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Writing Einstein's field equations as a hyperbolic first-order system in the CCZ4 formulation

Deriving boundary conditions and initial data for an asymptotically Anti-de Sitter spacetime

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Abstract

In this thesis, we are interested in finding time-dependent gravitational solutions of Einstein's field equations in asymptotically hyperbolic spacetimes. In the context of the anti-de Sitter/conformal field theory correspondence (AdS/CFT-correspondence), these are dual to non-equilibrium systems in quantum field theories. By foliating the n-dimensional spacetime into $(n-1)$-dimensional spatial Cauchy surfaces Σ_t, the system of Einstein's field equations can be formulated as a Cauchy-problem with constraints representing the conservation of momentum and energy. The strongly hyperbolic first-order conformally covariant Z4 system (FO-CCZ4) of Einstein's field equations, as a first-order formulation of the CCZ4-system, may be well suited as a time-evolving system for finding time-dependent solutions of the Einstein field equations in asymptotically AdS spacetimes.

The FO-CCZ4 system of Einstein's field equations in four spacetime dimensions with vanishing cosmological constant Λ_4 was derived by Dumbser et al. [20]. Furthermore, it was shown that this system is strongly hyperbolic for various standard gauge choices. As the anti-de Sitter space has a non-vanishing negative cosmological constant and as we are interested in higher-dimensional AdS spaces, the FO-CCZ4 system was extended by previous work of Grosvenor [37] to a first-order system of Einstein field equations in n spacetime dimensions with a non-trivial cosmological constant Λ_n. Then, we argued that the generalized FO-CCZ4 system for a three-dimensional and four-dimensional spacetime with a non-vanishing cosmological constant remains strongly hyperbolic. The generalized FO-CCZ4 system, with its $n^3/2 + n^2 + 5n/2$ unknown variables, is more complex than the Einstein field equations in their original formulation. However, it has the advantages of a strongly hyperbolic first-order system.

Furthermore, the AdS boundary is of utter importance for solving the FO-CCZ4 system of Einstein's field equations in an asymptotically, non-globally hyperbolic AdS space. In order for the entire space and not just a small neighbourhood in the causal future to be determined by some initial data on the Cauchy surface Σ_t at time $t = 0$, conditions for the variables of the FO-CCZ4 system at the AdS boundary must be calculated. To gain some intuition about how these fields behave near the boundary, we wrote the AdS metric $\hat{g}_{\mu\nu}$ as a perturbation series and derived the time-dependent conditions at the AdS boundary for the $n^3/2 + n^2 + 5n/2$ variables of the generalized FO-CCZ4 system. Moreover, we checked these derived boundary conditions for $n = 3$ and $n = 4$ with a Mathematica script. [36]

Since momentum and energy conservation of the FO-CCZ4 system must be satisfied for all times on the $(n-1)$-dimensional Cauchy surfaces Σ_t, setting up initial

data at time $t = 0$ on the initial surface Σ_0 becomes a non-trivial problem. For simplicity, we restricted ourselves to a time-symmetric surface Σ_t and used a scalar field ζ as a non-trivial deviation from the AdS space. To obtain physically relevant initial data on the Cauchy surface Σ_0, we derived a time-independent second-order elliptic initial value problem using a conformal decomposition. For this problem, we have found a solution for the trivial case of a vacuum AdS spacetime and simplified the initial value problem for a scalar field ζ as matter.

Moreover, the numerical discontinuous Galerkin (DG) method for the FO-CCZ4 system was implemented by Dumbser et al. [20] in ExaHyPE (Exascale Hyperbolic PDE Engine). ExaHyPE is open-source software for solving first-order hyperbolic partial differential equations using the ADER-DG (arbitrary high-order using derivatives - discontinuous Galerkin) or finite volume (FV) method. The simulations of various physical phenomena, such as the numerical simulation of earthquakes, tsunamis or the orbiting of two black holes around each other, have been successfully implemented within this framework. Based on this, the goal of this work was to extend the implementation of the FO-CCZ4 system to asymptotically hyperbolic spaces. However, a major challenge arose by explicitly setting the derived boundary conditions in the ExaHyPE code. To address this complication, we tried to solve first the simplified Cauchy-problem of a static vacuum three-dimensional AdS spacetime with time-symmetric initial data, i.e. $(E = 0, p_i = 0, K_{ij} = 0, \hat{\gamma}_{ij})$, and the anti-de Sitter metric $\hat{\gamma}_{ij}$ as a boundary condition within the ExaHyPE framework.

Zusammenfassung

In der vorliegenden Arbeit interessieren wir uns für zeitabhängige Gravitationslösungen der Einsteinschen Feldgleichungen im asymptotisch hyperbolischen Raum. Diese sind im Rahmen der Anti-de Sitter/Konformen Feldtheorie-Korrespondenz (engl. AdS/CFT-correspondence, für 'Anti-de Sitter/Conformal field theory') dual zu Nichtgleichgewichtssystemen in der Quantenfeldtheorie. Durch eine Zerlegung der n-dimensionalen Raumzeit in $(n-1)$-dimensionale räumliche Cauchy-Flächen Σ_t, kann das System der Einsteinschen Feldgleichungen als Cauchy-Problem mit physikalischen Nebenbedingen, welche die Impuls- und Energieerhaltung darstellen, formuliert werden. Das stark hyperbolische System erster Ordnung (engl. FO-CCZ4, für 'first-order CCZ4') der Feldgleichungen, welches auf der konform kovarianten Formulierung des Z4-Systems (engl. CCZ4, für 'conformal and covariant Z4') aufbaut, könnte als Zeitentwicklungssystem zum Finden von zeitabhängigen Lösungen der Einsteinschen Feldgleichungen im asymptotischen AdS Raum gut geeignet sein.

Das FO-CCZ4 System der Einsteinschen Feldgleichungen in 4 Raumzeitdimensionen mit verschwindender kosmologischer Konstante Λ_4 wurde von Dumbser et al. [20] hergeleitet. Es wurde gezeigt, dass dieses System für ausgewälte Standarteichungen stark hyperbolisch ist. Da der Anti-de Sitter Raum eine nichtverschwindende, negative kosmologische Konstante hat und wir uns für höherdimensionale AdS Räume interessieren, wurde das FO-CCZ4 System durch vorherige Arbeiten von Grosvenor [37] auf ein System erster Ordnung der Einsteinschen Feldgleichungen in n Raumzeitdimensionen mit nichttrivialer kosmologischer Konstante Λ_n erweitert. Das verallgemeinerte FO-CCZ4 System ist mit ihren $n^3/2 + n^2 + 5n/2$ unbekannten Variablen komplexer als die Einsteinschen Feldgleichungen in ihrer ursprünglichen Form. Da ein stark hyperbolisches System erster Ordnung große Vorteile bringt, haben wir argumentiert, dass die starke Hyperbolizität des FO-CCZ4 Systems mit nichtverschwindender kosmologischer Konstante in 3 und 4 Raumzeitdimensionen erhalten bleibt.

Des Weiteren ist der AdS Rand für das Lösen des FO-CCZ4 Systems der Einsteinschen Feldgleichungen in einem asymptotischen, nicht-global hyperbolischen AdS Raum von großer Bedeutung. Damit der gesamte Raum, und nicht nur eine kleine Umgebung in der kausalen Zukunft, durch die Angabe von Anfangsdaten auf der Cauchy-Fläche Σ_t zum Zeitpunkt $t = 0$ bestimmt werden kann, müssen Bedingungen für die Variablen auf dem AdS-Rand hergeleitet werden. Um eine gewisse Intuition zu gewinnen, wie sich diese Felder in der Nähe des Randes verhalten, haben wir die AdS Metrik $\hat{g}_{\mu\nu}$ als Störungsreihe geschrieben, und daraus die zeitabhängigen

Bedingungen auf dem AdS-Rand für die $n^3/2 + n^2 + 5n/2$ Variablen des verallgemeinerten FO-CCZ4 Systems hergeleitet. Diese Gleichungen haben wir für $n = 3$ und $n = 4$ durch ein Mathematica Skript [36] überprüft.

Da die Impuls- und Energieerhaltung des FO-CCZ4 Systems für alle Zeiten auf den $(n-1)$-dimensionalen Cauchy-Flächen Σ_t erfüllt sein müssen, wird das Aufstellen von Anfangsdaten zum Zeitpunkt $t = 0$ auf der Anfangsfäche Σ_0 ein nicht-triviales Problem. Zur Vereinfachung haben wir uns eine zeitsymetrische Fläche Σ_t beschränkt, und ein Skalarfeld ζ als nicht-triviale Abweichung vom AdS Raum genutzt. Um nun physikalische Anfangsdaten auf der Cauchy-Fläche Σ_0 zu erhalten, haben wir mit Hilfe einer konformalen Zerlegung ein zeitunabhängiges, elliptisches Anfangswertproblem zweiter Ordnung hergeleitet. Für dieses Problem haben wir eine Lösung für den trivialen Fall einer Vakuumraumzeit gefunden und das Anfangswertproblem für ein Skalarfeld ζ als Materie vereinfacht.

Des Weiteren wurde das numerischen diskontinuierliche Galerkin-Verfahren (engl. DG, für 'discontinuous Galerkin') für das FO-CCZ4 System von Dumbser et al. [20] im ExaHyPE-Code (engl., für 'Exascale Hyperbolic PDE Engine') implementiert. ExaHyPE ist eine open source Computersofteware zur Lösung von hyperbolischen partiellen Differentialgleichungen erster Ordnung mit Hilfe des ADER-DG (engl., für 'Arbitrary high order using Derivatives-Discontinuous Galerkin') oder Finite Volumen Verfahrens. Unterschiedlichste physikalische Phänomene, wie die numerische Simulation von Erdbeben, Tsunamis oder das Umkreisen von zwei Schwarzen Löchern, wurden erfolgreich in ExaHyPE implementiert. Darauf aufbauend war das Ziel dieser Arbeit, die Implementierung des FO-CCZ4 Systems auf asymptotisch hyperbolische Räume zu erweitern. Dafür müssen die von uns hergeleiteten Randbedingungen und Anfangsdaten implementiert und das FO-CCZ4 System mit den Termen der kosmologischen Konstante erweitert werden. Eine größere Schwierigkeit stellte das explizite Setzen der von uns hergeleiteten Randbedingungen im ExaHyPE-Code dar. Um diese Schwierigkeiten anzugehen, haben wir abschließend versucht das triviale Cauchy-Problem einer statischen Vakuumraumzeit mit zeitsymmetrischen Anfangsdaten, $(E = 0, p_i = 0, K_{ij} = 0, \hat{\gamma}_{ij})$, und der Anti-de Sitter Metrik $\hat{\gamma}_{ij}$ als Randbedinung für $n = 3$ mit ExaHyPE zu lösen.

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Table of Contents

Chapter 1

Introduction

General Relativity Einstein's theory of general relativity is by far the most successful gravitational theory. While a significant number of exact solutions of the Einstein field equations are known, only a few of them, derived from highly idealized approximations, are physically relevant [42]. Three astrophysically relevant solutions are the Kerr solution for stationary rotating black holes, the TOV solution of a spherically symmetric perfect fluid model of stars and the Friedman solution for the expansion describing the model of a homogeneous and isotropic universe.

Since general relativity is strongly non-linear, the superposition principle does not hold and certain approximations must be used to solve the two-body problem. Nevertheless, the geodesic equation describes the motion of a sufficiently small object in the gravitational field of a much bigger mass. This calculation is a reasonable approximation for the motion of photons in the sun's gravitational field. However, for calculating the perihelion precession of Mercury's orbit or the inspiral of a small black hole inspiral into a giant black hole, the Post-Newtonian expansion [10], or the gravitational self-force correction [63] has to be used, respectively. While these approximations apply to most physical problems, they fail for binary systems of high mass and rotation, such as two rotating black holes or neutron stars. In order to describe the motion of an astrophysical object in a strong gravitational field regime, one needs to solve Einstein's field equations numerically without any approximations.

Numerical Relativity Several different methods for numerically solving Einstein's field equations have been studied in the past. There is the characteristic formulation [65], the conformal field equations by Friedrich [25], the generalized-harmonic formulation by Pretorius [57], and the commonly used $3+1$ formalism, originating from works by Darmois [16], Lichnerowicz [46, 47] and Choquet-Bruhat [23, 24].

The space and time decomposition of general relativity relies on slicing the four-dimensional spacetime into three-dimensional spacelike hypersurfaces, projecting four-dimensional tensors onto submanifolds and time-evolving them from one hypersurface to another along a timelike vector field. This way, the Einstein field equations can be cast as a Cauchy evolution problem with constraints, representing the conservation of momentum and energy. The $3+1$ decomposition is discussed excessively in many modern books, such as [1, 6, 13, 27, 59, 61].

However, it took almost a hundred years to recast the Einstein field equations as a first-order, strongly hyperbolic system (known as FO-CCZ4 [20]) that is suitable for stable numerical time integration. Steady long-term simulations for the head-on collision of two puncture black holes and stable long-term evaluation of a neutron star in an anti-Cowling approximation were only presented recently, respectively, by Dumbser et al. [20, 19].

Anti-de Sitter/Conformal field theory The solutions of Einstein's field equations in hyperbolic space are in the context of AdS/CFT-correspondence dual to non-equilibrium systems in quantum field theory. But, what is this AdS/CFT-correspondence actually about? The most famous and original AdS/CFT duality by Maldacena [48] is a relation between the maximally supersymmetric conformal Yang-Mills theory as a conformal quantum field theory on our four-dimensional flat space without gravity to the type II_B string theory, a theory of gravity on the curved 10-dimensional manifold $AdS_5 \times S^5$. As the geometry of the boundary of the compactified AdS_5 spacetime is the four-dimensional Minkowski spacetime, we may say that the CFT lives on the boundary of the compactified AdS_5 spacetime. By taking a particular limit, the string theory reduces to a classical weakly coupled theory of gravity describing pointlike particles, while the conformal quantum field becomes strongly coupled. For a detailed introduction to AdS/CFT, there are several good books and lecture notes about AdS/CFT [4, 41, 49, 52, 53, 68].

The strongly coupled $\mathcal{N} = 4$ Superconformal Yang-Mills theory and the AdS spacetime are far from our real world, which can be described by quantum chromodynamics (QCD) and has a positive cosmological constant. But why is then this AdS/CFT-duality so interesting? While there is no universal approach for calculating observables in strongly coupled quantum field theories, perturbative methods facilitate the calculations in weakly coupled QFTs. Nevertheless, by studying AdS/CFT, many things about strong couplings in QFTs and even for QCD have been understood. An example of applying the AdS/CFT duality is the study of far-from-equilibrium dynamics of strongly coupled QFTs. Here, the time evolution of the combined cold-hot bath system of two identical copies of quantum critical systems at different temperatures and chemical potential can be analyzed. After joining the two systems, a non-equilibrium steady state forms between the shock, moving towards the cold bath, and the rarefaction wave, moving towards the hot bath [21]. Furthermore, the entanglement entropy as a measure of the entanglement of quantum states of different spatial subregions can be studied. While the increase in entropy of the steady-state region is small, it is given by the entropy production of the shock and rarefaction wave [21]. For small temperature, [22] derived an analytical formula for the time dependence for the entanglement entropy. As we want to numerically solve the FO-CCZ4 system for a hyperbolic AdS space and study the real-time dynamics and time evolution of the entanglement entropy of strongly coupled systems, we can use the analytic formula as a benchmark for solving the FO-CCZ4 system within the ExaHyPE framework.

Structure of the thesis This thesis is structured as follows: In Chapter 2, we state the various initial value problems of the Einstein field equations with main focus on the FO-CCZ4 system for a general n-dimensional spacetime with a non-vanishing cosmological constant. As we want to evolve forward in time some initial data for the generalized FO-CCZ4 system for an asymptotically Anti-de Sitter (AAdS_n) spacetime, we derive a time-independent, elliptic, second-order partial differential equation in Chapter 3. Solving this PDE will give us physical initial data that satisfies the conservation of momenta and energy. Thereafter, we introduce the main properties of an AdS spacetime and, in order to obtain a well-posed and deterministic initial value problem of the generalized FO-CCZ4 system of the Einstein's field equations for an asymptotically Anti-de Sitter spacetime, calculate boundary conditions for the unknowns of the generalized FO-CCZ4 system in Chapter 4. Moreover, in Chapter 5, we summarize the boundary conditions for an AAdS_3 and AAdS_4 spacetime, while in Chapter 6 we give a short introduction to the ExaHyPE (Exascale Hyperbolic PDE-Engine) software and present some output in 7. We have devoted much effort to defining the main objects for recasting Einstein's equations as a time-evolution problem in a mathematically rigorous manner in Appendix A. Furthermore, we stated the main properties of the Lie derivative and for a relativistic spacetime, respectively, in Appendix B and C. To characterize symmetric spacetimes, we need the notion of Killing vector fields as can be found in Appendix D, while the tremendous calculation for the boundary behaviour and the Jupyter notebook for the calculation of initial data can be found, respectively, in Appendix E and F.

Convention and Notation Whenever there are upper and lower indices repeated, we will, unless otherwise indicated, apply the Einstein summation convention, i.e.

$$\sum_{\nu=0}^{n-1} g_{\alpha\nu}g^{\nu\beta} = g_{\alpha\nu}g^{\nu\beta}.$$

Greek letters generally run over the n spacetime coordiantes and take the values $0, 1, 2, 3, \ldots, n-1$, where $x^0 = ct$ is the time coordinate. For example

$$g_{\alpha\nu}g^{\nu\beta} = \sum_{\nu=0}^{n-1} g_{\mu\nu}g^{\nu\delta}.$$

Every time we use Einsteins summation convention, Greek letters from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots$ are used as free indices, while Greek letters starting at $\mu, \nu, \ldots$ are used as dumb indices for contraction. In this way, the tensorial degree will be immediately clear. For example, we can easily see that

$$\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\gamma}{}_{\rho}\gamma^{\sigma}{}_{\delta}{}^{(n)}R^{\rho}{}_{\sigma\mu\nu}$$

is a ein $(1,3)$-tensor. Latin letters only run over the spatial coordinates $1, 2, 3, \ldots, n-1$, while Latin letters starting at $i, j, k, \ldots$ are purely spatial coordinates, Latin letters from the beginning of the alphabet $a, b, c, \ldots$ represent only angle coordinates and Latin letters starting at $m, n, \ldots$ are non-rodial coordinates. For example

$$\gamma_{ij}\gamma^{jk} = \sum_{j=1}^{n-1} \gamma_{ij}\gamma^{jk}$$

$$g_{ab}g^{bc} = g_{a\theta_1}g^{\theta_1 b} + g_{a\theta_2}g^{\theta_2 b} + \cdots$$

Be careful with the notation used in Chapter 4, i.e.

$$g^{\alpha\mu}g_{\mu\beta}\bar{h}_{\mu\beta} = g^{\alpha 0}g_{0\beta}\bar{h}_{0\beta} + g^{\alpha 1}g_{1\beta}\bar{h}_{1\beta} + \cdots\ ,$$

as we have to sum over $g_{\mu\beta}$ and $\bar{h}_{\mu\beta}$ simultaneasly. For the sake of simplicity, we will set the gravitational constant G and the speed of light c to one and symmetrize a 2-fold contravariant tensor by

$$A_{\alpha\beta} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right).$$

Chapter 2

First-Order Formulation of Einstein's Field Equations

"Spacetime tells matter how to move.
Matter tells spacetime how to curve." [62]
John Wheeler (1911 - 2008)
Physicist

Since the Einstein field equations in their original formulation are not suitable for high convergence order numerical integration, much effort has been devoted to rewriting them as a strongly hyperbolic time-evolution system of partial differential equations of first-order. This chapter states the different Cauchy-systems suitable for numerical implementation. However, we focus primarily on the newly derived generalized FO-CCZ4 system for an n-dimensional spacetime M with a non-vanishing cosmological constant Λ_n, as we aim to numerically solve the latter system for a hyperbolic AdS spacetime with the ExaHyPE software. Eventually, we show that the generalized FO-CCZ4 system will be stable for numerical time integration, as it remains strongly hyperbolic for the special cases of a three- and four-dimensional spacetime.

2.1 Motivation for rewriting Einsteins's field equations

The Einstein field equations, as the most essential part of general relativity, are given globally by

$$^{(n)}\mathrm{Ric} - \frac{1}{2}\,^{(n)}\mathrm{R}\,g + \Lambda_n g = 8\pi T, \tag{2.1}$$

where the Ricci tensor $^{(n)}\mathrm{Ric} \in \Gamma^\infty(T^*M^{\otimes 2})$, the metric tensor $g \in \Gamma^\infty(T^*M^{\otimes 2})$ and the matter stress-energy tensor $T \in \Gamma^\infty(T^*M^{\otimes 2})$ are 2-fold covariant tensor fields on the n-dimensional spacetime M, while their space is given by the set of smooth sections of the tangent bundel $T^*M^{\otimes 2}$. The Ricci curvature scalar $^{(n)}\mathrm{R} \in \mathcal{C}^\infty(M, \mathbb{R})$ is a smooth function on the manifold M and Λ_n is the cosmological constant. As the 2-fold covariant Ricci, metric and matter-stress tensor fields and the smooth function $^{(n)}\mathrm{R}$ of Einstein's field equations are of utter importance, we introduce them in a mathematically rigorous manner in Appendix A.

While the global formulation is not very usefule, we can write the Einstein field equations locally as

$$
{}^{(n)}R_{\mu\nu} - \frac{1}{2}{}^{(n)}Rg_{\mu\nu} + \Lambda_n g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.2}
$$

where ${}^{(n)}R_{\mu\nu}$, $g_{\mu\nu}$, ${}^{(n)}R$ and $T_{\mu\nu}$ are, respectively, the component functions of the Ricci tensor ${}^{(n)}\mathrm{Ric}$, the component functions of the Lorentzian metric g, the corresponding Ricci scalar ${}^{(n)}\mathrm{R}$, and the components of the matter energy-momentum tensor T.

The Einstein field equations are a set of $n(n+1)/2$ coupled, nonlinear partial differential equations of hyperbolic and elliptic type, written in a tensorial form that describes how mass and energy curves the geometry of spacetime. While a significant number of exact solutions of the Einstein field equations (2.2) are known, only a few of them are physical problems derived from highly idealized approximations. While these approximations apply to most physical problems, they fail for binary systems of high mass and rotation, such as two rotating black holes or neutron stars. In order to describe the motion of an astrophysical object in a strong gravitational field regime, one needs to solve Einstein's field equations numerically without any approximations. However, as the tensorial formulation of Einstein's field equations is not very suitable for high convergence order numerical integration, we need to recast the most important equation of general relativity. But, rewriting Einstein's field equations as a first-order hyperbolic system of partial differential equations suitable for stable numerical time integration has been a long way of almost a hundred years. Let us, therefore, summarize in the following sections the reformulations of Einstein's field equations as a time-evolution Cauchy-problem while we focus on the generalized FO-CCZ4 system with a non-vanishing cosmological constant Λ_n.

