 + \underline{v}_1) + (\widetilde{v}_2 + \underline{v}_2) \big)^2 + (\widetilde{u}_{12} + \underline{u}_{12}), \\ 0 &< \frac{C}{2} - \frac{1}{2}\big((\widetilde{v}_1 + \underline{v}_1) - (\widetilde{v}_2 + \underline{v}_2) \big)^2 - (\widetilde{u}_{12} + \underline{u}_{12}). \end{aligned}$$

These inequalities imply

$$\begin{aligned}C &> (\widetilde{v}_1+\underline{v}_1)^2+(\widetilde{v}_2+\underline{v}_2)^2=|\widetilde{v}+\underline{v}|^2,\\ |\widetilde{u}_{11}+\underline{u}_{11}| &< C,\\ |\widetilde{u}_{12}+\underline{u}_{12}| &< C.\end{aligned}$$

Let now $(\underline{v},\underline{u})\in X_0$. With the above computations we have[8]

$$\begin{aligned}||(\underline{v},\underline{u})||_{L^\infty} &= \operatorname*{ess\,sup}_{(t,x)\in\Omega} ||(\underline{v}(t,x),\underline{u}(t,x))||_{\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}}\\ &= \operatorname*{ess\,sup}_{(t,x)\in\Omega}\big(|\underline{v}(t,x)|+|\underline{u}_{11}(t,x)|+|\underline{u}_{12}(t,x)|\big)\\ &= \operatorname*{ess\,sup}_{(t,x)\in\Omega}\big(|\underline{v}(t,x)+\widetilde{v}-\widetilde{v}|+|\underline{u}_{11}(t,x)+\widetilde{u}_{11}-\widetilde{u}_{11}|+|\underline{u}_{12}(t,x)+\widetilde{u}_{12}-\widetilde{u}_{12}|\big)\\ &\le \operatorname*{ess\,sup}_{(t,x)\in\Omega}\big(|\underline{v}(t,x)+\widetilde{v}|+|\widetilde{v}|+|\underline{u}_{11}(t,x)+\widetilde{u}_{11}|+|\widetilde{u}_{11}|+|\underline{u}_{12}(t,x)+\widetilde{u}_{12}|+|\widetilde{u}_{12}|\big)\\ &< \sqrt{C}+|\widetilde{v}|+C+|\widetilde{u}_{11}|+C+|\widetilde{u}_{12}|,\end{aligned}$$

which shows that X_0 is bounded in L^∞.

Now we define X to be the closure of X_0 in the L^∞ weak* topology. Because X_0 is bounded, proposition A.3 yields that X is bounded, too. Therefore by proposition A.4 the weak* topology on X is metrizable and X is weak* compact. We will denote the metric, which induces the weak* topology on X, as d. So (X,d) is a complete metric space because of the compactness of X.

Note that $X\subset L^\infty(\Omega,\mathbb{R}^2\times\mathcal{S}_0^{2\times 2})$, i.e. the 1st property of proposition 4.4 is fulfilled[9] on X. It is not difficult to show that the 2nd property is true on X, too:

Let $(\underline{v},\underline{u})\in X$. Then there exists a sequence[10] $(\underline{v}_n,\underline{u}_n)_{n\in\mathbb{N}}\subset X_0$ that converges to $(\underline{v},\underline{u})$ in the L^∞ weak* topology, i.e. in the metric d. Let now $(\psi,\phi)\in C_c^\infty(\Omega,\mathbb{R}\times\mathbb{R}^2)$ test functions. Since the derivatives of the test functions are L^1 functions and $(\underline{v}_n,\underline{u}_n)\in X_0$ for all $n\in\mathbb{N}$ (i.e. $\operatorname{div}_x\underline{v}=0,\ \partial_t\underline{v}+\operatorname{div}_x\underline{u}=0$), we get

$$\begin{aligned}\iint_\Omega \underline{v}\cdot\nabla_x\psi\,dx\,dt &= \lim_{n\to\infty}\iint_\Omega \underline{v}_n\cdot\nabla_x\psi\,dx\,dt=0,\\ \iint_\Omega\big(\underline{v}\cdot\partial_t\phi+\underline{u}:D_x\phi\big)\,dx\,dt &= \lim_{n\to\infty}\iint_\Omega\underline{v}_n\cdot\partial_t\phi\,dx\,dt\\ &\quad+\lim_{n\to\infty}\iint_\Omega\underline{u}_n:D_x\phi\,dx\,dt=0,\end{aligned}$$

which proves that the 2nd property of proposition 4.4 holds on X.

To complete the proof of proposition 4.4 we want to show that the subset

$$Y:=\big\{(\underline{v},\underline{u})\in X\,\big|\,\text{the 3rd condition of proposition 4.4 holds}\big\}\subset X$$

[8] Note that the space $\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}$ has dimension 4 because the dimension of $\mathbb{R}^2$ is 2 and the dimension of $\mathcal{S}_0^{2\times 2}$ is also 2. Since all norms on finite dimensional vector spaces are equivalent, we can use any norm. For simplicity we choose here:

$$||(\underline{v},\underline{u})||:=|\underline{v}|+|\underline{u}_{11}|+|\underline{u}_{22}|=\sqrt{\underline{v}_1^2+\underline{v}_2^2}+|\underline{u}_{11}|+|\underline{u}_{22}|.$$

[9] To be precise, we identify each $f\in L^\infty(\Omega)$ as $f\in L^\infty(\mathbb{R}\times\mathbb{R}^2)$ via setting $f\equiv 0$ outside Ω.

[10] In general the weak* closedness of X guarantees the existence of a net that converges to $(\underline{v},\underline{u})$ in the L^∞ weak* topology. This net does not need to be a sequence. But in this case the weak* topology is metrizable, so X is closed with respect to the metric d. Therefore there is even a sequence converging to $(\underline{v},\underline{u})$.

is infinite. To do so we will show that this set Y is residual[11]. According to the Baire category theorem (theorem B.4) and since (X, d) is a complete metric space, this residual set Y is dense in X. We will then show that this implies that Y is infinite.
Now define $\Gamma_N := \Omega \cap \big((-N, N) \times B_N(0)\big) \subset \mathbb{R} \times \mathbb{R}^2$ for all $N \in \mathbb{N}$ and consider the maps[12] $I_N : (X, d) \to \big(L^\infty(\Gamma_N, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}), ||\cdot||_{L^2}\big)$, defined by $(\underline{v}, \underline{u}) \mapsto (\underline{v}, \underline{u})\big|_{\Gamma_N}$. The next step is to show that I_N is a Baire-1-function for all $N \in \mathbb{N}$, what can be done similar to [LS09, Proof of lemma 4.5]. Let $\Phi \in C_c^\infty\big((-1, 1) \times B_1(0), \mathbb{R}_0^+\big)$ be a function with

$$\int_{-1}^{1} \int_{B_1(0)} \Phi(t, x)\, dx\, dt = 1$$

and define $\Phi_r(t, x) := \frac{1}{r^3}\, \Phi\big(\frac{t}{r}, \frac{x}{r}\big)$, for $r > 0$.
Let $I_{N\,r} : (X, d) \to \big(L^\infty(\Gamma_N, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}), ||\cdot||_{L^2}\big)$ defined by $(\underline{v}, \underline{u}) \mapsto (\Phi_r * \underline{v}, \Phi_r * \underline{u})\big|_{\Gamma_N}$. The convolution here is meant component-wise. First we're going to show that $I_{N\,r}$ is continuous for all $N \in \mathbb{N}$ and $r > 0$. Let $(v, u) \in X$ and $(v_k, u_k)_{k\in\mathbb{N}} \subset X$ be a sequence which converges in the weak* topology, i.e. $(v_k, u_k) \overset{*}{\rightharpoonup} (v, u)$. We have to show that

$$(\Phi_r * v_k, \Phi_r * u_k)\big|_{\Gamma_N} \overset{k\to\infty}{\longrightarrow} (\Phi_r * v, \Phi_r * u)\big|_{\Gamma_N},$$

strongly in $L^2(\Gamma_N)$, which is equivalent to

$$\Big|\Big|\big(\Phi_r * (v_k - v), \Phi_r * (u_k - u)\big)\big|_{\Gamma_N}\Big|\Big|_{L^2} \overset{k\to\infty}{\longrightarrow} 0. \tag{4.6}$$

It holds that

$$\begin{aligned}
&\big(\Phi_r * (v_k - v), \Phi_r * (u_k - u)\big)(t, x) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi_r(t - \widetilde{t}, x - \widetilde{x})\, \big(v_k(\widetilde{t}, \widetilde{x}) - v(\widetilde{t}, \widetilde{x}), u_k(\widetilde{t}, \widetilde{x}) - u(\widetilde{t}, \widetilde{x})\big)\, d\widetilde{x}\, d\widetilde{t} \quad \overset{k\to\infty}{\longrightarrow} \quad 0,
\end{aligned}$$

because $(\widetilde{t}, \widetilde{x}) \mapsto \Phi_r(t - \widetilde{t}, x - \widetilde{x})$ is an L^1 function and $(v_k, u_k) \overset{*}{\rightharpoonup} (v, u)$. This implies (4.6) and hence the $I_{N\,r}$ are continuous.
In addition for fixed $(v, u) \in X$ we get that

$$(\Phi_r * v, \Phi_r * u)\big|_{\Gamma_N} \overset{r\to 0}{\longrightarrow} (v, u)\big|_{\Gamma_N}$$

strongly in $L^2(\Gamma_N)$ (see e.g. [Lan93, Chapter VIII, Corollary 3.4]), what shows the convergence $I_{N\,r}(v, u) \overset{r\to 0}{\longrightarrow} I_N(v, u)$ for all $(v, u) \in X$. Hence for all $N \in \mathbb{N}$, I_N is the pointwise limit of a sequence of continuous functions, in other words a Baire-1-function.
Now by proposition B.6 it follows that the sets

$$C_N := \big\{(\underline{v}, \underline{u}) \in X \,\big|\, I_N \text{ is continuous in } (\underline{v}, \underline{u})\big\} \subset X$$

[11] For the notions of Baire's theory we refer to appendix B.

[12] The original idea by C. De Lellis and L. Székelyhidi in [LS09, Lemma 4.5] is to consider the identity $I : (X, d) \to (L^\infty(\Omega), ||\cdot||_{L^2})$. But this works only for bounded sets Ω. The problem is that we cannot endow $L^\infty(\Omega)$ with the strong L^2-topology if Ω is unbounded. So we have to change over to the restriction to the sets Γ_N (see also [CK14, Proof of lemma 3.2] or [CLK15, Proof of lemma 3.7]).

are residual in X.
Next we are going to show that on C_N the 3rd property of proposition 4.4 is fulfilled almost everywhere on Γ_N. Let $(\underline{v}, \underline{u}) \in C_N$ be arbitrary. We first prove that $(\widetilde{v}+\underline{v}) \otimes (\widetilde{v}+\underline{v}) - (\widetilde{u}+\underline{u}) \leq \frac{C}{2}\,\mathrm{Id}$ a.e. on Γ_N. Since X is the closure of X_0 in the L^∞ weak* topology, there is a sequence $(\underline{v}_k, \underline{u}_k)_{k\in\mathbb{N}} \subset X_0$ converging to $(\underline{v}, \underline{u})$ in the L^∞ weak* topology. Because I_N is continuous in $(\underline{v}, \underline{u})$, the sequence $\big((\underline{v}_k, \underline{u}_k)\big|_{\Gamma_N}\big)_{k\in\mathbb{N}}$ converges to $(\underline{v}, \underline{u})\big|_{\Gamma_N}$ even strongly in the L^2 norm. Hence there exists a subsequence which converges almost everywhere on Γ_N pointwise to $(\underline{v}, \underline{u})$, what is showed in [Lan93, Chapter VII, Theorem 1.4]. Since the function which maps a pair $(v, u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ to the eigenvalues of the matrix $\frac{C}{2}\,\mathrm{Id} - (\widetilde{v}+v) \otimes (\widetilde{v}+v) + (\widetilde{u}+u)$ is continuous, and because the eigenvalues of $\frac{C}{2}\,\mathrm{Id} - (\widetilde{v}+\underline{v}_k) \otimes (\widetilde{v}+\underline{v}_k) + (\widetilde{u}+\underline{u}_k)$ are positive (since $(\underline{v}_k, \underline{u}_k) \in X_0$), we conclude that the eigenvalues of $\frac{C}{2}\,\mathrm{Id} - (\widetilde{v}+\underline{v}) \otimes (\widetilde{v}+\underline{v}) + (\widetilde{u}+\underline{u})$ are non-negative almost everywhere on Γ_N. This means that $(\widetilde{v}+\underline{v}) \otimes (\widetilde{v}+\underline{v}) - (\widetilde{u}+\underline{u}) \leq \frac{C}{2}\,\mathrm{Id}$ a.e. on Γ_N.
Assume by contradiction that the 3rd property of proposition 4.4 is not true almost everywhere on Γ_N. That means that there is a subset $\widetilde{\Gamma} \subset \Gamma_N$ of non-zero measure such that the matrix $\frac{C}{2}\,\mathrm{Id} - \big(\widetilde{v}+\underline{v}(t,x)\big) \otimes \big(\widetilde{v}+\underline{v}(t,x)\big) + \big(\widetilde{u}+\underline{u}(t,x)\big)$ is non-zero, and, if we remember what we showed recently, positive semi-definite for all $(t,x) \in \widetilde{\Gamma}$. In other words both eigenvalues are not negative and at least one is positive. Therefore the trace of the above matrix, which is equal to the sum of the eigenvalues, is positive for all $(t,x) \in \widetilde{\Gamma}$:

$$\mathrm{tr}\bigg(\frac{C}{2}\,\mathrm{Id} - \big(\widetilde{v}+\underline{v}(t,x)\big) \otimes \big(\widetilde{v}+\underline{v}(t,x)\big) + \big(\widetilde{u}+\underline{u}(t,x)\big)\bigg) = C - \big|\widetilde{v}+\underline{v}(t,x)\big|^2 > 0.$$

Hence we conclude that

$$\begin{aligned} ||\widetilde{v}+\underline{v}||^2_{L^2(\Gamma_N)} &= \int_{\Gamma_N} |\widetilde{v}+\underline{v}|^2\,dx\,dt \\ &= \underbrace{\int_{\widetilde{\Gamma}} |\widetilde{v}+\underline{v}|^2\,dx\,dt}_{<C\,|\widetilde{\Gamma}|} + \underbrace{\int_{\Gamma_N\setminus\widetilde{\Gamma}} |\widetilde{v}+\underline{v}|^2\,dx\,dt}_{\leq C\,|\Gamma_N\setminus\widetilde{\Gamma}|} \quad < \quad C\,|\Gamma_N|, \end{aligned} \tag{4.7}$$

what we will need later.
Now we state an important lemma which is known as *perturbation property* or *oscillatory lemma*. This lemma is the heart of the convex integration method. We postpone its proof.

Lemma 4.5. (perturbation property, see [CLK15, Section 4.1]) *Let $\Gamma \subset \Omega$ open and bounded, and $(\underline{v}, \underline{u}) \in X_0$. Then there exists a sequence $(v_k, u_k)_{k\in\mathbb{N}} \subset X_0$ with the following properties:*

- $(v_k, u_k) \overset{*}{\rightharpoonup} (\underline{v}, \underline{u})$.
- *There exists a constant $\beta > 0$ such that*

$$\liminf_{k\to\infty} ||\widetilde{v}+v_k||^2_{L^2(\Gamma)} \geq ||\widetilde{v}+\underline{v}||^2_{L^2(\Gamma)} + \beta\left(C\,|\Gamma| - ||\widetilde{v}+\underline{v}||^2_{L^2(\Gamma)}\right)^2.$$

Since $(\underline{v},\underline{u}) \in X$, there is a sequence $(\underline{v}_k,\underline{u}_k)_{k\in\mathbb{N}} \subset X_0$ converging L^∞ weakly* to $(\underline{v},\underline{u})$. We set now $\Gamma = \Gamma_N$ and apply the perturbation property (Lemma 4.5) to each $(\underline{v}_k,\underline{u}_k) \in X_0$ and find for every $k \in \mathbb{N}$ sequences $(v_{k,j},u_{k,j})_{j\in\mathbb{N}} \subset X_0$ such that

$$\liminf_{j\to\infty} ||\widetilde{v} + v_{k,j}||^2_{L^2(\Gamma_N)} \geq ||\widetilde{v} + \underline{v}_k||^2_{L^2(\Gamma_N)} + \beta\left(C\,|\Gamma_N| - ||\widetilde{v} + \underline{v}_k||^2_{L^2(\Gamma_N)}\right)^2 \tag{4.8}$$

and $(v_{k,j},u_{k,j}) \underset{j\to\infty}{\overset{*}{\rightharpoonup}} (\underline{v}_k,\underline{u}_k)$. With a standard diagonal argument we get a sequence $(v_{k,j(k)},u_{k,j(k)})_{k\in\mathbb{N}} \subset X_0$ with $(v_{k,j(k)},u_{k,j(k)}) \underset{k\to\infty}{\overset{*}{\rightharpoonup}} (\underline{v},\underline{u})$. Since $(\underline{v},\underline{u}) \in C_N$, we have that $(\underline{v}_k,\underline{u}_k)\big|_{\Gamma_N} \to (\underline{v},\underline{u})\big|_{\Gamma_N}$ and $(v_{k,j(k)},u_{k,j(k)})\big|_{\Gamma_N} \to (\underline{v},\underline{u})\big|_{\Gamma_N}$ strongly in $L^2(\Gamma_N)$ as $k\to\infty$. Therefore we can pass to the limit in (4.8) to obtain

$$||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma_N)} \geq ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma_N)} + \beta\left(C\,|\Gamma_N| - ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma_N)}\right)^2 > ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma_N)},$$

where the last inequality comes from (4.7). This is a contradiction, i.e. the assumption that the 3rd property of proposition 4.4 does not hold almost everywhere on Γ_N is not true.

Hence we proved that on C_N the 3rd property of proposition 4.4 is fulfilled a.e. on Γ_N. Therefore $Y \supset \bigcap_{N\in\mathbb{N}} C_N$ and lemma B.2 yields that Y contains a residual set since all the sets C_N are residual, what we showed above. Now because (X,d) is a complete metric space, the Baire category theorem (Theorem B.4) yields that Y is dense. To conclude we will prove that this implies that Y is infinite. Since $0 = (0,0) \in X_0$, $X_0 \neq \varnothing$. From the perturbation property (Lemma 4.5) it follows that $|X_0| = \infty$ what is explained in what follows. Because $0 \in X_0$ we can apply the perturbation property to find a sequence $(v_k,u_k)_{k\in\mathbb{N}} \subset X_0$ with

$$\liminf_{k\to\infty} ||\widetilde{v} + v_k||^2_{L^2(\Gamma_N)} \geq |\Gamma_N|\,|\widetilde{v}|^2 + \beta\,|\Gamma_N|^2\left(C - |\widetilde{v}|^2\right)^2 > |\Gamma_N|\,|\widetilde{v}|^2$$

since $|\widetilde{v}|^2 < C$. So there is a $K \in \mathbb{N}$ such that $||\widetilde{v} + v_K||^2_{L^2(\Gamma_N)} > |\Gamma_N|\,|\widetilde{v}|^2$, and we set $(\widehat{v}_1,\widehat{u}_1) := (v_K,u_K)$. Now applying the perturbation property to $(\widehat{v}_1,\widehat{u}_1)$ we obtain a sequence $(v_k,u_k)_{k\in\mathbb{N}} \subset X_0$ with

$$\liminf_{k\to\infty} ||\widetilde{v}+v_k||^2_{L^2(\Gamma_N)} \geq ||\widetilde{v}+\widehat{v}_1||^2_{L^2(\Gamma_N)} + \beta\left(C\,|\Gamma_N| - ||\widetilde{v}+\widehat{v}_1||^2_{L^2(\Gamma_N)}\right)^2 > ||\widetilde{v}+\widehat{v}_1||^2_{L^2(\Gamma_N)}$$

because $(\widehat{v}_1,\widehat{u}_1) \in X_0$ and therefore $||\widetilde{v} + \widehat{v}_1||^2_{L^2(\Gamma_N)} < C\,|\Gamma_N|$. This again yields a $K \in \mathbb{N}$ such that $||\widetilde{v} + v_K||^2_{L^2(\Gamma_N)} > ||\widetilde{v} + \widehat{v}_1||^2_{L^2(\Gamma_N)}$, and we set $(\widehat{v}_2,\widehat{u}_2) := (v_K,u_K)$. Repeating this successively we get a sequence $(\widehat{v}_k,\widehat{u}_k)_{k\in\mathbb{N}} \subset X_0$ where $||\widetilde{v} + \widehat{v}_k||^2_{L^2(\Gamma_N)} < ||\widetilde{v} + \widehat{v}_{k+1}||^2_{L^2(\Gamma_N)}$ for all $k \in \mathbb{N}$. This implies that $|X_0| = \infty$ and hence also $|X| = \infty$. Assume Y is finite. Then Y is closed, since finite subsets of metric spaces are closed. But this contradicts the denseness of Y in X, i.e. $|Y| = \infty$, what finishes the proof. □

4.3.3. Proof of the perturbation property (Lemma 4.5)

Let us prove the perturbation property, lemma 4.5.

Proof. Let $\Gamma \subset \Omega$ open and bounded, and $(\underline{v}, \underline{u}) \in X_0$. The proof consists of three steps, where the actual claim of lemma 4.5 is proved in step 3 and the first two steps deal with auxiliary statements.

Step 1 (see also [LS10, Lemma 3])
Define

$$\mathcal{U} := \left\{ (v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\middle|\, v \otimes v - u < \frac{C}{2}\,\mathrm{Id} \right\} \text{ and}$$
$$K := \left\{ (v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\middle|\, v \otimes v - u = \frac{C}{2}\,\mathrm{Id} \right\}.$$

The first step is to show that $\mathcal{U} = (K^{\mathrm{co}})^\circ$ where K^{co} denotes the convex hull of K. We define the function $e : \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \to \mathbb{R}$ through

$$e(v,u) := \lambda_{\max}\big(v \otimes v - u\big),$$

where $\lambda_{\max} : \mathcal{S}^{2\times 2} \to \mathbb{R}$ maps to the largest eigenvalue of a symmetric matrix.
Next we want to show that e is a convex function. It is not difficult to understand that

$$e(v,u) = \max_{y \in \mathbb{R}^2, ||y||=1} \langle\, y \,|\, (v \otimes v - u)\, y \,\rangle. \tag{4.9}$$

To prove this, let $y_{\min}, y_{\max} \in \mathbb{R}^2$ be the two normed eigenvectors of $v \otimes v - u$ and $\lambda_{\min}, \lambda_{\max} \in \mathbb{R}$ the corresponding eigenvalues[13] where $\lambda_{\min} \leq \lambda_{\max}$. Note that $v \otimes v - u \in \mathcal{S}^{2\times 2}$ and hence it is diagonalizable with orthogonal eigenvectors, i.e. $\langle\, y_{\max} \,|\, y_{\min} \,\rangle = 0$. Then

$$\begin{aligned} e(v,u) &= \lambda_{\max} = \lambda_{\max} \langle\, y_{\max} \,|\, y_{\max} \,\rangle = \langle\, y_{\max} \,|\, \lambda_{\max}\, y_{\max} \,\rangle \\ &= \langle\, y_{\max} \,|\, (v \otimes v - u)\, y_{\max} \,\rangle \leq \max_{y \in \mathbb{R}^2, ||y||=1} \langle\, y \,|\, (v \otimes v - u)\, y \,\rangle. \end{aligned}$$

Let $\overline{y} \in \mathbb{R}^2$ such that $\max_{y \in \mathbb{R}^2, ||y||=1} \langle\, y \,|\, (v \otimes v - u)\, y \,\rangle = \langle\, \overline{y} \,|\, (v \otimes v - u)\, \overline{y} \,\rangle$. Then there are coefficients $\alpha, \beta \in \mathbb{R}$ such that $\overline{y} = \alpha\, y_{\max} + \beta\, y_{\min}$, and $\alpha^2 + \beta^2 = 1$ because $||\overline{y}|| = ||y_{\max}|| = ||y_{\min}|| = 1$. So we obtain

$$\begin{aligned} \max_{y \in \mathbb{R}^2, ||y||=1} \langle\, y \,|\, (v \otimes v - u)\, y \,\rangle \; &= \; \langle\, \overline{y} \,|\, (v \otimes v - u)\, \overline{y} \,\rangle \\ &= \alpha^2 \langle\, y_{\max} \,|\, (v \otimes v - u)\, y_{\max} \,\rangle + \alpha\,\beta \,\langle\, y_{\max} \,|\, (v \otimes v - u)\, y_{\min} \,\rangle \\ &\quad + \alpha\,\beta\,\langle\, y_{\min} \,|\, (v \otimes v - u)\, y_{\max} \,\rangle + \beta^2 \langle\, y_{\min} \,|\, (v \otimes v - u)\, y_{\min} \,\rangle \\ &= \alpha^2\, \lambda_{\max} \langle\, y_{\max} \,|\, y_{\max} \,\rangle + \alpha\,\beta\,\lambda_{\min} \langle\, y_{\max} \,|\, y_{\min} \,\rangle \\ &\quad + \alpha\,\beta\,\lambda_{\max} \langle\, y_{\min} \,|\, y_{\max} \,\rangle + \beta^2\, \lambda_{\min} \langle\, y_{\min} \,|\, y_{\min} \,\rangle \\ &= \alpha^2\, \lambda_{\max} + \beta^2\, \lambda_{\min} \quad \leq \quad \lambda_{\max} \quad = \quad e(v,u), \end{aligned}$$

[13]Here we abused the notation a bit. Recently we wrote $\lambda_{\max}$ for the function that maps a symmetric matrix to its largest eigenvalue and here we use it for this function's value at $v \otimes v - u$.

which finishes the proof of (4.9). An easy calculation yields

$$\begin{aligned} e(v,u) &= \max_{y\in\mathbb{R}^2, ||y||=1} \langle\, y \,|\, (v\otimes v - u)\, y\,\rangle = \max_{y\in\mathbb{R}^2, ||y||=1} \Big(\langle\, y \,|\, v\, v^T\, y\,\rangle - \langle\, y \,|\, u\, y\,\rangle \Big) \\ &= \max_{y\in\mathbb{R}^2, ||y||=1} \Big(\langle\, y \,|\, v\,\rangle^2 - \langle\, y \,|\, u\, y\,\rangle \Big). \end{aligned}$$

We are ready to show now the convexity of e. Let $(v_1,u_1),(v_2,u_2) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ and $t \in [0,1]$. Furthermore, let $\overline{y} \in \mathbb{R}^2$ with $||\overline{y}|| = 1$ be such that

$$\begin{aligned} &\max_{y\in\mathbb{R}^2, ||y||=1} \Big(\langle\, y \,|\, (t\, v_1 + (1-t)\, v_2)\,\rangle^2 - \langle\, y \,|\, (t\, u_1 + (1-t)\, u_2)\, y\,\rangle \Big) \\ &\qquad = \langle\, \overline{y} \,|\, (t\, v_1 + (1-t)\, v_2)\,\rangle^2 - \langle\, \overline{y} \,|\, (t\, u_1 + (1-t)\, u_2)\, \overline{y}\,\rangle. \end{aligned}$$

Then

$$\begin{aligned} & e\big((t\, v_1 + (1-t)\, v_2), (t\, u_1 + (1-t)\, u_2)\big) \\ &\quad = \langle\, \overline{y} \,|\, (t\, v_1 + (1-t)\, v_2)\,\rangle^2 - \langle\, \overline{y} \,|\, (t\, u_1 + (1-t)\, u_2)\, \overline{y}\,\rangle \\ &\quad = \Big(t\, \langle\, \overline{y} \,|\, v_1\,\rangle + (1-t)\, \langle\, \overline{y} \,|\, v_2\,\rangle \Big)^2 - t\, \langle\, \overline{y} \,|\, u_1\, \overline{y}\,\rangle - (1-t)\, \langle\, \overline{y} \,|\, u_2\, \overline{y}\,\rangle \\ &\quad \le t\, \langle\, \overline{y} \,|\, v_1\,\rangle^2 + (1-t)\, \langle\, \overline{y} \,|\, v_2\,\rangle^2 - t\, \langle\, \overline{y} \,|\, u_1\, \overline{y}\,\rangle - (1-t)\, \langle\, \overline{y} \,|\, u_2\, \overline{y}\,\rangle \\ &\quad = t\, \Big(\langle\, \overline{y} \,|\, v_1\,\rangle^2 - \langle\, \overline{y} \,|\, u_1\, \overline{y}\,\rangle \Big) + (1-t)\, \Big(\langle\, \overline{y} \,|\, v_2\,\rangle^2 - \langle\, \overline{y} \,|\, u_2\, \overline{y}\,\rangle \Big) \\ &\quad \le t \max_{y\in\mathbb{R}^2, ||y||=1} \Big(\langle\, y \,|\, v_1\,\rangle^2 - \langle\, y \,|\, u_1\, y\,\rangle \Big) + (1-t) \max_{y\in\mathbb{R}^2, ||y||=1} \Big(\langle\, y \,|\, v_2\,\rangle^2 - \langle\, y \,|\, u_2\, y\,\rangle \Big) \\ &\quad = t\ e(v_1,u_1) + (1-t)\ e(v_2,u_2). \end{aligned}$$

We used hereby the inequality $\big(t\, a + (1-t)\, b\big)^2 \le t\, a^2 + (1-t)\, b^2$, which is true for all $a, b \in \mathbb{R}$, $t \in [0,1]$ and easy to recalculate.
Now we want to show that

$$\begin{aligned} v\otimes v - u < \frac{C}{2}\,\mathrm{Id} \quad &\Leftrightarrow \quad e(v,u) < \frac{C}{2} \qquad \text{and} \\ v\otimes v - u \le \frac{C}{2}\,\mathrm{Id} \quad &\Leftrightarrow \quad e(v,u) \le \frac{C}{2} \end{aligned}$$

for all $(v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$. With this we can characterize the elements of $\mathcal{U}$ better. Using easy linear algebra, one can show that λ is an eigenvalue of $v\otimes v - u$ if and only if $\frac{C}{2} - \lambda$ is an eigenvalue of $\frac{C}{2}\,\mathrm{Id} - v\otimes v + u$. Let first $v\otimes v - u < \frac{C}{2}\,\mathrm{Id}$. So the matrix $\frac{C}{2}\,\mathrm{Id} - v\otimes v + u$ is positive definite, which means that both eigenvalues of this matrix are positive. Especially $\lambda_{\min}(\frac{C}{2}\,\mathrm{Id} - v\otimes v + u) > 0$. So we get that

$$e(v,u) = \lambda_{\max}(v\otimes v - u) = \frac{C}{2} - \lambda_{\min}\Big(\frac{C}{2}\,\mathrm{Id} - v\otimes v + u\Big) < \frac{C}{2}.$$

Conversely let $e(v,u) = \lambda_{\max}(v\otimes v - u) < \frac{C}{2}$. With the above claim on the eigenvalues of $v\otimes v - u$ and $\frac{C}{2}\,\mathrm{Id} - v\otimes v + u$, this yields that

$$\lambda_{\min}\Big(\frac{C}{2}\,\mathrm{Id} - v\otimes v + u\Big) = \frac{C}{2} - \lambda_{\max}(v\otimes v - u) > 0.$$

Therefore $\frac{C}{2}\,\mathrm{Id} - v\otimes v + u$ is positive definite, which shows that $v\otimes v - u < \frac{C}{2}\,\mathrm{Id}$. The claim for semi-definiteness and $e(v,u) \le \frac{C}{2}$ can be showed analogously.

Remark. Note that

$$v \otimes v - u = \frac{C}{2}\,\mathrm{Id} \quad \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \quad e(v,u) = \frac{C}{2}.$$

In particular there are $(v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ with $e(v,u) = \frac{C}{2}$ and $v \otimes v - u \leq \frac{C}{2}\,\mathrm{Id}$ but $v \otimes v - u \neq \frac{C}{2}\,\mathrm{Id}$.

We're going to show that $\mathcal{U}$ is open. It is not difficult to understand that e is a continuous function. Easily one can check that $(v,u) \mapsto v \otimes v - u$ is continuous. The map $M \mapsto \lambda_{\max}(M)$ is also continuous[14] for $M \in \mathcal{S}^{2\times 2}$ and therefore e is continuous. Since $\mathcal{U}$ is the pre-image of the open set $\big(-\infty, \frac{C}{2}\big)$ under e, the set $\mathcal{U}$ is open.

Now we consider the closure of $\mathcal{U}$. We get

$$\overline{\mathcal{U}} = \left\{ (v,u) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \,\middle|\, v \otimes v - u \leq \frac{C}{2}\,\mathrm{Id} \right\} =: S,$$

what is shown in the sequel. Because S is the pre-image of the closed set $\big(-\infty, \frac{C}{2}\big]$ under e, it is closed. The fact that $\mathcal{U} \subset S$ is clear. Therefore we have $\overline{\mathcal{U}} \subset S$. To show that $S \subset \overline{\mathcal{U}}$ we let $(v,u) \in S$ be arbitrary and prove that there is a sequence $(v_k, u_k)_{k\in\mathbb{N}} \subset \mathcal{U}$ which converges to (v,u). If $(v,u) \in \mathcal{U}$ this is obvious. So assume $(v,u) \notin \mathcal{U}$, i.e. $e(v,u) = \frac{C}{2}$. Then define $(v_k, u_k) := \big((1-\frac{1}{k})\,v, (1-\frac{1}{k})^2\,u\big)$. Obviously $(v_k, u_k) \overset{k\to\infty}{\longrightarrow} (v,u)$ and

$$\begin{aligned} e(v_k, u_k) &= \lambda_{\max}\left[\left(1 - \frac{1}{k}\right)^2 v \otimes v - \left(1 - \frac{1}{k}\right)^2 u\right] \\ &= \left(1 - \frac{1}{k}\right)^2 \lambda_{\max}(v \otimes v - u) = \left(1 - \frac{1}{k}\right)^2 \frac{C}{2} \quad < \quad \frac{C}{2}, \end{aligned}$$

because $(1-\frac{1}{k})^2 < 1$ for all $k \in \mathbb{N}$. So we have $(v_k, u_k) \in \mathcal{U}$ for all $k \in \mathbb{N}$. Hence we proved that $\overline{\mathcal{U}} = S$.

Now we are ready to prove that $\overline{\mathcal{U}} = K^{\mathrm{co}}$. To do so, we use Minkowski's theorem C.1. Analogously to the proof that X_0 is bounded in the proof of proposition 4.4, it can be shown that $\overline{\mathcal{U}}$ is bounded. Since $\overline{\mathcal{U}}$ is closed and a subset of $\mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$, which has dimension 4 and is in particular finite dimensional, $\overline{\mathcal{U}}$ is compact by Heine-Borel. Additionally $\overline{\mathcal{U}}$ is a convex set which can be easily checked using the convexity of e: Let $(v_1, u_1), (v_2, u_2) \in \overline{\mathcal{U}}$ and $t \in [0,1]$. Then

$$e\big(t\,(v_1,u_1) + (1-t)\,(v_2,u_2)\big) \leq t\,e(v_1,u_1) + (1-t)\,e(v_2,u_2) \leq t\,\frac{C}{2} + (1-t)\,\frac{C}{2} = \frac{C}{2},$$

which shows that $t\,(v_1,u_1) + (1-t)\,(v_2,u_2) \in \overline{\mathcal{U}}$. So the assumptions of theorem C.1 are fulfilled and hence it suffices to show that the extreme points of $\overline{\mathcal{U}}$ are contained in K. Let $(v,u) \in \overline{\mathcal{U}} \setminus K$. We want to prove that (v,u) is not an extreme point

[14] This can be understood as follows: Let $M \in \mathcal{S}^{2\times 2}$. Then the characteristic polynomial reads $\lambda^2 - \mathrm{tr}(M)\,\lambda + \det(M)$, whose largest zero is $\lambda_{\max} = \frac{1}{2}\big(\mathrm{tr}(M) + \sqrt{(\mathrm{tr}(M))^2 - 4\det(M)}\big)$. Since $M \mapsto \mathrm{tr}(M)$ and $M \mapsto \det(M)$ are continuous, we conclude that $M \mapsto \lambda_{\max}(M)$ is also continuous.

of $\overline{\mathcal{U}}$. Since $v \otimes v - u \in \mathcal{S}^{2\times 2}$, it is diagonalizable with orthogonal eigenvalues, i.e. there exists an orthogonal matrix T such that

$$v \otimes v - u = T \begin{pmatrix} \lambda_{\max} & 0 \\ 0 & \lambda_{\min} \end{pmatrix} T^{-1}.$$

Because $(v,u) \in \overline{\mathcal{U}}$, we have that $\lambda_{\min} \leq \lambda_{\max} \leq \frac{C}{2}$. Additionally it holds that $\lambda_{\min} < \frac{C}{2}$, since otherwise we had $\lambda_{\min} = \lambda_{\max} = \frac{C}{2}$ and therefore

$$v \otimes v - u = T \, \frac{C}{2} \operatorname{Id} T^{-1} = \frac{C}{2} \operatorname{Id},$$

which contradicts $(v,u) \notin K$. Let $y_{\max}, y_{\min}$ be the normed eigenvectors which correspond to the eigenvalues $\lambda_{\max}, \lambda_{\min}$. Then there are unique coefficients $\alpha, \beta \in \mathbb{R}$ such that $v = \alpha \, y_{\max} + \beta \, y_{\min}$. Let now $(\widehat{v}, \widehat{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ defined by $\widehat{v} = y_{\min}$ and $\widehat{u} = \alpha \left(y_{\max} \otimes y_{\min} + y_{\min} \otimes y_{\max} \right)$. Then for $t \in \mathbb{R}$ one has

$$\begin{aligned}
&T^{-1} \big[(v + t\,\widehat{v}) \otimes (v + t\,\widehat{v}) - (u + t\,\widehat{u}) \big] \, T = \\
&= T^{-1} \big[v \otimes v + t\,(\widehat{v} \otimes v + v \otimes \widehat{v}) + t^2 \, \widehat{v} \otimes \widehat{v} - u - t\,\widehat{u} \big] \, T \\
&= T^{-1} \big[v \otimes v - u \big] \, T + t \, T^{-1} \big[y_{\min} \otimes v + v \otimes y_{\min} - \widehat{u} \big] \, T + t^2 \, T^{-1} \big[y_{\min} \otimes y_{\min} \big] \, T \\
&= T^{-1} \big[v \otimes v - u \big] \, T + t \, T^{-1} \big[\alpha \, y_{\min} \otimes y_{\max} + \beta \, y_{\min} \otimes y_{\min} + \alpha \, y_{\max} \otimes y_{\min} \\
&\quad + \beta \, y_{\min} \otimes y_{\min} - \alpha \, y_{\max} \otimes y_{\min} - \alpha \, y_{\min} \otimes y_{\max} \big] \, T + t^2 \, T^{-1} \big[y_{\min} \otimes y_{\min} \big] \, T \\
&= T^{-1} \big[v \otimes v - u \big] \, T + (2\,\beta\,t + t^2) \, T^{-1} \big[y_{\min} \otimes y_{\min} \big] \, T. \qquad (4.10)
\end{aligned}$$

Remember that the columns of the matrix T are exactly the eigenvectors $y_{\max}, y_{\min}$. Then we have that

$$\begin{aligned}
T^{-1} \big[y_{\min} \otimes y_{\min} \big] \, T &= T^T \, y_{\min} \, y_{\min}^T \, T = (y_{\min}^T \, T)^T \, (y_{\min}^T \, T) \\
&= (0,1)^T \, (0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

because

$$y_{\min}^T \, T = \big(y_{\min}^T \, y_{\max}, y_{\min}^T \, y_{\min} \big) = \big(\langle y_{\min} | y_{\max} \rangle, \langle y_{\min} | y_{\min} \rangle \big) = (0,1).$$

If we plug this in (4.10), we obtain

$$\begin{aligned}
&T^{-1} \big[(v + t\,\widehat{v}) \otimes (v + t\,\widehat{v}) - (u + t\,\widehat{u}) \big] \, T \\
&\quad = \begin{pmatrix} \lambda_{\max} & 0 \\ 0 & \lambda_{\min} \end{pmatrix} + (2\,\beta\,t + t^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Since $\lambda_{\max} \leq \frac{C}{2}$ and $\lambda_{\min} < \frac{C}{2}$, we conclude that for small $|t|$ we have that

$$e\big(v + t\,\widehat{v}, u + t\,\widehat{u} \big) \leq \frac{C}{2},$$

and hence using $\overline{\mathcal{U}} = S$ we find $\big(v + t\,\widehat{v}, u + t\,\widehat{u} \big) \in \overline{\mathcal{U}}$ for $|t|$ small enough. This proves that (v,u) can not be an extreme point because (v,u) can be expressed as a convex combination of $\big(v + t\,\widehat{v}, u + t\,\widehat{u} \big)$ and $\big(v - t\,\widehat{v}, u - t\,\widehat{u} \big)$. This shows that the extreme points of $\overline{\mathcal{U}}$ are contained in K. By Minkowski's theorem C.1 we get $\overline{\mathcal{U}} = K^{\mathrm{co}}$, which implies that $\overline{\mathcal{U}}^{\circ} = (K^{co})^{\circ}$. As explained above $\mathcal{U}$ is open and analogously to the proof of convexity of $\overline{\mathcal{U}}$ one can show that $\mathcal{U}$ is convex, too. Then proposition C.2 yields that $\mathcal{U} = \overline{\mathcal{U}}^{\circ}$ and therefore $\mathcal{U} = (K^{co})^{\circ}$.

Step 2 (see also [CLK15, Proposition 4.1 and Lemma 4.3], [LS10, Lemma 4 and 6]) Fix an arbitrary point $(t_0, x_0) \in \Gamma$. For convenience we define

$$(v^\star, u^\star) := \big(\widetilde{v} + \underline{v}(t_0, x_0), \widetilde{u} + \underline{u}(t_0, x_0)\big).$$

By assumption it holds that

$$v^\star \otimes v^\star - u^\star = \big(\widetilde{v} + \underline{v}(t_0, x_0)\big) \otimes \big(\widetilde{v} + \underline{v}(t_0, x_0)\big) - \big(\widetilde{u} + \underline{u}(t_0, x_0)\big) < \frac{C}{2}\,\mathrm{Id}. \tag{4.11}$$

The next step is to show that there exists a segment $\sigma = [-p, p] \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ with the following properties:

1. There exist $a, b \in \mathbb{R}^2$ with $|a| = |b| = \sqrt{C}$ and $a \neq \pm b$, and $\lambda > 0$ such that
$$p = \lambda\,[(a, a \otimes a) - (b, b \otimes b)].$$
2. For all $(\overline{v}, \overline{u}) \in \sigma$ it holds that
$$\big(\widetilde{v} + \underline{v}(t_0, x_0) + \overline{v}\big) \otimes \big(\widetilde{v} + \underline{v}(t_0, x_0) + \overline{v}\big) - \big(\widetilde{u} + \underline{u}(t_0, x_0) + \overline{u}\big) < \frac{C}{2}\,\mathrm{Id}.$$
3. For all $\varepsilon > 0$ there exists a pair $(v, u) \in C_c^\infty\big((-1, 1) \times B_1(0)\big)$ which solves
$$\begin{aligned} \operatorname{div}_x v &= 0, \\ \partial_t v + \operatorname{div}_x u &= 0, \end{aligned}$$
and such that
   a) for all $(t, x) \in (-1, 1) \times B_1(0)$ it holds that $\operatorname{dist}\big((v(t, x), u(t, x)), \sigma\big) < \varepsilon$,
   b) there exists a constant $c_1 > 0$ such that
$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \big|v(t, x)\big|\, dx\, dt \geq c_1 \left(C - \big|\widetilde{v} + \underline{v}(t_0, x_0)\big|^2\right)$$
   and
   c)
$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \big(v(t, x), u(t, x)\big)\, dx\, dt = 0,$$
   which is meant component-wise.

Remark. Note that (4.11) is equivalent to $(v^\star, u^\star) \in \mathcal{U}$, and the second claim is equivalent to $(v^\star, u^\star) + \sigma \subset \mathcal{U}$.

Using the result of step 1 we have that $(v^\star, u^\star) \in \mathcal{U} = (K^{co})^{\circ}$, i.e. there are finitely many $(v_i, u_i) \in K$ such that $(v^\star, u^\star)$ lies in the interior of the convex polytope spanned by the (v_i, u_i). Note that if $(v, u) \in K$ and $r > 0$ are arbitrary, then there exists another $(\widehat{v}, \widehat{u}) \in K$ with $||(\widehat{v}, \widehat{u}) - (v, u)|| < r$, where $||\cdot||$ can be any norm on $\mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ (since on finite dimensional spaces, every two norms are equivalent). This can be understood by looking for instance at

$$\begin{aligned}
\widehat{u}_{11} &:= u_{11} + \delta, \\
\widehat{v}_1 &:= \operatorname{sign}(v_1)\,\sqrt{v_1^2 + \delta}, \\
\widehat{v}_2 &:= \operatorname{sign}(v_2)\,\sqrt{v_2^2 - \delta}, \\
\widehat{u}_{12} &:= \widehat{v}_1\,\widehat{v}_2.
\end{aligned}$$

It is easy to check that $(\widehat{v}, \widehat{u}) \in K$ and one has $||(\widehat{v}, \widehat{u}) - (v, u)|| < r$ if $|\delta|$ is small enough.
Since $(v^\star, u^\star)$ lies in the interior, it is possible to slightly change the (v_i, u_i) as above to obtain $v_i \neq \pm v_j$ for all $i \neq j$. Now by Caratheodory's theorem C.4 and because $\dim(\mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}) = 4$, there are at most 5 points among the (v_i, u_i) and $\alpha_i \geq 0$ such that

$$(v^\star, u^\star) = \sum_{i=1}^{5} \alpha_i\,(v_i, u_i),$$

and $\sum_{i=1}^{5} \alpha_i = 1$. Since $(v^\star, u^\star) \notin K$, there are at least two indices i with $\alpha_i > 0$. Without loss of generality we can assume that the coefficients are ordered such that $\alpha_1 = \max_i \alpha_i$. Let j be such that $\alpha_j\,|v_j - v_1| = \max_i \alpha_i\,|v_i - v_1|$ and set $a = v_j$, $b = v_1$. Note that $j \neq 1$ and hence $a \neq \pm b$. We also obtain that $|a| = |b| = \sqrt{C}$ because $(v_i, u_i) \in K$ and therefore $|v_i|^2 = \operatorname{tr}(v_i \otimes v_i) = \operatorname{tr}(\frac{C}{2}\,\mathrm{Id} + u_i) = C$ (for all $i \in \{1, \dots, 5\}$). Further we set $\lambda = \frac{1}{2}\,\alpha_j$ and $p = \lambda\,[(a, a\otimes a) - (b, b \otimes b)]$. Then $p \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ since $\operatorname{tr}\big(\lambda(a \otimes a - b \otimes b)\big) = \lambda\,(|a|^2 - |b|^2) = 0$ and $\lambda(a\otimes a - b \otimes b)$ is obviously symmetric. Hence we showed the first claim.
Next we're going to prove that

$$(v^\star, u^\star) + \sigma \subset \mathcal{U},$$

which is, as already remarked, equivalent to the second claim. To show this, we first prove that $(v^\star, u^\star) \pm 2\,p \in K^{\mathrm{co}}$. Note that we can write p as follows:

$$\begin{aligned}
p &= \lambda\,[(a, a\otimes a) - (b, b\otimes b)] \\
&= \frac{1}{2}\,\alpha_j\,[(v_j, v_j \otimes v_j) - (v_1, v_1 \otimes v_1)] \\
&= \frac{1}{2}\,\alpha_j\,(v_j - v_1, v_j \otimes v_j - v_1 \otimes v_1) \\
&= \frac{1}{2}\,\alpha_j\,\left(v_j - v_1, v_j \otimes v_j - \frac{C}{2}\,\mathrm{Id} - v_1 \otimes v_1 + \frac{C}{2}\,\mathrm{Id}\right) \\
&= \frac{1}{2}\,\alpha_j\,(v_j - v_1, u_j - u_1) \\
&= \frac{1}{2}\,\alpha_j\,\big((v_j, u_j) - (v_1, u_1)\big),
\end{aligned}$$

because $(v_1, u_1), (v_j, u_j) \in K$. Let us now compute $(v^\star, u^\star) + 2\,p$ using the above identity for p:

$$\begin{aligned}(v^\star, u^\star) + 2\,p &= \sum_{i=1}^{5} \alpha_i\,(v_i, u_i) + \alpha_j\,\big((v_j, u_j) - (v_1, u_1)\big)\\ &= (\alpha_1 - \alpha_j)\,(v_1, u_1) + 2\,\alpha_j\,(v_j, u_j) + \sum_{i\in\{2,\dots,5\}\setminus\{j\}} \alpha_i\,(v_i, u_i).\end{aligned}$$