2.2 The ADM evolution system

We aim to rewrite the Einstein field equations (2.2) as an initial value Cauchy problem. To do so, we need to introduce the notion of a foliation of an n-dimensional spacetime into $(n-1)$-dimensional spacelike surfaces Σ_t. Then we need to project the objects of Einstein's field equation that are living on the n-dimensional spacetime onto the hypersurfaces Σ_t, along the vector field N, and onto the hypersurface Σ_t and along the vector field N.

Since we could not find a rigorous introduction to numerical relativity in physics textbooks, we have devoted much effort to defining the main objects from numerical relativity for recasting Einstein's equations as a time-evolution problem in a mathematically rigorous manner. The definitions of a spacelike hypersurfaces Σ_t, a foliation of spacetime M and the projection operator $\gamma^{\mu}{}_{\nu}$ can be found, respectively, in Section A.2, A.5 and A.6.

The derivation of the ADM equations

Using the projection operator $\gamma^{\mu}{}_{\nu}$, as defined rigorously in Appendix A.6, and the normal vector field $N \in \Gamma^{\infty}(TM)$, as defined rigorously in Appendix A.4, we can project any tensor field living on the generalized tangent bundle $TM^{\otimes \ell} \otimes T^*M^{\otimes k}$ completly onto the spatial surfaces Σ_t, completly along the normal vector field N or onto the spatial surfaces Σ_t and along the normal vector field N.

Let us first project the matter stress-energy tensor $T \in \Gamma^\infty(T^*M^{\otimes 2})$ of the Einstein field equations. Locally, we can write the 2-fold covariant tensor as

$$T\Big|_U = T_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{2.3}$$

where $T_{\mu\nu} \in \mathcal{C}^\infty(M, \mathbb{R})$ are the component functions. By the very definition of the matter stress-energy tensor, we can locally define the following four quantities as measured by the Eulerian observer with four-velocity N_p as in Section 7.2.2 in [59] by

$$E = T_{\mu\nu} N^\mu N^\nu \in \mathcal{C}^\infty(M, \mathbb{R}) \quad \text{the energy density} \tag{2.4a}$$
$$S_\alpha = -T_{\mu\nu} \gamma^\mu{}_\alpha N^\nu \in \mathcal{C}^\infty(M, \mathbb{R}) \quad \text{the momentum densitiy} \tag{2.4b}$$
$$S_{\alpha\beta} = T_{\mu\nu} \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \in \mathcal{C}^\infty(M, R) \quad \text{the spatial energy momentum tensor and} \tag{2.4c}$$
$$S = S^i_i \in \mathcal{C}^\infty(M, R) \quad \text{its trace.} \tag{2.4d}$$

As Gourgoulhon points out in Section 4.1.2 in [27], we can reconstruct the matter stress-energy tensor from these quantities by

$$T = S + N^\flat \otimes p + p \otimes N^\flat + E N^\flat \otimes N^\flat, \tag{2.5}$$

where $^\flat\colon TM \to T^*M$ is the musical isomorphism. Next, let us project the metric tensor $g \in \Gamma^\infty(T^*M^\otimes)$. Locally, we can write the tensor field as

$$g\Big|_U = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{2.6}$$

where $g_{\mu\nu} \in \mathcal{C}^\infty(M, \mathbb{R})$ are the component functions. Then, the projections of the component functions $g_{\mu\nu}$, as given in Appendix A, are given by

$$g_{\mu\nu} N^\mu N^\nu = -1 \in \mathcal{C}^\infty(M, \mathbb{R}), \tag{2.7}$$
$$g_{\mu\nu} \gamma^\mu{}_\alpha \gamma^\nu{}_\beta = \gamma_{\alpha\beta} \in \mathcal{C}^\infty(M, \mathbb{R}), \tag{2.8}$$
$$g_{\mu\nu} \gamma^\mu{}_\alpha N^\nu = 0 \in \mathcal{C}^\infty(M, \mathbb{R}). \tag{2.9}$$

The last 2-fold covariant tensor field of the Einstein field equations, the Ricci tensor $^{(n)}\mathrm{Ric} \in \Gamma^\infty(T^*M^{\otimes 2})$, can be written locally as

$$^{(n)}\mathrm{Ric}\,\Big|_U = {}^{(n)}R_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{2.10}$$

The projection of the component functions have been derived in Appendix A and are given by

$$^{(n)}R_{\mu\nu} N^\mu N^\nu = \frac{1}{2}\left(R + K^2 - K_{ij}K^{ij} - {}^{(n)}R\right) \in \mathcal{C}^\infty(M, \mathbb{R}), \tag{2.11}$$

$$^{(n)}R_{\mu\nu} \gamma^\mu_\alpha \gamma^\nu_\beta = -\frac{1}{\alpha}\left(\mathcal{L}_m K_{\alpha\beta} - D_\alpha D_\beta \alpha + R_{\alpha\beta}\right) + K K_{\alpha\beta} - 2K_{\alpha\mu} K^\mu{}_\beta \in \mathcal{C}^\infty(M, \mathbb{R}), \tag{2.12}$$

$$^{(n)}R_{\mu\nu}\gamma^{\mu}_{\alpha}N^{\nu} = D_{\alpha}K - D_{\mu}K^{\mu}{}_{\alpha} \in \mathcal{C}^{\infty}(M, \mathbb{R}), \tag{2.13}$$

where the projection (2.11) along the normal vector field N is given by the scalar Gaus equation (A.92), the projection (2.12) onto the hypersurface Σ_t is given by the combination of the contracted Gauss equation with the Ricci equation, i.e. Eq (A.99), and the projection (2.13) once along the normal vector field and once onto the hypersurface Σ_t is given by the contracted Peterson-Mainardi-Codazzi equation (A.95). The Levi-Civita connection D on $T\Sigma$ is rigorously defined in Section A.3.

Using the projections of the matter stress-engerg tensor T, metric tensor g and the Ricci tensor $^{(n)}\mathrm{Ric}$, we can fully project the Einstein field equations with a vanishing cosmological constant

$$G_{\mu\nu} - 8\pi T_{\mu\nu} = 0, \tag{2.14}$$

where $G_{\mu\nu} = {}^{(n)}R_{\mu\nu} - \frac{1}{2}{}^{(n)}Rg_{\mu\nu} \in \mathcal{C}^{\infty}(M, \mathbb{R})$ are the component functions of the Einstein Tensor $G \in \Gamma^{\infty}(T^*M^{\otimes 2})$. While a full projection along the normal vector field N yields the **Hamilton constraint** equation

$$H = (G_{\mu\nu} + \Lambda_n g_{\mu\nu} - 8\pi T_{\mu\nu})\, N^{\mu}N^{\nu} = R + K^2 - K_{ij}K^{ij} - 16\pi E = 0, \tag{2.15}$$

the projection once along the normal vector field N and once onto Σ_t yields the **momentum constraint** equations

$$M_{\alpha} = (G_{\mu\nu} - 8\pi T_{\mu\nu})\, \gamma^{\mu}{}_{\alpha}N^{\nu} = D_{\alpha}K - D_{\mu}K^{\mu}{}_{\alpha} + 8\pi S_{\alpha} = 0. \tag{2.16}$$

A full derivation of the constraint equations can be found in Section 4.1.3 in [27]. Note that these equations (2.15) and (2.16) are called constrained equations as they have to be fulfilled for all times $t \in \mathbb{R}$, i.e. for all hypersurfaces Σ_t. But, even if initially the constraint equations are satisfied, they will not be, up to numerical accuracy, for later times as explained in Section 10.3.3 in [27]. Moreover, the full projection of the Einstein equations onto Σ_t, i.e.

$$G_{\mu\nu}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta} = 8\pi T_{\mu\nu}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}, \tag{2.17}$$

yields the evolution equation of the extrinsic curvature $K_{\alpha\beta}$, namely

$$\mathcal{L}_M K_{\alpha\beta} = -D_{\alpha}D_{\beta}\alpha + \alpha\left\{R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}{}_{\beta} - 8\pi\left(S_{\alpha\beta} + \frac{E-S}{D-2}\gamma_{\alpha\beta}\right)\right\}. \tag{2.18}$$

The component functions of the extrinsic curvature, a measure of how the hypersurface Σ_t is embedded into the spacetime manifold M, is defined rigorously in Section A.3. A different, but equivalent definition of the extrinsic curvature, as calculated in Section 3.3.5 in [27], is given by the Lie derivative of the spatial metric $\gamma_{\mu\nu}$ in the direction of the normal evolution vector field $M \in \Gamma^{\infty}(TM)$, i.e.

$$\mathcal{L}_M \gamma_{\mu\nu} = -2\alpha K_{\mu\nu}. \tag{2.19}$$

Using the timelike evolution vector field $M \in \Gamma^{\infty}(TM)$, as defined rigorously in Appendix A.4, the shift vector field $\beta \in \Gamma^{\infty}(TM)$, as defined rigorously in Appendix

A.4 as $\beta = \partial_t - M$, we can write the Lie derivative in the direction of the vector field M as

$$\mathcal{L}_M = \mathcal{L}_{\partial_t} - \mathcal{L}_\beta = \partial_t - \mathcal{L}_\beta. \tag{2.20}$$

Note, as explained in Appendix A, each term of Eq. (2.15), (2.16), (2.18) and (2.19) is a tensor field tangent to Σ_t. Therefore, we can restrict the equations to only spatial indicies without loss of generality. Using the property (2.20) of the Lie derivative, we are ready to state the ADM equations, as given in Section 4.3.2 in [27], the first and original reformulatlion of the Einstein field equations. Then,

$$\left(\partial_t - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij} \tag{2.21a}$$

$$\left(\partial_t - \mathcal{L}_\beta\right)K_{ij} = -D_iD_j\alpha + \alpha\Big\{R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j - 8\pi\left(S_{ij} + \frac{E-S}{D-2}\gamma_{ij}\right)\Big\} \tag{2.21b}$$

$$H = R + K^2 - K_{ij}K^{ij} - 16\pi E - 2\Lambda = 0 \tag{2.21c}$$

$$M_i = D_iK - D_jK^j{}_i + 8\pi S_i = 0. \tag{2.21d}$$

For the sake of completeness, let us introduce spatial coordinates (x^i) and express the terms of the ADM equations within this coordinate system. The terms with a covariant derivative D associated with the hypersurface Σ_t can then be expressed via Eq. (A.16) by

$$D_iD_j\alpha = \partial_i\partial_j\alpha - \Gamma^k{}_{ij}\partial_k\alpha, \tag{2.22a}$$

$$D_jK^j{}_i = \partial_jK^j{}_i + \Gamma^j{}_{jk}K^k{}_i - K^j{}_k\Gamma^k_{ji}, \tag{2.22b}$$

$$D_i\alpha = \partial_iK, \tag{2.22c}$$

where the Christoffel symbolds are given by

$$\Gamma^k{}_{ij} = \frac{1}{2}\gamma^{kl}\left(\partial_i\gamma_{lj} + \partial_j\gamma_{il} - \partial_l\gamma_{ij}\right). \tag{2.23}$$

The Lie derivative of the spatial metric γ_{ij} and the extrinsic curvature K_{ij} in direction of the shift vector β can be written, respectively, by using Eq. (B.28) as

$$\mathcal{L}_\beta\gamma_{ij} = \partial_j\beta_i + \partial_i\beta_j - 2\Gamma^k{}_{ij}\beta_k \tag{2.24}$$

and

$$\mathcal{L}_\beta K_{ij} = \beta^k\partial_kK_{ij} + K_{kj}\partial_i\beta^k + K_{ik}\partial_j\beta^k. \tag{2.25}$$

The Ricci tensor and the Ricci scalar of the $(n-1)$-dimensional submanifold Σ_t are given, respectively, by

$$R_{ij} = \partial_k\Gamma^k{}_{ij} - \partial_j\Gamma^k{}_{ik} + \Gamma^k{}_{ij}\Gamma^l{}_{kl} - \Gamma^l{}_{ik}\Gamma^k{}_{lj} \quad \text{and} \tag{2.26a}$$

$$R = \gamma^{ij}R_{ij}. \tag{2.26b}$$

The equations (2.21a) - (2.21d) with all terms expanded by (2.22a) - (2.26b) are called the ADM equations, named after Arnowitt, Deser and Misner, even though it was Darmois, Lichnerowicz and Choquet-Bruhat who have first derived these equations for special cases for the lapse function α and the shift vector β [27]. Arnowitt, Deser and Misner are well-known for their Hamiltonian formulation of general relativity, where they derived a slightly different set of equations with the extrinsic curvature replaced by the momentum conjugate of the spatial metric [27]. In numerical relativity, the n degrees of freedom for gauge fixing are incorporated in the freedom to choose the lapse function α and the shift vector β freely. We will postpone the discussion about the different choices used in the past for the lapse function α and the shift vector β to Section 2.8.

Given initial fields that generate a certain matter distribution (E, p_i, S_{ij}), the ADM equations constitute a second-order nonlinear system of PDEs in the unknows (γ_{ij}, K_{ij}) that can be used on the same footing as the original Einstein equations and have the same 10 degrees of freedom. As all quantities are $(n-1)$-dimensional, we may forget about the ambient n-dimensional spacetime and consider these equations as a time evolution problem of $(n-1)$-dimensional tensor fields on the spatial hypersurfaces with the constraint equations

$$R + K^2 - K_{ij}K^{ij} - 2\Lambda = 16\pi E \tag{2.27a}$$

$$D_j K^j{}_i - D_i K = 8\pi S_i. \tag{2.27b}$$

Furthermore, Choquet-Bruhat and Geroch showed in 1969 the global existence and uniqueness of the Cauchy initial value problem with initial data (γ, K, E, p) on Σ_0 that obey the constraint equations. Finally, we refer the attentive reader to the summary of the Cauchy problem [15] by Choquet-Bruhat and reviews by York, Andersson and Rendall that can be found in [27].

However, the AMD equations are only weakly hyperbolic and therefore not very suitable for stable long-term numerical implementation [60]. We can achieve hyperbolicity of the Cauchy system by using several tricks. First, we decompose the variables into a traceless and trace part, conformally decompose the ADM state variables γ_{ij}, K_{ij}, add the constraint equations $H = M_i = 0$ to the evolution equations and evolve the evolution of the trace of the extrinsic curvature, the contracted Christoffel symbol and the conformal factor. We will introduce this new system in the following.

2.3 The BSSNOK evolution system

The ADM equations with a particular gauge fixing choice constitute a system of evolution equations that are not well-suited for stable numerical integration. The failure of a stable evolution is due to the weakly hyperbolicity property of this system, which can be seen by a first-order reduction [39]. Consequently, this means that the Cauchy initial value problem of the ADM equations is ill-posed, and it can not be ensured that the solutions are well-behaved. But then, to ensure stable numerical evolution, we should put these equations into a strongly hyperbolic form. We can solve this problem using the constraint equations, performing a conformal decomposition and separating the dynamical variables into trace and traceless parts. However, before we have a look at the derivation of the *BSSNOK system*, we note that well-posedness is a necessary condition to obtain a stable evolution scheme

[33], but it is not a sufficient condition. Symmetric or strongly hyperbolic equations are characterized by solutions that do not increase more rapidly than exponentially. However, from the numerical point of view, exponentially growing solutions, unless they can be controlled, are still very bad and can terminate the evolution system after a finite time [6].

Definition of the traceless decomposition

Any 2-fold covariant tensor field $S \in \Gamma^\infty(T^*\Sigma^{\otimes 2})$ can be decomposed into its trace

$$S = \gamma^{ij} S_{ij} \tag{2.28}$$

and tracefree part

$$S_{ij}^{TF} = S_{ij} - \frac{1}{D-1} S \gamma_{ij}, \tag{2.29}$$

where TF stands for tracefree or traceless. Therefore, let us first decompose the extrinsic curvature K_{ij} into a traceless part

$$A_{ij} = K_{ij}^{TF} = K_{ij} - \frac{1}{n-1} K \gamma_{ij}, \tag{2.30}$$

and a trace part, where the trace is being taking with respect to the spatial metric γ,

$$K = \gamma^{ij} K_{ij}. \tag{2.31}$$

Then, let us introduce an equivalence class of a conformal metric.

Definition of the conformal decomposition

Not only can we use the conformal decomposition to restore strong hyperbolicity of the Einstein equations, but it also provides a useful tool to get valid initial data for the Cauchy problem in Section 3.

Let g and $\tilde{g}$ be two metrics on the pseudo-Riemannian manifold M. Then g and $\tilde{g}$ are called conformally invariant if there exits a smooth real-valued function $\lambda \in \mathcal{C}^\infty(M, \mathbb{R})$ such that

$$\tilde{g} = \lambda g. \tag{2.32}$$

The function λ is called the conformal factor. The various definitions of a conformal equivalence class of the spatial metric γ in the literature, as stated in Section 12.2 in [43], are given by

$$\tilde{\gamma}_{ij} = \chi \gamma_{ij} \qquad \text{with} \qquad \chi = \gamma^{-\frac{1}{n-1}}, \tag{2.33a}$$

$$\tilde{\gamma}_{ij} = e^{-4\Phi} \gamma_{ij} \qquad \text{with} \qquad \Phi = -\tfrac{1}{4} \log \chi = \frac{1}{4(n-1)} \log \gamma, \tag{2.33b}$$

or

$$\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij} \qquad \text{with} \qquad \phi = \sqrt{\chi} = \gamma^{-\frac{1}{2(n-1)}}, \tag{2.33c}$$

where $\gamma = \det \gamma_{ij}$. In the original derivation, the conformal metric as in Eq. (2.33b) was used, while we use Eq. (2.33a) for the rest of this chapter. Mathematically speacking, the equations (2.33a) - (2.33c) define a class of metrics that are defined up to scale. As we want to work with the conformal equivalence class, we need to conformally decompose the related objects. First, we begin with the traceless part of the extrinsic curvature A_{ij}. Then, the conformal traceless part of the extrinsic curvature is defined as

$$\tilde{A}_{ij} = \chi A_{ij} = \chi \left(K_{ij} - \frac{1}{n-1} K \gamma_{ij} \right), \tag{2.34}$$

where we can raise and lower indices with respect to the conformal metric $\tilde{\gamma}^{ij}$ and $\tilde{\gamma}_{ij}$. Then, we decompose the Ricci tensor into two parts, namely

$$R_{ij} = \tilde{R}_{ij} + R^{\chi}_{ij}, \tag{2.35}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to the conformal spatial metric $\tilde{\gamma}_{ij}$ and R^{χ}_{ij} is the Ricci tensor with respect to the conformal factor χ. To overcome the numerical instabiliites caused by the first three terms of $\tilde{R}_{ij}$, we introduce the auxiliary variable

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk} = -\partial_j \tilde{\gamma}^{ij}, \tag{2.36}$$

where

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} \tilde{\gamma}^{ij} \left(\partial_j \tilde{\gamma}_{kl} + \partial_k \tilde{\gamma}_{jl} - \partial_l \tilde{\gamma}_{jk} \right) \tag{2.37}$$

are the Christoffel symbols associated with the conformally decomposed metric $\tilde{\gamma}$. Using the conformal connection function $\tilde{\Gamma}^i$, we can rewrite $\tilde{R}_{ij}$ and R^{χ}_{ij} as in [67] by

$$\begin{aligned} \tilde{R}_{ij} = &-\frac{1}{2} \left(\tilde{\gamma}^{kl} \partial_k \partial_l \tilde{\gamma}_{ij} + \partial_k \tilde{\gamma}_{il} \partial_j \tilde{\gamma}^{kl} + \partial_k \tilde{\gamma}_{jl} \partial_i \tilde{\gamma}^{kl} - \tilde{\Gamma}^l \partial_l \tilde{\gamma}_{ij} \right) \\ &+ \frac{1}{2} \left(\tilde{\gamma}_{ki} \partial_j \tilde{\Gamma}^k - \tilde{\gamma}_{kj} \partial_i \tilde{\Gamma}^k \right) - \tilde{\Gamma}^l_{ik} \tilde{\Gamma}^k_{jl} \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} R^{\chi}_{ij} = &\frac{n-3}{2\chi} \left(\partial_i \partial_j \chi - \tilde{\Gamma}^k_{ij} \partial_k \chi \right) - \frac{n-3}{4\chi^2} \partial_i \chi \partial_j \chi \\ &+ \tilde{\gamma}_{ij} \tilde{\gamma}^{kl} \left[\frac{\partial_k \partial_l}{2\chi} - (n-1) \frac{\partial_k \chi \partial_l \chi}{4\chi^2} \right] - \frac{1}{2} \tilde{\gamma}_{ij} \frac{\partial_m \chi}{\chi} \tilde{\Gamma}^m. \end{aligned} \tag{2.39}$$

The derivation of the BSSNOK system

There are different ways to derive the BSSNOK system. A full derivation by taking the time derivatives of the newly defined tensor quantities, inserting them into the ADM equations and using the definition of the conformal metric can be found in the paper [14]. A different way, by adding the constraint equations to the ADM equations and using tensor densities, can be found in the paper [66]. Then, after doing some calculation, the BSSNOK system is given by

$$\partial_t \chi = \frac{2}{n-1} \alpha \chi K - \frac{2}{n-1} \chi \partial_k \beta^k + \beta^k \partial_k \chi \tag{2.40a}$$

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - \frac{2}{n-1}\tilde{\gamma}_{ij}\partial_k\beta^k + \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{kj}\partial_i\beta^k \tag{2.40b}$$

$$\partial_t K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{n-1}\right) - D^iD_i\alpha + \frac{8\pi\alpha}{n-2}\left((n-3)E + S\right) + \beta^k\partial_k K \tag{2.40c}$$

$$\partial_t\tilde{A}_{ij} = \chi\left[\alpha\left(R^{TF}_{ij} - 8\pi S^{TF}_{ij}\right) - \left(D_iD_j\alpha\right)^{TF}\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) \tag{2.40d}$$

$$- \frac{2}{n-1}\tilde{A}_{ij}\partial_k\beta^k + 2\tilde{A}_{k(i}\partial_{j)}\beta^k + \beta^k\partial_k\tilde{A}_{ij} \tag{2.40e}$$

$$\partial_t\tilde{\Gamma}^i = -2\frac{n-2}{n-1}\alpha\tilde{\gamma}^{ik}\partial_k K - 2\tilde{A}^{ik}\partial_k\alpha - \alpha\frac{n-1}{\chi}\tilde{A}^{ik}\partial_k\chi + 2\alpha\tilde{\Gamma}^i_{kl}\tilde{A}^{kl} \tag{2.40f}$$

$$+ \frac{2}{n-1}\tilde{\Gamma}^i\partial_k\beta^k + \frac{n-3}{n-1}\tilde{\gamma}^{ik}\partial_k\partial_l\beta^l + \tilde{\gamma}^{kl}\partial_k\partial_l\beta^i - 16\pi\alpha\tilde{\gamma}^{ij}S_j \tag{2.40g}$$

$$+ \beta^k\partial_k\tilde{\Gamma}^i - \tilde{\Gamma}^k\partial_k\beta^i, \tag{2.40h}$$

where D_i and $\tilde{D}_i$ are, respectively, the covariant derivatives associated to the physical spatial metric γ_{ij} and the conformal spatial metric $\tilde{\gamma}_{ij}$, and (E, S_i, S_{ij}) the spatial parts of the energy momentum tensor as defined earlier.

We consider χ, $\tilde{\gamma}_{ij}$, $\tilde{\Gamma}^i$, K and $\tilde{A}_{ij}$ as fundamental variables and evolve them with the evolution equations (2.40a), (2.40b), (2.40c), (2.40d) and (2.40f). Note that the auxiliary algebraic and differential constraints to the system, namely

$$\det\tilde{\gamma}_{ij} = 1, \quad \tilde{\gamma}^{ij}\tilde{A}_{ij} = 0 \quad \text{and} \quad \tilde{\gamma}_{ij}\tilde{\Gamma}^j = \tilde{\gamma}^{jk}\partial_k\tilde{\gamma}_{ij} \tag{2.41}$$

are regarded as new constraint equations. They increased as the number of dynamical variables increased. While we evolve the BSSNOK system, we can use these conditions as a numerical check.

Furthermore, we can see that it is first-ordered in time and second-ordered in space. Moreover, while in [9], the BSSNOK system was rewritten as a first-order in time and space system to prove hyperbolicity, in [30], symmetric hyperbolicity for the second-order BSSNOK system was shown for the second-order in space system.

Let us remark at the end that we have succeeded in rewriting the Einstein equations as a strongly hyperbolic system by introducing conformally transformed variables. Physically, it does not matter if we evolve the physical variables or their representation from an equivalence class. In fact, it was shown by York [27] that the conformal equivalence class carries the proper degrees of freedom of the gravitational field.

Furthermore, let us notice that we can see from the very definition of a tensor density of weight $d \in \mathbb{Q}$, i.e. a quantity

$$s = \gamma^{d/2}S, \tag{2.42}$$

where S is an arbitrary tensor field on Σ_t [27], that the dynamical variables $(\chi, \tilde{\gamma}_{ij}, \tilde{A}_{ij})$ are tensor densities of weight $d = -2/(n-1)$, while the conformal connection function $\tilde{\Gamma}^i$ is the derivative of a tensor density and transforms as

$$\tilde{\Gamma}^i = -\gamma^{n/2}\left(n\gamma^{ij}\Gamma^k_{kj} + \partial_j\gamma^{ij}\right). \tag{2.43}$$

To avoid to work with tensor densities and to be able to work with spherical coordinates, we could introduce an extra structure on the hypersurface Σ_t, namely a background metric $\mathbf{f}$, with the following properties

- $\mathbf{f}$ is a Riemannian metric,
- $\mathcal{L}_{\partial_t}\mathbf{f} = 0$,
- the inverse metric is given by $f^{ik}f_{kj} = \delta^i{}_j$ with $f^{ij} \neq \gamma^{ik}\gamma^{jl}f_{kl}$ in general,
- $\mathcal{D}$ is the Levi-Civita connection associated with the background metric $\mathbf{f}$,
- $\bar{\Gamma}^k_{ij}$ are the Christoffel symbols associated with the background metric $\mathbf{f}$.

By using the background metric and the new definition of the conformal factors, i.e.

$$\chi = \left(\frac{\gamma}{f}\right)^{-\frac{1}{n-1}}, \quad \Phi = \frac{1}{4(n-1)}\log\frac{\gamma}{f} \quad \text{and} \quad W = \left(\frac{\gamma}{f}\right)^{-\frac{1}{2(n-1)}}, \tag{2.44}$$

the conformal metric $\tilde{\gamma}$, the conformal traceless part of the extrinsic curvature $\tilde{A}_{ij}$ and the conformal factors become tensor fields on Σ_t, and the contraction of the Christoffel symbolds associated with the conformal metric is just a partial derivative of tensor fields. Be aware, that the constraints with respect to the background metric changes to

$$\det\tilde{\gamma}_{ij} = f, \quad \tilde{\gamma}^{ij}\tilde{A}_{ij} = 0 \quad \text{and} \quad \tilde{\gamma}_{ij}\tilde{\Gamma}^j = \tilde{\gamma}^{jk}\partial_k\tilde{\gamma}_{ij}, \tag{2.45}$$

where $f = \det f_{ij}$.

2.4 The Z4 evolution system

As noted in the paper [12], general relativity is general covariant as the physical laws are invariant under arbitrary smooth coordinate transformations, i.e.

$$y^\mu = f^\mu(x^\nu). \tag{2.46}$$

As the original Einstein field equations are generally covariant, the solution space of the ADM equations will be as well. If the constraint equations are satisfied at $t = 0$, they will be, at least when all fields are analytical, satisfied for all times $t > 0$ [27]. Therefore, we could evolve the ADM equations without enforcing the constraint equations at each time, but only at $t = 0$. This so-called free-evolution scheme of the ADM equations breaks general covariance. Bona, Ledvinka, Palenzuela and Zacek proposed in [11] to extend the Einstein field equations without breaking general covariance by introducing an n-dimensional "zero" vector Z_μ and adding it to the Einstein field equations as a generalized Lagrangian multiplier (GLM) by

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{2.47}$$

The modified Einstein field equations can be derived from the covariant Lagrangian

$$\mathcal{L}_G = g^{\mu\nu}[R_{\mu\nu} + 2\nabla_\mu Z_\nu] \tag{2.48}$$

by a variational action principle [43] and represent a set partial differential equations of mixed-order: first-order in time, second-order in space in the metric components and first-order in space in the components of our newly defiend Z vector. These PDE's have to be solved for the field components $\{g_{\mu\nu}, Z_\mu\}$ and the physical solution can be restored by setting $Z_\mu = 0$.

The Z4 system can be derived by projecting the terms of the Einstein equations (2.47) as introduced in Section 2.2. As the component functions of the Ricci tensor $R_{\mu\nu}$, the metric tensor $g_{\mu\nu}$ and matter stress-energy tensor $T_{\mu\nu}$ have been projected, we only need to project the $\nabla_\mu Z_\nu + \nabla_\nu Z_\mu$ term. Let us first, decompose the n-vector Z_μ into

$$Z_\mu = (\theta/\alpha, Z_i)\,, \tag{2.49}$$

where $\theta = N_\mu Z^\mu$. Then, we can calculate the full projection onto the hypersurface Σ_t, namley

$$\begin{aligned}
\gamma^\mu{}_\alpha \gamma^\nu{}_\beta \nabla_\mu Z_\nu &= \gamma^\mu{}_\alpha \nabla_\mu \left(\gamma^\nu{}_\beta Z_\nu\right) - Z_\nu \gamma^\mu{}_\alpha \nabla_\mu \gamma^\nu{}_\beta \\
&= \gamma^\mu{}_\alpha \nabla_\mu \left(\gamma^\nu{}_\beta Z_\nu\right) - Z^\nu N_\nu \gamma^\mu{}_\alpha \nabla_\mu N_\beta - Z^\nu N_\beta \gamma^\mu{}_\alpha \nabla_\mu N_\nu \\
&= \gamma^\mu{}_\alpha \nabla_\mu \left(\gamma^\nu{}_\beta Z_\nu\right) - \theta K_{\alpha\beta} - K_{\alpha\mu} Z^\mu N_\beta .
\end{aligned}$$

The full projection along the vector field N is given by

$$\begin{aligned}
N^\mu N_\nu \nabla_\mu Z^\nu &= N^\mu \nabla_\mu \left(N_\nu Z^\nu\right) - Z^\nu N^\mu \nabla_\mu N_\nu \\
&= N^\mu \nabla_\mu \theta - Z^\mu A_\mu \\
&= N^\mu \partial_\mu \theta - Z^i A_i,
\end{aligned}$$

as the covariant derivative of a scalar function is just the partial derivative. And the mixed projection is given by

$$\begin{aligned}
\gamma^\mu{}_\alpha N^\nu \nabla_\mu Z_\nu &= \gamma^\mu{}_\alpha \nabla_\mu \left(N_\nu Z^\nu\right) - Z^\nu \gamma^\mu{}_\alpha \nabla_\mu N_\nu \\
&= \gamma^\mu{}_\alpha \nabla_\mu \Theta - K_{\alpha\beta} Z^\beta && (2.50a) \\
\gamma^\mu{}_\alpha N^\nu \nabla_\nu Z_\mu &= N^\nu \nabla_\nu \left(\gamma^\mu{}_\alpha Z_\mu\right) - Z^\mu N_\mu N^\nu \nabla_\nu N_\alpha - Z^\mu N_\alpha N^\nu \nabla_\nu N_\mu \\
&= N^\nu \nabla_\nu \left(\gamma^\mu{}_\alpha Z_\mu\right) - \Theta A_\alpha - Z^\mu A_\mu N_\alpha && (2.50b)
\end{aligned}$$

Putting the projections of all terms together, we derive the Z4 system

$$\left(\partial_t - \mathcal{L}_\beta\right) \gamma_{ij} = -2\alpha K_{ij}, \tag{2.51a}$$

$$\begin{aligned}
\left(\partial_t - \mathcal{L}_\beta\right) K_{ij} = -D_i D_j \alpha + \alpha \Big[& R_{ij} + D_i Z_j + D_j Z_i + (K - 2\theta) K_{ij} \\
& - 2K_{ik} K^k{}_j - 8\pi \left(S_{ij} + \frac{E - S}{n-2} \gamma_{ij}\right)\Big],
\end{aligned} \tag{2.51b}$$

$$\left(\partial_t - \mathcal{L}_\beta\right) \theta = \frac{\alpha}{2} \left[R + 2D_k Z^k + (K - 2\theta) K - K_{ij} K^{ij} - 16\pi E\right] - Z^k \partial_k \alpha, \tag{2.51c}$$

$$\left(\partial_t - \mathcal{L}_\beta\right) Z_i = \alpha \left[D_j K^j{}_i - D_i K + \partial_i \theta - 2K^j{}_i Z_j - \theta \frac{\partial_i \alpha}{\alpha} - 8\pi S_i\right]. \tag{2.51d}$$

The evolution equations for γ_{ij} and K_{ij} are still at our disposal, though in a slightly different form. The n evolution equation for the algebraic constraint $Z_\mu = 0$ replace the n constraint equations representing the physical Hamiltonian and momentum constraints. However, setting $Z_\mu = 0$ in Eq. (2.51c) and (2.51d), we can restore the Hamiltonian and momentum constraint equation (2.21c) and (2.21d). Consequently, we have to use the evolution equations for θ and Z_i on the same footing as the evolution equations for γ_{ij} and K_{ij}. This way, general covariance will not be broken. Numerically, non-zero values for Z_μ are possible, and they tell us how physical our numerical solution will be [43].

The system (2.51a)-(2.51d) with γ_{ij}, K_{ij}, θ and Z_i as evolution variables is called the Z4 sytem[1]. Even though the $Z4$ system is strongly hyperbolic, Bernuzzi introduced a conformal extension of the Z4 system.

2.5 The Z4c evolution system

A conformal extension of the $Z4$ system was introduced by Bernuzzi and Hilditch in [8] and is known as $Z4c$ (conformal $Z4$). This system can be derived from the modified Einstein field equations

$$\begin{aligned} R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu + \kappa_1 \Big[N_\mu Z_\nu + N_\nu Z_\mu \\ - (1+\kappa_2) g_{\mu\nu} N_\sigma Z^\sigma \Big] = 8\pi \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \end{aligned} \tag{2.52}$$

where N_μ is the timelike normal vector, and κ_1 and κ_2 are algebraic damping terms used to hold the constraint violations modes in numerical applications small. Consequently, using damping terms will keep the numerical solution of the system as physical as possible. Bernuzzi and Hilditch derived the $Z4c$ system by projecting the terms of the damped Einstein equations (2.52) as introduced in Section 2.2. As the component functions of the Ricci tensor $R_{\mu\nu}$, the metric tensor $g_{\mu\nu}$, the matter stress-energy tensor $T_{\mu\nu}$ and the Z4-vector term have been projected respectively, we only need to project

$$\kappa_1 \Big[N_\mu Z_\nu + N_\nu Z_\mu - (1+\kappa_2) g_{\mu\nu} N_\sigma Z^\sigma \Big].$$

Therefore, as $N_\sigma Z^\sigma$ is a constant, we get

$$-(1+\kappa_2) g_{\mu\nu} N_\sigma Z^\sigma N^\mu N^\nu = (1+\kappa_2) N_\sigma Z^\sigma, \tag{2.53}$$
$$-(1+\kappa_2) g_{\mu\nu} N_\sigma Z^\sigma \gamma^\mu{}_\alpha \gamma^\nu{}_\beta = -(1+\kappa_2) N_\sigma Z^\sigma \gamma_{\alpha\beta}, \tag{2.54}$$
$$-(1+\kappa_2) g_{\mu\nu} N_\sigma Z^\sigma \gamma^\mu{}_\alpha N^\nu = 0, \tag{2.55}$$

and

$$\begin{aligned} \gamma^\mu{}_\alpha \gamma^\nu{}_\beta N_\mu Z_\nu &= 0, \\ N^\mu N^\nu N_\mu Z_\nu &= -\theta, \\ \gamma^\mu{}_\alpha N^\nu N_\nu Z_\mu &= -\gamma^\mu{}_\alpha Z_\mu, \\ \gamma^\mu{}_\alpha N^\nu N_\mu Z_\nu &= 0. \end{aligned} \tag{2.56}$$

[1] In their original formulation, this general covariant extension was studied for the four-dimensional Einstein field equations. We will still call it Z4, even though we have written down the system in general n dimensions.

Then, the space and time projections yield evolution equations for the fundamental variables γ_{ij}, K_{ij}, θ and Z_i given by given by

$$(\partial_t - \mathcal{L}_\beta)\,\gamma_{ij} = -2\alpha K_{ij}, \tag{2.57a}$$

$$\begin{aligned}(\partial_t - \mathcal{L}_\beta)\,K_{ij} = -D_iD_j\alpha + \alpha\Big[R_{ij} + D_iZ_j + D_jZ_i + (K-2\theta)K_{ij} - 2K_{ik}K^k{}_j \\ +\kappa_1(1+\kappa_2)\theta\gamma_{ij} - 8\pi\left(S_{ij} + \frac{E-S}{n-2}\gamma_{ij}\right)\Big],\end{aligned} \tag{2.57b}$$

$$\begin{aligned}(\partial_t - \mathcal{L}_\beta)\,\theta = \frac{\alpha}{2}\left[R + 2D_kZ^k + (K-2\theta)K - K_{ij}K^{ij} - 16\pi E\right] \\ -\alpha\kappa_1\left(\frac{n}{2} + \frac{n-2}{2}\kappa_2\right)\theta - Z^k\partial_k\alpha,\end{aligned} \tag{2.57c}$$

$$(\partial_t - \mathcal{L}_\beta)\,Z_i = \alpha\left[D_jK^j{}_i - D_iK + \partial_i\theta - 2K^j{}_iZ_j - \theta\frac{\partial_i\alpha}{\alpha} - 8\pi S_i - \kappa_1 Z_i\right], \tag{2.57d}$$

where the red terms represent the decomposed terms with respect to the damping parameters κ_1 and κ_2 introduced in Eq. (2.52). As we can see, these terms constitute exactly the differences to the decomposed $Z4$ system. A full derivation of these equations for a general n-dimensional manifold M can be found in the notes by Grosvenor in [37], while they agree with the equations for a four-dimensional manifold in the paper [2].

The system (2.57a)-(2.57d) with γ_{ij}, K_{ij}, θ and Z_i as evolution variables is called the Z4c sytem.

2.6 The CCZ4 evolution system

This system was derived to adress the non-covariant part of the Z4c System. It has to be understood as beeing more covariant as the Z4c system, as the it wil luse the non-covariant conformal connection functions $\tilde{\Gamma}^i$. The derivation starts with the Einstein field equations (2.52). First, we define the following quantities

$$\phi = \gamma^{-\frac{1}{2(n-1)}}, \tag{2.58a}$$

$$\tilde{\gamma}_{ij} = \phi^2\gamma_{ij}, \tag{2.58b}$$

$$\tilde{A}_{ij} = \phi^2 A_{ij}, \tag{2.58c}$$

$$\hat{\Gamma}^i = \tilde{\Gamma}^i + 2\tilde{\gamma}^{ij}Z_j, \tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}. \tag{2.58d}$$

Then, by beforming a change of variables in the Z4c system, the full CCZ4 evolution equations are then given by

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{n-1}\tilde{\gamma}_{ij}\partial_k\beta^k + \beta^k\partial_k\tilde{\gamma}_{ij}, \tag{2.59a}$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} = \phi^2\left[-D_iD_j\alpha + \alpha\left(R_{ij} + 2D_{(i}Z_{j)} - 8\pi S_{ij}\right)\right]^{TF} + \alpha\tilde{A}_{ij}(K-2\theta) \\ - 2\alpha\tilde{A}_{ik}\tilde{A}^k_j + 2\tilde{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{n-1}\tilde{A}_{ij}\partial_k\beta^k + \beta^k\partial_k\tilde{A}_{ij},\end{aligned} \tag{2.59b}$$

$$\partial_t\phi = \frac{1}{n-1}\phi\left(\alpha K - \partial_i\beta^i\right) + \beta^i\partial_i\phi, \tag{2.59c}$$

$$\partial_t K = -D_i D^i \alpha + \alpha \Big[R + 2D_i Z^i + (K - 2\theta)K\Big] - (n-1)\alpha\kappa_1(1+\kappa_2)\theta$$
$$+ \frac{8\pi\alpha}{n-2}\left[S - (n-1)E\right] + \beta^i \partial_i K, \tag{2.59d}$$
$$\partial_t \theta = \frac{\alpha}{2}\left(R + 2D_i Z^i - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{n-2}{n-1}K^2 - 2\theta K\right) - Z^i \partial_i \alpha + \beta^i \partial_i \theta$$
$$- \alpha\kappa_1 \left(\frac{n}{2} + \frac{n-2}{2}\kappa_2\right)\theta - 8\pi\alpha E, \tag{2.59e}$$
$$\partial_t \hat{\Gamma}^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - (n-1)\tilde{A}^{ij}\frac{\partial_j \phi}{\phi} - \frac{n-2}{n-1}\tilde{\gamma}^{ij}\partial_j K\right) - 2\tilde{A}^{ij}\partial_j \alpha$$
$$+ 2\alpha\tilde{\gamma}^{ij}\left(\partial_j \theta - \theta\frac{\partial_j \alpha}{\alpha} - \frac{2}{n-1}K Z_j - \kappa_1 Z_j - 8\pi S_j\right)$$
$$+ \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{n-3}{n-1}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k + \frac{2}{n-1}\hat{\Gamma}^i\partial_j\beta^j - \hat{\Gamma}^j\partial_j\beta^i + \beta^k\partial_k\hat{\Gamma}^i. \tag{2.59f}$$

Again, these equations were derived for an n-dimensional manifold M by Grosvenor in [37]. By setting $n = 4$, we can see that these equations agree with the system of equation that can be found in [2]. Furthermore, the $\hat{\Gamma}^i$ evolution equation for black-hole-spacetime evolution was modified due to numerical instabilites by an extra parameter κ_3 in [2] by

$$\frac{2}{n-1}\hat{\Gamma}^i\partial_j\beta^j - \hat{\Gamma}^j\partial_j\beta^i = \frac{2}{n-1}\tilde{\Gamma}^i\partial_j\beta^j + 2\frac{2}{n-1}\tilde{\gamma}^{ij}Z_j\partial_k\beta^k - \tilde{\Gamma}^j\partial_j\beta^i - 2\tilde{\gamma}^{kj}Z_j\partial_k\beta^i$$
$$\to \frac{2}{n-1}\tilde{\Gamma}^i\partial_j\beta^j + 2\kappa_3\frac{2}{n-1}\tilde{\gamma}^{ij}Z_j\partial_k\beta^k - \tilde{\Gamma}^j\partial_j\beta^i - 2\kappa_3\tilde{\gamma}^{kj}Z_j\partial_k\beta^i. \tag{2.60}$$

General covariance for the $CCZ4$ system is broken if $\kappa_3 \neq 1$ is choosen, but as argued above, [2] uses a non-covariant ($\kappa_3 = 1/2$) and conformal formulation of $Z4$ for black-hole spacetime evolutions.

The system (2.59a) - (2.59f) with $\tilde{\gamma}_{ij}$, $\tilde{A}_{ij}$, ϕ, K, θ and $\hat{\Gamma}^i$ as evolution variables is called the CCZ4 system. To prove strong hyperbolicity of the CCZ4 system, we need to rewrite it as a first-order system. This will be done in the next section.

2.7 The generalized FO-CCZ4 evolution system

As we want to evolve initial data for a hyperbolic Anti-de Sitter spacetime, we need to derive the $CCZ4$ system from the modified Einstein field equations with a non-vanishing cosmological constant $\Lambda_n \neq 0$

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu + \kappa_1\left[N_\mu Z_\nu + N_\nu Z_\mu - (1+\kappa_2)g_{\mu\nu}N_\sigma Z^\sigma\right]$$
$$= 8\pi\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) + \frac{2}{n-2}\Lambda_n g_{\mu\nu}, \tag{2.61}$$

Here, we will only state the equations of the CCZ4 system that will change with respect to the non-vanishing cosmological constant Λ_n:

$$\partial_t K = [CCZ4] - \frac{2(n-1)}{n-2}\alpha\Lambda_n \tag{2.62a}$$

$$\partial_t \theta = [CCZ4] - \alpha \Lambda_n, \tag{2.62b}$$

where $[CCZ4]$ represents the right-hand sight of the respective equations of the CCZ4 system. A full derivation of the FO-CCZ4 system for a general n-dimensional manifold can be found in the notes by Grosvenor in [37].

To prove the strong hyperbolicity of the second-order $CCZ4$ system, we need to rewrite it as a first-order system and analyze the eigenstructure of the coefficient matrices. Although there are many different ways of performing a first-order reduction, we will follow the ideas introduced in [20] by making maximum use of auxiliary variables and second-order ordering constraints.