Note that the coefficients are all non-negative, since $\alpha_1 = \max_i \alpha_i$ and therefore $\alpha_1 \geq \alpha_j$, and the sum of all coefficients is 1:

$$\alpha_1 - \alpha_j + 2\,\alpha_j + \sum_{i\in\{2,\dots,5\}\setminus\{j\}} \alpha_i = \sum_{i=1}^{5} \alpha_i = 1.$$

That means that $(v^\star, u^\star) + 2\,p$ is a convex combination of elements of K, in other words $(v^\star, u^\star) + 2\,p \in K^{\mathrm{co}}$. Analogously we compute $(v^\star, u^\star) - 2\,p$:

$$\begin{aligned}(v^\star, u^\star) - 2\,p &= \sum_{i=1}^{5} \alpha_i\,(v_i, u_i) - \alpha_j\,\big((v_j, u_j) - (v_1, u_1)\big)\\ &= (\alpha_1 + \alpha_j)\,(v_1, u_1) + \sum_{i\in\{2,\dots,5\}\setminus\{j\}} \alpha_i\,(v_i, u_i).\end{aligned}$$

All the coefficients are non-negative and the sum of the coefficients is 1:

$$\alpha_1 + \alpha_j + \sum_{i\in\{2,\dots,5\}\setminus\{j\}} \alpha_i = \sum_{i=1}^{5} \alpha_i = 1.$$

That means that $(v^\star, u^\star) - 2\,p$ is a convex combination of elements of K and hence $(v^\star, u^\star) - 2\,p \in K^{\mathrm{co}}$.
Since $(v^\star, u^\star) \in \mathcal{U} = (K^{\mathrm{co}})^\circ$, there exists a positive radius $r > 0$ such that the whole ball $B_r(v^\star, u^\star) \subset K^{\mathrm{co}}$. Because K^{co} is convex, the convex hull of the set $B_r(v^\star, u^\star) \cup \{(v^\star, u^\star) \pm 2\,p\}$ is contained in K^{co}. So we have $(v^\star, u^\star) + \sigma \subset (K^{\mathrm{co}})^\circ = \mathcal{U}$ since σ lies obviously in the interior of the convex hull of $B_r(v^\star, u^\star) \cup \{(v^\star, u^\star) \pm 2\,p\}$, see figure 4.
Additionally the following estimates hold: Since $\alpha_j\,|v_j - v_1| = \max_i \alpha_i\,|v_i - v_1|$, we get that

$$\begin{aligned}|v^\star - v_1| &= \Big|\sum_{i=1}^{5} \alpha_i\,v_i - \sum_{i=1}^{5} \alpha_i\,v_1\Big|\\ &= \Big|\sum_{i=1}^{5} \alpha_i\,(v_i - v_1)\Big|\\ &\leq \sum_{i=1}^{5} \alpha_i\,|v_i - v_1|\\ &\leq 5\,\alpha_j\,|v_j - v_1|.\end{aligned}$$

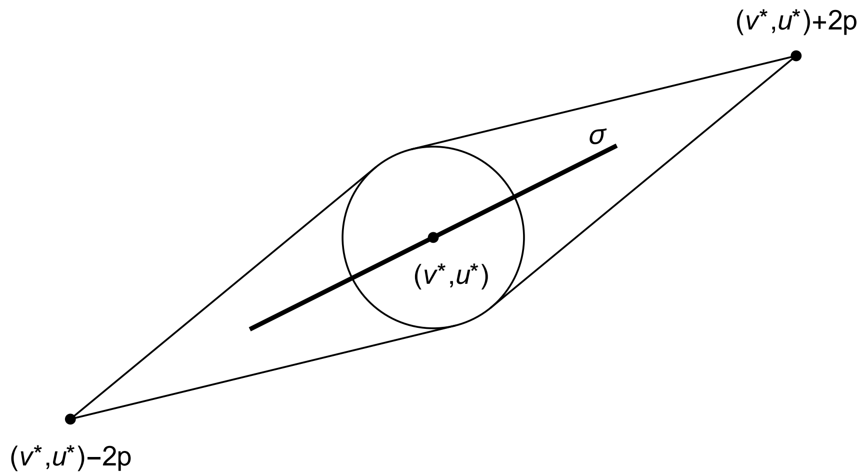

Figure 4: $(v^\star, u^\star) + \sigma$ is contained in the convex hull of $B_r(v^\star, u^\star) \cup \{(v^\star, u^\star) \pm 2\,p\}$.

With this estimate we get an inequality for the velocity part of p:

$$\begin{aligned}\frac{1}{2}\,\alpha_j\,|v_j - v_1| &\geq \frac{1}{2}\,\frac{1}{5}\,|v^\star - v_1| \\ &\geq \frac{1}{10}\,\big(|v_1| - |v^\star|\big) \\ &> \frac{1}{10}\,\big(\sqrt{C} - |v^\star|\big)\,\frac{\sqrt{C} + |v^\star|}{2\,\sqrt{C}} = \frac{1}{20\,\sqrt{C}}\,\big(C - |v^\star|^2\big). \end{aligned} \tag{4.12}$$

Here we used the reverse triangular inequality and $\frac{\sqrt{C}+|v^\star|}{2\sqrt{C}} < 1$, which can be proved as follows: We have $(v^\star, u^\star) \in \mathcal{U}$ and hence $v^\star \otimes v^\star - u^\star < \frac{C}{2}\,\mathrm{Id}$. In other words $\frac{C}{2}\,\mathrm{Id} - v^\star \otimes v^\star + u^\star$ is positive definite, what implies that

$$C - |v^\star|^2 = \mathrm{tr}\Big(\frac{C}{2}\,\mathrm{Id} - v^\star \otimes v^\star + u^\star\Big) > 0.$$

Hence we obtain $|v^\star| < \sqrt{C}$ and therefore $\frac{\sqrt{C}+|v^\star|}{2\sqrt{C}} < 1$.
To finish step 2 of the proof it remains to show that the third claim is true. Let $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ be a real-valued function, which we will specify later. Define $(u, v) \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ by

$$\begin{aligned} v_i(t,x) &:= \frac{1}{2}\Bigg(\sum_{k=1}^{2}(a_k\,b_i - a_i\,b_k)\,\partial_k\,(\partial_1^2 + \partial_2^2)\Bigg)\,\Phi(t,x) \\ u_{ij}(t,x) &:= \frac{1}{2}\Bigg(\sum_{k=1}^{2}(a_i\,b_k - a_k\,b_i)\,\partial_k\,\partial_j\,\partial_t + \sum_{k=1}^{2}(a_j\,b_k - a_k\,b_j)\,\partial_k\,\partial_i\,\partial_t\Bigg)\,\Phi(t,x) \end{aligned} \tag{4.13}$$

for $i, j \in \{1, 2\}$. Here a, b are defined as above. It is obvious that u is symmetric and an easy computation shows that

$$\begin{aligned}\operatorname{tr}\big(u(t,x)\big) &= u_{11}(t,x) + u_{22}(t,x) \\ &= \bigg(\sum_{k=1}^{2}(a_1\, b_k - a_k\, b_1)\,\partial_k\,\partial_1\,\partial_t + \sum_{k=1}^{2}(a_2\, b_k - a_k\, b_2)\,\partial_k\,\partial_2\,\partial_t\bigg)\,\Phi(t,x) \\ &= \Big((a_1\, b_2 - a_2\, b_1)\,\partial_2\,\partial_1\,\partial_t + (a_2\, b_1 - a_1\, b_2)\,\partial_1\,\partial_2\,\partial_t\Big)\,\Phi(t,x) \quad = \quad 0.\end{aligned}$$

Another computation yields that $\operatorname{div}_x v = 0$ and $\partial_t v + \operatorname{div}_x u = 0$:

$$\begin{aligned}\operatorname{div}_x v(t,x) &= \sum_{i=1}^{2}\partial_i\, v_i(t,x) \\ &= \frac{1}{2}\bigg(\sum_{i,k=1}^{2}(a_k\, b_i - a_i\, b_k)\,\partial_i\,\partial_k\bigg)\,(\partial_1^2 + \partial_2^2)\,\Phi(t,x) \quad = \quad 0,\end{aligned}$$

$$\begin{aligned}\big(\partial_t\, v(t,x) + \operatorname{div}_x u(t,x)\big)_i &= \partial_t\, v_i(t,x) + \sum_{j=1}^{2}\partial_j\, u_{ij}(t,x) \\ = \frac{1}{2}\bigg(\sum_{k=1}^{2}(a_k\, b_i - a_i\, b_k)\,\partial_k\,(\partial_1^2 + \partial_2^2) &+ \sum_{j,k=1}^{2}(a_i\, b_k - a_k\, b_i)\,\partial_k\,\partial_j^2 \\ &+ \sum_{j,k=1}^{2}(a_j\, b_k - a_k\, b_j)\,\partial_k\,\partial_i\,\partial_j\bigg)\,\partial_t\,\Phi(t,x) \quad = \quad 0.\end{aligned}$$

Concerning the function Φ, we are interested in two different choices. We define

$$\eta := \frac{-1}{\big(|a|\,|b| + a\cdot b\big)^{2/3}}\,\big(a + b - (|a|\,|b| + a\cdot b)\, e_3\big) \in \mathbb{R}^3,$$

where we consider the 2-dimensional vectors $a, b \in \mathbb{R}^2$ as 3-dimensional by setting the third component 0, i.e. if in the prequel $a = (a_1, a_2)^T$ we mean here $a = (a_1, a_2, 0)^T$, and $e_3 = (0,0,1)^T$. Note that the denominator above is non-zero because $a \neq -b$, and hence η is well-defined.

Let $\Psi \in C^\infty(\mathbb{R},\mathbb{R})$ be a function, which we will define precisely later. The first interesting choice for Φ will be $\Phi(t,x) := \Psi\big((x,t)\cdot\eta\big)$, where here $\cdot$ denotes the scalar product in $\mathbb{R}^3$. More precisely by $(x,t)\cdot\eta$ for a vector $\eta \in \mathbb{R}^3$ we mean $(x,t)\cdot\eta = x_1\,\eta_1 + x_2\,\eta_2 + t\,\eta_3 \in \mathbb{R}$. We define $(\widehat{v},\widehat{u})$ as in (4.13) with the choice $\Phi(t,x) := \Psi\big((x,t)\cdot\eta\big)$. Simple but long calculations show, that

$$\begin{aligned}\widehat{v} &= (a - b)\;\Psi'''\big((x,t)\cdot\eta\big), \\ \widehat{u} &= (a\otimes a - b\otimes b)\;\Psi'''\big((x,t)\cdot\eta\big)\end{aligned}$$

are true. Since we want (v,u) to be compactly supported, our second and final choice of Φ will be slightly different. Let $\varphi \in C_c^\infty\big((-1,1)\times B_1(0), [-1,1]\big)$ be a cutoff function which is identically 1 inside $(-\frac{1}{2},\frac{1}{2})\times B_{1/2}(0)$. Define Φ as $\Phi(t,x) := \varphi(t,x)\,\Psi\big((x,t)\cdot\eta\big)$ and (v,u) as in (4.13) with this choice of Φ.

Let us fix Ψ by $\Psi(y) := -\lambda\, N^{-3}\,\sin(N\,y)$ where λ was defined above and $N > 0$ is

a large number, which will be specified later.
Obviously $(v,u) \in C_c^\infty\big((-1,1)\times B_1(0)\big)$. Next we want to show that

$$\operatorname{dist}\big((v(t,x),u(t,x)),\sigma\big) < \varepsilon$$

for all $(t,x) \in (-1,1)\times B_1(0)$. It is not difficult to check that

$$||(v,u) - \varphi\cdot(\widehat{v},\widehat{u})||_\infty \le c\,\frac{1}{N},$$

where $||\cdot||_\infty$ denotes the maximum norm on $C_c^\infty\big((-1,1)\times B_1(0)\big)$ and $c>0$ is a suitable constant. We can choose N large such that $c\,\frac{1}{N} < \varepsilon$. On the other hand we know that

$$\begin{aligned}(\widehat{v},\widehat{u}) &= \big((a-b),(a\otimes a - b\otimes b)\big)\,\Psi'''\big((x,t)\cdot\eta\big)\\ &= \big((a-b),(a\otimes a - b\otimes b)\big)\,\lambda\,\cos\big(N\,(x,t)\cdot\eta\big)\\ &= p\,\cos\big(N\,(x,t)\cdot\eta\big) \in \sigma,\end{aligned}$$

and therefore also $\varphi\cdot(\widehat{v},\widehat{u}) \in \sigma$, because φ takes values in $[-1,1]$. So for all $(t,x)\in(-1,1)\times B_1(0)$ we obtain

$$\begin{aligned}\operatorname{dist}\big((v(t,x),u(t,x)),\sigma\big) &\le \operatorname{dist}\big((v(t,x),u(t,x)),\varphi(t,x)\cdot(\widehat{v}(t,x),\widehat{u}(t,x))\big)\\ &\le ||(v,u)-\varphi\cdot(\widehat{v},\widehat{u})||_\infty < \varepsilon.\end{aligned}$$

It remains to show some estimates. We denote the ball with radius $\frac{1}{2}$ and center 0 in the 3-dimensional space-time $\mathbb{R}\times\mathbb{R}^2$ as $B^3_{1/2}(0)$. Because $B^3_{1/2}(0) \subset (-\frac{1}{2},\frac{1}{2})\times B_{1/2}(0)$, we have that $\varphi(t,x)=1$ for $(t,x)\in B^3_{1/2}(0)$. Hence for $(t,x)\in B^3_{1/2}(0)$ it holds that $v(t,x) = \widehat{v}(t,x) = (a-b)\,\Psi'''\big((x,t)\cdot\eta\big)$ and therefore

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\big|v(t,x)\big|\,dx\,dt &\ge \iint_{B^3_{1/2}(0)}\big|v(t,x)\big|\,dx\,dt\\ &= \iint_{B^3_{1/2}(0)}\lambda\,|a-b|\,\big|\cos\big(N\,(x,t)\cdot\eta\big)\big|\,dx\,dt\\ &= \lambda\,|a-b|\iint_{B^3_{1/2}(0)}\big|\cos\big(N\,t\,|\eta|\big)\big|\,dx\,dt,\end{aligned}\tag{4.14}$$

where we turned the coordinate system in the last step. In order to estimate the integral in (4.14) further, we consider the cube $[-1/4,1/4]^3 \subset B^3_{1/2}(0)$ and obtain

$$\begin{aligned}\iint_{B^3_{1/2}(0)}\big|\cos\big(N\,t\,|\eta|\big)\big|\,dx\,dt &\ge \int_{-1/4}^{1/4}\int_{-1/4}^{1/4}\int_{-1/4}^{1/4}\big|\cos\big(N\,t\,|\eta|\big)\big|\,dx_1\,dx_2\,dt\\ &= \frac{1}{4}\int_{-1/4}^{1/4}\big|\cos\big(N\,t\,|\eta|\big)\big|\,dt\\ &= \frac{1}{4\,N\,|\eta|}\int_{-1/4\,N\,|\eta|}^{1/4\,N\,|\eta|}\big|\cos(t)\big|\,dt\\ &\ge \frac{1}{4\,N\,|\eta|}\int_{-1/4\,N\,|\eta|}^{1/4\,N\,|\eta|}\cos^2(t)\,dt\\ &= \frac{1}{16}+\frac{1}{8\,N\,|\eta|}\sin\Big(\frac{1}{2}\,N\,|\eta|\Big)\\ &\ge \frac{1}{16}-\frac{1}{8\,N\,|\eta|}.\end{aligned}$$

For large N, say $N \geq \frac{10}{|\eta|}$, and using (4.12) and (4.14) we get with the definition $c_1 = \frac{1}{400\sqrt{C}}$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \big|v(t,x)\big| \, dx \, dt \geq \frac{\lambda}{20} \, |a-b| \geq \frac{1}{400\sqrt{C}} \left(C - |v^\star|^2\right) = c_1 \left(C - \big|\widetilde{v} + \underline{v}(t_0,x_0)\big|^2\right).$$

Now it remains to show that $\int_{\mathbb{R}} \int_{\mathbb{R}^2} (v,u) \, dx \, dt = 0$. This can be seen quite easily keeping in mind the definition (4.13) of (v,u). Since $(v,u) \in C_c^\infty\big((-1,1) \times B_1(0)\big)$ we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} (v,u) \, dx \, dt = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (v,u) \, dx_1 \, dx_2 \, dt.$$

Applying Fubini's theorem we have to solve integrals like

$$\int_{-1}^{1} \partial_y \Phi(t,x) \, dy,$$

where $y \in \{x_1, x_2, t\}$. Because of the compact support of Φ we get the claimed statement.

Step 3 (see also [CLK15, Section 4.1])

Now we are ready to prove the actual claim of the perturbation property. Let again $(t_0, x_0) \in \Gamma$ be any point. Step 2 yields a segment σ such that

$$(\widetilde{v}, \widetilde{u}) + \big(\underline{v}(t_0,x_0), \underline{u}(t_0,x_0)\big) + \sigma \subset \mathcal{U}.$$

Since $(\underline{v}, \underline{u})$ is continuous and $\mathcal{U}$ is open, we find a radius $r > 0$ such that

$$(\widetilde{v}, \widetilde{u}) + \big(\underline{v}(t,x), \underline{u}(t,x)\big) + \sigma \subset \mathcal{U}$$

for all $(t,x) \in (t_0 - r, t_0 + r) \times B_r(x_0)$ and such that $(t_0 - r, t_0 + r) \times B_r(x_0) \subset \Gamma$. For all $\varepsilon > 0$ step 2 yields a pair $(v,u) \in C_c^\infty\big((-1,1) \times B_1(0)\big)$ that fulfills

$$\operatorname{dist}\big((v(t,x), u(t,x)), \sigma\big) < \varepsilon$$

for all $(t,x) \in (-1,1) \times B_1(0)$. Define

$$\big(v_{t_0,x_0,r}, u_{t_0,x_0,r}\big)(t,x) := (v,u)\left(\frac{t-t_0}{r}, \frac{x-x_0}{r}\right),$$

then $\operatorname{supp}\big(v_{t_0,x_0,r}, u_{t_0,x_0,r}\big) \subset (t_0 - r, t_0 + r) \times B_r(x_0)$. In addition to that we get that

$$\operatorname{dist}\big((v_{t_0,x_0,r}(t,x), u_{t_0,x_0,r}(t,x)), \sigma\big) < \varepsilon$$

for all $(t,x) \in (t_0 - r, t_0 + r) \times B_r(x_0)$. Because $\mathcal{U}$ is open, we can choose ε so small that

$$(\widetilde{v}, \widetilde{u}) + \big(\underline{v}(t,x), \underline{u}(t,x)\big) + \big(v_{t_0,x_0,r}(t,x), u_{t_0,x_0,r}(t,x)\big) \in \mathcal{U}$$

for all $(t,x) \in (t_0 - r, t_0 + r) \times B_r(x_0)$. From step 2 we also obtain the following estimate:

$$\begin{aligned} \iint_{\Gamma} \big|v_{t_0,x_0,r}(t,x)\big| \, dx \, dt &= r^3 \int_{-1}^{1} \int_{B_1(0)} \big|v(t,x)\big| \, dx \, dt \\ &\geq r^3 \, c_1 \left(C - \big|\widetilde{v} + \underline{v}(t_0,x_0)\big|^2\right). \end{aligned} \tag{4.15}$$

Because of the uniform continuity of $(\underline{v}, \underline{u})$ there is a radius $r_1 > 0$ such that the above construction works for all radii $0 < r < r_1$ and all $(t_0, x_0) \in \Gamma$ with $(t_0 - r, t_0 + r) \times B_r(x_0) \subset \Gamma$.
There exists a radius $r_2 > 0$ and a constant $c_2 > 0$ such that for all $0 < r < r_2$ there are finitely many points $(t_j, x_j) \in \Gamma$ with the following properties:

- The sets $(t_j - r, t_j + r) \times B_r(x_j)$ are contained in Γ and pairwise disjoint.
- The inequality

$$r^3 \sum_j \Big(C - \big|\widetilde{v} + \underline{v}(t_j, x_j)\big|^2 \Big) \geq c_2 \iint_\Gamma \Big(C - \big|\widetilde{v} + \underline{v}(t, x)\big|^2 \Big)\, dx\, dt$$
$$= c_2 \Big(C\, |\Gamma| - \iint_\Gamma \big|\widetilde{v} + \underline{v}(t, x)\big|^2\, dx\, dt \Big) \qquad (4.16)$$

  holds.

Let now $r = \frac{1}{k}$ for $k \in \mathbb{N}$ such that $\frac{1}{k} < \min\{r_1, r_2\}$ and do the above constructions for this radius. Define $(v_k, u_k) = (\underline{v}, \underline{u}) + \sum_j (v_{t_j, x_j, r}, u_{t_j, x_j, r})$. It is not difficult to check that $(v_k, u_k) \in X_0$: Obviously $(v_k, u_k) \in C_c^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ and with the 3rd claim in step 2 we have $\operatorname{div}_x v_k = 0$ and $\partial_t v_k + \operatorname{div}_x u_k = 0$. Furthermore, since the sets $(t_j - r, t_j + r) \times B_r(x_j)$ are pairwise disjoint, we obtain $(\widetilde{v}, \widetilde{u}) + \big(v_k(t, x), u_k(t, x)\big) \in \mathcal{U}$ for all $(t, x) \in \Omega$ and hence $(\widetilde{v} + v_k) \otimes (\widetilde{v} + v_k) - (\widetilde{u} + u_k) < \frac{C}{2}\,\mathrm{Id}$. So $(v_k, u_k) \in X_0$.
Moreover we get $(v_k, u_k) \overset{*}{\rightharpoonup} (\underline{v}, \underline{u})$. This can be proved using proposition A.2. First note that $||(v_k, u_k)||_{L^\infty}$ is bounded because $(v_k, u_k) \in X_0$ and X_0 is bounded in L^∞ as already shown. Let $g \in C_c^\infty(\Omega)$. Then we get

$$\iint_\Omega \big((v_k, u_k) - (\underline{v}, \underline{u}) \big)\, g\, dx\, dt = \iint_\Omega \sum_j (v_{t_j, x_j, r}, u_{t_j, x_j, r})\, g\, dx\, dt$$
$$= \sum_j \int_{t_j - r}^{t_j + r} \int_{B_r(x_j)} (v_{t_j, x_j, r}, u_{t_j, x_j, r})\, g\, dx\, dt. \qquad (4.17)$$

Let us use Taylor's expansion for g to obtain

$$\begin{aligned}
&\int_{t_j - r}^{t_j + r} \int_{B_r(x_j)} (v_{t_j, x_j, r}, u_{t_j, x_j, r})\, g\, dx\, dt \\
&= \int_{t_j - r}^{t_j + r} \int_{B_r(x_j)} \Big(v\Big(\frac{t - t_j}{r}, \frac{x - x_j}{r}\Big), u\Big(\frac{t - t_j}{r}, \frac{x - x_j}{r}\Big) \Big)\, g(t, x)\, dx\, dt \\
&= \int_{-1}^{1} \int_{B_1(0)} \big(v(t, x), u(t, x) \big)\, g(r\, t + t_j, r\, x + x_j)\, r^3\, dx\, dt \\
&= \int_{-1}^{1} \int_{B_1(0)} \big(v(t, x), u(t, x) \big)\, \big[g(t_j, x_j) \\
&\qquad + r\, t\, \partial_t g(t_j, x_j) + r\, x_1\, \partial_1 g(t_j, x_j) + r\, x_2\, \partial_2 g(t_j, x_j) + O(r^2) \big]\, r^3\, dx\, dt \\
&\leq g(t_j, x_j)\, r^3 \underbrace{\int_{-1}^{1} \int_{B_1(0)} \big(v(t, x), u(t, x) \big)\, dx\, dt}_{=0} + M\, r^4 \\
&= M\, r^4 \quad = \quad M\, \frac{1}{k^4}, \qquad (4.18)
\end{aligned}$$

with a suitable constant M and r sufficiently small. In particular M does not depend on j: First of all the derivatives of g are bounded since $g \in C_c^\infty(\Omega)$. Hence we can estimate $\partial_y g(t_j, x_j)$ for $y \in \{t, x_1, x_2\}$ by a constant which doesn't depend on (t_j, x_j). The bigger problem are the functions (v, u) which depend on (t_j, x_j) according to the construction above. So the bound of (v, u) may depend on j. However we can find a bound which does not depend on j:

$$\begin{aligned}
||(v,u)||_\infty &= \max_{(t,x)\in(-1,1)\times B_1(0)} ||\big(v(t,x), u(t,x)\big)||_{\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}} \\
&\leq \max_{(t,x)\in(-1,1)\times B_1(0)} \operatorname{dist}\big((v(t,x), u(t,x)), \sigma\big) + ||p||_{\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}} \\
&< \varepsilon + |\lambda|\, ||(a-b, a\otimes a - b\otimes b)||_{\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}}.
\end{aligned}$$

Again note that ε, λ, a and b depend on (t_j, x_j). But we can choose $\varepsilon < 1$ and it is easy to check that $|\lambda|$ is bounded by 1 and $||(a-b, a\otimes a - b\otimes b)||_{\mathbb{R}^2\times\mathcal{S}_0^{2\times 2}}$ is bounded by a constant which depends only on C since $|a| = |b| = \sqrt{C}$. Hence M does not depend on j.