2.7.1 Auxiliary variables and ordering constraints

From here on, we will closely follow chapter II of the paper [20] by Dumbser et al. In order to rewrite the CCZ4 system as a first-order system, we use the 33 auxiliary variables

$$\begin{aligned} A_i &= \partial_i \ln \alpha = \partial_i \alpha / \alpha, & B^i_k &= \partial_k \beta^i, \\ D_{kij} &= \partial_k \tilde{\gamma}_{ij}/2, & P_i &= \partial_i \ln \phi = \partial_i \phi / \phi, \end{aligned} \tag{2.63}$$

the second-order ordering constraints

$$\begin{aligned} \mathcal{A}_{ki} &= \partial_k A_i - \partial_i A_k = 0, & \mathcal{B}^i_{kl} &= \partial_k B^i_l - \partial_l B^i_k = 0, \\ \mathcal{D}_{klij} &= \partial_k D_{lij} - \partial_l D_{kij} = 0, & \mathcal{P}_{ki} &= \partial_k P_i - \partial_i P_k = 0, \end{aligned} \tag{2.64}$$

and the hereafter defined constraints from the paper [20]. As $\tilde{\gamma}^{ij}\tilde{A}_{ij} = 0$, Dumbser et al. derived the constraint equation

$$\mathcal{T}_k = \partial_k \left(\tilde{\gamma}^{ij}\tilde{A}_{ij}\right) = \tilde{A}_{ij}\partial_k \tilde{\gamma}^{ij} + \tilde{\gamma}^{ij}\partial_k \tilde{A}_{ij} = 0. \tag{2.65}$$

Moreover, from the constraint $\det \tilde{\gamma}_{ij} = f$, we can derive via the general law of variation

$$\delta(\ln \det A = \operatorname{tr}\left(A^{-1} \times \delta A\right), \tag{2.66}$$

where δ denotes any derivative that fulfills the Leibniz rule, the following constraint

$$\tilde{\gamma}^{ij} D_{kij} = \frac{1}{2}\partial_k \ln f \tag{2.67}$$

for arbitrary coordinates. Note that for Cartesian coordinates, i.e. $f = 1$, this simplifies to

$$\tilde{\gamma}^{ij} D_{kij} = 0, \tag{2.68}$$

as was derived by Dumbser et al. in [20].

2.7.2 First-order formulation of the CCZ4 system

As was done in the PhD thesis by Köppel [2], Grosvenor made maximum use of the above introduced auxiliary variables and ordering constraints in order to evolve the variables defining the physical metric $g_{\mu\nu}$ as a nonlinear system of ordinary differential equations and wrote the other dynamical partial differential equations in terms of non-conservative products with vanishing flux.

We will not get into detail, but refer the reader to the papers [43], [20] and [19] to understand the proper use of the auxiliary variables and ordering constraints. For example, as mentioned in Chapter II in [20], a naive first-order formulation or the use of the first- and second-order ordering constraints alone are not enough, and the system would lose its strong hyperbolicity.

After some tremendous calculations, that was done by Grosvenor [37], the FO-CCZ4 system in general n dimensions and with nonvanishing cosmological constant Λ_n then reads

$$\partial_t \tilde{\gamma}_{ij} = 2\beta^k D_{kij} + 2\tilde{\gamma}_{k(i}B^k_{j)} - \tfrac{2}{n-1}\tilde{\gamma}_{ij}B^k_k - 2\alpha\tilde{A}^{TF}_{ij} - \tau^{-1}(\tilde{\gamma} - f)\tilde{\gamma}_{ij} \tag{2.69a}$$

$$\partial_t \ln\phi = \beta^k P_k + \frac{1}{n-1}(\alpha K - B^k_k) \tag{2.69b}$$

$$\begin{aligned}\partial_t K = {}& \beta^k\partial_k K - \nabla^i\nabla_i\alpha + \alpha(R + 2\nabla_i Z^i) + \alpha K(K - 2\theta c) \\ & - (n-1)\alpha\kappa_1(1+\kappa_2)\theta - \tfrac{2(n-1)}{n-2}\alpha\Lambda + \tfrac{8\pi\alpha}{n-2}\left[S - (n-1)E\right]\end{aligned} \tag{2.69c}$$

$$\begin{aligned}\partial_t \tilde{A}_{ij} = {}& \beta^k\partial_k\tilde{A}_{ij} + \phi^2[-\nabla_i\nabla_j\alpha + \alpha(R_{ij} + \nabla_i Z_j + \nabla_j Z_i - 8\pi S_{ij})] \\ & - \tfrac{1}{n-1}\tilde{\gamma}_{ij}[-\nabla^k\nabla_k\alpha + \alpha(R + 2\nabla_k Z^k - 8\pi S)] + \tilde{A}_{ki}B^k_j + \tilde{A}_{kj}B^k_i \\ & - \tfrac{2}{n-1}\tilde{A}_{ij}B^k_k + \alpha\tilde{A}_{ij}(K - 2\theta c) - 2\alpha\tilde{A}_{il}\tilde{\gamma}^{lm}\tilde{A}_{mj} - \tau^{-1}\tilde{\gamma}_{ij}\tilde{A}^k_k\end{aligned} \tag{2.69d}$$

$$\begin{aligned}\partial_t\theta = {}& \beta^k\partial_k\theta + \tfrac{1}{2}\alpha e^2\left(R + 2\nabla_i Z^i + \tfrac{n-2}{n-1}K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - 2\Lambda - 16\pi E\right) \\ & - \alpha\theta K c - \alpha Z^i A_i - \alpha\kappa_1(\tfrac{n}{2} + \tfrac{n-2}{2}\kappa_2)\theta\end{aligned} \tag{2.69e}$$

$$\begin{aligned}\partial_t\hat{\Gamma}^i = {}& \beta^k\partial_k\hat{\Gamma}^i - \tfrac{2(n-2)}{n-1}\alpha\tilde{\gamma}^{ij}\partial_j K + 2\alpha\tilde{\gamma}^{ki}\partial_k\theta + s\tilde{\gamma}^{kl}\partial_{(k}B^i_{l)} + \tfrac{n-3}{n-1}s\tilde{\gamma}^{ik}\partial_{(k}B^l_{l)} \\ & + 2s\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} + \tfrac{2}{n-1}\tilde{\Gamma}^i B^k_k - \tilde{\Gamma}^k B^i_k + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - (n-1)\tilde{A}^{ij}P_j) \\ & - 2\alpha\tilde{\gamma}^{ki}(\theta A_k + \tfrac{2}{n-1}KZ_k) - 2\alpha\tilde{A}^{ij}A_j - 4s\alpha\tilde{\gamma}^{ik}D^{nm}_k\tilde{A}_{nm} \\ & + 2\kappa_3(\tfrac{2}{n-1}\tilde{\gamma}^{ij}Z_jB^k_k - \tilde{\gamma}^{jk}Z_jB^i_k) - 2\alpha\kappa_1\tilde{\gamma}^{ij}Z_j - 16\pi\alpha\tilde{\gamma}_{ij}p_j\end{aligned} \tag{2.69f}$$

$$\begin{aligned}\partial_t A_k = {}& \beta^l\partial_l A_k - \alpha f(\alpha)(\partial_k K - \partial_k K_0 - 2c\partial_k\theta) - s\alpha f(\alpha)\Big[\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} \\ & + 2D^{nm}_k\tilde{A}_{nm}\Big] - \alpha A_k(K - K_0 - 2\theta c)(f(\alpha) + \alpha f'(\alpha)) + B^l_k A_l\end{aligned} \tag{2.69g}$$

$$\partial_t B^i_k = s\Big(\beta^l\partial_l B^i_k + k\partial_k b^i + \alpha^2\mu\tilde{\gamma}^{ij}(\partial_k P_j - \partial_j P_k) + B^l_k B^i_l \tag{2.69h}$$

$$- \alpha^2\mu\tilde{\gamma}^{ij}\tilde{\gamma}^{nl}(\partial_k D_{ljn} - \partial_l D_{kjn})\Big) \tag{2.69i}$$

$$\begin{aligned}\partial_t D_{kij} = {}& \beta^l\partial_l D_{kij} + \tfrac{s}{2}\tilde{\gamma}_{mi}\partial_{(k}B^m_{j)} + \tfrac{s}{2}\tilde{\gamma}_{mj}\partial_{(k}B^m_{i)} - \tfrac{s}{n-1}\tilde{\gamma}_{ij}\partial_{(k}B^m_{m)} - \alpha\partial_k\tilde{A}_{ij} \\ & + \tfrac{1}{n-1}\alpha\tilde{\gamma}_{ij}\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} + B^l_k D_{lij} + B^l_j D_{kli} + B^l_i D_{klj} - \tfrac{2}{n-1}B^l_l D_{kij} \\ & - \tfrac{2}{n-1}\alpha\tilde{\gamma}_{ij}D^{nm}_k\tilde{A}_{nm} - \alpha A_k\tilde{A}^{TF}_{ij}\end{aligned} \tag{2.69j}$$

$$\begin{aligned}\partial_t P_k = {}& \beta^l\partial_l P_k + \tfrac{1}{n-1}\alpha\partial_k K + \tfrac{s}{n-1}\partial_{(k}B^i_{i)} + \tfrac{1}{n-1}\alpha A_k K + B^l_k P_l \\ & + \tfrac{s}{n-1}\alpha\left(\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} - 2D^{nm}_k\tilde{A}_{nm}\right)\end{aligned} \tag{2.69k}$$

Let us give some remarks about the coloured terms that can be found in the FO-CCZ4 system:

- the red terms have been added by using the second-order ordering constraints

(2.64) and the constraint (2.65) to symmetrize the sparsity pattern of the system matrices in order to avoid Jordan blocks as they cannot be diagonilazied

- the constant e is introduced in front of the Hamiltonian constraint only for numerical reasons: $e > 1$ achvies better numerical constraint values, but breaks covariance
- the constant τ is a relaxation time and is used to impose the constraints $\det \tilde{\gamma}_{ij} = f$ and $\tilde{\gamma}^{ij}\tilde{A}_{ij} = 0$ weakly
- the constant μ is used to adjust the second-oder odering constraints
- the constant s is used to turn on or off the evolution of the shift vector β: While $s = 0$ corresponds to $\partial_t \beta^i = 0$, $s = 1$ corresponds to the Gamma-driver shift condition
- the constant c is used to remove the algebraic source of the modified $1+\log$-slicing
- the blue terms are the terms with a non-vanishing cosmological constant Λ_n

Finally, the equations (2.69a)-(2.69k) with $\tilde{\gamma}_{ij}$, $\ln\phi$, K, $\tilde{A}_{ij}$, θ, $\hat{\Gamma}^i$, A_k, B_k^i, D_{ijk} and P_k as evolution variables is called the generalized FO-CCZ4 system for an n-dimensional manifold with a non-vanishing cosmological constant Λ_n. By setting $n = 4$ and $\Lambda_n = 0$, these equations will simplify to the original formulation by Dumbser et al. in Chapter II in [20].

2.7.3 Hyperbolicity analysis of the generalized FO-CCZ4 system

As strong hyperbolicity for the FO-CCZ4 sytem in four dimensions with vanishing cosmological constant $\Lambda_n = 0$ was shown by Dumser et al. in Section D of Chapter II in [20], we need to show strong hyperbolicity for the generalized FO-CCZ4 system with a non-vanishing cosmological constant. In compact matrix-form, we can write the FO-CCZ4 system as

$$\frac{\partial \mathbf{Q}}{\partial t} + \sum_{i=1}^{n-1} \mathbf{A}_i(\mathbf{Q}) \frac{\partial \mathbf{Q}}{\partial x_i} = \mathbf{S}(\mathbf{Q}), \tag{2.70}$$

where $\mathbf{A}_i \in \mathbb{R}^{\xi\times\xi}$ are the system-matrices and

$$\mathbf{Q} = \left(\tilde{\gamma}_{ij}, \ln\alpha, \beta^i, \ln\phi, \tilde{A}_{ij}, K, \theta, \hat{\Gamma}^i, b^i, A_k, B_k^i, D_{kij}, P_k\right) \in \mathbb{R}^{\xi} \tag{2.71}$$

is the state vector containing $\xi = \frac{n^3}{2} + n^2 + \frac{5}{2}n$ variables. By splitting the state vector $\mathbf{Q}$ and the algebraic source $\mathbf{S}(\mathbf{Q})$, respectively, into $\mathbf{Q} = (\mathbf{V}, \mathbf{U})$ with

$$\mathbf{V} = \left(\tilde{\gamma}_{ij}, \ln\alpha, \beta^i, \ln\phi\right) \tag{2.72}$$

$$\mathbf{U} = \left(\tilde{A}_{ij}, K, \theta, \hat{\Gamma}^i, b^i, A_k, B_k^i, D_{kij}, P_k\right) \tag{2.73}$$

and $\mathbf{S}(\mathbf{Q}) = (\mathbf{S}'(\mathbf{Q}), \mathbf{S}''(\mathbf{Q})) \in \left(\mathbb{R}^{\xi_1}, \mathbb{R}^{\xi_2}\right)$, we can write the system (2.70) as

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{V}\\ \mathbf{U}\end{bmatrix} + \sum_{i=1}^{n-1}\begin{bmatrix}0 & 0\\ 0 & B_i(\mathbf{V})\end{bmatrix}\frac{\partial}{\partial x_i}\begin{bmatrix}\mathbf{V}\\ \mathbf{U}\end{bmatrix} = \begin{bmatrix}\mathbf{S}'(\mathbf{Q})\\ \mathbf{S}''(\mathbf{Q})\end{bmatrix}. \tag{2.74}$$

Thus, this equation can be decomposed, respectively, into the $\xi_1 = \frac{n^2}{2} + \frac{n}{2} + 1$ and $\xi_2 = \frac{n^3}{2} + \frac{n^2}{2} + 2n - 1$ equations

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{S}'(\mathbf{Q}), \tag{2.75}$$

and

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{i=1}^{n-1}\mathbf{B}_i(\mathbf{V})\frac{\partial \mathbf{U}}{\partial x_i} = \mathbf{S}''(\mathbf{Q}). \tag{2.76}$$

As the system matrices for the ODE subsystems are vanishing, the eigenvalues will be trivially zero, and the eigenvectors are just the unit vectors. Now, we need to calculate the eigenvalues and the eigenvectors for the reduced system (2.76). This has been done in the appendix of [20] for a four-dimensional manifold with vanishing cosmological constant Λ_n for the two coordinate gauge choices:

- zero shift $\beta^i = 0$ and harmonic slicing as well as
- Gamma-driver shift and $1 +$ log-slicing.

As the two blue terms proportional to the cosmological constant that were added are purely algebraic, they will not affect the hyperbolicity analysis and the four-dimensional generalized FO-CCZ4 system will remain strongly hyperbolic. The full eigenstructure of the coefficient matrices

$$\mathcal{P} = \omega^i B_i$$

for the unit normal vector $\omega^i = (1, 0, 0)$, such that $\mathcal{P} = B_1$, with all their eigenvalues and eigenvectors can be found in the appendix of [20]. As the Einstein field equations are isotropic, we do not have to analyze the eigenstructure for all unit normal vectors, but we can restrict ourselves to just one. Now, the non-trivial eigenvectors of the system (2.70) are given by "glueing" the unit vectors obtained from the ODE subsystem to the eigenvectors obtained from the reduced system (2.76). Therefore, the generalized FO-CCZ4 system for a four-dimensional spacetime with a non-vanishing cosmological constant remains strongly hyperbolic. Furthermore, as the FO-CCZ4 system restricted to a three-dimensional embedded surface, by setting the z-component of the four-dimensional spacetime to zero, is strongly hyperbolic, the FO-CCZ4 system for a three-dimensional spacetime will be strongly hyperbolic. This way, the FO-CCZ4 system for $n = 3$ and $n = 4$ will be strongly hyperbolic, while the eigenvectors and the eigenvalues of the coefficient matrices of the generalized FO-CCZ4 system with a non-vanishing cosmological constant for a $n > 4$-dimensional manifold still have to be calculated.

As noted in [43], the FO-CCZ4 system admits constraint violations. Thus, the hyperbolicity analysis is only valid for the augmented solution space. However, as we have noted earlier, if the constraint equations are satisfied for the initial data, they will be satisfied for all times $t > 0$ and thus, the hyperbolicity analysis in [20] is also valid for the solutions that satisfy the constraint equations.

Fixing the gauge variables (α, β) is equivalent to choosing a coordinate system in general relativity and therefore of utter importance for numerical implementation. Therefore, let us discuss in the next section the various gauge choices used in the past. After that, we will mainly follow and summarize the notes in Chapter 9 by [27].

2.8 The n degrees of freedom of gauge fixing

In numerical relativity, the 4 degrees of freedom for gauge fixing are incorporated in the freedom to choose the lapse function α and the shift vector β freely. Thus, they are the same n degrees of gauge freedom fixing a reference frame as in general relativity. While the lapse function α and the shift vector β, respectively, are defined in a differential geometric rigorous manner in Section A.4, they tell us how far a slice is located perpendicular above another and how to propagate the spatial coordinates (x^i) from one slice to another. In this way, the lapse function and the shift vector reflect the choice of foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of the spacetime and the choice of spatial coordinates on each leaf Σ_t, respectively. If we want to numerically solve the dynamical evolution equations, every term must be given a specific numerical value or solved via a PDE. While a numerical value is straightforward to implement, evolving the lapse function α and the shift vector β via a PDE extends the overall evolution system, and the system can be made more hyperbolic or elliptic [27].

In the following two sections, we will only state the choices of foliation and spatial coordinates used in the past and state their main properties, but refer the attentive reader to Chapter 9 in [27] and references within.

2.8.1 The different choices of foliation of the spacetime M

The most simple way of choosing the lapse function is called the *geodesic slicing*, i.e.

$$\alpha = 1. \tag{2.77}$$

It is called geodesic as for $\alpha = 1$, the n-acceleration of the Eulerian observer is zero, and thus the word lines of the Eulerian observer are geodesics. [27] From equation (A.55), we can see that the eigentime coincides with the coordinate time. As argued in [27], this type of foliation will collapse after a limited range of time due to the tendency of timelike geodesics without vorticity to focus and eventually cross.

The *maximal slicing* foliation of the spacetime maximizes the volume of the hypersurfaces Σ_t by the condition that the trace of the extrinsic curvature is zero, i.e.

$$K = 0. \tag{2.78}$$

This foliation has a singularity avoidance property, which means that the entire spacetime outside the event horizon is covered, while near this region, the slices pile up. A consequence is that the lapse goes to zero if time increases. This choice of foliation yields an elliptic equation for the lapse α that is time-consuming if one does not use fast elliptic solvers. However, it could be transferred into a parabolic equation to approximate the maximal slicing.

The *harmonic slicing* gauge can be imposed by requiring that the harmonic condition holds for the time coordinate t, i.e.

$$\nabla_\mu \nabla^\mu t = 0, \tag{2.79}$$

leaving the freedom to choose the spatial coordintes. This gauge has a weaker singularity avoidance property than the maximal slicing and has been generelized by Bona, Massó, Seidel and Stela [27] to

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right) \alpha = -\left(K - K_0\right) \alpha^2 f(\alpha), \tag{2.80}$$

where f is an arbitrary function and $K_0 = K(t = 0)$. This equation reduces to the geodesic slicing for $f(\alpha) = 0$, and to the harmonic slicing for $f(\alpha) = 1$ and $K_0 = 0$. We call this dynamical evolution equation the *"$1 + \log$"-slicing*, since the solution for $K_0 = 0$ and $\beta = 0$ are of the form $\alpha = 1 + \ln \gamma$. The *"$1 + \log$"-slicing* has even better singularity avoidance properties then the harmonic slicing and is hyperbolic. In [2], the $1 + \log$-slicing condition was slighlty modified to

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right) \alpha = -\alpha^2 f(\alpha) \left(K - K_0 - 2\theta\right), \tag{2.81}$$

where $f(\alpha) = 2/\alpha$ and $K_0 = 0$ was choosen, while [20] uses due to numerical reasons

$$\partial_t \ln \alpha = \beta^k A_k - \alpha f(\alpha)(K - K_0 - 2\theta). \tag{2.82}$$

From the very definitions of these gauges, we can see that the maximal slicing can be defined on a single hypersurface Σ_t, while the other foliations are only meaningful for a foliation $(\Sigma_t)_{t\in\mathbb{R}}$.

2.8.2 The different choices of spatial coordinates

Once some coordinates are set in the initial slice Σ_0, the shift vector β tells us how these spatial coordinates propagate to all other slices. As for the lapse function, the most simple way to choose the shift vector is to set it to zero, i.e.

$$\beta = 0. \tag{2.83}$$

These coordinates are called *normal coordinates* since the lines $x^i = const$ are orthogonal to all hypersurfaces Σ_t. However, as soon as we want to simulate rotating star spacetimes, the field lines of the stationary Killing vector are not orthogonal to the hypersurfaces [27], and we need to use $\beta \neq 0$.

The *minimal distortion* tries to minimize the time derivative of the conformal spatial metric $\tilde{\gamma}_{ij}$ that we will introduce in the subsequent section. This condition yields an elliptic equation for the shift vector β that has to be solved or could be simplified by considering the *pseudo-minimal distortion* or *approximate minimal distortion*.

Most interesting for us will be the *Gamma-freezing* and *Gamma-driver* conditions. The *Gamma-freezing* is just a simplification of the *pseudo-minimal distortion* and is given by

$$\mathcal{L}_{\partial_t}\tilde{\Gamma}^i = 0, \text{ with } \tilde{\Gamma}^i := \tilde{\gamma}^{jk}\left(\tilde{\Gamma}^i{}_{jk} - \bar{\Gamma}^i{}_{jk}\right), \tag{2.84}$$

where $\bar{\Gamma}^i{}_{jk}$ are the Christoffel symbols with respect to a background metric f and the tilde objects are conformal quantities that will be introduced as well in the next section. This condition will yield an elliptic equation for the shift vector and can be modified by

$$\partial_t\beta^i = k\partial_t\tilde{\Gamma}^i \tag{2.85}$$

yielding a parabolic equation and

$$\partial_t^2\beta^i = k\partial_t\tilde{\Gamma}^i - (\eta - \partial_t \ln k)\,\partial_t\beta^i \tag{2.86}$$

yielding an hyperbolic equation that is given by the first-order system

$$\partial_t\beta^i = kb^i + \beta^k\partial_k b^i \tag{2.87a}$$
$$\partial_t b^i = \partial_t\tilde{\Gamma}^i - \beta^k\partial_k\tilde{\Gamma}^i - \eta b^i + \beta^k\partial_k b^i, \tag{2.87b}$$

where $k = 3/4$ is usually used, a damping term η and an auxiliary field b^i in order to rewrite the hyperbolic equation as a first order system. In [2], the Gamma-driver shift condition was slighlty modified to

$$\partial_t\beta^i = kb^i + \beta^k\partial_k\beta^i \tag{2.88a}$$
$$\partial_t b^i = \partial_t\hat{\Gamma}^i - \beta^k\partial_k\hat{\Gamma}^i - \eta b^i + \beta^k\partial_k b^i, \tag{2.88b}$$

where $\tilde{\Gamma}^i$ was replaced with the newly defined variable $\hat{\Gamma}^i$. As noted in [2], the choise of $k = 3/4$ can lead to weak hyperbolicity of the system, when the lapse function is close to one. The *Gamma-driver shift* condition will be nowadays used as a standard choice for the spatial coordinates.

All choices for spatial coordinates mentioned above tell us how the coordinates propagate from Σ_t to $\Sigma_{t'}$ with $t, t' \neq 0$, but do not tell us how to choose the coordinates on the initial hypersurface Σ_0. We can choose an initial coordinate system by so-called spatial coordinate-fixing choices, but instead, refer the attentive reader to the well-explained section in [27] and the references within.