Because the number of points (t_j, x_j) grows with k^3, we get from (4.17) and (4.18):

$$\begin{aligned}
\iint_\Omega \big((v_k, u_k) - (\underline{v}, \underline{u})\big)\, g\, dx\, dt &= \sum_j \int_{t_j-r}^{t_j+r} \int_{B_r(x_j)} (v_{t_j,x_j,r}, u_{t_j,x_j,r})\, g\, dx\, dt \\
&\leq \sum_j M\, \frac{1}{k^4} \\
&\leq k^3\, \widetilde{M}\, \frac{1}{k^4} \quad \overset{k\to\infty}{\longrightarrow} \quad 0,
\end{aligned}$$

which proves with proposition A.2 that $(v_k, u_k) \overset{*}{\rightharpoonup} (\underline{v}, \underline{u})$.

Additionally we have the following estimate

$$\begin{aligned}
||v_k - \underline{v}||_{L^1(\Gamma)} &= \iint_\Gamma \big|v_k(t,x) - \underline{v}(t,x)\big|\, dx\, dt \\
&= \iint_\Gamma \Big|\sum_j v_{t_j,x_j,r}(t,x)\Big|\, dx\, dt \\
&= \sum_j \iint_\Gamma \big|v_{t_j,x_j,r}(t,x)\big|\, dx\, dt,
\end{aligned}$$

because the supports of the functions $v_{t_j,x_j,r}$ are disjoint. Using (4.15) and (4.16) we arrive at

$$\begin{aligned}
||v_k - \underline{v}||_{L^1(\Gamma)} &= \sum_j \iint_\Gamma \big|v_{t_j,x_j,r}(t,x)\big|\, dx\, dt \\
&\geq \sum_j r^3\, c_1 \Big(C - \big|\widetilde{v} + \underline{v}(t_j, x_j)\big|^2\Big) \\
&\geq c_1\, c_2 \Big(C\, |\Gamma| - \iint_\Gamma \big|\widetilde{v} + \underline{v}(t,x)\big|^2\, dx\, dt\Big) \\
&= c_1\, c_2 \Big(C\, |\Gamma| - ||\widetilde{v} + \underline{v}||_{L^2(\Gamma)}^2\Big).
\end{aligned}$$

Hölder's inequality yields

$$||v_k - \underline{v}||_{L^1(\Gamma)} \leq ||v_k - \underline{v}||_{L^2(\Gamma)} \, ||1||_{L^2(\Gamma)} = ||v_k - \underline{v}||_{L^2(\Gamma)} \, \sqrt{|\Gamma|},$$

and therefore

$$\frac{1}{|\Gamma|} ||v_k - \underline{v}||^2_{L^1(\Gamma)} \leq ||v_k - \underline{v}||^2_{L^2(\Gamma)}.$$

Putting the previous inequalities together we obtain

$$||v_k - \underline{v}||^2_{L^2(\Gamma)} \geq \frac{c_1^2 \, c_2^2}{|\Gamma|} \Big(C \, |\Gamma| - ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} \Big)^2.$$

Hence

$$\begin{aligned} ||\widetilde{v} + v_k||^2_{L^2(\Gamma)} &= ||\widetilde{v} + \underline{v} + v_k - \underline{v}||^2_{L^2(\Gamma)} \\ &= ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} + ||v_k - \underline{v}||^2_{L^2(\Gamma)} \\ &\qquad + 2 \iint_\Gamma \big(\widetilde{v} + \underline{v}(t,x)\big) \big(v_k(t,x) - \underline{v}(t,x)\big) \, dx \, dt \\ &\geq ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} + \frac{c_1^2 \, c_2^2}{|\Gamma|} \Big(C \, |\Gamma| - ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} \Big)^2 \\ &\qquad + 2 \iint_\Gamma \big(\widetilde{v} + \underline{v}(t,x)\big) \big(v_k(t,x) - \underline{v}(t,x)\big) \, dx \, dt. \end{aligned}$$

Since $v_k \overset{*}{\rightharpoonup} \underline{v}$ and $\widetilde{v} + \underline{v} \in L^1(\Gamma)$, the integral tends to 0 as $k \to \infty$. Therefore we obtain

$$\liminf_{k \to \infty} ||\widetilde{v} + v_k||^2_{L^2(\Gamma)} \geq ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} + \frac{c_1^2 \, c_2^2}{|\Gamma|} \Big(C \, |\Gamma| - ||\widetilde{v} + \underline{v}||^2_{L^2(\Gamma)} \Big)^2,$$

what finishes the proof of the perturbation property, lemma 4.5.

□

5. Existence of a subsolution

In this section we want to construct admissible fan subsolutions to the compressible Euler equations (1.1) with initial data (1.6). If we could find such a fan subsolution, theorem 4.3 claims non-uniqueness of admissible weak solutions to the system (1.1), (1.6).
We try to find easy admissible fan subsolutions in the following sense.

Definition 5.1. *(simple fan subsolution)* An admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ where the corresponding fan partition consists of three open sets P_-, P_1, P_+, i.e. $N = 1$, $I_C = \{1\}$ and $I_= = \varnothing$, is denoted as a simple fan subsolution.

In the case of such a simple fan subsolution we can simplify the properties given in definition 4.2 to the following system of algebraic equations and inequalities:

Proposition 5.2. (see [CLK15, Proposition 5.1] or [CK14, Proposition 4.1]) *Let $\rho_-, \rho_+ \in \mathbb{R}^+$, $v_-, v_+ \in \mathbb{R}^2$ with $v_{-1} = v_{+1}$ be given (see initial condition* (1.6)*). The constants $\nu_0, \nu_1 \in \mathbb{R}$ (with $\nu_0 < \nu_1$), $\rho_1 \in \mathbb{R}^+$, $v_1 \in \mathbb{R}^2$, $u_1 \in \mathcal{S}_0^{2\times 2}$ and $C_1 \in \mathbb{R}^+$ define a simple fan subsolution to the Cauchy problem* (1.1), (1.6) *if and only if they fulfill the following algebraic equations and inequalities:*

- *Rankine Hugoniot conditions on the left interface:*

$$\nu_0\,(\rho_- - \rho_1) = \rho_-\,v_{-\,2} - \rho_1\,v_{1\,2} \tag{5.1}$$
$$\nu_0\,(\rho_-\,v_{-\,1} - \rho_1\,v_{1\,1}) = \rho_-\,v_{-\,1}\,v_{-\,2} - \rho_1\,u_{1\,12} \tag{5.2}$$
$$\nu_0\,(\rho_-\,v_{-\,2} - \rho_1\,v_{1\,2}) = \rho_-\,v_{-\,2}^2 + \rho_1\,u_{1\,11} + p(\rho_-) - p(\rho_1) - \rho_1\,\frac{C_1}{2} \tag{5.3}$$

- *Rankine Hugoniot conditions on the right interface:*

$$\nu_1\,(\rho_1 - \rho_+) = \rho_1\,v_{1\,2} - \rho_+\,v_{+\,2} \tag{5.4}$$
$$\nu_1\,(\rho_1\,v_{1\,1} - \rho_+\,v_{+\,1}) = \rho_1\,u_{1\,12} - \rho_+\,v_{+\,1}\,v_{+\,2} \tag{5.5}$$
$$\nu_1\,(\rho_1\,v_{1\,2} - \rho_+\,v_{+\,2}) = -\rho_1\,u_{1\,11} - \rho_+\,v_{+\,2}^2 + p(\rho_1) - p(\rho_+) + \rho_1\,\frac{C_1}{2} \tag{5.6}$$

- *Subsolution condition:*

$$v_{1\,1}^2 + v_{1\,2}^2 < C_1 \tag{5.7}$$
$$\left(\frac{C_1}{2} - v_{1\,1}^2 + u_{1\,11}\right)\left(\frac{C_1}{2} - v_{1\,2}^2 - u_{1\,11}\right) - (u_{1\,12} - v_{1\,1}\,v_{1\,2})^2 > 0 \tag{5.8}$$

- *Admissibility condition on the left interface:*

$$\begin{aligned}&\nu_0\left(\rho_-\,\varepsilon(\rho_-) + \rho_-\,\frac{|v_-|^2}{2} - \rho_1\,\varepsilon(\rho_1) - \rho_1\,\frac{C_1}{2}\right)\\ &\leq \big(\rho_-\,\varepsilon(\rho_-) + p(\rho_-)\big)\,v_{-\,2} - \big(\rho_1\,\varepsilon(\rho_1) + p(\rho_1)\big)\,v_{1\,2} + \rho_-\,v_{-\,2}\,\frac{|v_-|^2}{2} - \rho_1\,v_{1\,2}\,\frac{C_1}{2}\end{aligned} \tag{5.9}$$

- *Admissibility condition on the right interface:*

$$\begin{aligned}&\nu_1\left(\rho_1\,\varepsilon(\rho_1) + \rho_1\,\frac{C_1}{2} - \rho_+\,\varepsilon(\rho_+) - \rho_+\,\frac{|v_+|^2}{2}\right)\\ &\leq \big(\rho_1\,\varepsilon(\rho_1) + p(\rho_1)\big)\,v_{1\,2} - \big(\rho_+\,\varepsilon(\rho_+) + p(\rho_+)\big)\,v_{+\,2} + \rho_1\,v_{1\,2}\,\frac{C_1}{2} - \rho_+\,v_{+\,2}\,\frac{|v_+|^2}{2}\end{aligned} \tag{5.10}$$

Proof. For simple fan subsolutions the left-hand sides of the equations (4.3), (4.4) and of the inequality (4.5) can be simplified as follows. We get

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^2}\big(\overline{\rho}\,\partial_t\psi + \overline{\rho}\,\overline{v}\cdot\nabla_x\psi\big)dx\,dt + \int_{\mathbb{R}^2}\rho_0(x)\,\psi(0,x)\,dx\\ &= \underbrace{\iint_{P_-}\big(\rho_-\,\partial_t\psi + \rho_-\,v_-\cdot\nabla_x\psi\big)dx\,dt}_{=:J_-} + \underbrace{\iint_{P_1}\big(\rho_1\,\partial_t\psi + \rho_1\,v_1\cdot\nabla_x\psi\big)dx\,dt}_{=:J_1}\\ &\qquad + \underbrace{\iint_{P_+}\big(\rho_+\,\partial_t\psi + \rho_+\,v_+\cdot\nabla_x\psi\big)dx\,dt}_{=:J_+} + \int_{\mathbb{R}^2}\rho_0(x)\,\psi(0,x)\,dx\end{aligned} \tag{5.11}$$

and

$$\int_0^\infty \int_{\mathbb{R}^2} \Bigg[\overline{\rho}\, \overline{v} \cdot \partial_t \phi + \overline{\rho}\, \overline{u} : D_x \phi + \bigg(p(\overline{\rho}) + \frac{1}{2} \overline{\rho} \Big(|v_-|^2 \mathbf{1}_{P_-} + C_1 \mathbf{1}_{P_1} + |v_+|^2 \mathbf{1}_{P_+} \Big) \bigg) \mathrm{div}_x \phi \Bigg] dx\, dt + \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx$$

$$= \underbrace{\iint_{P_-} \Bigg[\rho_- \, v_- \cdot \partial_t \phi + \rho_- \, u_- : D_x \phi + \Big(p(\rho_-) + \frac{1}{2} \rho_- \, |v_-|^2 \Big) \mathrm{div}_x \phi \Bigg] dx\, dt}_{=:K_-}$$

$$+ \underbrace{\iint_{P_1} \Bigg[\rho_1 \, v_1 \cdot \partial_t \phi + \rho_1 \, u_1 : D_x \phi + \Big(p(\rho_1) + \frac{1}{2} \rho_1 \, C_1 \Big) \mathrm{div}_x \phi \Bigg] dx\, dt}_{=:K_1}$$

$$+ \underbrace{\iint_{P_+} \Bigg[\rho_+ \, v_+ \cdot \partial_t \phi + \rho_+ \, u_+ : D_x \phi + \Big(p(\rho_+) + \frac{1}{2} \rho_+ \, |v_+|^2 \Big) \mathrm{div}_x \phi \Bigg] dx\, dt}_{=:K_+}$$

$$+ \int_{\mathbb{R}^2} \rho_0(x)\, v_0(x) \cdot \phi(0,x)\, dx. \tag{5.12}$$

In the same way we obtain

$$\int_0^\infty \int_{\mathbb{R}^2} \Bigg[\bigg(\overline{\rho}\, \varepsilon(\overline{\rho}) + \frac{1}{2} \overline{\rho} \Big(|v_-|^2 \mathbf{1}_{P_-} + C_1 \mathbf{1}_{P_1} + |v_+|^2 \mathbf{1}_{P_+} \Big) \bigg) \partial_t \varphi$$
$$+ \bigg(\overline{\rho}\, \varepsilon(\overline{\rho}) + p(\overline{\rho}) + \frac{1}{2} \overline{\rho} \Big(|v_-|^2 \mathbf{1}_{P_-} + C_1 \mathbf{1}_{P_1} + |v_+|^2 \mathbf{1}_{P_+} \Big) \bigg) \overline{v} \cdot \nabla_x \varphi \Bigg] dx\, dt$$
$$+ \int_{\mathbb{R}^2} \rho_0(x) \left(\varepsilon(\rho_0(x)) + \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx$$

$$= \underbrace{\iint_{P_-} \Bigg[\Big(\rho_- \, \varepsilon(\rho_-) + \rho_- \frac{|v_-|^2}{2} \Big) \partial_t \varphi + \Big(\rho_- \, \varepsilon(\rho_-) + p(\rho_-) + \rho_- \frac{|v_-|^2}{2} \Big) v_- \cdot \nabla_x \varphi \Bigg] dx\, dt}_{=:L_-}$$

$$+ \underbrace{\iint_{P_1} \Bigg[\Big(\rho_1 \, \varepsilon(\rho_1) + \rho_1 \frac{C_1}{2} \Big) \partial_t \varphi + \Big(\rho_1 \, \varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2} \Big) v_1 \cdot \nabla_x \varphi \Bigg] dx\, dt}_{=:L_1}$$

$$+ \underbrace{\iint_{P_+} \Bigg[\Big(\rho_+ \, \varepsilon(\rho_+) + \rho_+ \frac{|v_+|^2}{2} \Big) \partial_t \varphi + \Big(\rho_+ \, \varepsilon(\rho_+) + p(\rho_+) + \rho_+ \frac{|v_+|^2}{2} \Big) v_+ \cdot \nabla_x \varphi \Bigg] dx\, dt}_{=:L_+}$$

$$+ \int_{\mathbb{R}^2} \rho_0(x) \left(\varepsilon(\rho_0(x)) + \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx. \tag{5.13}$$

The next goal is to compute the integrals J_i, K_i and L_i for $i \in \{-, 1, +\}$ using integration by parts.

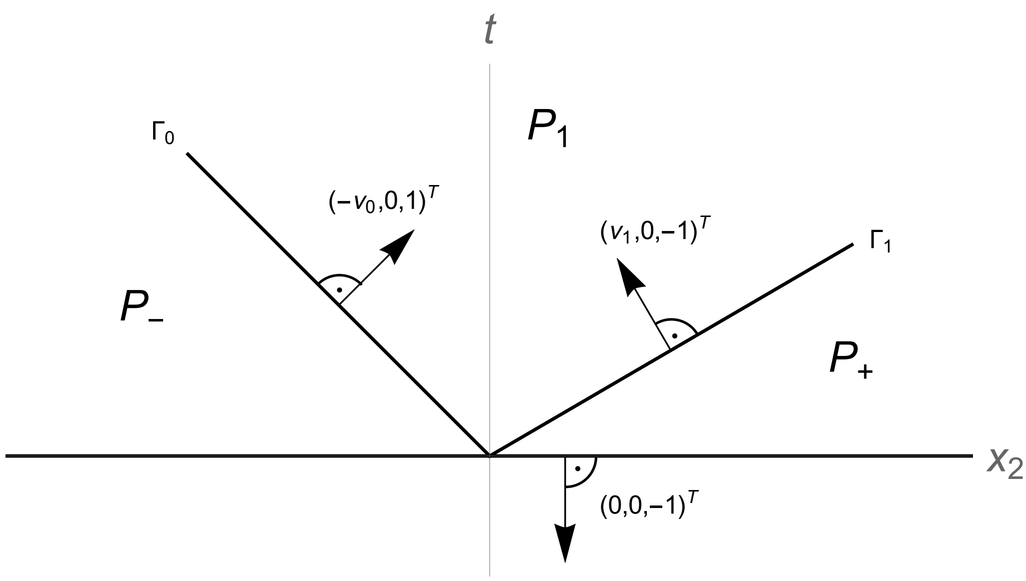

Figure 5: The normal vectors on the boundary surfaces Γ_0 and Γ_1.

Let Γ_0, Γ_1 be the two boundary surfaces in space-time between P_- and P_1, resp. between P_1 and P_+, see figure 5. More precisely

$$\Gamma_0 = \big\{(t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^2 \big| x_2 = \nu_0\, t\big\},$$
$$\Gamma_1 = \big\{(t, x_1, x_2) \in [0, \infty) \times \mathbb{R}^2 \big| x_2 = \nu_1\, t\big\}.$$

Then the unit normal vectors on Γ_0, Γ_1 which point into P_1 are

$$n_0 = \frac{1}{\sqrt{\nu_0^2+1}}\,(-\nu_0, 0, 1)^T, \qquad n_1 = \frac{1}{\sqrt{\nu_1^2+1}}\,(\nu_1, 0, -1)^T,$$

see also figure 5.
Additionally we define the half planes in $\mathbb{R}^2$ as $H_- := \big\{(x_1, x_2) \in \mathbb{R}^2 \big| x_2 < 0\big\}$ and $H_+ := \big\{(x_1, x_2) \in \mathbb{R}^2 \big| x_2 > 0\big\}$. Integration by parts yields:

$$J_- = \int_{\Gamma_0} \left(\rho_-\, \psi\, \frac{-\nu_0}{\sqrt{\nu_0^2+1}} + \rho_-\, v_{-\,2}\, \psi\, \frac{1}{\sqrt{\nu_0^2+1}}\right) dS + \int_{H_-} \rho_-\, \psi(0, x)\, (-1)\, dx,$$
$$J_1 = \int_{\Gamma_0} \left(\rho_1\, \psi\, \frac{\nu_0}{\sqrt{\nu_0^2+1}} + \rho_1\, v_{1\,2}\, \psi\, \frac{-1}{\sqrt{\nu_0^2+1}}\right) dS$$
$$+ \int_{\Gamma_1} \left(\rho_1\, \psi\, \frac{-\nu_1}{\sqrt{\nu_1^2+1}} + \rho_1\, v_{1\,2}\, \psi\, \frac{1}{\sqrt{\nu_1^2+1}}\right) dS,$$
$$J_+ = \int_{\Gamma_1} \left(\rho_+\, \psi\, \frac{\nu_1}{\sqrt{\nu_1^2+1}} + \rho_+\, v_{+\,2}\, \psi\, \frac{-1}{\sqrt{\nu_1^2+1}}\right) dS + \int_{H_+} \rho_+\, \psi(0, x)\, (-1)\, dx.$$

In the sequel we denote the components of the test function ϕ as ϕ_1, ϕ_2 what leads to:

$$
\begin{aligned}
K_- = \int_{\Gamma_0} \bigg[&\rho_- \left(v_{-\,1}\,\phi_1 + v_{-\,2}\,\phi_2\right) \frac{-\nu_0}{\sqrt{\nu_0^2+1}} + \rho_- \left(u_{-\,12}\,\phi_1 + u_{-\,22}\,\phi_2\right) \frac{1}{\sqrt{\nu_0^2+1}} \\
&+ \left(p(\rho_-) + \rho_- \frac{|v_-|^2}{2}\right) \phi_2 \frac{1}{\sqrt{\nu_0^2+1}} \bigg] \, dS \\
&+ \int_{H_-} \rho_- \left(v_{-\,1}\,\phi_1(0,x) + v_{-\,2}\,\phi_2(0,x)\right) (-1)\, dx, \\
K_1 = \int_{\Gamma_0} \bigg[&\rho_1 \left(v_{1\,1}\,\phi_1 + v_{1\,2}\,\phi_2\right) \frac{\nu_0}{\sqrt{\nu_0^2+1}} + \rho_1 \left(u_{1\,12}\,\phi_1 + u_{1\,22}\,\phi_2\right) \frac{-1}{\sqrt{\nu_0^2+1}} \\
&+ \left(p(\rho_1) + \rho_1 \frac{C_1}{2}\right) \phi_2 \frac{-1}{\sqrt{\nu_0^2+1}} \bigg] \, dS \\
&+ \int_{\Gamma_1} \bigg[\rho_1 \left(v_{1\,1}\,\phi_1 + v_{1\,2}\,\phi_2\right) \frac{-\nu_1}{\sqrt{\nu_1^2+1}} + \rho_1 \left(u_{1\,12}\,\phi_1 + u_{1\,22}\,\phi_2\right) \frac{1}{\sqrt{\nu_1^2+1}} \\
&\quad + \left(p(\rho_1) + \rho_1 \frac{C_1}{2}\right) \phi_2 \frac{1}{\sqrt{\nu_1^2+1}} \bigg] \, dS, \\
K_+ = \int_{\Gamma_1} \bigg[&\rho_+ \left(v_{+\,1}\,\phi_1 + v_{+\,2}\,\phi_2\right) \frac{\nu_1}{\sqrt{\nu_1^2+1}} + \rho_+ \left(u_{+\,12}\,\phi_1 + u_{+\,22}\,\phi_2\right) \frac{-1}{\sqrt{\nu_1^2+1}} \\
&+ \left(p(\rho_+) + \rho_+ \frac{|v_+|^2}{2}\right) \phi_2 \frac{-1}{\sqrt{\nu_1^2+1}} \bigg] \, dS \\
&+ \int_{H_+} \rho_+ \left(v_{+\,1}\,\phi_1(0,x) + v_{+\,2}\,\phi_2(0,x)\right) (-1)\, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
L_- = \int_{\Gamma_0} \bigg[&\left(\rho_-\,\varepsilon(\rho_-) + \rho_- \frac{|v_-|^2}{2}\right) \varphi \frac{-\nu_0}{\sqrt{\nu_0^2+1}} \\
&+ \left(\rho_-\,\varepsilon(\rho_-) + p(\rho_-) + \rho_- \frac{|v_-|^2}{2}\right) v_{-\,2}\,\varphi \frac{1}{\sqrt{\nu_0^2+1}} \bigg] dS \\
&+ \int_{H_-} \left(\rho_-\,\varepsilon(\rho_-) + \rho_- \frac{|v_-|^2}{2}\right) \varphi(0,x)\,(-1)\, dx, \\
L_1 = \int_{\Gamma_0} \bigg[&\left(\rho_1\,\varepsilon(\rho_1) + \rho_1 \frac{C_1}{2}\right) \varphi \frac{\nu_0}{\sqrt{\nu_0^2+1}} \\
&+ \left(\rho_1\,\varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2}\right) v_{1\,2}\,\varphi \frac{-1}{\sqrt{\nu_0^2+1}} \bigg] dS \\
&+ \int_{\Gamma_1} \bigg[\left(\rho_1\,\varepsilon(\rho_1) + \rho_1 \frac{C_1}{2}\right) \varphi \frac{-\nu_1}{\sqrt{\nu_1^2+1}} \\
&\quad + \left(\rho_1\,\varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2}\right) v_{1\,2}\,\varphi \frac{1}{\sqrt{\nu_1^2+1}} \bigg] dS,
\end{aligned}
$$

$$
\begin{aligned}
L_+ = \int_{\Gamma_1} \Bigg[& \left(\rho_+ \, \varepsilon(\rho_+) + \rho_+ \, \frac{|v_+|^2}{2} \right) \varphi \, \frac{\nu_1}{\sqrt{\nu_1^2 + 1}} \\
& + \left(\rho_+ \, \varepsilon(\rho_+) + p(\rho_+) + \rho_+ \, \frac{|v_+|^2}{2} \right) v_{+\,2} \, \varphi \, \frac{-1}{\sqrt{\nu_1^2 + 1}} \Bigg] dS \\
& + \int_{H_+} \left(\rho_+ \, \varepsilon(\rho_+) + \rho_+ \, \frac{|v_+|^2}{2} \right) \varphi(0, x) \, (-1) \, dx.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
& J_- + J_1 + J_+ + \int_{\mathbb{R}^2} \rho_0(x) \, \psi(0, x) \, dx \\
& = \frac{1}{\sqrt{\nu_0^2 + 1}} \Big(\nu_0 \, (\rho_1 - \rho_-) - (\rho_1 \, v_{1\,2} - \rho_- \, v_{-\,2}) \Big) \int_{\Gamma_0} \psi \, dS \\
& \quad + \frac{1}{\sqrt{\nu_1^2 + 1}} \Big(\nu_1 \, (\rho_+ - \rho_1) - (\rho_+ \, v_{+\,2} - \rho_1 \, v_{1\,2}) \Big) \int_{\Gamma_1} \psi \, dS.
\end{aligned} \tag{5.14}
$$

Likewise we have

$$
\begin{aligned}
& K_- + K_1 + K_+ + \int_{\mathbb{R}^2} \rho_0(x) \, v_0(x) \cdot \phi(0, x) \, dx \\
& = \frac{1}{\sqrt{\nu_0^2 + 1}} \Bigg[\Big(\nu_0 \, (\rho_1 \, v_{1\,1} - \rho_- \, v_{-\,1}) - (\rho_1 \, u_{1\,12} - \rho_- \, u_{-\,12}) \Big) \int_{\Gamma_0} \phi_1 \, dS \\
& \quad + \Bigg(\nu_0 \, (\rho_1 \, v_{1\,2} - \rho_- \, v_{-\,2}) - \rho_1 \, u_{1\,22} - p(\rho_1) \\
& \quad - \rho_1 \, \frac{C_1}{2} + \rho_- \, u_{-\,22} + p(\rho_-) + \rho_- \, \frac{|v_-|^2}{2} \Bigg) \int_{\Gamma_0} \phi_2 \, dS \Bigg] \\
& \quad + \frac{1}{\sqrt{\nu_1^2 + 1}} \Bigg[\Big(\nu_1 \, (\rho_+ \, v_{+\,1} - \rho_1 \, v_{1\,1}) - (\rho_+ \, u_{+\,12} - \rho_1 \, u_{1\,12}) \Big) \int_{\Gamma_1} \phi_1 \, dS \\
& \quad + \Bigg(\nu_1 \, (\rho_+ \, v_{+\,2} - \rho_1 \, v_{1\,2}) - \rho_+ \, u_{+\,22} - p(\rho_+) \\
& \quad - \rho_+ \, \frac{|v_+|^2}{2} + \rho_1 \, u_{1\,22} + p(\rho_1) + \rho_1 \, \frac{C_1}{2} \Bigg) \int_{\Gamma_1} \phi_2 \, dS \Bigg],
\end{aligned} \tag{5.15}
$$

and

$$
\begin{aligned}
&L_- + L_1 + L_+ + \int_{\mathbb{R}^2} \rho_0(x) \left(\varepsilon(\rho_0(x)) + \frac{|v_0(x)|^2}{2} \right) \varphi(0,x)\, dx \\
&= \frac{1}{\sqrt{\nu_0^2+1}} \Bigg[\nu_0 \left(\rho_1\, \varepsilon(\rho_1) + \rho_1 \frac{C_1}{2} - \rho_-\, \varepsilon(\rho_-) + \rho_- \frac{|v_-|^2}{2} \right) \\
&\qquad - \left(\rho_1\, \varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2} \right) v_{1\,2} \\
&\qquad + \left(\rho_-\, \varepsilon(\rho_-) + p(\rho_-) + \rho_- \frac{|v_-|^2}{2} \right) v_{-\,2} \Bigg] \int_{\Gamma_0} \varphi\, dS \\
&\quad + \frac{1}{\sqrt{\nu_1^2+1}} \Bigg[\nu_1 \left(\rho_+\, \varepsilon(\rho_+) + \rho_+ \frac{|v_+|^2}{2} - \rho_1\, \varepsilon(\rho_1) + \rho_1 \frac{C_1}{2} \right) \\
&\qquad - \left(\rho_+\, \varepsilon(\rho_+) + p(\rho_+) + \rho_+ \frac{|v_+|^2}{2} \right) v_{+\,2} \\
&\qquad + \left(\rho_1\, \varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2} \right) v_{1\,2} \Bigg] \int_{\Gamma_1} \varphi\, dS. \qquad (5.16)
\end{aligned}
$$

Putting (5.11) and (5.14) together we obtain that, if we deal with simple fan sub-solutions, the equation (4.3) holds if and only if

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{\nu_0^2+1}} \Big(\nu_0\, (\rho_1 - \rho_-) - (\rho_1\, v_{1\,2} - \rho_-\, v_{-\,2}) \Big) \int_{\Gamma_0} \psi\, dS \\
&\quad + \frac{1}{\sqrt{\nu_1^2+1}} \Big(\nu_1\, (\rho_+ - \rho_1) - (\rho_+\, v_{+\,2} - \rho_1\, v_{1\,2}) \Big) \int_{\Gamma_1} \psi\, dS, \qquad (5.17)
\end{aligned}
$$

Similarly we get from (5.12) and (5.15) that (4.4) is equivalent to

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{\nu_0^2+1}} \Bigg[\Big(\nu_0\, (\rho_1\, v_{1\,1} - \rho_-\, v_{-\,1}) - (\rho_1\, u_{1\,12} - \rho_-\, u_{-\,12}) \Big) \int_{\Gamma_0} \phi_1\, dS \\
&\quad + \Big(\nu_0\, (\rho_1\, v_{1\,2} - \rho_-\, v_{-\,2}) - \rho_1\, u_{1\,22} - p(\rho_1) \\
&\quad - \rho_1 \frac{C_1}{2} + \rho_-\, u_{-\,22} + p(\rho_-) + \rho_- \frac{|v_-|^2}{2} \Big) \int_{\Gamma_0} \phi_2\, dS \Bigg] \\
&\quad + \frac{1}{\sqrt{\nu_1^2+1}} \Bigg[\Big(\nu_1\, (\rho_+\, v_{+\,1} - \rho_1\, v_{1\,1}) - (\rho_+\, u_{+\,12} - \rho_1\, u_{1\,12}) \Big) \int_{\Gamma_1} \phi_1\, dS \\
&\quad + \Big(\nu_1\, (\rho_+\, v_{+\,2} - \rho_1\, v_{1\,2}) - \rho_+\, u_{+\,22} - p(\rho_+) \\
&\quad - \rho_+ \frac{|v_+|^2}{2} + \rho_1\, u_{1\,22} + p(\rho_1) + \rho_1 \frac{C_1}{2} \Big) \int_{\Gamma_1} \phi_2\, dS \Bigg] \qquad (5.18)
\end{aligned}
$$

and with (5.13) and (5.16) we have that (4.5) is true if and only if

$$\begin{aligned}0 \leq \frac{1}{\sqrt{\nu_0^2+1}} \Bigg[&\nu_0 \left(\rho_1\, \varepsilon(\rho_1) + \rho_1 \frac{C_1}{2} - \rho_-\, \varepsilon(\rho_-) + \rho_- \frac{|v_-|^2}{2} \right) \\ &- \left(\rho_1\, \varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2} \right) v_{1\,2} \\ &+ \left(\rho_-\, \varepsilon(\rho_-) + p(\rho_-) + \rho_- \frac{|v_-|^2}{2} \right) v_{-\,2} \Bigg] \int_{\Gamma_0} \varphi\, dS \\ + \frac{1}{\sqrt{\nu_1^2+1}} \Bigg[&\nu_1 \left(\rho_+\, \varepsilon(\rho_+) + \rho_+ \frac{|v_+|^2}{2} - \rho_1\, \varepsilon(\rho_1) + \rho_1 \frac{C_1}{2} \right) \\ &- \left(\rho_+\, \varepsilon(\rho_+) + p(\rho_+) + \rho_+ \frac{|v_+|^2}{2} \right) v_{+\,2} \\ &+ \left(\rho_1\, \varepsilon(\rho_1) + p(\rho_1) + \rho_1 \frac{C_1}{2} \right) v_{1\,2} \Bigg] \int_{\Gamma_1} \varphi\, dS. \end{aligned} \tag{5.19}$$

Now we start with the actual proof of proposition 5.2.
Suppose there exists a simple subsolution to (1.1), (1.6). Hence, with the above consideration, the equations (5.17), (5.18) and the inequality (5.19) are true for all test functions $(\psi, \phi, \varphi) \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_0^+)$. We first look at (5.17) and choose a test function ψ such that $\mathrm{supp}(\psi) \cap \Gamma_0 \neq \varnothing$ and $\mathrm{supp}(\psi) \cap \Gamma_1 = \varnothing$. Then we obtain

$$\nu_0\, (\rho_1 - \rho_-) - (\rho_1\, v_{1\,2} - \rho_-\, v_{-\,2}) = 0,$$

which is equivalent to (5.1). Equation (5.4) is achieved by choosing ψ such that $\mathrm{supp}(\psi) \cap \Gamma_0 = \varnothing$ and $\mathrm{supp}(\psi) \cap \Gamma_1 \neq \varnothing$. Similarly in (5.18) we set the test function ϕ successively in a way such that

- $\phi_2 \equiv 0$, $\mathrm{supp}(\phi_1) \cap \Gamma_0 \neq \varnothing$ and $\mathrm{supp}(\phi_1) \cap \Gamma_1 = \varnothing$,
- $\phi_2 \equiv 0$, $\mathrm{supp}(\phi_1) \cap \Gamma_0 = \varnothing$ and $\mathrm{supp}(\phi_1) \cap \Gamma_1 \neq \varnothing$,
- $\phi_1 \equiv 0$, $\mathrm{supp}(\phi_2) \cap \Gamma_0 \neq \varnothing$ and $\mathrm{supp}(\phi_2) \cap \Gamma_1 = \varnothing$,
- $\phi_1 \equiv 0$, $\mathrm{supp}(\phi_2) \cap \Gamma_0 = \varnothing$ and $\mathrm{supp}(\phi_2) \cap \Gamma_1 \neq \varnothing$,

to obtain respectively (5.2), (5.5), (5.3) and (5.6), where we have to remember that $u_{1\,22} = -u_{1\,11}$, since $\mathrm{tr}(u) = 0$, and $u_{\pm\,12} = v_{\pm\,1} \cdot v_{\pm\,2}$, $u_{\pm\,22} = v_{\pm\,2}^2 - \frac{|v_\pm|^2}{2}$ (see definition 4.2).
In the same way we get (5.9) and (5.10) from (5.19) setting the test function φ such that $\mathrm{supp}(\varphi) \cap \Gamma_0 \neq \varnothing$ and $\mathrm{supp}(\varphi) \cap \Gamma_1 = \varnothing$, respectively $\mathrm{supp}(\varphi) \cap \Gamma_0 = \varnothing$ and $\mathrm{supp}(\varphi) \cap \Gamma_1 \neq \varnothing$.
It remains to show the subsolution conditions (5.7) and (5.8). From (4.2) we obtain that the symmetric matrix $\frac{C_1}{2}\,\mathrm{Id} - v_1 \otimes v_1 + u_1$ is positive definite. Since positive definiteness of a symmetric $2{\times}2$ matrix S is equivalent to $\det(S) > 0$ and $\mathrm{tr}(S) > 0$[15],

[15]Proof: It is easy to check that two real numbers are positive if and only if their sum and product is positive. Since a matrix is positive definite if and only if the two eigenvalues are positive we have that positive definiteness is equivalent to sum and product of the eigenvalues is positive. Because S is symmetric, it is diagonalizable. Then the trace of S is equal to the sum and the determinant is equal to the product of the eigenvalues what finishes the proof of the claim.

we get

$$\begin{aligned}
0 &< \operatorname{tr}\left(\frac{C_1}{2}\operatorname{Id} - v_1 \otimes v_1 + u_1\right) = C_1 - |v_1|^2, \\
0 &< \det\left(\frac{C_1}{2}\operatorname{Id} - v_1 \otimes v_1 + u_1\right) \\
&= \left(\frac{C_1}{2} - v_{1\,1}^2 + u_{1\,11}\right)\left(\frac{C_1}{2} - v_{1\,2}^2 - u_{1\,11}\right) - (u_{1\,12} - v_{1\,1}\,v_{1\,2})^2,
\end{aligned}$$

which are exactly (5.7), respectively (5.8). Here we used, that $u_1 \in \mathcal{S}_0^{2\times 2}$ has the form

$$u_1 = \begin{pmatrix} u_{1\,11} & u_{1\,12} \\ u_{1\,12} & -u_{1\,11} \end{pmatrix}.$$

To prove the converse direction suppose that the constants $\nu_0, \nu_1 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^+$, $v_1 \in \mathbb{R}^2$, $u_1 \in \mathcal{S}_0^{2\times 2}$ and $C_1 \in \mathbb{R}^+$ fulfill (5.1) - (5.10). Then (5.17), (5.18) and (5.19) hold for all test functions $(\psi, \phi, \varphi) \in C_c^\infty([0,\infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_0^+)$. Hence (4.3), (4.4) and (4.5) are fulfilled. Furthermore (5.7) and (5.8) and above computations lead to

$$\begin{aligned}
\operatorname{tr}\left(\frac{C_1}{2}\operatorname{Id} - v_1 \otimes v_1 + u_1\right) &> 0, \\
\det\left(\frac{C_1}{2}\operatorname{Id} - v_1 \otimes v_1 + u_1\right) &> 0,
\end{aligned}$$

which yields that $\frac{C_1}{2}\operatorname{Id} - v_1 \otimes v_1 + u_1$ is positive definite. So (4.2) holds, what finishes the proof. □

The equations and inequalities in proposition 5.2 can be simplified further, what is the content of the following proposition.

Proposition 5.3. (see [CK14, Lemma 4.4]) *Let $\rho_-, \rho_+ \in \mathbb{R}^+$, $v_-, v_+ \in \mathbb{R}^2$ with $v_{-\,1} = v_{+\,1}$ be given (see initial condition (1.6)). There exists a simple fan subsolution to the Cauchy problem (1.1), (1.6) if and only if there exist constants $\nu_0, \nu_1 \in \mathbb{R}$ (with $\nu_0 < \nu_1$), $\rho_1 \in \mathbb{R}^+$, $v_{1\,2} \in \mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{R}$ such that the following algebraic equations and inequalities hold:*

- *Rankine Hugoniot conditions on the left interface*

$$\nu_0\,(\rho_- - \rho_1) = \rho_-\,v_{-\,2} - \rho_1\,v_{1\,2} \tag{5.20}$$
$$\nu_0\,(\rho_-\,v_{-\,2} - \rho_1\,v_{1\,2}) = \rho_-\,v_{-\,2}^2 - \rho_1\,(v_{1\,2}^2 + \delta_1) + p(\rho_-) - p(\rho_1) \tag{5.21}$$

- *Rankine Hugoniot conditions on the right interface*

$$\nu_1\,(\rho_1 - \rho_+) = \rho_1\,v_{1\,2} - \rho_+\,v_{+\,2} \tag{5.22}$$
$$\nu_1\,(\rho_1\,v_{1\,2} - \rho_+\,v_{+\,2}) = \rho_1\,(v_{1\,2}^2 + \delta_1) - \rho_+\,v_{+\,2}^2 + p(\rho_1) - p(\rho_+) \tag{5.23}$$

- *Subsolution condition*

$$\delta_1 > 0 \tag{5.24}$$
$$\delta_2 > 0 \tag{5.25}$$

- *Admissibility condition on the left interface*

$$(v_{1\,2} - v_{-\,2}) \left(p(\rho_-) + p(\rho_1) - 2\,\rho_-\,\rho_1\,\frac{\varepsilon(\rho_-) - \varepsilon(\rho_1)}{\rho_- - \rho_1} \right) \\ \le \delta_1\,\rho_1\,(v_{1\,2} + v_{-\,2}) - (\delta_1 + \delta_2)\,\frac{\rho_-\,\rho_1\,(v_{1\,2} - v_{-\,2})}{\rho_- - \rho_1} \tag{5.26}$$

- *Admissibility condition on the right interface*

$$(v_{+\,2} - v_{1\,2}) \left(p(\rho_1) + p(\rho_+) - 2\,\rho_1\,\rho_+\,\frac{\varepsilon(\rho_1) - \varepsilon(\rho_+)}{\rho_1 - \rho_+} \right) \\ \le -\delta_1\,\rho_1\,(v_{+\,2} + v_{1\,2}) + (\delta_1 + \delta_2)\,\frac{\rho_1\,\rho_+\,(v_{+\,2} - v_{1\,2})}{\rho_1 - \rho_+} \tag{5.27}$$

Proof. Suppose there is a simple fan subsolution to the Cauchy problem (1.1), (1.6), i.e. by proposition 5.2 there exist constants $\nu_0, \nu_1 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^+$, $v_1 \in \mathbb{R}^2$, $u_1 \in \mathcal{S}_0^{2\times 2}$ and $C_1 \in \mathbb{R}^+$ such that (5.1) - (5.10) hold. Then obviously (5.20) and (5.22) are fulfilled.
Let us add (5.2) and (5.5) to obtain

$$\nu_0\,(\rho_-\,v_{-\,1} - \rho_1\,v_{1\,1}) + \nu_1\,(\rho_1\,v_{1\,1} - \rho_+\,v_{+\,1}) = \rho_-\,v_{-\,1}\,v_{-\,2} - \rho_+\,v_{+\,1}\,v_{+\,2}.$$

Using the fact that $v_{-\,1} = v_{+\,1}$, we get

$$v_{1\,1}\,\rho_1\,(\nu_1 - \nu_0) = v_{-\,1}\,(\rho_-\,v_{-\,2} - \rho_+\,v_{+\,2}) + v_{-\,1}\,(\nu_1\,\rho_+ - \nu_0\,\rho_-). \tag{5.28}$$

Adding (5.1) and (5.4) yields

$$\nu_0\,(\rho_- - \rho_1) + \nu_1\,(\rho_1 - \rho_+) = \rho_-\,v_{-\,2} - \rho_+\,v_{+\,2}.$$

We put this into (5.28):

$$\begin{aligned} v_{1\,1}\,\rho_1\,(\nu_1 - \nu_0) &= v_{-\,1}\,\big(\nu_0\,(\rho_- - \rho_1) + \nu_1\,(\rho_1 - \rho_+) + \nu_1\,\rho_+ - \nu_0\,\rho_-\big) \\ &= v_{-\,1}\,\rho_1\,(\nu_1 - \nu_0), \end{aligned}$$

which leads to

$$v_{1\,1} = v_{-\,1} = v_{+\,2}, \tag{5.29}$$

since $\rho_1 > 0$ and $\nu_0 < \nu_1$. Starting with (5.5) and using (5.29) and (5.4), we figure out that

$$\begin{aligned} \rho_1\,u_{1\,12} &= \nu_1\,(\rho_1 - \rho_+)\,v_{1\,1} + \rho_+\,v_{+\,2}\,v_{1\,1} \\ &= v_{1\,1}\,(\rho_1\,v_{1\,2} - \rho_+\,v_{+\,2}) + \rho_+\,v_{+\,2}\,v_{1\,1} \\ &= \rho_1\,v_{1\,1}\,v_{1\,2} \end{aligned}$$

holds, which yields

$$u_{1\,12} = v_{1\,1}\,v_{1\,2}, \tag{5.30}$$

because $\rho_1 > 0$.
Define

$$\begin{aligned} \delta_1 &= \frac{C_1}{2} - v_{1\,2}^2 - u_{1\,11}, \\ \delta_2 &= \frac{C_1}{2} - v_{1\,1}^2 + u_{1\,11}. \end{aligned}$$

It is easy to see, that with this definition (5.3) implies (5.21) and (5.6) implies (5.23). Using the subsolution conditions (5.7) and (5.8) we obtain

$$\begin{aligned} \delta_1 + \delta_2 &> 0, \\ \delta_1 \cdot \delta_2 &> 0, \end{aligned}$$

where we also used (5.30). These are equivalent to $\delta_1, \delta_2 > 0$, i.e. (5.24), (5.25) are proven.
Before we prove the admissibility conditions (5.26) and (5.27) we show that $\rho_1 \neq \rho_-$ and $\rho_1 \neq \rho_+$ such that the expressions in (5.26) and (5.27) are well-defined. Assume $\rho_1 = \rho_-$ then (5.1) yields $v_{1\,2} = v_{-\,2}$ and therefore from (5.3) we get

$$0 = \rho_1 \left(v_{1\,2}^2 + u_{1\,11} - \frac{C_1}{2} \right).$$

Using this and (5.30) we find out that (5.8) is not true, which is a contradiction. Analogously one can prove that $\rho_1 \neq \rho_+$.
Let's turn our attention to the admissibility conditions. We start with (5.9) and subtract on both sides $\frac{v_{1\,1}}{2}$ times equation (5.20) to obtain

$$\begin{aligned} &\nu_0 \left(\rho_- \, \varepsilon(\rho_-) - \rho_1 \, \varepsilon(\rho_1) + \rho_- \, \frac{v_{-\,2}^2}{2} - \rho_1 \, \frac{C_1 - v_{-\,1}^2}{2} \right) \\ &\leq \big(\rho_- \, \varepsilon(\rho_-) + p(\rho_-)\big) \, v_{-\,2} - \big(\rho_1 \, \varepsilon(\rho_1) + p(\rho_1)\big) \, v_{1\,2} + \rho_- \, v_{-\,2} \, \frac{v_{-\,2}^2}{2} - \rho_1 \, v_{1\,2} \, \frac{C_1 - v_{-\,1}^2}{2}. \end{aligned}$$