We aimed to fully specify the generalized FO-CCZ4 system of the Einstein field equations for a n-dimensional AdS spacetime with a non-vanishing cosmological constant Λ_n. We have shown that the system for a three- and four-dimensional AdS spacetime remains strongly hyperbolic and we derived gauge variables for gauge fixing our reference frame. From here on, we will use the generalized FO-CCZ4 system (2.69a)-(2.69k) with the gauge fixing (2.82), (2.88a)-(2.88b).

Chapter 3

The Initial Data Problem for an Asymptotically AdS$_n$ Spacetime

"The resolution of Einstein equation amounts to solving a Cauchy problem, namely to evolve *forward in time* some initial data. However, this is a Cauchy problem with constraints. This makes the set up of initial data a non-trivial task, because these data must obey the constraints" [27]
Éric Gourgoulhon
Physicist

As we want to find a unique solution to the generalized FO-CCZ4 system (2.69a)-(2.69k) with the gauge fixing (2.82), (2.88a)-(2.88b), we need to set initial conditions. However, the Einstein system constitutes a time evolution problem with constraints that have to be fulfilled for all times $t \in \mathbb{R}$. Thus, the initial data that we want to evolve forward in time has to satisfy the n constraint equations, making the set-up a quite non-trivial task. Mathematically, we have the following problem taken from [27]:

Given some hypersurface Σ_0 at $t = 0$, we need to find a Riemannian metric γ, a symmetric bilinear form K and some energy-matter distribution (E, p) on Σ_0 such that the Hamiltonian and momentum constraints

$$R + K^2 - K_{ij}K^{ij} - 2\Lambda_n = 16\pi E \tag{3.1}$$

$$D_j K^j{}_i - D_i K = 8\pi p_i, \tag{3.2}$$

are satiesfied. Let us remember that R is the spatial Riemannian scalar curvature with respect to the spatial metric γ_{ij}.

There is no natural way of freely choosing between these variables [27], though, Lichnerowicz, Choquet-Bruhat, York and Ó Murchadha, and York and Pfeiffer have shown that we can split the initial data into freely choosable parts and parts obtained by solving the constrained equations by performing a conformal decomposition. However, we will not get into detail but refer the attentive reader to the respective chapter in [27] and references within.

For the sake of simplicity, we restricted ourselves to a time-symmetric surface Σ_t, i.e. as given in [5] equivalent to demanding that $K_{ij}|_{t=0}$ on Σ_0. Thus, the momentum density p_i must vanish everywhere on Σ_0 in order to satisfy the momentum constraint. This leaves us only with the simplified Hamiltionian constraint

$$R - 2\Lambda_n = 16\pi E. \tag{3.3}$$

By performing a conformal decomposition, i.e.

$$\gamma_{ij} = \zeta^2 \hat{\gamma}_{ij} \quad \text{and} \quad \gamma^{ij} = \zeta^{-2}\hat{\gamma}^{ij}, \tag{3.4}$$

for the spatial part of the AdS$_n$ metric $\hat{\gamma}_{ij}$ with the boundary condition $\zeta|_{\partial\Sigma} = 1$, we can write the scalar curvature for an n-dimensional manifold as

$$R = \zeta^{-2}\left[\hat{R} - 2(n-2)\hat{D}_k\hat{D}^k \ln\zeta + (n-3)(2-n)\hat{D}_k \ln\zeta \hat{D}^k \ln\zeta\right], \tag{3.5}$$

where

$$\hat{R} = \hat{\gamma}^{ij}\hat{R}_{ij} \tag{3.6}$$

is the Riemannian scalar curvature associated with the conformal spatial metric $\hat{\gamma}_{ij}$. Now, we can rewrite the scalar curvature R by using

$$\hat{D}_i\hat{D}^i \ln\zeta = \hat{D}_i\left(\frac{1}{\zeta}\hat{D}^i\zeta\right) = -\frac{1}{\zeta^2}\hat{D}_i\zeta\hat{D}^i\zeta + \frac{1}{\zeta}\hat{D}_i\hat{D}^i\zeta = -\hat{D}_i \ln\zeta\hat{D}^i \ln\zeta + \frac{1}{\zeta}\hat{D}_i\hat{D}^i\zeta \tag{3.7}$$

to

$$R = \zeta^{-2}\left[\hat{R} + (n-2)(5-n)\hat{D}_i \ln\zeta\hat{D}^i \ln\zeta - 2(n-2)\zeta^{-1}\hat{D}_i\hat{D}^i\zeta\right]. \tag{3.8}$$

The Hamiltonian constraint for an n-dimensional manifold is then given by

$$\hat{R} + (n-2)(5-n)\hat{D}_i \ln\zeta\hat{D}^i \ln\zeta - 2(n-2)\zeta^{-1}\hat{D}_i\hat{D}^i\zeta - 2\Lambda_n\zeta^2 = 16\pi\zeta^2 E, \tag{3.9}$$

where

$$\hat{R} = 2\Lambda_n = -\frac{(n-1)(n-2)}{L^2}. \tag{3.10}$$

By using Eq. (3.10) and by setting the AdS$_n$ radius $L = 1$, thereby fixing the cosmological constant to

$$\Lambda_n = -\frac{(n-1)(n-2)}{2}, \tag{3.11}$$

we can at last simplify the Hamiltonian constraint to

$$\hat{D}_i\hat{D}^i\zeta - \left(\frac{1}{n-2}\Lambda_n + \frac{5-n}{2}\hat{D}_i \ln\zeta\hat{D}^i \ln\zeta\right)\zeta + \frac{1}{n-2}\left(\Lambda_n + 8\pi E\right)\zeta^3 = 0. \tag{3.12}$$

Now, as $\hat{D}$ is the covariant derivative with respect to the conformally decomposed spatial AdS$_n$ metric $\hat{\gamma}_{ij}$, let us explicitely write out the $\hat{D}$-derivative terms. Then,

$$\begin{aligned}
\hat{D}_i\hat{D}^i\zeta &= \hat{D}_i\left(\hat{\gamma}^{ij}\hat{D}_j\zeta\right) = \hat{\gamma}^{ij}\hat{D}_i\hat{D}_j\zeta + \hat{D}_i\hat{\gamma}^{ij}\hat{D}_j\zeta = \hat{\gamma}^{ij}\hat{D}_i\left(\partial_j\zeta\right) \\
&= \hat{\gamma}^{ij}\left(\partial_i\partial_j\zeta - \partial_k\zeta\hat{\Gamma}^k_{ij}\right) = \hat{\gamma}^{ij}\partial_j\partial_i\zeta - \hat{\gamma}^{ij}\partial_k\zeta\hat{\Gamma}^k_{ji} \\
&= \hat{\gamma}^{\rho\rho}\partial^2_\rho\zeta + \hat{\gamma}^{ab}\partial_a\partial_b\zeta - \hat{\gamma}^{\rho\rho}\hat{\Gamma}^k_{\rho\rho}\partial_k\zeta - \hat{\gamma}^{ab}\hat{\Gamma}^k_{ab}\partial_k\zeta \\
&= \hat{\gamma}^{\rho\rho}\partial^2_\rho\zeta + \hat{\gamma}^{ab}\partial_a\partial_b\zeta - \hat{\gamma}^{\rho\rho}\hat{\Gamma}^\rho_{\rho\rho}\partial_\rho\zeta \\
&\quad - \hat{\gamma}^{\rho\rho}\hat{\Gamma}^a_{\rho\rho}\partial_a\zeta - \hat{\gamma}^{ab}\hat{\Gamma}^\rho_{ab}\partial_\rho\zeta - \hat{\gamma}^{ab}\hat{\Gamma}^c_{ab}\partial_c\zeta,
\end{aligned} \tag{3.13}$$

where we have used that the covariant derivative $\hat{D}$ is compatible with the metric $\hat{\gamma}$, i.e. $\hat{D}_i \hat{\gamma}^{ij} = 0$. Furthermore,

$$\hat{D}_i \ln \zeta = \partial_i \ln \zeta \tag{3.14a}$$

$$\hat{D}^i \ln \zeta = \hat{\gamma}^{ij} \partial_j \ln \zeta = \hat{\gamma}^{i\rho} \partial_\rho \ln \zeta + \hat{\gamma}^{ia} \partial_a \ln \zeta \tag{3.14b}$$

and therefore

$$\begin{aligned} \hat{D}_i \ln \zeta \hat{D}^i \ln \zeta &= \partial_i \ln \zeta \left(\hat{\gamma}^{i\rho} \partial_\rho \ln \zeta + \hat{\gamma}^{ia} \partial_a \ln \zeta \right) \\ &= \hat{\gamma}^{\rho\rho} \partial_\rho \ln \zeta \partial_\rho \ln \zeta + \hat{\gamma}^{ab} \partial_a \ln \zeta \partial_b \ln \zeta \end{aligned} \tag{3.15}$$

As we want the spatial metric γ_{ij} to be asymptotically AdS, the ζ function has to be one on the boundary, i.e. $\zeta\Big|_{\partial\Sigma_0} = 1$. Therefore, we can now state the initial value problem

$$\begin{aligned} \hat{D}_i \hat{D}^i \zeta + \left(\frac{n-1}{2} - \frac{5-n}{2} \hat{D}_i \ln \zeta \hat{D}^i \ln \zeta \right) \zeta & \\ - \frac{1}{n-2} \left(\frac{(n-1)(n-2)}{2} - 8\pi E \right) \zeta^3 = 0 &\quad \text{on} \quad \Sigma_0 \\ \zeta = 1 &\quad \text{on} \quad \partial\Sigma_0, \end{aligned} \tag{3.16}$$

where the covariant derivative terms $\hat{D}_i \hat{D}^i \zeta$ and $\hat{D}_i \ln \zeta \hat{D}^i \ln \zeta$ are, respectivley, given by Eq. (3.13) and (3.15). After solving the initial value problem Eq. (3.16), we can reconstruct the spatial metric γ_{ij} by

$$\gamma_{ij} = \zeta^2 \hat{\gamma}_{ij}, \tag{3.17}$$

while we choose the energy-density E and set

$$p_i = 0 \quad \text{and} \quad K_{ij} = 0 \quad \forall\, i, j. \tag{3.18}$$

on the initial slice Σ_0 at $t = 0$.

Let us note that Eq. (3.16) is a second-order, time-independent elliptic partial differential equation, which can be solved numerically by so-called spectral methods. Readers not familiar with the initial value problem in numerical relativity can have a look at the respective chapter in [27], and read about the spectral method for elliptic partial differential equations in [29] and [38]. Even though getting valid initial physical data sets in numerical relativity is a whole domain to itself, let us take a look at the initial data sets for a matter-free distribution, i.e. $E = 0$, and a symmetric scalar field as matter.

3.1 Initial data for a matter-free distribution

Assuming a matter-free distribution at time $t = 0$, i.e. $E = 0$, we can simplify the initial value problem to

$$\hat{D}_i \hat{D}^i \zeta + \left(\frac{n-1}{2} - \frac{5-n}{2} \hat{D}_i \ln \zeta \hat{D}^i \ln \zeta \right) \zeta - \frac{n-1}{2} \zeta^3 = 0 \quad \text{on} \quad \Sigma_0$$

$$\zeta = 1 \quad \text{on} \quad \partial\Sigma_0. \tag{3.19}$$

A closer look shows that for the simple case $\Sigma_0 = \mathbb{R}^{n-1}$

$$\zeta = 1 \tag{3.20}$$

is a solution. This way, we can choose

$$\gamma_{ij} = \hat{\gamma}_{ij}, \quad p_i = 0, \qquad E = 0 \quad \text{and} \quad K_{ij} = 0 \tag{3.21}$$

as initial data. If we are taking the topology of the initial leave as $\Sigma_0 = \mathbb{R}^{n-1} \setminus B_\rho(0)$ or as $\Sigma_0 = \mathbb{R}^{n-1} \setminus O$, we need to set boundary conditions for ζ, respectively, on the sphere or at the puncture O.

By choosing the latter initial data for $\Sigma_0 = \mathbb{R}^{n-1}$, the state vector at $t = 0$, i.e.

$$\mathbf{Q}\Big|_{t=0} = \left(\hat{\tilde{\gamma}}_{ij}, \ln\hat{\alpha}, \hat{\beta}^i, \ln\hat{\phi}, \hat{\tilde{A}}_{ij}, \hat{K}, \hat{\theta}, \hat{\tilde{\Gamma}}^i, \hat{b}^i, \hat{A}_k, \hat{B}^i_k, \hat{D}_{kij}, \hat{P}_k\right), \tag{3.22}$$

is given by the purely AdS$_n$ part of the state variables.

3.2 Initial data for scalar field as matter

In the following, we will derive an explicit formula for the energy density E, constructed from a scalar field $\xi \in \mathcal{C}^\infty(M, \mathbb{R})$. As noted in [5], scalar fields are of utter importance, as we can use their energy density E as a parameter to tune the desired initial data.

The energy-momentum tensor, constructed from the Lagrangian density of a scalar field ξ, is given by

$$T_{\alpha\beta} = \partial_\alpha\xi\partial_\beta\xi - g_{\alpha\beta}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + V(\xi)\right). \tag{3.23}$$

By substituting Eq. (3.23) into Eq. (2.4a), we can derive the energy density of a scalar field ξ

$$\begin{aligned}
E = T_{\mu\nu}N^\mu N^\nu &= \partial_\mu\xi\partial_\nu\xi N^\mu N^\nu - \underbrace{g_{\mu\nu}N^\mu N^\nu}_{=-1}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\xi\partial_\sigma\xi + V(\xi)\right) \\
&= \partial_\mu\xi\partial_\nu\xi N^\mu N^\nu + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + V(\xi) \\
&= \left(N^\mu N^\nu + g^{\mu\nu}\right)\partial_\mu\xi\partial_\nu\xi - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + V(\xi) \\
&= \gamma^{\mu\nu}\partial_\mu\xi\partial_\nu\xi - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + V(\xi) \\
&= \gamma^{ij}\partial_i\xi\partial_j\xi - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + V(\xi) \\
&= \gamma^{ij}\partial_i\xi\partial_j\xi - \tfrac{1}{2}\left(g^{tt}\partial_t\xi\partial_t + 2g^{ti}\partial_t\xi\partial_i\xi + g^{ij}\partial_i\partial_j\xi\right) + V(\xi) \\
&= \gamma^{ij}\partial_i\xi\partial_j\xi - \frac{1}{2}\left(-\frac{1}{\alpha^2}\partial_t\xi\partial_t\xi + 2\frac{\beta^i}{\alpha^2}\partial_t\xi\partial_i\xi + \gamma^{ij}\partial_i\xi\partial_j\xi - \frac{1}{\alpha^2}\beta^i\beta^j\partial_i\xi\partial_j\xi\right) + V(\xi) \\
&= \frac{1}{2}\gamma^{ij}\partial_i\xi\partial_j\xi + \frac{1}{2\alpha^2}\left(\partial_t\xi\partial_t\xi - 2\beta^i\partial_i\xi\partial_t\xi + \beta^i\beta^j\partial_i\xi\partial_j\xi\right) + V(\xi)
\end{aligned}$$

$$= \frac{1}{2}\gamma^{ij}\partial_i\xi\partial_j\xi + \frac{1}{2\alpha^2}\left[\left(\partial_t - \beta^i\partial_i\right)\xi\right]^2 = \frac{1}{2}\zeta^{-2}\hat{\gamma}^{ij}\partial_i\xi\partial_j\xi + V(\xi), \tag{3.24}$$

where we have set $\left(\partial_t - \beta^i\partial_i\right)\xi\Big|_{t=0} = 0$. If one only sets $\partial_t\xi\Big|_{t=0} = 0$, as was done in Eq. (35) in [5], there is one term missing. Furthermore, Eq. (35) in [5] is missing a factor of 1/2. Plugging the energy density (3.24) into the Hamilton constraint (3.12), we get the initial value problem that needs to be solved

$$\begin{aligned}\hat{D}_i\hat{D}^i\zeta + \left(\frac{n-1}{2} - \frac{5-n}{2}\hat{D}_i\ln\zeta\hat{D}^i\ln\zeta + \frac{1}{n-2}8\pi\hat{\gamma}^{ij}\partial_i\xi\partial_j\xi\right)\zeta \\ -\frac{1}{n-2}\left(\frac{(n-1)(n-2)}{2} - 8\pi V(\xi)\right)\zeta^3 = 0 \quad \text{on} \quad \Sigma_0 \\ \zeta = 1 \quad \text{on} \quad \partial\Sigma_0,\end{aligned} \tag{3.25}$$

As the scalar field ξ on the initial data slice is completely arbitrary, we can choose whatever suits us best. Furthermore, for the sake of simplicity, we can even restrict ourselves to free and massless fields by setting $V(\xi) = 0$.

Let us at last write down the explicit formulas for the derivatives $\hat{D}_i\hat{D}^i\zeta$ and $\hat{D}_i\ln\zeta\hat{D}^i\ln\zeta$ in three and four dimensions. The Derivatives for an AdS_3 spacetime are given by

$$\begin{aligned}\hat{D}_i\hat{D}^i\zeta &= \hat{\gamma}^{\rho\rho}\partial_\rho^2\zeta + \hat{\gamma}^{\chi\chi}\partial_\chi^2\zeta - \hat{\gamma}^{\rho\rho}\partial_\rho\zeta\hat{\Gamma}^\rho_{\rho\rho} - \hat{\gamma}^{\rho\rho}\partial_\chi\zeta\hat{\Gamma}^\chi_{\rho\rho} \\ &\quad - \hat{\gamma}^{\chi\chi}\partial_\rho\zeta\hat{\Gamma}^\rho_{\chi\chi} - \hat{\gamma}^{\chi\chi}\partial_\chi\zeta\hat{\Gamma}^\chi_{\chi\chi} \\ &= \hat{\gamma}^{\rho\rho}\partial_\rho^2\zeta + \hat{\gamma}^{\chi\chi}\partial_\chi^2\zeta - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^\rho_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^\rho_{\chi\chi}\right)\partial_\rho\zeta \\ &\quad - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^\chi_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^\chi_{\chi\chi}\right)\partial_\chi\zeta \\ &= \hat{\gamma}^{\rho\rho}\partial_\rho^2\zeta + \hat{\gamma}^{\chi\chi}\partial_\chi^2\zeta - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^\rho_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^\rho_{\chi\chi}\right)\partial_\rho\zeta \\ &= q^2a\partial_\rho^2\zeta + \frac{q^2}{\rho^2}\partial_\chi^2\zeta - \left(\frac{qa}{\ell} + \frac{q^3}{\ell} - \frac{\rho q^2}{L^2} - \frac{qa}{\rho}\right)\partial_\rho\zeta,\end{aligned} \tag{3.26}$$

where we have used in the second to the last line the results from the Jupyter notebook F.1, and

$$\hat{D}_i\ln\zeta\hat{D}^i\ln\zeta = q^2a\partial_\rho\ln\zeta\partial_\rho\ln\zeta + \frac{q^2}{\rho^2}\partial_\chi\ln\zeta\partial_\chi\ln\zeta, \tag{3.27}$$

where we have used the spatial AdS metric $\hat{\gamma}_{ij}$. The Derivatives for an AdS_4 spacetime are given by

$$\begin{aligned}\hat{D}_i\hat{D}^i\zeta = &\ \hat{\gamma}^{\rho\rho}\partial_\rho^2\zeta + \hat{\gamma}^{\chi\chi}\partial_\chi^2\zeta + \hat{\gamma}^{\theta\theta}\partial_\theta^2\zeta \\ &- \hat{\gamma}^{\rho\rho}\partial_\rho\zeta\hat{\Gamma}^\rho_{\rho\rho} - \hat{\gamma}^{\rho\rho}\partial_\chi\zeta\hat{\Gamma}^\chi_{\rho\rho} - \hat{\gamma}^{\rho\rho}\partial_\theta\zeta\hat{\Gamma}^\theta_{\rho\rho} \\ &- \hat{\gamma}^{\chi\chi}\partial_\rho\zeta\hat{\Gamma}^\rho_{\chi\chi} - \hat{\gamma}^{\chi\chi}\partial_\chi\zeta\hat{\Gamma}^\chi_{\chi\chi} - \hat{\gamma}^{\chi\chi}\partial_\theta\zeta\hat{\Gamma}^\theta_{\chi\chi} \\ &- \hat{\gamma}^{\theta\theta}\partial_\rho\zeta\hat{\Gamma}^\rho_{\theta\theta} - \hat{\gamma}^{\theta\theta}\partial_\chi\zeta\hat{\Gamma}^\chi_{\theta\theta} - \hat{\gamma}^{\theta\theta}\partial_\theta\zeta\hat{\Gamma}^\theta_{\theta\theta} \\ = &\ \hat{\gamma}^{\rho\rho}\partial_\rho^2\zeta + \hat{\gamma}^{\chi\chi}\partial_\chi^2\zeta + \hat{\gamma}^{\theta\theta}\partial_\theta^2\zeta \\ &- \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^\rho_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^\rho_{\chi\chi} + \hat{\gamma}^{\theta\theta}\hat{\Gamma}^\rho_{\theta\theta}\right)\partial_\rho\zeta\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^{\chi}_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^{\chi}_{\chi\chi} + \hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\chi}_{\theta\theta}\right)\partial_\chi\zeta \\
&\quad - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^{\theta}_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^{\theta}_{\chi\chi} + \hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\theta}_{\theta\theta}\right)\partial_\theta\zeta \\
&= \hat{\gamma}^{\rho\rho}\partial^2_\rho\zeta + \hat{\gamma}^{\chi\chi}\partial^2_\chi\zeta + \hat{\gamma}^{\theta\theta}\partial^2_\theta\zeta \\
&\quad - \left(\hat{\gamma}^{\rho\rho}\hat{\Gamma}^{\rho}_{\rho\rho} + \hat{\gamma}^{\chi\chi}\hat{\Gamma}^{\rho}_{\chi\chi} + \hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\rho}_{\theta\theta}\right)\partial_\rho\zeta - \hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\chi}_{\theta\theta}\partial_\chi\zeta \\
&= q^2 a\partial^2_\rho\zeta + \frac{q^2}{\rho^2}\partial^2_\chi\zeta + \frac{q^2}{\rho^2\sin^2\chi}\partial^2_\theta\zeta \\
&\qquad\qquad - \left(\frac{qa}{\ell} + \frac{q^3}{\ell} - \frac{\rho q^2}{L^2} - \frac{2qa}{\rho}\right)\partial_\rho\zeta + \frac{q^2\cot\chi}{\rho^2}\partial_\chi\zeta,
\end{aligned}
\tag{3.28}
$$

where we have used in the second to the last line the results from the Jupyter notebook F.2, and

$$
\hat{D}_i \ln\zeta \hat{D}^i \ln\zeta = q^2 a\partial_\rho \ln\zeta\partial_\rho \ln\zeta + \frac{q^2}{\rho^2}\partial_\chi \ln\zeta\partial_\chi \ln\zeta + \frac{q^2}{\rho^2\sin^2\chi}\partial_\theta \ln\zeta\partial_\theta \ln\zeta, \tag{3.29}
$$

where we have used the spatial AdS metric $\hat{\gamma}_{ij}$.

Finally, we have found physical relevant initial data for a matter-free distribution, while the second-order, time-independent elliptic PDE (3.25) still needs to be solved. As we first aim to numerically solve the FO-CCZ4 system (2.69a)-(2.69k) with the gauge fixing (2.82), (2.88a)-(2.88b) for a static vacuum asymptotic AdS spacetime, we will postpone finding initial data for a symmetric scalar field for the moment.

Chapter 4

The Boundary Behaviour for an Asymptotically AdS$_n$ Spacetime

"Holographic duality between gauge theory and gravity
is the best we have now for a nonperturbative
description of quantum gravity." [26]
Edward Witten
Physicist

Even though Anti-de Sitter spacetimes can be characterised as maximally symmetric spacetimes which are geodesically complete, they will not be globally hyperbolic. Consequently, an Anti-de Sitter spacetime does not admit a Cauchy surface, i.e. an achronal subset of the spacetime, which is met by every inextendible causal curve. As we want the entire manifold to be determined by some specified initial data on the initial Cauchy surface Σ_0 at $t = 0$, the spacetime must be globally hyperbolic. Only in this case, one can find a globally unique solution. Therefore, to obtain a well-posed, unique and deterministic initial value problem for the FO-CCZ4 system (2.69a)-(2.69k) with the gauge fixing (2.82), (2.88a)-(2.88b) for an asymptotically Anti-de Sitter spacetime, we need to specify boundary conditions at spacelike infinity, i.e. $r \to \infty$.

Before we state the boundary conditions for the variables of state vector $\mathbf{Q}$, we will give a short introduction to Anti-de Sitter spacetimes. Most of the results will be taken from the book [4] by Erdmenger and the paper [5] by Bantilan, Pretorius and Gubser, whereas the ideas on how to obtain a well-posed and deterministic Cauchy problem for a non-globally hyperbolic Anti-de Sitter spacetimes can be found in the paper [35].

4.1 Maximally symmetric spacetimes

As symmetries are of utter importance in physics, we will focus on maximally symmetric solutions to the vacuum Einstein field equations, i.e. solutions without any matter. We can characterise symmetries of spacetimes locally by so-called Killing vector fields $X \in \Gamma^\infty(TM)$, which are vector fields satisfying the following condition

$$\mathcal{L}_g X = 0. \tag{4.1}$$

For further characterisations of Killing vector fields, see Appendix D. Furthermore, we want to know how many linear independent Killing vector fields, and therefore symmetries, a manifold can have. Moreover, exactly those spacetimes with maximal linear independent Killing vector fields are of interest and we call them *maximally symmetric spacetimes*. As noted in [4], an n-dimensional manifold can have at most $n(n+1)/2$ independent symmetries. The manifold M we are interested in will be pseudo-Riemannian, putting it more precisely Lorentzian, and we can find three maximally symmetric spacetimes depending on the sign of the Ricci scalar ${}^{n}R$:

- ${}^{(n)}R = 0$: Minkowski spacetime
- ${}^{(n)}R > 0$: de Sitter spacetime
- ${}^{(n)}R < 0$: Anti-de Sitter spacetime.

Maximally symmetric spacetimes will have the same curvature ${}^{(n)}R$ everywhere and we can see by taking the trace of the vacuum Einstein field equations, i.e.

$$ {}^{(n)}R = \frac{2\Lambda_n n}{n-2}, \tag{4.2} $$

that the cosmological constant Λ_n will be zero for Minkowski spacetimes, positive for de Sitter spacetimes, and negative for Anti-de Sitter spacetimes. We refer the attentive reader to [51], [31] or [40], respectively, for some introduction to Minkowski and de Sitter spacetimes.

4.2 The Anti-de Sitter spacetime AdS$_n$

As mentioned in the latter section, the maximally symmetric spacetime with negative scalar curvature ${}^{(n)}R$ and therefore negative cosmological constant $\Lambda_n < 0$ is called an Anti-de Sitter spacetime. Such spacetimes can occur as the maximally symmetric solution to the vacuum Einstein equations

$$ {}^{(n)}R_{\mu\nu} - \frac{1}{2}{}^{(n)}R g_{\mu\nu} + \Lambda_n g_{\mu\nu} = 0. \tag{4.3} $$

Geometrically, we can view the AdS$_n$ spacetime as the hypersurface defined by

$$ -\left(x^0\right)^2 + \sum_{i=1}^{n-1}\left(x^i\right)^2 - \left(x^4\right)^2 = -L^2, \tag{4.4} $$

embedded in a higher-dimensional pseudo-Euclidean spacetime $\mathbb{R}^{n-1,2}$ with two time dimensions and the metric

$$ g_{\mathbb{R}^{n-1,2}} = -\left(dx^0\right)^2 + \sum_{i=1}^{n-1}\left(dx^i\right)^2 - \left(dx^4\right)^2. \tag{4.5} $$

For $x^i = \text{const.}$, the temporal coordinates x^0 and x^4 define a circle by

$$ \left(x^0\right)^2 + \left(x^4\right)^2 = \text{const.} \tag{4.6} $$

This way, the AdS_n spacetime is homeomorphic to $S^1 \times \mathbb{R}^{n-1}$ [56], where the product topology is given by the subspace topology of S^1 and the standard topology of $\mathbb{R}^{n-1}$. As the hyperbolic space $\mathbb{H}^{n-1}$, the unique, simply connected and non-compact $(n-1)$-dimensional Riemannian manifold with sectional curvature -1, is diffeomorphic to $\mathbb{R}^{n-1}$, the product topology of AdS_n spacetime is given by the product topology of $S^1 \times \mathbb{H}^{n-1}$.

But, as the temporal coordinates x^0 and x^4 define a circle in the (x^0, x^4)-plane, closed timelike curves (CTC) exist. For closed timelike curves, light cones are arranged so that particles moving in free fall on a CTC can loop back to the same spacetime event on the manifold. As this is very uncomfortable for us as we do not yet know if such CTC's are possible, we will pass to the universal covering spacetime denoted by $\widetilde{AdS_n}$, which is defined in the same way as AdS_n but instead has the product topology of

$$\mathbb{R} \times \mathbb{H}^{n-1}, \tag{4.7}$$

where we have unwrapped the circle S^1 to $\mathbb{R}$. This way, time t does not run from 0 to $2\pi L$, but instead from $-\infty$ to ∞.

The AdS_n hyperboloid can be parametrized by a chart (U, x) that cover the spacetime by

$$\begin{aligned}
x^0 &= \sqrt{r^2 + L^2} \sin(t/L) \\
x^1 &= r \sin\theta_{n-2} \cdots \sin\theta_4 \sin\theta_3 \sin\theta_2 \cos\theta_1 \\
x^2 &= r \sin\theta_{n-2} \cdots \sin\theta_4 \sin\theta_3 \sin\theta_2 \sin\theta_1 \\
x^3 &= r \sin\theta_{n-2} \cdots \sin\theta_4 \sin\theta_3 \cos\theta_2 \\
&\vdots \\
x^{n-2} &= r \sin\theta_{n-2} \cos\theta_{n-3} \\
x^{n-1} &= r \cos\theta_{n-2} \\
x^n &= \sqrt{r^2 + L^2} \cos(t/L),
\end{aligned} \tag{4.8}$$

where $(t, r, \theta_{n-2}, \ldots, \theta_1) \in (-\infty, \infty) \times (0, \infty) \times (0, \pi)^{n-3} \times (0, 2\pi)$ are the coordinates. By plugging Eq. (4.8) into Eq. (4.5), we get the solution of the vacuum Einstein field equation by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{n-2} \equiv \hat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{4.9}$$

where we have defined $f(r) := 1 + r^2/L^2$ and where $d\Omega^2_{n-2}$ is the line element of the $(n-2)$-unit sphere S^{n-2} parametrized by the angels $(\theta_{n-2}, \ldots, \theta_1)$. Furthermore, the Eq. 4.2 corresponds for $n = 5$ with the metric given in [5], and L is the AdS radius related to the cosmological constant via

$$\Lambda_n = \frac{-(n-1)(n-2)}{2L^2}. \tag{4.10}$$

As the AdS_n boundary is important for the AdS/CFT duality and for the sake of simplicity for further calculations, let us compactify the "radial" coordinate r to a finite value ρ, as used in [5], by

$$r = \frac{\rho}{1 - \rho/\ell}, \tag{4.11}$$

where ℓ is an arbitrary compactification scale that is independent of the AdS$_n$ radius L. This way, the AdS$_n$ boundary is at the finite value $\rho = \ell$, even though we will set $\ell = 1$ in the code, but write it out explicitly in all the calculations in order to dimensionally check the equations. Transforming to the coordinates $x^\mu = (t, \rho, \theta_{n-2}, \ldots, \theta_1)$, we can write the line element of AdS$_n$ by

$$ds^2 = -\left(1 + \frac{\rho^2}{q^2 L^2}\right) dt^2 + \frac{1}{q^2\left(q^2 + \frac{\rho^2}{L^2}\right)} d\rho^2 + \frac{\rho^2}{q^2} d\Omega^2_{n-2}, \tag{4.12}$$

where we have defined the smooth scalar field

$$q = 1 - \frac{\rho}{\ell} \in \mathcal{C}^\infty(M, \mathbb{R}) \tag{4.13}$$

with simple zero at the AdS$_n$ boundary $\rho = \ell$. As the AdS$_n$ boundary is of utter importance for us, let us state the most important properties in the following section.

4.3 The AdS$_n$ boundary

The AdS$_n$ boundary is given by the set of all lines on the light-cone originating from the point $0 \in \mathbb{R}^{n-1,2}$ and is properly defined in [4] by

$$\partial\text{AdS}_n = \left\{[x]\middle| x \in \mathbb{R}^{n-1,2}, x \neq 0, -\left(x^0\right)^2 + \sum_{i=1}^{n-1}\left(x^i\right)^2 - \left(x^4\right)^2 = 0\right\}, \tag{4.14}$$

where $[x]$ is an equivalence class such that two parametrizations are equivalent if they only differ by a real number. Furthermore, as noted in [4], the topology of ∂AdS$_n$ is given by the quotient topology of

$$\left(S^1 \times S^{n-2}\right)/\mathbb{Z}_2, \tag{4.15}$$

where we take the quotient space as $x \in \mathbb{R}^{n-1,2}$ and $-x \in \mathbb{R}^{n-1,2}$ on ∂AdS$_n$ are different points in $S^1 \times S^{n-2}$, but the same points in ∂AdS$_n$. By conformally compactifing AdS$_n$ with the coordinate transformation

$$r/L = \tan R \tag{4.16}$$

as discussed in [5], the infinite region $r \in [0, \infty]$ is compactified to the region $R \in [0, \pi/2]$. Therefore, AdS$_n$ is conformal to one-half of the Einstein static universe. Furthermore, we note that the time coordinate has not been compactified and thus spatial infinity runs along the time coordiate as one can see in Fig. 4.1. Now, an essential consequence is that spatial infinity is timelike and causally connected to the interior [5].

As AdS$_n$ is not globally hyperbolic, a Cauchy surface does not exist. Therefore, we can not just set initial date at $t = 0$, but have to specify boundary conditions at spacelike infinity. These boundary conditions will therefore be time-dependent and derived in 4.5.

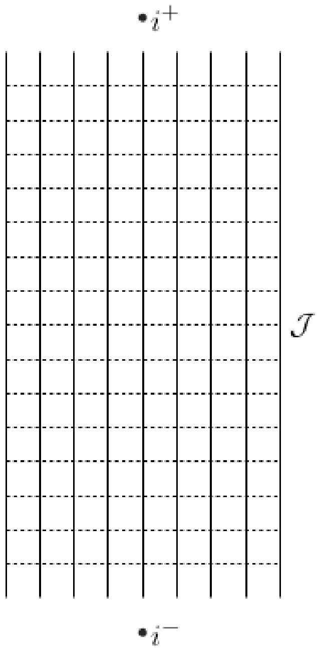

Figure 4.1: The conformal diagram of an Anti-de Sitter spacetime take from [5]. The boundary is the timelike surface $\mathcal{J}$. Dashed lines represent constant time t surfaces, and solid lines are constant r surfaces. Past and future timelike infinity are given by i^- and i^+, respectively. Note that the angles have been suppressed and that we have to add a uni-sphere S^{n-2} at each point of the diagram.

4.4 The asymptotically AdS$_n$ spacetime

As we want to solve the Cauchy problem of the Einstein equations for an asymptotically AdS$_n$ spacetime, let us state the main properties in the following.

An asymptotically AdS$_n$ spacetime has the same structure as the AdS$_n$ spacetime near the boundary, i.e. at $q = 0$ or $\rho = \ell$. And, as we want to find a spacetime that can be characterised by the metric $g_{\mu\nu}$ within the bulk and the AdS$_n$ metric $\hat{g}_{\mu\nu}$ on the boundary, we need to find physical fall-off conditions near the boundary $\rho = \ell$ such that we can decompose the full metric $g_{\mu\nu}$ near the horizon by

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}. \tag{4.17}$$

The matter-free asymptotics of $h_{\mu\nu}$ for an asymptotically AdS$_n$ spacetime were found in [32] by using that the boundary conditions are invariant under the AdS$_n$ symmetry group. Now, taken from [5], the physical fall-off conditions $h_{\mu\nu}$ near the boundary are given by

$$\begin{aligned}
h_{rr}(t, r, \theta_{n-2} \dots \theta_1) &= f_{rr}(t, \theta_{n-2} \dots \theta_1)\frac{1}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}) \\
h_{rm}(t, r, \theta_{n-2} \dots \theta_1) &= f_{rm}(t, \theta_{n-2} \dots \theta_1)\frac{1}{r^{n}} + \mathcal{O}(r^{-(n+1)}) \\
h_{mn}(t, r, \theta_{n-2} \dots \theta_1) &= f_{mn}(t, \theta_{n-2} \dots \theta_1)\frac{1}{r^{n-3}} + \mathcal{O}(r^{-(n-2)}),
\end{aligned} \tag{4.18}$$

where m, n are the non-radial coordiantes $(t, \theta_{n-2} \dots \theta_1)$. As we want to use the compactified "radial coordinate", let us transform the fall-off metric $h_{\mu\nu}$ from the coordinates $(t, r, \theta_{n-2} \dots \theta_1)$ to the coordinates $(t, \rho, \theta_{n-2} \dots \theta_1)$. The transformation law of the metric tensor $g_{\mu\nu}$ is given by

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}, \tag{4.19}$$

and due to Eq. (4.17), the fall-off metric $h_{\mu\nu}$ transforms in exaclty the same way

$$h_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} h_{\mu\nu}. \tag{4.20}$$

Furthermore, the spatial part of the full metric $g_{\mu\nu}$ transforms as

$$\begin{aligned} \gamma_{kl} &= \frac{\partial x^\mu}{\partial x'^k} \frac{\partial x^\nu}{\partial x'^l} g_{\mu\nu} \\ &= \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} \gamma_{ij} + 2 \frac{\partial x^i}{\partial x'^k} \frac{\partial x^0}{\partial x'^l} \beta_i + \frac{\partial x^0}{\partial x'^k} \frac{\partial x^0}{\partial x'^l} g_{00}, \end{aligned} \tag{4.21}$$

where we have used $g_{ij} = \gamma_{ij}$ and $g_{i0} = \beta_i$ as in Eq. A.85. As we are only interested in coordinate transformation that does not involve a time transformation, we can simplify the latter equation to

$$\gamma_{kl} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} \gamma_{ij}. \tag{4.22}$$

Now, as $g_{ij} = \gamma_{ij}$, we can write

$$\gamma_{ij} = \hat{\gamma}_{ij} + h_{ij}, \tag{4.23}$$

where $\hat{\gamma}_{ij}$ and h_{ij} are the spatial parts, respectively, of the AdS$_n$ metric and the fall-off metric $h_{\mu\nu}$. Therefore, we know that the spatial part h_{ij} transforms in exactly the same way

$$h_{kl} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} h_{ij} \tag{4.24}$$

and we can finally transform the fall-off metric components to

$$\begin{aligned} h_{\rho\rho}(t, \rho, \theta_{n-2} \dots \theta_1) &= \frac{\partial r}{\partial \rho} \frac{\partial r}{\partial \rho} h_{rr}(t, r(\rho), \theta_{n-2} \dots \theta_1) \\ &= \frac{1}{(1 - \rho/\ell)^4} h_{rr}(t, r(\rho), \theta_{n-2} \dots \theta_1) \\ &= \frac{(1 - \rho/\ell)^{n-3}}{\rho^{n+1}} f_{\rho\rho} + \mathcal{O}\left(\frac{(1 - \rho/\ell)^{n-2}}{\rho^{n+2}} \right) \\ &= \frac{q^{n-3}}{\rho^{n+1}} f_{\rho\rho} + \mathcal{O}\left(\frac{q^{n-2}}{\rho^{n+2}} \right) \\ &= q^{n-3} f_{\rho\rho} + \mathcal{O}(q^{n-2}) \\ h_{\rho m}(t, \rho, \theta_{n-2} \dots \theta_1) &= \frac{\partial r}{\partial \rho} h_{rm}(t, r, \theta_{n-2} \dots \theta_1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-\rho/\ell)^2} h_{rm}(t, r(\rho), \theta_{n-2} \dots \theta_1) \\
&= \frac{(1-\rho/\ell)^{n-2}}{\rho^n} f_{\rho m} + \mathcal{O}\left(\frac{(1-\rho/\ell)^{n-1}}{\rho^{n+1}}\right) \\
&= \frac{q^{n-2}}{\rho^n} f_{\rho m} + \mathcal{O}\left(\frac{q^{n-1}}{\rho^{n+1}}\right) \\
&= q^{n-2} f_{\rho m} + \mathcal{O}(q^{n-1}) \\
h_{mn}(t, \rho, \theta_{n-2} \dots \theta_1) &= h_{mn}(t, r, \theta_{n-2} \dots \theta_1) \\
&= \frac{(1-\rho/\ell)^{n-3}}{\rho^{n-3}} f_{mn} + \mathcal{O}\left(\frac{(1-\rho/\ell)^{n-2}}{\rho^{n-2}}\right) \\
&= \frac{q^{n-3}}{\rho^{n-3}} f_{mn} + \mathcal{O}\left(\frac{q^{n-2}}{\rho^{n-2}}\right) \\
&= q^{n-3} f_{mn} + \mathcal{O}(q^{n-2}),
\end{aligned} \tag{4.25}
$$

where we have renamed in the third line the f-functions $f_{rr} \rightarrow f_{\rho\rho}$ and $f_{rm} \rightarrow f_{\rho m}$ as these are just labels and wrote them for the sake of simplicity without coordinates, used Eq. 4.13 in the fourth line and performed a Taylor-expansion in q around the boundary $q = 0$ in the last line. As mentioned in [5], the fall-off conditions (4.18) and (4.25) hold for vacuum asymptotically AdS spacetimes and for spacetimes containing localized matter distributions with sufficiently rapid fall-off near the boundary. To use general energy-matter distributions, see the fall-off conditions obtained in the references noted by Gubser et al. in [5].

As the coordinates of the components functions $h_{ij}(t, \rho, \theta_{n-2} \dots \theta_1)$ are clear, we will omit them from here on and only write h_{ij}.

4.5 Calculation of the timelike boundary conditions

To find solutions for the Cauchy problem of the FO-CCZ4 system in an asymptotically AdS$_n$ spacetime, we need to specify boundary conditions at timelike infinity for the variables of the state vector $\mathbf{Q}$. In order to gain some intuition on how these fields behave near the boundary $\rho = \ell$, we will write the spacetime metric $g_{\mu\nu}$ as

$$
g_{\mu\nu} = \hat{g}_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{4.26}
$$

where $\epsilon h_{\mu\nu}$ represents a small perturbation, while ϵ is an auxiliary variable according to which we expand. By using Eq. (4.26), we can derive the boundary behaviour of the evolution variables of the state vector $\mathbf{Q}$, writing them as a power series in ϵ.

We will use the regularization scheme proposed by Gubser et al. [5], although they have not proven that it is a correct and complete characterization of the asymptotically AdS$_n$ boundary behaviour, but rather argued that it is consistent with the considered initial data, as they obtained stable and convergent numerical solutions. However, as we have not yet obtained any numerical output, we can not say if this holds for the considered FO-CCZ4 system, i.e. Eqs. (2.69a) - (2.69k).

The regularization scheme by Gubser et al. [5] is given by writing the fall-off metric components $h_{\mu\nu}$ as a power series in q around the boundary $q = 0$, i.e.

$$h_{\mu\nu} = \sum_{i=0}^{\infty} h_{\mu\nu,i} q^i. \tag{4.27}$$

For a regular solution, a solution that has the desired fall-off as given in Eq. (4.25), the first $n-3$ terms for $h_{\rho\rho}$, the first $n-2$ terms for $h_{\rho m}$ and the first $n-3$ terms for h_{mn} on the right hand sight have to vanish, while the $n-2$, $n-1$ and $n-2$ term describes, respectively, the leading-order behaviour of the vacuum boundary conditions. This seems that we would need to supply, respectively, $n-2$, $n-1$ and $n-2$ boundary conditions for the fall-off conditions. Following the discussion in [5], the next step will be to define a new evolution variable $\bar{h}_{\mu\nu}$ via

$$h_{\rho\rho} = q^{n-4}\bar{h}_{\rho\rho}, \quad h_{\rho m} = q^{n-3}\bar{h}_{\rho m} \quad \text{and} \quad h_{mn} = q^{n-4}\bar{h}_{mn}, \tag{4.28}$$

where we demand that the new evolution variable $\bar{h}_{\mu\nu}$ satisfies the Dirichlet boundary problem

$$\bar{h}_{\mu\nu}(t, q = 0, \theta_{n-2} \dots \theta_1) = 0 \tag{4.29}$$

at $q = 0$. Inserting Eq. (4.28) into Eq. (4.27), we can write

$$\bar{h}_{\mu\nu} = \dots + h_{\mu\nu,n-2}q^{-1} + h_{\mu\nu,n-1} + h_{\mu\nu,n}q + \dots$$

Now, we see that if we choose regular initial data for $\bar{h}_{\mu\nu}$ that fulfils

$$\bar{h}_{\mu\nu}(t = 0, q = 0, \theta_{n-2} \dots \theta_1) = 0, \tag{4.30}$$

the first terms with undesired fall-off at $t = 0$ vanish.