Using $\nu_0 = \frac{\rho_- \, v_{-\,2} - \rho_1 \, v_{1\,2}}{\rho_- - \rho_1}$ and $C_1 - v_{-\,1}^2 = \delta_1 + \delta_2 + v_{1\,2}^2$, and multiplying by 2 we get after some computations

$$\begin{aligned} &(v_{1\,2} - v_{-\,2}) \left(p(\rho_-) + p(\rho_1) - 2 \, \rho_- \, \rho_1 \, \frac{\varepsilon(\rho_-) - \varepsilon(\rho_1)}{\rho_- - \rho_1} \right) \\ &\leq (v_{1\,2} + v_{-\,2}) \, \big(p(\rho_-) - p(\rho_1)\big) + (v_{1\,2} + v_{-\,2}) \, \frac{\rho_- \, \rho_1}{\rho_- - \rho_1} \, \Big(v_{-\,2}^2 - (\delta_1 + \delta_2 + v_{1\,2}^2) \Big). \end{aligned}$$

To eliminate $p(\rho_-) - p(\rho_1)$ we use (5.21). Thereafter we again use $\nu_0 = \frac{\rho_- \, v_{-\,2} - \rho_1 \, v_{1\,2}}{\rho_- - \rho_1}$ and finally arrive at (5.26). Analogously one proves (5.27).
Up to now we proved one direction. For the converse suppose there are constants $\nu_0, \nu_1 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^+$, $v_{1\,2} \in \mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{R}$ which fulfill (5.20) - (5.27). Define

$$\begin{aligned} v_{1\,1} &= v_{-\,1}, \\ u_{1\,12} &= v_{1\,1} \, v_{1\,2}, \\ C_1 &= \delta_1 + \delta_2 + v_{1\,1}^2 + v_{1\,2}^2, \\ u_{1\,11} &= \frac{\delta_2 - \delta_1 + v_{1\,1}^2 - v_{1\,2}^2}{2}. \end{aligned}$$

Just by computation one can show that then (5.1) - (5.10) are fulfilled, what finishes the proof. □

In order to show existence of an admissible fan subsolution we will apply proposition 5.3. To do this we have to find six real numbers that fulfill a set of four equations and some inequalities. As in [CK14] the idea is now to choose two parameters and try to express the other four values as functions of these parameters, since there are four equations available. Because δ_2 doesn't appear in the equations (5.20) - (5.23), it is a good choice to take δ_2 as one parameter. We determine ρ_1 to be the other parameter. Hence we will be able to express ν_0, ν_1, $v_{1\,2}$ and δ_1 as functions of ρ_1. To do the mentioned transformations we discuss three different cases, depending on the standard solution covered in section 3.
First we consider initial data such that the corresponding standard solution consists of two shocks. In this case E. Chiodaroli and O. Kreml proved in [CK14] that there always exist constants $\nu_0, \nu_1 \in \mathbb{R}$, $\rho_1 \in \mathbb{R}^+$, $v_{1\,2} \in \mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{R}$ such that the equations and inequalities of proposition 5.3 hold. That means that in this case there exists always a subsolution and therefore by theorem 4.3 infinitely many admissible weak solutions. In other words initial data, which are such that the corresponding standard solution consists of two shocks, are wild. We will briefly repeat their results.
In the case where the standard solution consists of two rarefactions, E. Feireisl and O. Kreml showed in [FK15] that the standard solution is unique, i.e. in this case there are no wild solutions. So initial data, which is such that the corresponding standard solution consists of two rarefactions, is not wild. To prove this uniqueness they used the relative entropy inequality.
The most interesting case is last one. Here the initial data is such that the standard solution consists of a 1-shock and a 3-rarefaction. Note that the case where we have a 1-rarefaction and a 3-shock does not need to be considered as another case. Because of the rotational invariance of the Euler equations we can easily rotate the initial data by 180 degrees and get a standard solution consisting of a 1-shock and a 3-rarefaction.
The reason why this case is the most interesting one, is the fact that a general statement on the existence of wild solutions is not yet known. In [CLK15] an example of a wild initial condition which belongs to this case is given by E. Chiodaroli, C. De Lellis and O. Kreml. We will show that there are also initial data, where there is no simple fan subsolution. Note that this does not prove that the solution is unique in this case. We will also give a criterion which allows us, given initial values such that the corresponding standard solution consists of a 1-shock and a 3-rarefaction, to check if there exists a simple fan subsolution or not.

5.1. The standard solution consists of a 1-shock and a 3-shock

Let the initial values $\rho_\pm \in \mathbb{R}^+$ and $v_\pm \in \mathbb{R}^2$ be such that the standard solution consists of two shocks. Proposition 3.3 states that this means that

$$v_{+\,2} - v_{-\,2} < -\sqrt{\frac{(\rho_- - \rho_+)\left(p(\rho_-) - p(\rho_+)\right)}{\rho_-\,\rho_+}}. \tag{5.31}$$

Theorem 5.4. *Let first* $\rho_- = \rho_+$*. There exists a simple fan subsolution to the Cauchy problem* (1.1), (1.6) *if there exist constants* $\rho_1, \delta_2 \in \mathbb{R}^+$ *that fulfill*

$$\rho_1 > \rho_-, \tag{5.32}$$
$$\delta_1 > 0, \tag{5.33}$$

and the admissibility conditions (5.26), (5.27)*, where we define*

$$v_{1\,2} = \frac{v_{-\,2} + v_{+\,2}}{2}, \tag{5.34}$$
$$\delta_1 = \frac{\rho_-\,(v_{-\,2} - v_{+\,2})^2}{4\,(\rho_1 - \rho_-)} - \frac{p(\rho_1) - p(\rho_-)}{\rho_1}. \tag{5.35}$$

Let now $\rho_- \neq \rho_+$*. There exists a simple fan subsolution to the Cauchy problem* (1.1), (1.6) *if there exist constants* $\rho_1, \delta_2 \in \mathbb{R}^+$ *that fulfill*

$$\rho_1 > \max\{\rho_-, \rho_+\}, \tag{5.36}$$
$$\delta_1 > 0, \tag{5.37}$$

and the admissibility conditions (5.26), (5.27)*, where we define*[16]

$$\begin{aligned} v_{1\,2} = \frac{1}{\rho_1\,(\rho_- - \rho_+)} \Bigg(& \rho_-\, v_{-\,2}\,(\rho_1 - \rho_+) - \rho_+\, v_{+\,2}\,(\rho_1 - \rho_-) \\ & - \sqrt{\Big[\rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big)\Big]\,(\rho_1 - \rho_-)\,(\rho_1 - \rho_+)} \Bigg), \end{aligned} \tag{5.38}$$

$$\begin{aligned} \delta_1 = -\frac{p(\rho_1) - p(\rho_-)}{\rho_1} + \frac{\rho_-\,(\rho_1 - \rho_-)}{\rho_1^2\,(\rho_- - \rho_+)^2} \Bigg(& \rho_+\,(v_{+\,2} - v_{-\,2}) \\ & + \sqrt{\Big[\rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big)\Big]\,\frac{\rho_1 - \rho_+}{\rho_1 - \rho_-}} \Bigg)^2. \end{aligned} \tag{5.39}$$

Proof. (see also [CK14, Section 4])
We start with the case that $\rho_- = \rho_+$ and assume there exist constants $\rho_1, \delta_2 \in \mathbb{R}^+$ such that (5.32), (5.33) and (5.26), (5.27) hold. We then define in addition to (5.34) and (5.35)

$$\nu_0 = \frac{v_{-\,2} + v_{+\,2}}{2} - \frac{\rho_-}{2\,(\rho_1 - \rho_-)}\,(v_{-\,2} - v_{+\,2}), \tag{5.40}$$
$$\nu_1 = \frac{v_{-\,2} + v_{+\,2}}{2} + \frac{\rho_-}{2\,(\rho_1 - \rho_-)}\,(v_{-\,2} - v_{+\,2}). \tag{5.41}$$

Now one can apply proposition 5.3 to show that this definition yields a simple fan subsolution: First of all the condition $\nu_0 < \nu_1$ holds because $v_{-\,2} - v_{+\,2} > 0$, which is a consequence of (5.31), and $\rho_1 > \rho_-$ according to (5.32). It is easy to check that the

[16] In the proof we will also show that everything is well-defined, in other words that the occurring radicands are not negative.

Rankine Hugoniot conditions (5.20) - (5.23) in proposition 5.3 hold. Furthermore (5.24) and (5.25) are true, because of (5.33), resp. $\delta_2 \in \mathbb{R}^+$. Hence the given values ρ_1, δ_2 together with $v_{1\,2}, \delta_1, \nu_0, \nu_1$ from (5.34), (5.35), (5.40) and (5.41) define a simple fan subsolution according to proposition 5.3.
Let now $\rho_- \neq \rho_+$ and suppose there are constants ρ_1, δ_2 such that (5.36), (5.37) and (5.26), (5.27) are true. In addition to (5.38) and (5.39) define

$$\nu_0 = \frac{\rho_- v_{-\,2} - \rho_+ v_{+\,2}}{\rho_- - \rho_+} - \frac{1}{\rho_- - \rho_+} \sqrt{\Big[\rho_+ \rho_- (v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)\Big] \frac{\rho_1 - \rho_+}{\rho_1 - \rho_-}}, \tag{5.42}$$

$$\nu_1 = \frac{\rho_- v_{-\,2} - \rho_+ v_{+\,2}}{\rho_- - \rho_+} - \frac{1}{\rho_- - \rho_+} \sqrt{\Big[\rho_+ \rho_- (v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)\Big] \frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}}. \tag{5.43}$$

Again with proposition 5.3 we prove that ρ_1, δ_2, which are given by assumption, together with $v_{1\,2}, \delta_1, \nu_0, \nu_1$ from (5.38), (5.39), (5.42) and (5.43) define a simple fan subsolution. Let us first check if the above definitions are well-defined, i.e. if the occurring radicands are non-negative: We get from (5.31)

$$\frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_- \rho_+} < (v_{-\,2} - v_{+\,2})^2,$$

which is equivalent to

$$\rho_+ \rho_- (v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) > 0.$$

Because of this and (5.36) all the radicands in the definitions above are positive, hence everything is well-defined.
Next we check if $\nu_0 < \nu_1$: We first assume that $\rho_- > \rho_+$ and obtain

$$\sqrt{\frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}} < 1 < \sqrt{\frac{\rho_1 - \rho_+}{\rho_1 - \rho_-}}.$$

Since in the equations for ν_0 and ν_1, i.e. in (5.42) and (5.43), there is a $-\frac{1}{\rho_- - \rho_+}$ in front of the square root and $\rho_- - \rho_+ > 0$ by assumption, we conclude that $\nu_0 < \nu_1$.
Let now $\rho_- < \rho_+$. Then

$$\sqrt{\frac{\rho_1 - \rho_+}{\rho_1 - \rho_-}} < 1 < \sqrt{\frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}}.$$

Because now we have $\rho_- - \rho_+ < 0$, we also get $\nu_0 < \nu_1$.
The next step is to check if the Rankine Hugoniot conditions (5.20) - (5.23) are fulfilled, what is not difficult to verify and hence omitted here. In order to check these conditions, it can be useful to realize that there is another expression for δ_1,

namely

$$\delta_1 = -\frac{p(\rho_1) - p(\rho_+)}{\rho_1} + \frac{\rho_+ (\rho_1 - \rho_+)}{\rho_1^2 (\rho_- - \rho_+)^2} \Bigg(\rho_- (v_{+2} - v_{-2}) + \sqrt{\left[\rho_+ \rho_- (v_{-2} - v_{+2})^2 - (\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)\right] \frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}} \Bigg)^2 . \qquad (5.44)$$

We leave the computation of this second expression to the reader, too.
Moreover the conditions (5.24) and (5.25) are true because of (5.37) and $\delta_2 > 0$ by assumption. □

The next result is proven by E. Chiodaroli and O. Kreml in [CK14].

Theorem 5.5. *There exists a simple fan subsolution and therefore infinitely many admissible weak solutions to the Cauchy problem* (1.1), (1.6).

Proof. We want to give a rough sketch of the proof by E. Chiodaroli and O. Kreml [CK14, Section 4]. The idea is to show that there always exists a pair of constants $\rho_1, \delta_2 \in \mathbb{R}^+$ as in theorem 5.4. Then this theorem states that there exists a simple fan subsolution and by theorem 4.3 there are infinitely many admissible weak solutions. To prove existence of such two constants ρ_1, δ_2 one first can check that the condition (5.37), resp. (5.33) is true if for all $\max\{\rho_\pm\} < \rho_1 < \rho_M$ and false for all $\rho_1 \geq \rho_M$, where ρ_M is the density of the intermediate state of the standard solution and given in proposition 3.3. With some continuity arguments one can show that there are positive constants $c_1, c_2 > 0$ such that the admissibility conditions (5.26) and (5.27) are fulfilled for all $(\rho_1, \delta_2) \in (\rho_M - c_1, \rho_M) \times (0, c_2)$. Hence if we choose c_1 small enough we have that the conditions (5.36), (5.37), resp. (5.32), (5.33) in theorem 5.4 and the admissibility conditions (5.26) and (5.27) hold for all $(\rho_1, \delta_2) \in (\rho_M - c_1, \rho_M) \times (0, c_2)$. This finishes the proof. □

Remark. In [CK14] the pressure is given by $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$. Note that this is a small difference to our pressure law which is given by $p(\rho) = K \rho^\gamma$ with $K > 0$ and $\gamma > 1$. However the proofs in [CK14] are still true for our pressure law.

Next we're going to investigate some properties of the wild solutions produced by a simple fan subsolution like in the above theorems. More precisely we want to compare the wild solutions to the standard solution presented in proposition 3.3. An obvious accordance is that both consist of three parts. While the standard solution is piecewise constant with the constant states (ρ_-, v_-), (ρ_M, v_M) and (ρ_+, v_+), a wild solution is constant in open sets $P_-, P_+ \subset [0, \infty) \times \mathbb{R}^2$ with constant states (ρ_-, v_-) and (ρ_+, v_+) and not constant on a set P_1. More precisely on the set P_1 the density $\rho \equiv \rho_1$ is constant, where ρ_1 is slightly smaller than ρ_M, and the velocity is not constant. We want to compare the numbers ν_0 and ν_1 to the shock speeds of the standard solution σ_0 and σ_1. As mentioned in the proof of theorem 5.5 the value δ_1 defined in the proof of theorem 5.4 is positive for $\rho_1 < \rho_M$ and negative for $\rho_1 > \rho_M$. Hence by continuity it is zero for $\rho_1 = \rho_M$. Let us use this fact to compute ν_0 and ν_1 for $\rho_1 = \rho_M$: Let first $\rho_- = \rho_+$. From $\delta_1 = 0$ we get

$$v_{-2} - v_{+2} = \pm 2 \sqrt{\frac{\big(p(\rho_M) - p(\rho_-)\big)(\rho_M - \rho_-)}{\rho_- \rho_M}},$$

and because of (5.31) the proper sign is here is "+". Using this we obtain

$$\begin{aligned}
\nu_0 &= \frac{v_{-2}+v_{+2}}{2} - \frac{\rho_-}{2\,(\rho_M-\rho_-)}\,(v_{-2}-v_{+2}) \\
&= v_{-2} - \frac{v_{-2}-v_{+2}}{2} - \frac{\rho_-}{2\,(\rho_M-\rho_-)}\,(v_{-2}-v_{+2}) \\
&= v_{-2} - \frac{\rho_M}{\rho_M-\rho_-}\sqrt{\frac{\big(p(\rho_M)-p(\rho_-)\big)\,(\rho_M-\rho_-)}{\rho_-\,\rho_M}} \\
&= v_{-2} - \sqrt{\frac{p(\rho_M)-p(\rho_-)}{\rho_M-\rho_-}\,\frac{\rho_M}{\rho_-}},
\end{aligned}$$

and similarly

$$\nu_1 = v_{+2} + \sqrt{\frac{p(\rho_M)-p(\rho_+)}{\rho_M-\rho_+}\,\frac{\rho_M}{\rho_+}}.$$

Now assume $\rho_- < \rho_+$. Since $\delta_1 = 0$ we get from (5.39)

$$\begin{aligned}
&\rho_+\,(v_{+2}-v_{-2}) + \sqrt{\Big[\rho_+\,\rho_-\,(v_{-2}-v_{+2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-)-p(\rho_+)\big)\Big]\,\frac{\rho_M-\rho_+}{\rho_M-\rho_-}} \\
&= \pm\sqrt{\frac{\big(p(\rho_M)-p(\rho_-)\big)\,\rho_M\,(\rho_- - \rho_+)^2}{\rho_-\,(\rho_M-\rho_-)}}.
\end{aligned}$$

The right sign in the equation above is "−" because the left-hand side is negative, what can be proved as follows. Since $v_{+2} - v_{-2} < 0$ (because of (5.31)), we have that $v_{+2} - v_{-2} = -\sqrt{(v_{+2}-v_{-2})^2}$ and hence

$$\begin{aligned}
&\rho_+\,(v_{+2}-v_{-2}) + \sqrt{\Big[\rho_+\,\rho_-\,(v_{-2}-v_{+2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-)-p(\rho_+)\big)\Big]\,\frac{\rho_M-\rho_+}{\rho_M-\rho_-}} \\
&\le -\sqrt{\rho_+^2\,(v_{+2}-v_{-2})^2} + \sqrt{\rho_+\,\rho_-\,(v_{-2}-v_{+2})^2\,\frac{\rho_M-\rho_+}{\rho_M-\rho_-}} \\
&= \sqrt{\frac{\rho_+\,(v_{+2}-v_{-2})^2}{\rho_M-\rho_-}}\,\Big(\sqrt{\rho_-\,(\rho_M-\rho_+)} - \sqrt{\rho_+\,(\rho_M-\rho_-)}\Big) \quad < \quad 0.
\end{aligned}$$

In the last step we used that $\rho_- < \rho_+$. Since $\sqrt{(\rho_- - \rho_+)^2} = -(\rho_- - \rho_+)$, we obtain

$$\begin{aligned}
&\rho_+\,(v_{+2}-v_{-2}) + \sqrt{\Big[\rho_+\,\rho_-\,(v_{-2}-v_{+2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-)-p(\rho_+)\big)\Big]\,\frac{\rho_M-\rho_+}{\rho_M-\rho_-}} \\
&= (\rho_- - \rho_+)\sqrt{\frac{\big(p(\rho_M)-p(\rho_-)\big)}{(\rho_M-\rho_-)}\,\frac{\rho_M}{\rho_-}}. \qquad (5.45)
\end{aligned}$$

With (5.45) it is easy to check that

$$\nu_0 = v_{-2} - \sqrt{\frac{p(\rho_M)-p(\rho_-)}{\rho_M-\rho_-}\,\frac{\rho_M}{\rho_-}}.$$

Because ρ_M is the density of the intermediate state of the standard solution, we get from proposition 3.3 that

$$v_{+\,2} - v_{-\,2} = -\sqrt{\frac{(\rho_M - \rho_+)\,\big(p(\rho_M) - p(\rho_+)\big)}{\rho_M\,\rho_+}} - \sqrt{\frac{(\rho_M - \rho_-)\,\big(p(\rho_M) - p(\rho_-)\big)}{\rho_M\,\rho_-}}.$$

This together with (5.45) yields

$$\nu_1 = v_{+\,2} + \sqrt{\frac{p(\rho_M) - p(\rho_+)}{\rho_M - \rho_+}\,\frac{\rho_M}{\rho_+}}.$$

In the case $\rho_- > \rho_+$ we get the same results if we use the expression (5.44) for δ_1.
On the other hand using proposition 3.3 it is easy to compute the shock speeds σ_0, σ_1:

$$\sigma_0 = v_{-\,2} - \sqrt{\frac{p(\rho_M) - p(\rho_-)}{\rho_M - \rho_-}\,\frac{\rho_M}{\rho_-}},$$
$$\sigma_1 = v_{+\,2} + \sqrt{\frac{p(\rho_M) - p(\rho_+)}{\rho_M - \rho_+}\,\frac{\rho_M}{\rho_+}}.$$

In other words we have that $\nu_0 = \sigma_0$ and $\nu_1 = \sigma_1$ for $\rho_1 = \rho_M$.
Next we want to investigate what happens if we consider ρ_1 to be a bit smaller than ρ_M. To do this we differentiate (5.40) and (5.41), resp. (5.42) and (5.43) with respect to ρ_1 to obtain in the case $\rho_- = \rho_+$

$$\frac{\partial}{\partial \rho_1}\nu_0 = \frac{\rho_-}{2\,(\rho_1 - \rho_-)^2}\,(v_{-\,2} - v_{+\,2}),$$
$$\frac{\partial}{\partial \rho_1}\nu_1 = -\frac{\rho_-}{2\,(\rho_1 - \rho_-)^2}\,(v_{-\,2} - v_{+\,2}),$$

and for $\rho_- \neq \rho_+$

$$\frac{\partial}{\partial \rho_1}\nu_0 = \frac{1}{2\,(\rho_1 - \rho_-)^2}\sqrt{\Big[\rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big)\Big]\,\frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}},$$
$$\frac{\partial}{\partial \rho_1}\nu_1 = \frac{-1}{2\,(\rho_1 - \rho_+)^2}\sqrt{\Big[\rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2 - (\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big)\Big]\,\frac{\rho_1 - \rho_+}{\rho_1 - \rho_-}}.$$

Hence if we remember (5.31), in both cases we have $\frac{\partial}{\partial \rho_1}\nu_0 > 0$ and $\frac{\partial}{\partial \rho_1}\nu_1 < 0$. Therefore we can conclude that for $\rho_1 < \rho_M$ we obtain $\nu_0 < \sigma_0$ and $\nu_1 > \sigma_1$.
Finally we are going to illustrate the results of this subsection by presenting an example.