Using the newly defined evolution variable $\bar{h}_{ij}$ that asymptotically falls of with $\bar{h}_{\mu\nu} \sim q$, we can write the perturbated metric field $g_{\mu\nu}$ as

$$g_{\mu\nu} = \begin{pmatrix} \hat{g}_{tt} + \epsilon q^{n-4}\bar{h}_{tt} & \epsilon q^{n-3}\bar{h}_{t\rho} & \epsilon q^{n-4}\bar{h}_{tb} \\ \epsilon q^{n-3}\bar{h}_{t\rho} & \hat{g}_{\rho\rho} + \epsilon q^{n-4}\bar{h}_{\rho\rho} & \epsilon q^{n-3}\bar{h}_{\rho b} \\ \epsilon q^{n-4}\bar{h}_{at} & \epsilon q^{n-3}\bar{h}_{a\rho} & \hat{g}_{ab} + \epsilon q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab} \end{pmatrix}, \tag{4.31}$$

where we have, similiar as Gubser et al. in [5], added $\rho^2 g_{ab,S^{n-2}}$ in order to ensure regularity at the origin $\rho = 0$. By using this ansatz, motivated by perturbation theory, we can gain some intuition on how the $n^3/2 + n^2 + 5n/2$ variables of the state vector $\mathbf{Q}$ behave near the boundary.

As we are only considering solutions that preserve a $SO(n-2)$ symmetry, let us apply the $SO(1)$, $SO(2)$ and $SO(3)$ symmetry to the metric tensor field in Eq. (4.31), respectively, for an AAdS$_3$, AAdS$_4$ and AAdS$_5$ spacetime.

The ansatz for an asymptotically AdS$_3$ spacetime

The components of the perturbated metric $g_{\mu\nu}$ around the purely AdS$_3$ metric $g_{\mu\nu}$ by some small value $\epsilon\bar{h}_{\mu\nu}$ is given by

$$g_{tt} = \hat{g}_{tt} + \epsilon q^{-1}\bar{h}_{tt}$$

$$
\begin{aligned}
g_{t\rho} &= \epsilon \bar{h}_{t\rho} \\
g_{t\chi} &= \epsilon q^{-1} \bar{h}_{t\chi} \\
g_{\rho\rho} &= \hat{g}_{\rho\rho} + \epsilon q^{-1} \bar{h}_{\rho\rho} \\
g_{\rho\chi} &= \epsilon \bar{h}_{\rho\chi} \\
g_{\chi\chi} &= \hat{g}_{\chi\chi} + \epsilon q^{-1} \rho^2 \bar{h}_{\chi\chi},
\end{aligned} \tag{4.32}
$$

where we have used $(t, \rho, \chi) \in (-\infty, \infty) \times (0, \ell) \times (0, 2\pi)$ as coordinates.

The ansatz for an asymptotically AdS_4 spacetime
The components of the perturbated metric $g_{\mu\nu}$ around the purely AdS_4 metric $g_{\mu\nu}$ by some small value $\epsilon\bar{h}_{\mu\nu}$ with an $SO(2)$ symmetry is given by

$$
\begin{aligned}
g_{tt} &= \hat{g}_{tt} + \epsilon \bar{h}_{tt} \\
g_{t\rho} &= \epsilon q \bar{h}_{t\rho} \\
g_{t\chi} &= \epsilon \bar{h}_{t\chi} \\
g_{t\theta} &= 0 \\
g_{\rho\rho} &= \hat{g}_{\rho\rho} + \epsilon \bar{h}_{\rho\rho} \\
g_{\rho\chi} &= \epsilon q \bar{h}_{\rho\chi} \\
g_{\rho\theta} &= 0 \\
g_{\chi\chi} &= \hat{g}_{\chi\chi} + \epsilon \rho^2 \bar{h}_{\chi\chi} \\
g_{\chi\theta} &= 0 \\
g_{\theta\theta} &= \hat{g}_{\theta\theta} + \epsilon \rho^2 \sin^2\chi \bar{h}_{\theta\theta},
\end{aligned} \tag{4.33}
$$

where we have used $(t, \rho, \chi, \theta) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi) \times (0, 2\pi)$ as coordinates.

The ansatz for an asymptotically AdS_5 spacetime
The components of the perturbated metric $g_{\mu\nu}$ around the purely AdS_5 metric $g_{\mu\nu}$ by some small value $\epsilon\bar{h}_{\mu\nu}$ with an $SO(3)$ symmetry is given by

$$
\begin{aligned}
g_{tt} &= \hat{g}_{tt} + \epsilon q \bar{h}_{tt} \\
g_{t\rho} &= \epsilon q^2 \bar{h}_{t\rho} \\
g_{t\chi} &= \epsilon q \bar{h}_{t\chi} \\
g_{t\theta} &= 0 \\
g_{t\phi} &= 0 \\
g_{\rho\rho} &= \hat{g}_{\rho\rho} + \epsilon q \bar{h}_{\rho\rho} \\
g_{\rho\chi} &= \epsilon q^2 \bar{h}_{\rho\chi} \\
g_{\rho\theta} &= 0 \\
g_{\rho\phi} &= 0 \\
g_{\chi\chi} &= \hat{g}_{\chi\chi} + \epsilon \rho^2 q \bar{h}_{\chi\chi} \\
g_{\chi\theta} &= 0 \\
g_{\chi\phi} &= 0
\end{aligned}
$$

$$\begin{aligned}
g_{\theta\theta} &= \hat{g}_{\theta\theta} + \epsilon\rho^2 \sin^2\chi q\bar{h}_\psi \\
g_{\theta\phi} &= 0 \\
g_{\phi\phi} &= \hat{g}_{\phi\phi} + \epsilon\rho^2 \sin^2\chi \sin^2\theta q\bar{h}_\psi,
\end{aligned} \tag{4.34}$$

where we have used $(t, \rho, \chi, \theta, \phi) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi)^2 \times (0, 2\pi)$ as coordinates. As we are considering solutions that preserve an $SO(3)$ symmetry, we can use a single term that rotates the S^2 [5].

As the newly defined evolution variables $\bar{h}_{\mu\nu} \in \mathcal{C}^\infty(M, \mathbb{R})$ have to satisfy a couple of constraint equations, let us state them next.

Boundary Conditions for the $\bar{h}_{\mu\nu}$ functions

Now, by using the regularized fall-off metric variables $\bar{h}_{\mu\nu}$, we can fully capture the boundary conditions given in Eq. (4.18) or Eq. (4.25) by a Dirichlet boundary problem

$$\begin{aligned}
\bar{h}_{\rho\rho}\Big|_{\rho=\ell} &= 0 \\
\bar{h}_{\rho m}\Big|_{\rho=\ell} &= 0 \\
\bar{h}_{mn}\Big|_{\rho=\ell} &= 0.
\end{aligned} \tag{4.35}$$

Furthermore, the origin and axis regularity conditions, taken from [5], are given, respectively, by

$$\begin{aligned}
\partial_\rho\bar{h}_{\rho\rho}\Big|_{\rho=0} &= 0 \\
\bar{h}_{\rho m}\Big|_{\rho=0} &= 0 \\
\partial_\rho\bar{h}_{mn}\Big|_{\rho=0} &= 0,
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
\partial_\chi\bar{h}_{\rho\rho}\Big|_{\chi=0,\pi} &= 0 \\
\partial_\chi\bar{h}_{\rho m'}\Big|_{\chi=0,\pi} &= 0 \\
\partial_\chi\bar{h}_{m'n'}\Big|_{\chi=0,\pi} &= 0 \\
\bar{h}_{\mu'\chi}\Big|_{\chi=0,\pi} &= 0,
\end{aligned} \tag{4.37}$$

where m', n' are non-radial coordiantes with $m', n' \neq \chi$ unless $m' = n'$ in the first line and $\mu' \neq \chi$.

As we have stated the perturbational ansatz of the full metric in Eq. (4.31), the precise ansatz of the full metric for an AAdS$_3$, AAdS$_4$ and AAdS$_5$ manifold is given, respectively, in Eq. (4.32), Eq. (4.33) and Eq. (4.34). The boundary, origin and regularity conditions for the evolution variables $\bar{h}_{\mu\nu}$ are written down, respectively, in Eq. (4.35), Eq. (4.36) and Eq. (4.37).

Now, we are ready to calculate the boundary conditions for the evolutions variables of the state vector $\mathbf{Q}$ in the following section.

Let us already note here that our Mathematica script [36] proves the following equations for a three- and four-dimensional manifold, i.e. setting $n = 3$ and $n = 4$. This gives us a good hint that the general formulas might be as well correct for $n > 4$.

4.5.1 The conformal factor ϕ

The conformal factor ϕ was first introduced in Chapter 2.3 for the derivation of the *BSSNOK* system to ensure stable evolution and remains a variable of the generalized FO-CCZ4 system. Let us, therefore, begin by calculating the boundary behaviour of ϕ by using the ansatz in Eq. (4.31).

Due to Eq. (A.85), i.e. $\hat{g}_{ij} = \hat{\gamma}_{ij}$, and Eq. (A.86) with a vanishing AdS$_n$ shift vector $\hat{\beta}$, i.e. $\hat{g}^{ij} = \hat{\gamma}^{ij}$, the spatial AdS$_n$ metric $\hat{\gamma}_{ij}$ and the inverse spatial AdS$_n$ metric $\hat{\gamma}^{ij}$ are given, respectively, by

$$\hat{\gamma}_{ij} = \operatorname{diag}\left(\frac{1}{q^2\left(q^2+\frac{\rho^2}{L^2}\right)}, \frac{\rho^2}{q^2} g_{ab,S^{n-2}}\right), \tag{4.38}$$

and

$$\hat{\gamma}^{ij} = \operatorname{diag}\left(q^2\left(q^2+\frac{\rho^2}{L^2}\right), \frac{q^2}{\rho^2} g^{ab}_{S^{n-2}}\right), \tag{4.39}$$

where $g_{ab,S^{n-2}}$ is the round metric of a $n-2$-dimensional unit sphere S^{n-2}. Note that, as the shift vector $\hat{\beta}_i = 0$, the dual shift vector $\hat{\beta}^i$, as it is defined by

$$\hat{\beta}^j = \hat{\gamma}^{ij}\hat{g}_{ti} = \hat{\gamma}^{ij}\hat{\beta}_i, \tag{4.40}$$

vanishes as well. Let furthermore $\gamma = \det(\gamma_{ij})$ be the determinant of the spatial part of the full metric $g_{\mu\nu}$,

$$\hat{\gamma} = \det(\hat{\gamma}_{ij}) = \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}\left(q^2+\frac{\rho^2}{L^2}\right)} \tag{4.41}$$

be the determinant of the spatial metric of the AdS$_n$ metric $\hat{g}_{\mu\nu}$ and h_{ij} be the spatial part of $h_{\mu\nu}$. By using the latter equations, the determinant of γ_{ij} near the boundary $\rho = \ell$ can be written as

$$\begin{aligned}
\det(\gamma_{ij}) = \det(\hat{\gamma}_{ij}+\epsilon h_{ij}) &= \det(\hat{\gamma}_{ij}) + \epsilon h_{ij}\frac{\partial \det\gamma_{ij}}{\partial\gamma_{ij}}\bigg|_{\gamma=\hat{\gamma}} + \frac{1}{2}\epsilon h_{kl}\epsilon h_{ij}\frac{\partial^2\det\gamma_{ij}}{\partial\gamma_{kl}\partial\gamma_{ij}}\bigg|_{\gamma=\hat{\gamma}} + ...\\
&= \det(\hat{\gamma}_{ij}) + \epsilon h_{ij}\det(\hat{\gamma}_{ij})\hat{\gamma}^{ij} + \mathcal{O}(\epsilon^2)\\
&= \det(\hat{\gamma}_{ij})\left(1+\epsilon\hat{\gamma}^{ij}h_{ij}\right) + \mathcal{O}(\epsilon^2)\\
&= \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}\left(q^2+\frac{\rho^2}{L^2}\right)}\left(1+\hat{\gamma}^{\rho\rho}\epsilon h_{\rho\rho}+\hat{\gamma}^{ab}\epsilon h_{ab}\right) + \mathcal{O}(\epsilon^2)\\
&= \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}\left(q^2+\frac{\rho^2}{L^2}\right)}\left(1+\hat{\gamma}^{\rho\rho}\epsilon h_{\rho\rho}+\hat{\gamma}^{ab}\epsilon h_{ab}\right) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}\left(q^2+\frac{\rho^2}{L^2}\right)}\left(1+q^2\left(q^2+\rho^2/L^2\right)\epsilon h_{\rho\rho}+\frac{q^2}{\rho^2}g^{ab}_{S^{n-2}}\epsilon h_{ab}\right)+\mathcal{O}(\epsilon^2)\\
&= \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}a}\left(1+q^{n-2}a\epsilon\bar{h}_{\rho\rho}+\frac{q^2}{\rho^2}g^{ab}_{S^{n-2}}q^{n-4}\rho^2 g_{ab,S^{n-2}}\epsilon\bar{h}_{ab}\right)+\mathcal{O}(\epsilon^2)\\
&= \frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{q^{2(n-1)}a}\left(1+q^{n-2}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)+\mathcal{O}(\epsilon^2), \qquad (4.42)
\end{aligned}
$$

where we have Taylor-expandet the determinant of γ_{ij} in ϵh_{ij} in the first line and set

$$a = q^2 + \frac{\rho^2}{L^2}. \tag{4.43}$$

By defining

$$h_{\gamma_{ij}} = q^{n-2}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right), \tag{4.44}$$

we can decompose Eq. (4.42) into a purely AdS$_n$ and a perturbational part

$$\det(\gamma_{ij}) = \det(\hat{\gamma}_{ij})\left(1+\epsilon h_{\gamma_{ij}}\right)+\mathcal{O}(\epsilon^2). \tag{4.45}$$

As the conformal factor ϕ is defined in Eq. (2.33c) by

$$\phi = \det\left(\gamma_{ij}\right)^{-\frac{1}{2(n-1)}}, \tag{4.46}$$

we can simplify this equation by using the binomial approximation $(1+x)^\alpha \approx 1+\alpha x$ if $x \in \mathbb{R}$ is small to

$$\phi = \hat{\gamma}^{-\frac{1}{2(n-1)}}\left(1-\frac{1}{2(n-1)}\epsilon h_{\gamma_{ij}}\right)+\mathcal{O}(\epsilon^2). \tag{4.47}$$

By defining the conformal factor $\hat{\phi}$ for an AdS$_n$ spacetime as

$$\hat{\phi} = \hat{\gamma}^{-\frac{1}{2(n-1)}} \tag{4.48}$$

and the perturbation h_ϕ for the conformal factor as

$$h_\phi = -\frac{1}{2(n-1)}h_{\gamma_{ij}} = -\frac{q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right), \tag{4.49}$$

we can write the boundary behaviour of the conformal factor ϕ as a purely AdS$_n$ part plus a small deviation, i.e. given by

$$\phi = \hat{\phi}\left(1+\epsilon h_\phi\right)+\mathcal{O}(\epsilon^2). \tag{4.50}$$

This equation tells us how the conformal factor ϕ behaves near the boundary for the perturbated AdS$_n$ metric as given in Eq. (4.31). In order to preserve the positivity of the conformal factor ϕ, we are time evolving the logarithm of the conformal factor. Let us, therefore, write down the corresponding boundary behaviour of the logarithm of the conformal factor. Thus,

$$\ln\phi = \ln\hat{\phi}+\ln\left(1+\epsilon h_\phi+\mathcal{O}(\epsilon^2)\right) = \frac{1}{2(n-1)}\ln\det(\hat{\gamma}^{ij})+\epsilon h_\phi+\mathcal{O}(\epsilon^2)$$

$$
\begin{aligned}
&= \frac{\ln\left(q^{2(n-1)}a\right)}{2(n-1)} - \frac{\ln\left(\rho^{2(n-2)}\det(g_{ab,S^{n-2}})\right)}{2(n-1)} \\
&\qquad\qquad\qquad - \frac{\epsilon q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) + \mathcal{O}(\epsilon^2) \\
&= \ln q + \frac{1}{2(n-1)}\ln a - \frac{n-2}{n-1}\ln\rho - \frac{1}{2(n-1)}\ln\det(g_{ab,S^{n-2}}) \\
&\qquad\qquad\qquad - \frac{q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{4.51}
$$

where we have used the approximation of the logarithm, i.e. $\ln(1+x) = x + \mathcal{O}(x^2)$ if $x \in \mathbb{R}$ is small. By defining the purely AdS_n part

$$
\ln\hat{\phi} = \ln q + \frac{1}{2(n-1)}\ln a - \frac{n-2}{n-1}\ln\rho - \frac{1}{2(n-1)}\ln\det(g_{ab,S^{n-2}}) \tag{4.52}
$$

and the deviation

$$
h_{\ln\phi} = -\frac{q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right), \tag{4.53}
$$

we can write the boundary behaviour of the logarithm of the conformal factor as

$$
\ln\phi = \ln\hat{\phi} + h_{\ln\phi}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.54}
$$

The logarithm of the conformal factor $\ln\phi$ has to behave near the boundary $\rho = \ell$ or $q = 0$ precisely as given in Eq. (4.51). It can be seen that the first three terms are purely AdS_n and that the first term diverges at $q = 0$, while the last term, reflecting the physical fall-off, vanishes at the boundary.

In order to calculate the boundary behaviour of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$, we will need the square of the conformal factor ϕ. Therefore,

$$
\begin{aligned}
\phi^2 &= \hat{\phi}^2\left(1 + \epsilon h_\phi + \mathcal{O}(\epsilon^2)\right)^2 = \hat{\phi}^2\left(1 + 2\epsilon h_\phi + \mathcal{O}(\epsilon^2)\right) \\
&= \sqrt[n-1]{\frac{aq^{2(n-1)}}{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}}\left(1 - \frac{\epsilon q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{4.55}
$$

and

$$
\begin{aligned}
\phi^{-2} &= \hat{\phi}^{-2}\left(1 + \epsilon h_\phi + \mathcal{O}(\epsilon^2)\right)^{-2} = \hat{\phi}^{-2}\left(1 - 2\epsilon h_\phi + \mathcal{O}(\epsilon^2)\right) \\
&= \sqrt[n-1]{\frac{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}{aq^{2(n-1)}}}\left(1 + \frac{\epsilon q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.56}
$$

By defining $h_{\phi^2} = 2h_\phi$ and $h_{\phi^{-2}} = 2h_\phi$, we can write the boundary behaviour of the smooth function ϕ^2 as

$$
\phi^2 = \hat{\phi}^2\left(1 + \epsilon h_{\phi^2}\right) + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \phi^{-2} = \hat{\phi}^{-2}\left(1 - \epsilon h_{\phi^{-2}}\right) + \mathcal{O}(\epsilon^2). \tag{4.57}
$$

These equations give us the behaviour of the square of the conformal factor near the boundary of the perturbated AdS$_n$ metric tensor field. The square of the conformal factor behaves as the square of the purely AdS$_n$ conformal factor plus a small perturbation $\epsilon\hat{\phi}^2 h_{\phi^2}$.

Now, we will be able to state the boundary conditions for the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ in the following section.

4.5.2 The conformally decomposed spatial metric $\tilde{\gamma}_{ij}$

The conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ was introduced in Chapter 2.3 as an equivalence class of conformally related metrics on pseudo-Riemannian manifolds to restore hyperbolicity of the evolution system. However, as it remains a variable of the generalized FO-CCZ4 system, let us calculate the boundary behaviour by using the boundary behaviour of the spatial metric γ_{ij} and the boundary behaviour conformal factor ϕ.

Now, the spatial metric γ_{ij} is given, as $g_{ij} = \gamma_{ij}$, by the spatial part of the full metric $g_{\mu\nu}$ in Eq. (4.31), i.e.

$$\gamma_{ij} = \begin{pmatrix} \hat{\gamma}_{\rho\rho} + \epsilon q^{n-4}\bar{h}_{\rho\rho} & \epsilon q^{n-3}\bar{h}_{\rho b} \\ \epsilon q^{n-3}\bar{h}_{a\rho} & \hat{\gamma}_{ab} + \epsilon q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab} \end{pmatrix}, \tag{4.58}$$

while the inverse of the spatial perturbated metric is given by the formula

$$\gamma^{ij} = \hat{\gamma}^{ij} - \epsilon\hat{\gamma}^{il}\hat{\gamma}^{jk}h_{kl} + \mathcal{O}(\epsilon^2). \tag{4.59}$$

Using the latter, the components of the inverse of the spatial metric can then be written as

$$\gamma^{ij} = \begin{pmatrix} \hat{\gamma}^{\rho\rho} - \epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{\rho\rho}q^{n-4}\bar{h}_{\rho\rho} & -\epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{ab}q^{n-3}\bar{h}_{\rho a} \\ -\epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{ba}q^{n-3}\bar{h}_{\rho b} & \hat{\gamma}^{ab} - \epsilon\hat{\gamma}^{ac}\hat{\gamma}^{bd}q^{n-4}\rho^2 g_{cd,S^{n-2}}\bar{h}_{cd} \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{4.60}$$

In order to calculate the boundary behaviour of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$, we have to plug in, respectively, the boundary behaviour of the spatial metric γ_{ij} and the square of the conformal factor ϕ^2, and the boundary behaviour of its inverse γ^{ij} and ϕ^{-2} into

$$\tilde{\gamma}_{ij} = \phi^2\gamma_{ij} \qquad \text{and} \qquad \tilde{\gamma}^{ij} = \phi^{-2}\gamma^{ij}. \tag{4.61}$$

Then, the boundary behaviour is given by

$$\begin{aligned} \tilde{\gamma}_{ij} &= \hat{\phi}^2\left(1 + \epsilon h_{\phi^2}\right)\left(\hat{\gamma}_{ij} + \epsilon h_{ij}\right) + \mathcal{O}(\epsilon^2) \\ &= \hat{\phi}^2\hat{\gamma}_{ij} + \hat{\phi}^2\left(h_{ij} + h_{\phi^2}\hat{\gamma}_{ij}\right)\epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} \tilde{\gamma}^{ij} &= \hat{\phi}^{-2}\left(1 + \epsilon h_{\phi^{-2}}\right)\left(\hat{\gamma}^{ij} + \epsilon h^{ij}\right) + \mathcal{O}(\epsilon^2) \\ &= \hat{\phi}^{-2}\hat{\gamma}^{ij} + \hat{\phi}^{-2}\left(h^{ij} + h_{\phi^{-2}}\hat{\gamma}^{ij}\right)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.63}$$

By defining the conformaly decomposed spatial metric related to the purely AdS$_n$ spacetime

$$\hat{\tilde{\gamma}}_{ij} = \hat{\phi}^2 \hat{\gamma}_{ij} \quad \text{and} \quad \hat{\tilde{\gamma}}^{ij} = \hat{\phi}^{-2} \hat{\gamma}^{ij} \tag{4.64}$$

and the small perturbation near the boundary

$$h_{\tilde{\gamma}_{ij}} = \hat{\phi}^2 \left(h_{ij} + h_{\phi^2} \hat{\gamma}_{ij} \right), \quad \text{and} \quad h_{\tilde{\gamma}^{ij}} = \hat{\phi}^{-2} \left(h^{ij} + h_{\phi^{-2}} \hat{\gamma}^{ij} \right), \tag{4.65}$$

we can write the boundary behaviour as

$$\tilde{\gamma}_{ij} = \hat{\tilde{\gamma}}_{ij} + \epsilon h_{\tilde{\gamma}_{ij}} + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \tilde{\gamma}^{ij} = \hat{\tilde{\gamma}}^{ij} + \epsilon h_{\tilde{\gamma}^{ij}} + \mathcal{O}(\epsilon^2) \tag{4.66}$$

While a complete calculation for the boundary behaviour of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$ can be found in Appendix E.1, the components are given, respectively, by

$$\begin{aligned}
\tilde{\gamma}_{\rho\rho} =& \frac{1}{a^{\frac{n-2}{n-1}} \rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(1 + \frac{q^{n-2}}{n-1} \left((n-2) a \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \right) \\
&+ \mathcal{O}(\epsilon^2) && \text{(4.67a)} \\
\tilde{\gamma}_{\rho b} =& \frac{a^{\frac{1}{n-1}} q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \bar{h}_{\rho b} \epsilon + \mathcal{O}(\epsilon^2) && \text{(4.67b)} \\
\tilde{\gamma}_{ab} =& \frac{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(1 - \frac{q^{n-2}}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} - (n-1) \bar{h}_{ab} \right) \epsilon \right) \\
&+ \mathcal{O}(\epsilon^2), && \text{(4.67c)}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\gamma}^{\rho\rho} =& a^{\frac{n-2}{n-1}} \rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} \left(1 + \frac{q^{n-2}}{n-1} \left((2-n) a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \right) \\
&+ \mathcal{O}(\epsilon^2) && \text{(4.68a)} \\
\tilde{\gamma}^{\rho a} =& - \frac{a^{\frac{n-2}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}} g^{ba}_{S^{n-2}} q^{n-1} \bar{h}_{\rho b} \epsilon + \mathcal{O}(\epsilon^2) && \text{(4.68b)} \\
\tilde{\gamma}^{ab} =& \frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}} a^{\frac{1}{n-1}}} \Bigg(g^{ab}_{S^{n-2}} - q^{n-2} g^{ac}_{S^{n-2}} g^{bd}_{S^{n-2}} g_{cd,S^{n-2}} \bar{h}_{cd} \epsilon \\
&+ \frac{q^{n-2}}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \epsilon \Bigg) + \mathcal{O}(\epsilon^2). && \text{(4.68c)}
\end{aligned}$$

This gives us the boundary behaviour of the components of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ and its inverse $\tilde{\gamma}^{ij}$. We can see that the purely AdS$_n$ metric $\hat{\gamma}_{ij}$, as given in Eq. (4.38), diverges at the boundary $q = 0$, whereas the conformally decomposed spatial metric $\hat{\tilde{\gamma}}_{ij}$ takes finite values unless for some angle values. Note that we have to sum over the whole object $g^{bd}_{S^{n-2}} g_{cd,S^{n-2}} \bar{h}_{cd}$.