Example 5.6. Let $K = 1$ and $\gamma = 2$ such that the pressure law turns into $p(\rho) = \rho^2$. We consider the initial data

$$\rho_- = 1, \qquad \rho_+ = 4,$$
$$v_- = \begin{pmatrix} 0 \\ \frac{8}{3}\sqrt{10} \end{pmatrix}, \qquad v_+ = \begin{pmatrix} 0 \\ -\frac{5}{6}\sqrt{13} \end{pmatrix}.$$

It is easy to check that (5.31) holds, i.e. the standard solution consists of two shocks. With proposition 3.3 one can compute the intermediate state:

$$\rho_M = 9, \qquad v_M = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Rankine Hugoniot condition (3.10) yields the shock speeds:

$$\sigma_0 = -\frac{1}{3}\sqrt{10} \approx -1.05, \qquad \sigma_1 = \frac{2}{3}\sqrt{13} \approx 2.40.$$

Next we want to investigate if there exist constants $\rho_1, \delta_2 \in \mathbb{R}^+$ as in theorem 5.4, which should be the case because of theorem 5.5. To do this we plot the regions in the (ρ_1, δ_2)-plane where the conditions (5.36), (5.37), (5.26) and (5.27) hold, see figure 6. The condition (5.36) simply means that $\rho_1 > 4$, so we don't need to plot it if we plot the other conditions only for $\rho_1 > 4$. In accordance with theorem 5.5 there are constants $c_1, c_2 > 0$ such that the conditions (5.36), (5.37), (5.26) and (5.27) hold for all $(\rho_1, \delta_2) \in (\rho_M - c_1, \rho_M) \times (0, c_2)$. We can also observe that condition (5.37) holds for all $\rho_1 < 9 = \rho_M$, what is predicted in the proof of theorem 5.5. Now

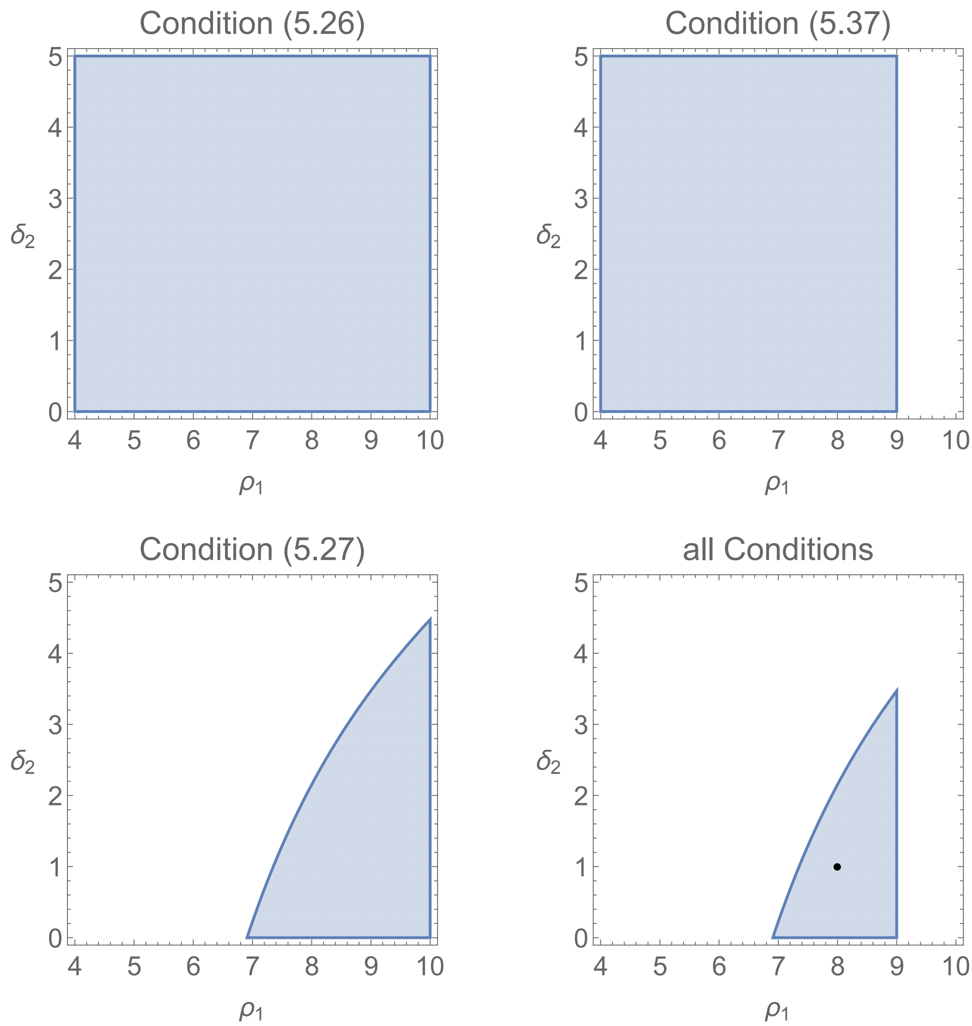

Figure 6: Example 5.6: The region in the ρ_1, δ_2-plane where the conditions of theorem 5.4 hold.

we're going to consider one particular simple fan subsolution namely the one which belongs to $(\rho_1, \delta_2) = (8, 1)$. These values for ρ_1 and δ_2 really fulfill the conditions (5.36), (5.37), (5.26) and (5.27), what can be easily checked by looking at figure 6. Then one can compute

$$\nu_0 \approx -1.31, \qquad \qquad \nu_1 \approx 2.83,$$

by (5.42) and (5.43), what verifies the above proved fact $\nu_0 < \sigma_0$, resp. $\nu_1 > \sigma_1$. A comparison of the structure of the standard solution to the structure of a wild solution can be found in figure 7.

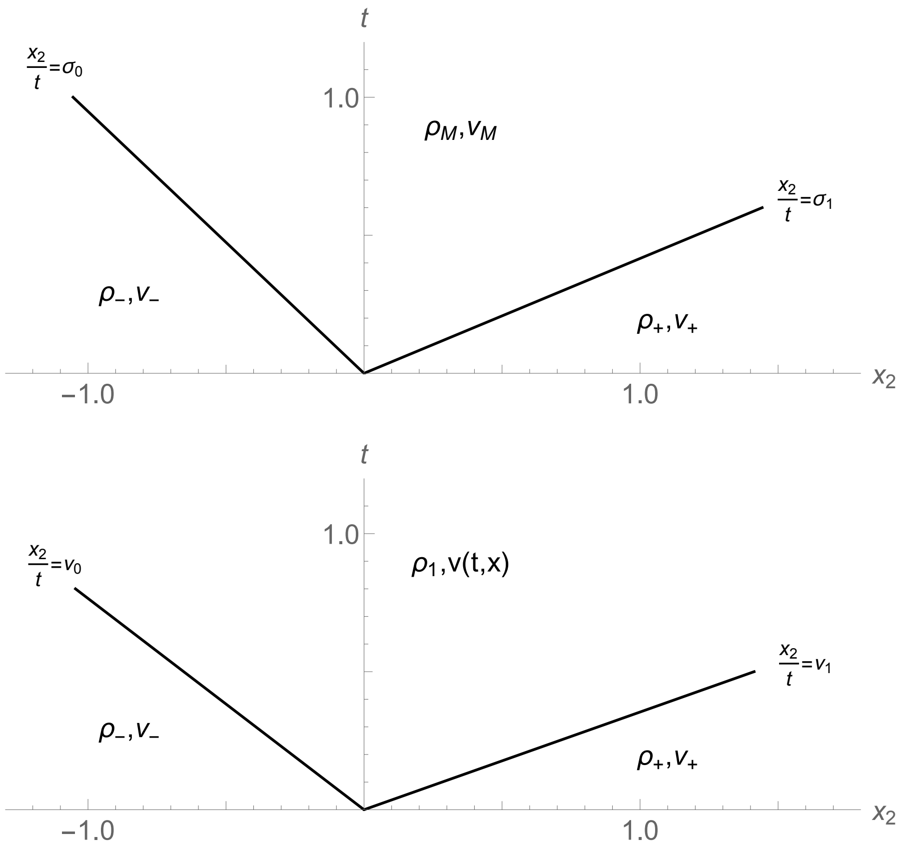

Figure 7: Example 5.6: The standard solution, which consists of two shocks (top) and a wild solution (bottom).

5.2. The standard solution consists of a 1-rarefaction and a 3-rarefaction

Let the initial values $\rho_\pm \in \mathbb{R}^+$ and $v_\pm \in \mathbb{R}^2$ be such that the standard solution consists of two rarefactions, i.e. by proposition 3.3

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\, dr \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(r)}}{r}\, dr + \int_0^{\rho_+} \frac{\sqrt{p'(r)}}{r}\, dr.$$

As already mentioned E. Feireisl and O. Kreml proved that in this case the standard solution is the unique solution to the problem (1.1), (1.6), see [FK15]. To prove this they used the relative entropy inequality, which is also used to show the well-known weak-strong uniqueness property. Here we don't want to redo their prove.

5.3. The standard solution consists of a 1-shock and a 3-rarefaction

Let now the initial values $\rho_\pm \in \mathbb{R}^+$ and $v_\pm \in \mathbb{R}^2$ be such that the standard solution consists of a 1-shock and a 3-rarefaction. By proposition 3.3 this means, that $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+}} < v_{+\,2} - v_{-\,2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\,dr. \tag{5.46}$$

In [CLK15] E. Chiodaroli, C. De Lellis and O. Kreml essentially consider the following example: $\rho_- = 1$, $\rho_+ = 4$, $v_- = (-1/4, 2\sqrt{2})^T$ and $v_+ = (-1/4, 0)^T$. Additionally they suppose the pressure law to be $p(\rho) = \rho^2$. It is easy to check that these values in fact fulfill (5.46). Furthermore in [CLK15] it is proven that there exists a simple fan subsolution. We will later on repeat this result.
Analoguosly to theorem 5.4 we want to find a general condition, which allows to check easily if there is a simple fan subsolution to given initial values or not.

Theorem 5.7. *There exists a simple fan subsolution to the Cauchy problem* (1.1), (1.6) *if and only if there exist constants* $\rho_1, \delta_2 \in \mathbb{R}^+$ *that fulfill*

$$\rho_- < \rho_1 < \rho_+, \tag{5.47}$$

$$\delta_1^\star > 0, \tag{5.48}$$

$$\begin{aligned}(v^\star_{1\,2} - v_{-\,2})\left(p(\rho_-) + p(\rho_1) - 2\,\rho_-\,\rho_1\,\frac{\varepsilon(\rho_-) - \varepsilon(\rho_1)}{\rho_- - \rho_1}\right)&\\ \leq \delta_1^\star\,\rho_1\,(v^\star_{1\,2} + v_{-\,2}) - (\delta_1^\star + \delta_2)\,\frac{\rho_-\,\rho_1\,(v^\star_{1\,2} - v_{-\,2})}{\rho_- - \rho_1}&,\end{aligned} \tag{5.49}$$

$$\begin{aligned}(v_{+\,2} - v^\star_{1\,2})\left(p(\rho_1) + p(\rho_+) - 2\,\rho_1\,\rho_+\,\frac{\varepsilon(\rho_1) - \varepsilon(\rho_+)}{\rho_1 - \rho_+}\right)&\\ \leq -\delta_1^\star\,\rho_1\,(v_{+\,2} + v^\star_{1\,2}) + (\delta_1^\star + \delta_2)\,\frac{\rho_1\,\rho_+\,(v_{+\,2} - v^\star_{1\,2})}{\rho_1 - \rho_+}&,\end{aligned} \tag{5.50}$$

where we define

$$\begin{aligned}v^\star_{1\,2} = \frac{1}{\rho_1\,(\rho_- - \rho_+)}\Bigg(& -\rho_-\,v_{-\,2}\,(\rho_+ - \rho_1) - \rho_+\,v_{+\,2}\,(\rho_1 - \rho_-)\\ &+ \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2\Big]\,(\rho_1 - \rho_-)\,(\rho_+ - \rho_1)}\Bigg),\end{aligned} \tag{5.51}$$

$$\begin{aligned}\delta_1^\star = -\frac{p(\rho_1) - p(\rho_-)}{\rho_1} + \frac{\rho_-\,(\rho_1 - \rho_-)}{\rho_1^2\,(\rho_- - \rho_+)^2}\Bigg(&\rho_+\,(v_{-\,2} - v_{+\,2})\\ &+ \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2\Big]\,\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}}\Bigg)^2.\end{aligned} \tag{5.52}$$

Remark. Theorems 5.4 and 5.7 are slightly different. In theorem 5.7 we state an equivalence to the existence of a simple fan subsolution, i.e. an "if and only if", whereas in theorem 5.4 we only proved a sufficient criterion for the existence of a simple fan subsolution ("if"). The converse in theorem 5.4 is also true. To prove the converse one has to assume that there is a simple fan subsolution and then show that the conditions (5.36), (5.37), resp. (5.32), (5.33) together with the admissibility conditions (5.26), (5.27) hold. The crucial point is to exclude the case $\rho_1 < \min\{\rho_-, \rho_+\}$, which can be done by showing that this causes a contradiction. However we don't need a converse statement in theorem 5.4 because we already showed that in the two shock case there always exist $\rho_1, \delta_2 \in \mathbb{R}^+$ that fulfill all the conditions of theorem 5.4. In the current case, i.e. the standard solution consists of a shock and a rarefaction, we want to show that there may be no simple fan subsolution. To prove this fact we will need a converse statement.
This "if and only if" is also the reason why we denote the variables defined in (5.51) and (5.52) as $v^\star_{1\,2}, \delta^\star_1$ and not simply $v_{1\,2}, \delta_1$. In what we called above converse statement, we have a simple fan subsolution and want the conditions (5.47) - (5.50) to be true. A priori it is not clear that the $v_{1\,2}, \delta_1$ which are given by the simple fan subsolution are equal to the $v^\star_{1\,2}, \delta^\star_1$ defined in (5.51) and (5.52).

Proof. Suppose there is a simple fan subsolution. By proposition 5.3 there exist constants $\nu_0, \nu_1 \in \mathbb{R}$ (with $\nu_0 < \nu_1$), $\rho_1 \in \mathbb{R}^+$, $v_{1\,2} \in \mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{R}$ such that (5.20)-(5.27) hold. Adding (5.20) and (5.22) and solving the result for ν_1 leads to

$$\nu_1 = \frac{\rho_-\, v_{-\,2} - \rho_+\, v_{+\,2} - \nu_0\, (\rho_- - \rho_1)}{\rho_1 - \rho_+}. \tag{5.53}$$

Next we add (5.21) and (5.23) and use (5.20) and (5.22) to obtain

$$\nu_0^2\, (\rho_- - \rho_1) + \nu_1^2\, (\rho_1 - \rho_+) = \rho_-\, v_{-\,2}^2 - \rho_+\, v_{+\,2}^2 + p(\rho_-) - p(\rho_+).$$

If we use (5.53) to eliminate ν_1 and solve for ν_0 we get

$$\begin{aligned}\nu_0 = {}& \frac{\rho_-\, v_{-\,2} - \rho_+\, v_{+\,2}}{\rho_- - \rho_+} \\ & \pm \frac{1}{\rho_- - \rho_+} \sqrt{\Big[(\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big) - \rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2\Big]\, \frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}}.\end{aligned} \tag{5.54}$$

Using this result and (5.53) one has

$$\begin{aligned}\nu_1 = {}& \frac{\rho_-\, v_{-\,2} - \rho_+\, v_{+\,2}}{\rho_- - \rho_+} \\ & \mp \frac{1}{\rho_- - \rho_+} \sqrt{\Big[(\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big) - \rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2\Big]\, \frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}},\end{aligned} \tag{5.55}$$

where the signs in the last two equations have to be different.
In the case of a 1-shock and a 3-rarefaction it holds that

$$(\rho_- - \rho_+)\,\big(p(\rho_-) - p(\rho_+)\big) - \rho_+\,\rho_-\,(v_{-\,2} - v_{+\,2})^2 > 0, \tag{5.56}$$

what can be proved as follows.
We first show that for all $\rho_- < \rho_+$ the inequality

$$\int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(r)}}{r}\, dr < \sqrt{\frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+}} \tag{5.57}$$

is fulfilled. Using (3.26) this turns into

$$\frac{2}{\gamma - 1}\Big(\sqrt{p'(\rho_+)} - \sqrt{p'(\rho_-)}\Big) < \sqrt{\frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+}}.$$

Because $p''(\rho) > 0$ for all $\rho > 0$ and $\gamma > 1$, both sides of the above inequality are positive and hence it is equivalent to

$$\frac{4}{(\gamma - 1)^2}\Big(\sqrt{p'(\rho_+)} - \sqrt{p'(\rho_-)}\Big)^2 < \frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+}.$$

Remember that $p(\rho) = K\,\rho^\gamma$ and $p'(\rho) = K\,\gamma\,\rho^{\gamma-1}$. Divide the inequality above by K and $\rho_-^{\gamma-1}$, and define $z := \frac{\rho_+}{\rho_-}$:

$$\frac{4\,\gamma}{(\gamma - 1)^2}\big(z^{\gamma-1} - 2\,z^{\frac{\gamma-1}{2}} + 1\big) < \frac{1}{z}\,(z - 1)\,(z^\gamma - 1).$$

Let

$$f(z) = (z - 1)\,(z^\gamma - 1) - \frac{4\,\gamma}{(\gamma - 1)^2}\big(z^\gamma - 2\,z^{\frac{\gamma+1}{2}} + z\big),$$

then it is sufficient to prove that $f(z) > 0$ for all $z > 1$. It is easy to recalculate that

$$\begin{aligned}
f'(z) &= (z^\gamma - 1) + (z - 1)\,\gamma\,z^{\gamma-1} - \frac{4\,\gamma}{(\gamma - 1)^2}\big(\gamma\,z^{\gamma-1} - (\gamma + 1)\,z^{\frac{\gamma-1}{2}} + 1\big)\\
f''(z) &= \gamma\,z^{\gamma-1} + (z - 1)\,\gamma\,(\gamma - 1)\,z^{\gamma-2} + \gamma\,z^{\gamma-1}\\
&\qquad - \frac{4\,\gamma}{(\gamma - 1)^2}\Big(\gamma\,(\gamma - 1)\,z^{\gamma-2} - (\gamma + 1)\,\frac{\gamma - 1}{2}\,z^{\frac{\gamma-3}{2}}\Big)\\
&= \underbrace{\gamma\,(\gamma + 1)\,z^{\frac{\gamma-3}{2}}}_{>0}\ \underbrace{\left[z^{\frac{\gamma-1}{2}}\Big(z - \frac{\gamma + 1}{\gamma - 1}\Big) + \frac{2}{\gamma - 1}\right]}_{=:g(z)}
\end{aligned}$$

Finally

$$\begin{aligned}
g'(z) &= \frac{\gamma - 1}{2}\,z^{\frac{\gamma-3}{2}}\Big(z - \frac{\gamma + 1}{\gamma - 1}\Big) + z^{\frac{\gamma-1}{2}}\\
&= \frac{\gamma + 1}{2}\,z^{\frac{\gamma-3}{2}}\,\big(z - 1\big) \quad > \quad 0,
\end{aligned}$$

what implies with $g(1) = 0$ that $g(z) > 0$ for $z > 1$. Hence $f''(z) > 0$ and with $f'(1) = f(1) = 0$ we obtain the wanted property $f(z) > 0$ for all $z > 1$. Now from (5.57) and (5.46) we get

$$\frac{(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big)}{\rho_-\,\rho_+} > (v_{-\,2} - v_{+\,2})^2,$$

what is equivalent to (5.56).
Hence (5.54) and (5.55) yield that $\rho_+ - \rho_1$ and $\rho_1 - \rho_-$ have the same sign. Because $\rho_- < \rho_+$, we have $\rho_- < \rho_1 < \rho_+$, i.e. (5.47). Now we want to choose the correct signs in the equations for ν_0 and ν_1, i.e. in (5.54) and (5.55). Assume we had a "$-$" in (5.54) and therefore a "$+$" in (5.55). Then

$$\nu_0 > \frac{\rho_- \, v_{-\,2} - \rho_+ \, v_{+\,2}}{\rho_- - \rho_+} > \nu_1,$$

since $\rho_- - \rho_+ < 0$. This is a contradiction. Hence the proper sign in (5.54) is "$+$" and in (5.55) it is "$-$", i.e.

$$\begin{aligned}\nu_0 = & \frac{\rho_- \, v_{-\,2} - \rho_+ \, v_{+\,2}}{\rho_- - \rho_+} \\ & + \frac{1}{\rho_- - \rho_+} \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+ \, \rho_- \, (v_{-\,2} - v_{+\,2})^2\Big] \frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}},\end{aligned} \tag{5.58}$$

$$\begin{aligned}\nu_1 = & \frac{\rho_- \, v_{-\,2} - \rho_+ \, v_{+\,2}}{\rho_- - \rho_+} \\ & - \frac{1}{\rho_- - \rho_+} \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+ \, \rho_- \, (v_{-\,2} - v_{+\,2})^2\Big] \frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}}.\end{aligned} \tag{5.59}$$

Next we compute $v_{1\,2}$ using our result for ν_0 and (5.20) and get

$$\begin{aligned} v_{1\,2} = & \frac{1}{\rho_1 \, (\rho_- - \rho_+)} \Bigg(- \rho_- \, v_{-\,2} \, (\rho_+ - \rho_1) - \rho_+ \, v_{+\,2} \, (\rho_1 - \rho_-) \\ & + \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+ \, \rho_- \, (v_{-\,2} - v_{+\,2})^2\Big] (\rho_1 - \rho_-) \, (\rho_+ - \rho_1)} \Bigg). \end{aligned}$$

With (5.21) we finally find

$$\begin{aligned} \delta_1 = & -\frac{p(\rho_1) - p(\rho_-)}{\rho_1} + \frac{\rho_- \, (\rho_1 - \rho_-)}{\rho_1^2 \, (\rho_- - \rho_+)^2} \Bigg(\rho_+ \, (v_{-\,2} - v_{+\,2}) \\ & + \sqrt{\Big[(\rho_- - \rho_+)\big(p(\rho_-) - p(\rho_+)\big) - \rho_+ \, \rho_- \, (v_{-\,2} - v_{+\,2})^2\Big] \frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} \Bigg)^2. \end{aligned}$$

Hence we have $\delta_1 = \delta_1^\star$ and $v_{1\,2} = v_{1\,2}^\star$. From (5.24) we obtain (5.48) and admissibility conditions (5.26) and (5.27) yield (5.49) and (5.50).
It remains to prove the converse. Let $\rho_1, \delta_2 \in \mathbb{R}^+$ such that (5.47) - (5.50) hold. Define $v_{1\,2} = v_{1\,2}^\star$, $\delta_1 = \delta_1^\star$ and ν_0, ν_1 through (5.58), resp. (5.59). Again by easy computations one can check that ρ_1, δ_2 together with $\nu_0, \nu_1, v_{1\,2}, \delta_1$ fulfill the conditions (5.20) - (5.27) and therefore define a simple fan subsolution by proposition 5.3. □

A result which is analogous to theorem 5.5 does not exist in the case of a 1-shock and a 3-rarefaction. We will show this by exposing examples. First of all we start with the above mentioned example in [CLK15].

Example 5.8. Let $K = 1$ and $\gamma = 2$ such that the pressure law turns into $p(\rho) = \rho^2$. We consider the initial data

$$\rho_- = 1, \qquad \rho_+ = 4,$$
$$v_- = \begin{pmatrix} -\frac{1}{4} \\ 2\sqrt{2} \end{pmatrix}, \qquad v_+ = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix}.$$

Since $\rho_- < \rho_+$ and (5.46) holds (what is easy to check), the standard solution consists of a 1-shock and a 3-rarefaction. With proposition 3.3 one can compute the intermediate state:

$$\rho_M \approx 3.70, \qquad v_M \approx \begin{pmatrix} -\frac{1}{4} \\ -0.22 \end{pmatrix}.$$

With the Rankine Hugoniot condition (3.10) we can compute the shock speed

$$\sigma_0 \approx -1.34,$$

and in addition to that we calculate the values

$$\lambda_3(\rho_M, v_M) \approx 2.51, \qquad \lambda_3(\rho_+, v_+) = 2\sqrt{2} \approx 2.83,$$

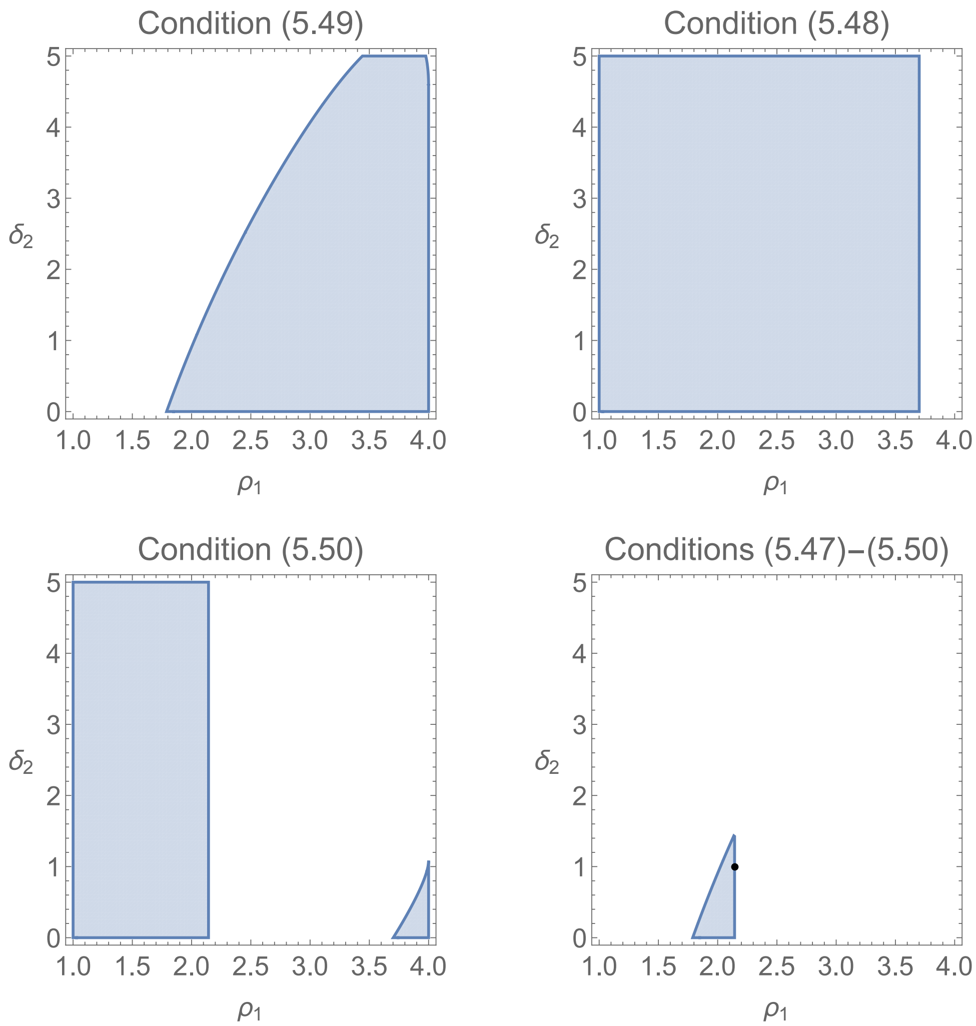

Figure 8: Example 5.8: The region in the ρ_1, δ_2-plane where the conditions of theorem 5.7 hold.

which yield the edges of the rarefaction[17], see also figure 9.
Next we want to investigate if there exist constants $\rho_1, \delta_2 \in \mathbb{R}^+$ as in theorem 5.7. In order to do this, we plot the regions in the (ρ_1, δ_2)-plane where the conditions (5.47) - (5.50) are true, see figure 8. The condition (5.47) simply means that $1 < \rho_1 < 4$, so we don't need to plot it if we plot the other conditions only for $\rho_1 \in (1, 4)$. Obviously the intersection of the regions is non-empty, which means that there are constants $\rho_1, \delta_2 \in \mathbb{R}^+$ that fulfill all the conditions of theorem 5.7. Hence there are infinitely many admissible weak solutions.
Now we consider one particular simple fan subsolution. As in [CLK15] we choose $\rho_1 = \frac{15}{7} \approx 2.14$ and δ_2 small enough, say $\delta_2 = 1$. It is not easy to realize by looking at figure 8 that for this choice the conditions (5.47) - (5.50) hold. Anyway they can be easily checked by plugging in all the values. Additionally one can compute

$$\nu_0 \approx -2.47, \qquad \nu_1 = 0,$$

by (5.58) and (5.59). A comparison of the structure of the standard solution to the structure of a wild solution can be found in figure 9.

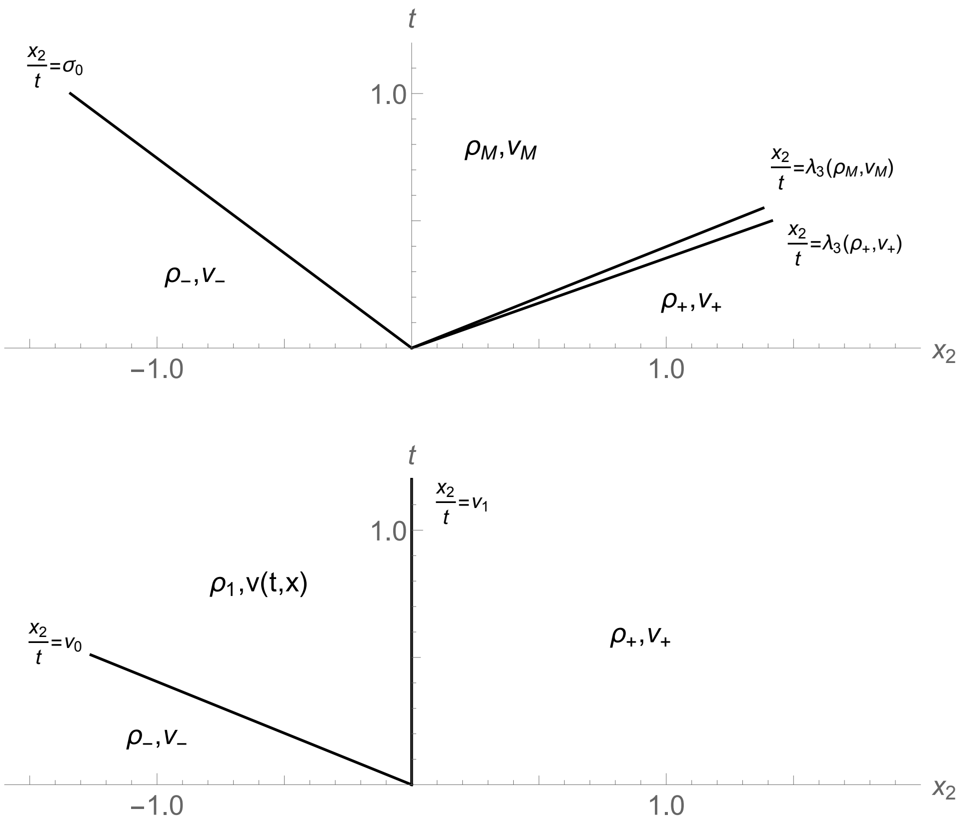

Figure 9: Example 5.8: The standard solution, which consists of a 1-shock and a 3-rarefaction (top) and a wild solution (bottom).

So far we have seen an example for initial values $\rho_\pm, v_\pm$ for which the standard solution consists of a 1-shock and a 3-rarefaction and there exists a simple fan subsolution. Next we will see an example where there is no simple fan subsolution.

[17] according to [LeV04, Section 13.8.5]

Example 5.9. Let again $K = 1$ and $\gamma = 2$ such that the pressure law turns into $p(\rho) = \rho^2$. We consider the initial data

$$\rho_- = 1, \qquad \rho_+ = 4,$$
$$v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_+ = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}.$$

Again it is easy to check that the standard solution consists of a 1-shock and a 3-rarefaction. Now we are not interested in the standard solution and try to find constants ρ_1, δ_2 as in theorem 5.7 immediately. As in the example above we plot the regions where the conditions (5.48) - (5.50) hold, see figure 10. It can be observed that there are no constants ρ_1, δ_2 that fulfill the conditions (5.47) - (5.50).

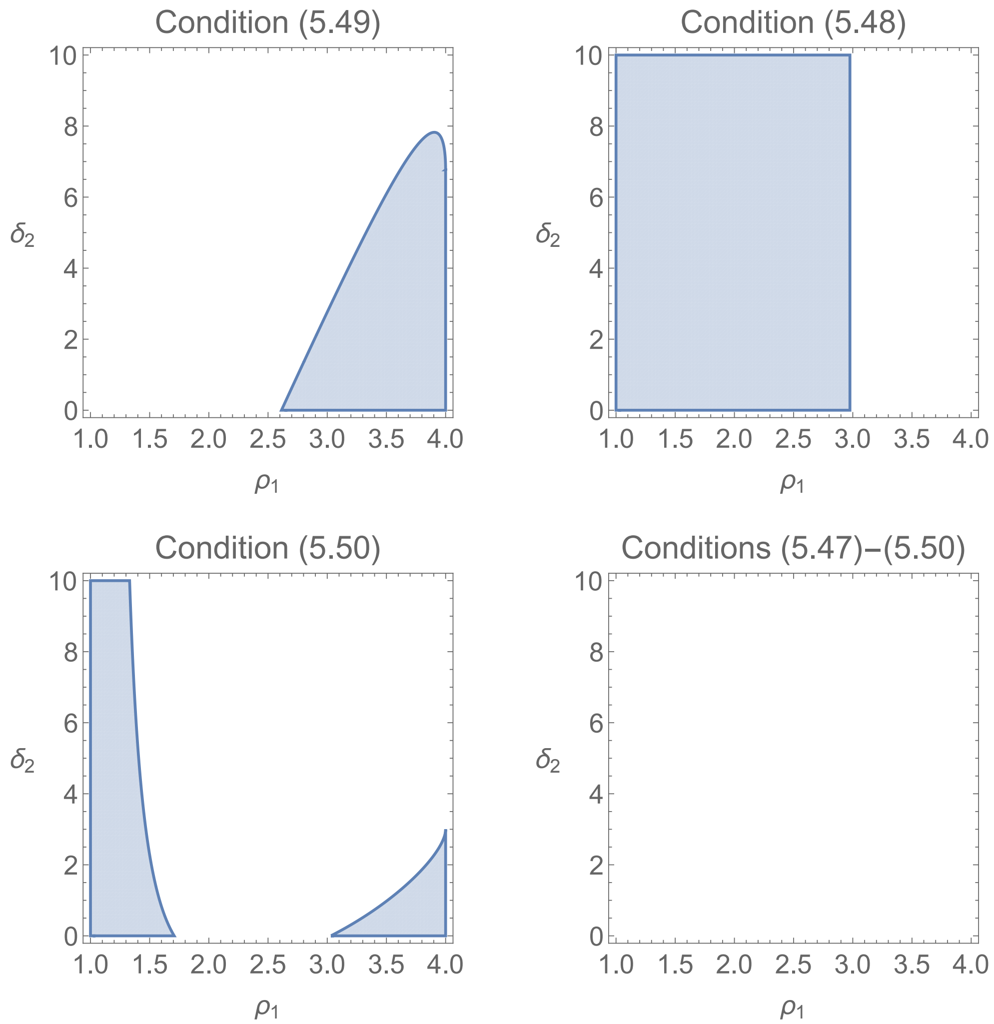

Figure 10: Example 5.9: The region in the ρ_1, δ_2-plane where the conditions of theorem 5.7 hold.

One can also prove this fact using basic skills in calculus. For instance one shows what figure 10 suggest, namely

- for $3 \le \rho_1 < 4$ the condition (5.48) is violated,
- inequality (5.49) is not true for $1 < \rho_1 \le 2$ and finally

- condition (5.50) does not hold for $2 < \rho_1 \leq 3$.

This together with theorem 5.7 proves that there is no simple fan subsolution to (1.1), (1.6) with initial values $\rho_\pm$, $v_\pm$ as above.

6. Outlook and open problems

In this thesis we showed that the initial value problem (1.1), (1.6) may have infinitely many admissible weak solutions, what depends on the initial values $\rho_\pm, v_\pm$. To prove non-uniqueness we used the convex integration method applied to the pressureless incompressible Euler equations. Of course the most interesting open problem is to find out if these wild solutions are physically relevant, resp. if there is a suitable criterion which singles out the unique physically relevant solution. As shown by E. Chiodaroli and O. Kreml there are values for $\rho_\pm$, $v_\pm$ and for the adiabatic coefficient γ such that the standard solution consists of two shocks and such that there exist wild solutions which dissipate more energy than the standard solution, see [CK14]. Hence the entropy rate admissibility criterion does not favor the standard solution. Hopefully there is another criterion which is physically reasonable and leads to a unique solution.

Apart from this very important open problem there are others. We showed that among initial data which lead to a standard solution consisting of a shock and a rarefaction, there are some to which there exist simple fan subsolutions (hence these initial data are wild) and some others to which there is no simple fan subsolution. The question is if these initial data are wild, too, or not. We conjecture that they are but there is no proof yet. To show this it could be useful to do the convex integration directly to the compressible Euler system instead of using the results for the pressureless incompressible Euler equations.

Another problem one could work on, is to apply the convex integration method to other partial differential equations, e.g. to the full Euler system including a balance law for the total energy, or the equations of magnetohydrodynamics.

A. On weak and weak* topologies

What is presented in this section can be found in textbooks on topology or functional analysis, e.g. the the book by J. B. Conway, [Con85]. The difference of what is shown here and [Con85] is that we adapted Conway's general results to our subject.
Let $(V, ||\cdot||)$ be a real normed vector space and denote its topological dual as V'. We write $\langle\cdot,\cdot\rangle$ for the dual pairing. One can define a weak topology on V and a weak* topology on the dual V'. We don't want to define these topologies properly (see [Con85, Chapter V, Definition 1.1] for an exact definition), but we state the convergence in these topologies.

Proposition A.1. *Let I be an index set.*

- *A net $(x_i)_{i\in I}$ in V converges to $x \in V$ with respect to the weak topology if and only if $\langle x_i, y\rangle \to \langle x, y\rangle$ for all $y \in V'$. One writes $x_i \rightharpoonup x$.*
- *A net $(y_i)_{i\in I}$ in V' converges to $y \in V'$ with respect to the weak* topology if and only if $\langle x, y_i\rangle \to \langle x, y\rangle$ for all $x \in V$. One writes $y_i \overset{*}{\rightharpoonup} y$*

Above proposition's claim is well known. For the proof we refer to the literature.

What we need is the weak* topology on $L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}\times\mathbb{R}^2$. So for us $V' = L^\infty(\Omega)$ and, since[18] $(L^1)' = L^\infty$, $V = L^1(\Omega)$. With the above proposition we get that a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ converges to $f \in L^\infty(\Omega)$ in the weak* topology if and only if it holds that

$$\lim_{n\to\infty} \iint_\Omega f_n(t,x)\, g(t,x)\, dx\, dt = \iint_\Omega f(t,x)\, g(t,x)\, dx\, dt \tag{A.1}$$

for all $g \in L^1(\Omega)$. We can replace $L^1(\Omega)$ by $C_c^\infty(\Omega)$, what is the content of the following proposition.

Proposition A.2. *A sequence $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ converges to $f \in L^\infty(\Omega)$ in the weak* topology if $||f_n||_{L^\infty}$ is bounded and* (A.1) *holds for all $g \in C_c^\infty(\Omega)$.*

Proof. We want to show that (A.1) holds for a given $g \in L^1(\Omega)$. Since $||f_n||_{L^\infty}$ is bounded, we find a constant $M > 0$ such that $||f_n - f||_{L^\infty} < M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ given. Because $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$ (see e.g. [Lan93, Chapter VI, Theorem 9.6]), we find $h \in C_c^\infty(\Omega)$ with $||g - h||_{L^1} < \frac{\varepsilon}{2M}$. As (A.1) holds for this $h \in C_c^\infty(\Omega)$, we obtain

$$\left| \iint_\Omega \big(f_n(t,x) - f(t,x)\big)\, h(t,x)\, dx\, dt \right| < \frac{\varepsilon}{2}$$

[18] Here we abused the meaning of "=" a little bit. To be more precise by "=" we mean here that there is an isometric isomorphism between $(L^1)'$ and L^∞.

for all $n \in \mathbb{N}$ sufficiently large. Applying Hölder's inequality we get

$$\begin{aligned}
&\left| \iint_\Omega \big(f_n(t,x) - f(t,x)\big)\, g(t,x)\, dx\, dt \right| \\
&\quad = \left| \iint_\Omega \big(f_n(t,x) - f(t,x)\big)\, \big(g(t,x) - h(t,x) + h(t,x)\big)\, dx\, dt \right| \\
&\quad \le \iint_\Omega \big|f_n(t,x) - f(t,x)\big|\, \big|g(t,x) - h(t,x)\big|\, dx\, dt \\
&\qquad\quad + \left| \iint_\Omega \big(f_n(t,x) - f(t,x)\big)\, h(t,x)\, dx\, dt \right| \\
&\quad < ||f_n - f||_{L^\infty}\, ||g-h||_{L^1} + \frac{\varepsilon}{2} \\
&\quad \le M\, \frac{\varepsilon}{2\,M} + \frac{\varepsilon}{2} \quad = \quad \varepsilon,
\end{aligned}$$

for all $n \in \mathbb{N}$ large enough. This is equivalent to (A.1). □

Proposition A.3. *Let $X_0 \subset L^\infty(\Omega)$ be bounded and X be the closure of X_0 in the L^∞ weak* topology. Then X is bounded, too.*

Proof. Let $f \in X$. Since X is the weak* closure of X_0, there is a net $(f_i)_{i\in I}$ in X_0 (I is an index set), which converges to f with respect to the weak* topology. So we get for all $g \in L^1(\Omega)$

$$\begin{aligned}
\left| \iint_\Omega f(t,x)\, g(t,x)\, dx\, dt \right| &= \lim_{i\in I} \left| \iint_\Omega f_i(t,x)\, g(t,x)\, dx\, dt \right| \\
&\le \lim_{i\in I} ||f_i||_{L^\infty(\Omega)}\, ||g||_{L^1(\Omega)} \;\le\; M\, ||g||_{L^1(\Omega)},
\end{aligned}$$

with a constant $M > 0$. Here we used that X_0 is bounded and Hölder's inequality. This shows that f, more precisely the operator $\langle \cdot, f\rangle$ is bounded in $(L^1)'$. Since there is an isometric isomorphism between $(L^1)'$ and L^∞, f is bounded in $L^\infty(\Omega)$, too. □

Proposition A.4. *Let $X \subset L^\infty(\Omega)$ be bounded and weak* closed. Then the weak* topology of L^∞ is metrizable on X. Furthermore X is compact in the weak* topology.*

The proof is a variant of two proofs that can be found in [Con85].

Proof. We start with the compactness claim. Anaoglu's theorem (see [Con85, Chapter V, Theorem 3.1]) states, that if V is normed space, the closed unit ball of V' is compact with respect to the weak* topology. With a small adaption in the proof the claim holds for closed balls with arbitrary radius, too. Since X is bounded, there is a Radius $r > 0$ such that $X \subset \overline{B_r(0)}$. Setting $V = L^1(\Omega)$ Anaoglu's theorem proves that the closed ball $\overline{B_r(0)}$ of $V' = L^\infty(\Omega)$ is compact in the weak* topology. Since X is a weak* closed subset of $\overline{B_r(0)}$, X is weak* compact, too.
Let's turn our attention to the metrizability. J. B. Conway proved that if V is a separable Banach space, then the weak* topology of V' is metrizable on the closed unit of the dual V' (see [Con85, Chapter V, Theorem 5.1]). Since $V = L^1(\Omega)$ is a separable Banach space, this theorem can by directly applied to the closed unit ball $\overline{B_1(0)}$ of $V' = L^\infty(\Omega)$. With a simple modification in the proof the claim is also true on closed balls $\overline{B_r(0)}$ with arbitrary radius $r > 0$. So we can set the radius to be as above, i.e. such that $X \subset \overline{B_r(0)}$. Since the L^∞ weak* topology is metrizable on $\overline{B_r(0)}$, it is metrizable on X, too. □

B. Results of Baire's theory used in this thesis

The content of this section can be found in textbooks on topology, e.g. the one by S. Waldmann [Wal14].

Definition B.1. *(basic notions)* Let $(M, \mathcal{T})$ be a topological space. A subset $A \subset M$ is called

- *nowhere dense* if the interior of the closure of A is empty:

$$(\overline{A})^\circ = \varnothing,$$

- *meager* (or *of first category*) if A is the countable union of nowhere dense sets,
- *residual* if the complement of A is meager.

Lemma B.2. *Let $(A_n)_{n\in\mathbb{N}}$ be a set of countably many residual subsets of a topological space $(M, \mathcal{T})$. Then the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ is residual, too.*

Proof. Let A be the above intersection. We have to show that the complement of A is meager. We know that the complements of all A_n are meager, i.e. for each $n \in \mathbb{N}$ there are countably many nowhere dense sets $(B_{n,j})_{j\in\mathbb{N}}$ with $X \setminus A_n = \bigcup_{j\in\mathbb{N}} B_{n,j}$. It follows that

$$X \setminus A = X \setminus \bigcap_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} X \setminus A_n = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} B_{n,j},$$

which finishes the proof since $\mathbb{N}^2$ is countable. □

Definition B.3. *(Baire space)* A topological space $(M, \mathcal{T})$ is denoted as a Baire space if every residual subset of M is dense.

Theorem B.4. (Baire category theorem, e.g. see [Wal14, Theorem 7.2.1]) *Every complete metric space (M, d) is a Baire space.*

For the proof of the Baire category theorem we refer to the literature, e.g. [Wal14].

Remark. There are several versions of the Baire category theorem and also several equivalent definitions of a Baire space (see [Wal14, Definition 7.1.6]). Here we chose the one which is most useful for our matter.

Definition B.5. *(Baire-1-function)* Let $(M_1, \mathcal{T})$ be a topological and (M_2, d) a metric space. We call a function $f : M_1 \to M_2$ Baire-1-function if it is the pointwise limit of a sequence of continuous functions. In other words f is a Baire-1-function, if there is a sequence $(f_n)_{n\in\mathbb{N}}$ with the properties that $f_n : (M_1, \mathcal{T}) \to (M_2, d)$ is continuous for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(m) = f(m)$, for all $m \in M_1$.

Proposition B.6. *Let $(M_1, \mathcal{T})$ be a topological and $(M_2, ||\cdot||)$ a normed space and consider a Baire-1-function $f : (M_1, \mathcal{T}) \to (M_2, ||\cdot||)$. Then the set $C \subset M_1$ of the points in which f is continuous is residual in M_1.*

This proposition is essentially [Wal14, Proposition 7.3.2]. The difference is that S. Waldmann states and proves the claim only for $(M_2, ||\cdot||) = (\mathbb{R}, |\cdot|)$. However the proof can be easily adapted to general normed spaces just by replacing the occurring absolute values $|\cdot|$ by the norm $||\cdot||$.

C. Convex Geometry

We need the following theorems about convex sets.

Theorem C.1. (Minkowski's theorem, see [Brø83, Theorem 5.10]) *Let $C \subset \mathbb{R}^d$ be a compact convex set and let $M \subset C$. Then*

$$C = M^{co} \Leftrightarrow ext(C) \subset M,$$

where $ext(C)$ denotes the set of extreme points of C.

Proof. For the proof we refer to textbooks on convex geometry, e.g. the book by A. Brøndsted, [Brø83]. □

Proposition C.2. *Let $C \subset \mathbb{R}^d$ be an open and convex set. Then $C = \overline{C}^{\circ}$.*

Proof. We first show $C \subset \overline{C}^{\circ}$. Let $x \in C$. Since C open, there is a radius $r > 0$ such that $B_r(x) \subset C$. Because $C \subset \overline{C}$, we know that $B_r(x) \subset \overline{C}$. This proves that x is an inner point of $\overline{C}$, i.e. $x \in \overline{C}^{\circ}$.
Now we're going to prove $C \supset \overline{C}^{\circ}$. Let $x \in \overline{C}^{\circ}$. In other words x is an inner point of $\overline{C}$ and hence there is a radius $r > 0$ such that $B_r(x) \subset \overline{C}$. Assume that $x \notin C$. It follows that x lies on the boundary of C. So there is a point $y \in C \cap B_r(x)$. Now consider the point $2\,x - y$. We know that $|x - y| < r$ and thus $|(2\,x - y) - x| = |x - y| < r$, i.e. $2\,x - y \in B_r(x) \subset \overline{C}$. If $2\,x - y \in C$, the convexity of C leads to $\frac{1}{2}\,(2\,x - y) + \frac{1}{2}\,y = x \in C$ which is a contradiction. So we can assume that $2\,x - y \in \overline{C} \setminus C$. Since $C \cap B_r(x)$ is open, there is another radius $\widehat{r} > 0$ such that $B_{\widehat{r}}(y) \subset C \cap B_r(x)$. Because $2\,x - y$ lies on the boundary of C, there exists a $z \in C \cap B_{\widehat{r}}(2\,x - y)$. For the point $2\,x - z$ we obtain $|(2\,x - z) - y| = |(2\,x - y) - z| < \widehat{r}$. Hence $2\,x - z \in B_{\widehat{r}}(y) \subset C$ and then using the convexity of C we get $\frac{1}{2}\,(2\,x - z) + \frac{1}{2}\,z = x \in C$ which is a contradiction. □

Theorem C.3. (Caratheodory's theorem, see [Brø83, Corollary 2.4]) *For any subset $M \subset \mathbb{R}^d$ with $\dim(\mathit{aff}\, M) = n$ the convex hull M^{co} is the set of all convex combinations of at most $n+1$ points of M. Here $\mathit{aff}\, M$ denotes the smallest affine subspace of $\mathbb{R}^d$ which contains M.*

Proof. For the proof we refer to textbooks on convex geometry, e.g. the book by A. Brøndsted, [Brø83]. □

We will need another version of Caratheodory's theorem, which is just a simple corollary of the version stated above.

Corollary C.4. (Version of Caratheodory's theorem) *For any subset $M \subset \mathbb{R}^d$ the convex hull M^{co} is the set of all convex combinations of at most $d+1$ points of M.*

Proof. The fact that $n := \dim(\mathrm{aff}\ M) \leq d$ and theorem C.3 show, that M^{co} is the set of all convex combinations of at most $d+1$ points of M. □

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig angefertigt und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt habe. Ferner versichere ich, dass diese Arbeit keiner anderen Prüfungsbehörde vorgelegt wurde.

Würzburg, 08.12.2016 ....