4.5.3 The shift vector β

We will, furthermore, need to calculate the boundary behaviour of the gauge variable β^i. The spatial vector field $\beta \in \Gamma^\infty(TM)$ was first introduced in Subsection A.4 to decompose the timelike section $\partial_t \in \Gamma^\infty(TM)$ into time and space. While a numerical value is straightforward to implement, evolving the shift vector β via a PDE extends the overall evolution system, and the system can be made more hyperbolic. For this reason, the shift vector β is a state variable of the generalized FO-CCZ4 system.

Plugging the boundary behaviour of γ^{ij}, as given in Eq. (4.58), and the boundary behaviour of g_{ti}, as given in Eq. (4.31), into the definition of the components of dual shift vector

$$\beta^j = \gamma^{ij} g_{ti}, \tag{4.69}$$

we can calculate the boundary behaviour of β. Then,

$$\begin{aligned}
\beta^\rho &= \gamma^{\rho\rho} g_{t\rho} + \gamma^{a\rho} g_{ta} \\
&= q^{n-3}\left(\hat{\gamma}^{\rho\rho} + \epsilon q^{n-4} \bar{h}_{\hat{\gamma}^{\rho\rho}}\right)\epsilon \bar{h}_{t\rho} - \epsilon^2 q^{n-4} q^{n-3} \hat{\gamma}^{\rho\rho}\hat{\gamma}^{ab}\bar{h}_{\rho b}\bar{h}_{ta} \\
&= q^{n-1} a \bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2) \qquad (4.70a)\\
\beta^a &= \gamma^{\rho a} g_{t\rho} + \gamma^{ba} g_{tb} \\
&= -\epsilon^2 \hat{\gamma}^{\rho\rho}\hat{\gamma}^{aa} q^{n-3}\bar{h}_{\rho a} q^{n-3}\bar{h}_{t\rho} + \left(\hat{\gamma}^{ba} + \epsilon q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab}\right)\epsilon q^{n-4}\bar{h}_{tb} \\
&= q^{n-4}\hat{\gamma}^{ba}\bar{h}_{tb}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{q^{n-2}}{\rho^2} g^{ba}_{S^{n-2}}\bar{h}_{tb}\epsilon + \mathcal{O}(\epsilon^2). \qquad (4.70b)
\end{aligned}$$

As the purely AdS$_n$ part of the shift vector $\hat{\beta}$ vanishes, the boudary behaviour of the varibale can be written as

$$\beta^i = \hat{\beta}^i + \epsilon h_{\beta^i} + \mathcal{O}(\epsilon^2), \tag{4.71}$$

where $\hat{\beta}^i = 0$. This way, the boundary behaviour of the shift vector β^i is only determined by the deviation and vanishes, as excpected, at the boundary $q = 0$.

4.5.4 The lapse function α

Another gauge variable, reflecting the coordinate degrees of freedom, is the lapse function $\alpha \in C^\infty(M, \mathbb{R})$. It was first introduced in Subsection A.4 to decompose the timelike section $\partial_t \in \Gamma^\infty(TM)$ into time and space. While a numerical value is straightforward to implement, evolving the lapse function α via a PDE extends the overall evolution system, and the system can be made more hyperbolic. For this reason, the lapse function α is still a variable of the generalized FO-CCZ4 system.

Plugging the boundary behaviour for the shift vector β and the time-time component of perturbated metric $g_{\mu\nu}$ into Eq. (A.84a), we can calculate the boundary behaviour of α to

$$\alpha = \sqrt{\beta_i \beta^i - g_{tt}} = \sqrt{g_{t\rho}\beta^\rho + \sum_{i=1}^{n-2} g_{t\theta_i}\beta^{\theta_i} - g_{tt}}$$

$$
\begin{aligned}
&= \sqrt{q^{n-1}q^{n-3}a\bar{h}_{t\rho}\bar{h}_{t\rho}\epsilon^2 + q^{n-4}q^{n-3}\bar{h}_{\rho\theta_i}\hat{\gamma}^{\theta_i\theta_i}\bar{h}_{t\theta_i}\epsilon^2 - \hat{g}_{tt} - q^{n-4}\bar{h}_{tt}\epsilon} \\
&= \sqrt{\frac{a}{q^2} - q^{n-4}\bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2)}. \qquad (4.72)
\end{aligned}
$$

By defining, respectively, the purely AdS$_n$ part

$$
\hat{\alpha} = \frac{\sqrt{a}}{q} \qquad (4.73)
$$

and the deviation

$$
h_\alpha = -\frac{q^{n-2}}{2a}h_{tt}, \qquad (4.74)
$$

we can decompose the lapse function α, using the binomial approximation, into the purely AdS$_n$ lapse function $\hat{\alpha}$ plus some small perturbation h_α

$$
\begin{aligned}
\alpha &= \sqrt{\frac{a}{q^2}\left(1 - \frac{\epsilon q^{n-2}}{a}\bar{h}_{tt} + \mathcal{O}(\epsilon^2)\right)} \\
&= \frac{\sqrt{a}}{q}\sqrt{1 - \frac{\epsilon q^{n-2}}{a}\bar{h}_{tt} + \mathcal{O}(\epsilon^2)} \\
&= \frac{\sqrt{a}}{q}\left(1 - \frac{\epsilon q^{n-2}}{2a}\bar{h}_{tt}\right) + \mathcal{O}(\epsilon^2) \\
&= \hat{\alpha}\left(1 + \epsilon h_\alpha\right) + \mathcal{O}(\epsilon^2). \qquad (4.75)
\end{aligned}
$$

As we are time evolving the logarithm of the lapse function, let us derive the boundary behaviour of the logarithm of the lapse function α. Then,

$$
\begin{aligned}
\ln\alpha &= \ln\left(\hat{\alpha}\left(1 + \epsilon h_\alpha + \mathcal{O}(\epsilon^2)\right)\right) = \ln\hat{\alpha} + \ln\left(1 + \epsilon h_\alpha + \mathcal{O}(\epsilon^2)\right) = \ln\hat{\alpha} + \epsilon h_\alpha + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{2}\ln\left(q^2 + \frac{\rho^2}{L^2}\right) - \ln q - \frac{q^{n-2}}{2a}\bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{2}\ln\left(\frac{\rho^2}{L^2}\left(1 + \frac{q^2L^2}{\rho^2}\right)\right) - \ln q - \frac{q^{n-2}}{2a}\bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \ln(1-q) + \ln\frac{\ell}{L} + \frac{1}{2}\ln\left(1 + \frac{q^2L^2}{\rho^2}\right) - \ln q - \frac{q^{n-2}}{2a}\bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2). \qquad (4.76)
\end{aligned}
$$

The logarithm of the lapse function α has to behave near the boundary $q = 0$ exactly as in Eq. (4.76). We see that the first four terms are purely AdS$_n$, while the last term describes the fall-off condition near the boundary. Thus, at the boundary, the logarithm of the purely AdS$_n$ lapse function $\hat{\alpha}$ diverges, whereas the perturbation vanishes.

4.5.5 Calculation for the auxiliary variables

The 33 auxiliary variables A_i, B^i_k, D_{kij} and P_i were only introduced in [20] to rewrite the CCZ4 system as a first-order evolution system and have no physical meaning. As they are evolution variables of the generalized FO-CCZ4 system, we need to derive their boundary behaviour.

The auxiliary variable A_i

Let us first calculate the boundary behaviour for the auxiliary variable

$$A_i := \partial_i \ln \alpha \tag{4.77}$$

as defined in Eq. (2.63). As the boundary behaviour of α is given in Eq. (4.76) by

$$\ln \alpha = \ln \hat{\alpha} + \epsilon h_\alpha + \mathcal{O}(\epsilon^2), \tag{4.78}$$

where $\hat{\alpha} = \hat{\alpha}(\rho)$ and $h_\alpha = h_\alpha(\rho, \theta_{n-2}, \ldots, \theta_1)$, we can write the boundary behaviour of A_i as

$$A_i = \partial_i \ln \hat{\alpha} + \epsilon \partial_i h_\alpha + \mathcal{O}(\epsilon^2). \tag{4.79}$$

By defining the purely AdS$_n$ auxiliary variable

$$\hat{A}_i = \partial_i \ln \hat{\alpha} \tag{4.80}$$

and the deviation

$$h_{A_i} = \partial_i h_\alpha, \tag{4.81}$$

we can write the boundary behaviour of the auxiliary variable A_i as

$$A_i = \hat{A}_i + \epsilon h_{A_i} + \mathcal{O}(\epsilon^2). \tag{4.82}$$

The individual derivatives, as can be found in Appendix E.2, are given by

$$\begin{aligned} A_\rho = \frac{\rho}{L^2 q a} + \left(\frac{q^{n-2}\rho}{a^2 L^2} - \frac{q^{n-1}}{a^2 l} + \frac{(n-2)q^{n-3}}{2la} \right) \bar{h}_{tt}\epsilon & \\ - \frac{q^{n-2}}{2a} \partial_\rho \bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2) & \qquad (4.83a) \\ A_a = - \frac{q^{n-2}}{2a} \partial_a \bar{h}_{tt}\epsilon + \mathcal{O}(\epsilon^2). & \qquad (4.83b) \end{aligned}$$

In terms of the perturbational split, the purely AdS$_n$ part of A_i is given by

$$\hat{A}_\rho = \frac{\rho}{L^2 q a} \quad \text{and} \quad \hat{A}_a = 0, \tag{4.84}$$

while the perturbation of the auxiliary variable takes the form

$$\begin{aligned} h_{A_\rho} &= \left(\frac{q^{n-2}\rho}{a^2 L^2} - \frac{q^{n-1}}{a^2 l} + \frac{(n-2)q^{n-3}}{2la} \right) \bar{h}_{tt} - \frac{q^{n-2}}{2a} \partial_\rho \bar{h}_{tt} && (4.85a) \\ h_{A_a} &= - \frac{q^{n-2}}{2a} \partial_a \bar{h}_{tt}. && (4.85b) \end{aligned}$$

Thus, the ρ-component of the auxiliary variable A_i behaves near the horizon as the purely AdS$_n$ part of the auxiliary variable $\hat{A}_\rho$ plus the small perturbation ϵh_{A_ρ}, while the boundary behaviour of the angle components is governed by just the small perturbation ϵh_{A_a}.

The auxiliary variable P_i

Let us next calculate the boundary conditions for the auxiliary variable

$$P_i = \partial_i \ln \phi, \tag{4.86}$$

as defined in Eq. (2.63). As the boundary behaviour of the logarithm of the conformal factor ϕ is given in Eq. (4.51) by

$$\ln \phi = \ln \hat{\phi} + h_\phi + \mathcal{O}(h_\phi^2), \tag{4.87}$$

where $\hat{\phi} = \hat{\phi}(\rho, \theta_{n-3}, \theta_{n-4}, \dots)$ and $h_\phi = h_\phi(\rho, \theta_{n-2}, \dots, \theta_1)$, we can write the boundary behaviour of P_i as

$$P_i = \partial_i \ln \hat{\phi} + \epsilon \partial_i h_\phi + \mathcal{O}(\epsilon^2). \tag{4.88}$$

By defining the purely AdS$_n$ auxiliary variable

$$\hat{P}_i = \partial_i \ln \hat{\phi} \tag{4.89}$$

and the deviation of $\hat{P}_i$

$$h_{P_i} = \partial_i h_\phi \tag{4.90}$$

we can write the boundary behaviour of the auxiliary variable P_i as

$$P_i = \hat{P}_i + \epsilon h_{P_i} + \mathcal{O}(\epsilon^2). \tag{4.91}$$

The individual derivatives, as can be found in Appendix E.2, are given by

$$\begin{aligned} P_\rho = &-\frac{1}{ql} + \frac{\frac{2\rho}{L^2} - \frac{2q}{l}}{2(n-1)a} - \frac{n-2}{(n-1)\rho} + \frac{(n-2)q^{n-3}}{2(n-1)\ell}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right)\epsilon \\ &- \frac{q^{n-2}}{2(n-1)}\left(\partial_\rho a\bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.92a}$$

$$\begin{aligned} P_a = &-\frac{1}{2(n-1)}\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})} \\ &- \frac{q^{n-2}}{2(n-1)}\left(a\partial_a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_a \bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.92b}$$

In terms of the perturbational split, the purely AdS$_n$ part of the auxiliary variable P_i is given by

$$\hat{P}_\rho = -\frac{1}{ql} + \frac{1}{n-1)L^2}\frac{\rho}{a} - \frac{1}{(n-1)a\ell}\frac{q}{a} - \frac{n-2}{(n-1)}\frac{1}{\rho}, \tag{4.93a}$$

$$\hat{P}_a = -\frac{1}{2(n-1)}\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}, \tag{4.93b}$$

while the perturbation of P_i given by

$$h_{A_\rho} = \frac{(n-2)q^{n-3}}{2(n-1)\ell}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right)$$

$$
- \frac{q^{n-2}}{2(n-1)} \left(\partial_\rho a \bar{h}_{\rho\rho} + a \partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \right), \tag{4.94a}
$$

$$
h_{A_a} = -\frac{q^{n-2}}{2(n-1)} \left(a \partial_a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_a \bar{h}_{\theta_i\theta_i} \right). \tag{4.94b}
$$

We note that the ρ-component of P_i behaves near the horizon as the purely AdS$_n$ part of the auxiliary variable $\hat{P}_\rho$ plus the small perturbation ϵh_{P_ρ}, while the boundary behaviour of the angle components is governed by just the perturbation ϵh_{P_a} if the partial derivative of the determinant of the round metric $g_{ab,S^{n-2}}$ vanishes.

The auxiliary variable B_k^l

Next, let us calculate the boundary behaviour of the auxiliary variable

$$
B_k^l = \partial_k \beta^l \tag{4.95}
$$

as defined in Eq. (2.63). Using the boundary behaviour of the shift vector β, as given in Eq. (4.71), we can derive the boundary behaviour of the auxiliary variable B_k^l. Thus,

$$
\begin{aligned}
B_k^l &= \partial_k \left(\hat{\beta}^l + \epsilon h_{\beta^l} + \mathcal{O}(\epsilon^2) \right) \\
&= \partial_k \hat{\beta}^l + \epsilon \partial_k h_{\beta^l} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{4.96}
$$

By defining the purely AdS$_n$ part of the auxiliary variable B_k^l as

$$
\hat{B}_k^l = \partial_k \hat{\beta}^l \tag{4.97}
$$

and the perturbational part of B_k^l as

$$
h_{B_k^l} = \partial_k h_{\beta^l}, \tag{4.98}
$$

we can write the boundary behaviour of the auxiliary variable B_k^l as

$$
B_k^l = \hat{B}_k^l + \epsilon h_{B_k^l} + \mathcal{O}(\epsilon^2). \tag{4.99}
$$

By using Eq. (4.70a) and Eq. (4.70b), we calculate explicit the components of B_k^l. Then,

$$
B_\rho^\rho = -\frac{(n-1)q^{n-2}}{\ell} a \bar{h}_{t\rho} \epsilon + q^{n-1} \partial_\rho a \bar{h}_{t\rho} \epsilon + q^{n-1} a \partial_\rho \bar{h}_{t\rho} \epsilon + \mathcal{O}(\epsilon^2), \tag{4.100a}
$$

$$
B_a^\rho = q^{n-1} a \partial_a \bar{h}_{t\rho} \epsilon + \mathcal{O}(\epsilon^2), \tag{4.100b}
$$

$$
B_\rho^a = -q^{n-4} \left(\frac{(n-2)q}{\rho^2 \ell} + \frac{2q^2}{\rho^3} \right) g_{S^{n-2}}^{ba} \bar{h}_{tb} \epsilon + \frac{q^{n-2}}{\rho^2} g_{S^{n-2}}^{ba} \partial_\rho \bar{h}_{tb} \epsilon + \mathcal{O}(\epsilon^2), \tag{4.100c}
$$

$$
B_a^b = \frac{q^{n-2}}{\rho^2} \partial_a g_{S^{n-2}}^{cb} \bar{h}_{tc} \epsilon + \frac{q^{n-2}}{\rho^2} g_{S^{n-2}}^{cb} \partial_a \bar{h}_{tc} \epsilon + \mathcal{O}(\epsilon^2). \tag{4.100d}
$$

The boundary behaviour of the auxiliary variable B_k^l is governed explicitly by the small perturbation $\epsilon h_{B_k^l}$, as the shift vector β is already purely perturbational.

The auxiliary variable D_{ijk}

And finally, let us calculate the boundary behaviour of the auxiliary variable

$$D_{ijk} = \frac{1}{2}\partial_i \tilde{\gamma}_{jk}, \tag{4.101}$$

as defined in Eq. (2.63). Using the boundary behaviour of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ as defined in Eq. (4.62), we can write

$$\begin{aligned} D_{kij} &= \frac{1}{2}\partial_k \tilde{\gamma}_{ij} = \frac{1}{2}\partial_k \Big[\hat{\tilde{\gamma}}_{ij} + \epsilon h_{\tilde{\gamma}_{ij}} + \mathcal{O}(\epsilon^2)\Big] \\ &= \frac{1}{2}\partial_k \hat{\tilde{\gamma}}_{ij} + \frac{1}{2}\epsilon \partial_k h_{\tilde{\gamma}_{ij}} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.102}$$

By defining the purely AdS$_n$ part of D_{ijk} as

$$\hat{D}_{kij} = \frac{1}{2}\partial_k \hat{\tilde{\gamma}}_{ij} \tag{4.103}$$

and the perturbational part as

$$\begin{aligned} h_{D_{kij}} &= \partial_k h_{\tilde{\gamma}_{ij}} = \partial_k \left(\hat{\phi}^2 \left(h_{ij} + h_{\phi^2}\hat{\gamma}_{ij}\right)\right) \\ &= \frac{1}{2} h_{ij}\partial_k \hat{\phi}^2 + \frac{1}{2} h_{\phi^2}\hat{\gamma}_{ij}\partial_k \hat{\phi}^2 + \frac{1}{2}\hat{\phi}^2 \partial_k h_{ij} + \frac{1}{2}\hat{\phi}^2\hat{\gamma}_{ij}\partial_k h_{\phi^2} + \frac{1}{2}\hat{\phi}^2 h_{\phi^2}\partial_k \hat{\gamma}_{ij}. \end{aligned} \tag{4.104}$$

we can decompose the boundary behaviour of the auxiliary varibale D_{kij} by

$$D_{kij} = \hat{D}_{kij} + \epsilon h_{D_{kij}} + \mathcal{O}(\epsilon^2). \tag{4.105}$$

After some tremendous calculations, that can be found in Appendix C, we can summarize the components of the auxiliary variable D_{kij} by

$$\begin{aligned} D_{cab} = &-\frac{1}{2} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \left(\frac{1}{n-1}\frac{g_{ab,S^{n-2}}\partial_c \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\right) \\ &+ \frac{q^{n-2} a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2(n-1)^2}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab}\right)\frac{g_{ab,S^{n-2}}\partial_c \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon \\ &- \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}} q^{n-2}}{2(n-1)}\frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left(a\partial_c \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_c \bar{h}_{\theta_i\theta_i} - (n-1)\partial_c \bar{h}_{ab}\right)\epsilon \\ &- \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}} q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab}\right)\frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\epsilon \\ &+ \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.106a}$$

$$\begin{aligned} D_{a\rho\rho} = &-\frac{a^{\frac{n-2}{1-n}}\rho^{\frac{2(n-2)}{1-n}}}{2(n-1)}\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} \\ &+ \frac{a^{\frac{n-2}{1-n}}\rho^{\frac{2(n-2)}{1-n}}}{2(n-1)}\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon \end{aligned}$$

$$
\begin{aligned}
&- \frac{q^{n-2}}{2(n-1)} \frac{a^{\frac{n-2}{1-n}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left((2-n) a \partial_a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_a \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&+ \mathcal{O}(\epsilon^2), \qquad (4.106b)
\end{aligned}
$$

$$
D_{a\rho b} = - \frac{a^{\frac{1}{n-1}} q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}} \left(\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} \frac{\bar{h}_{\rho b}}{n-1} - \frac{\partial_a \bar{h}_{\rho b}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \right) \epsilon + \mathcal{O}(\epsilon^2), \qquad (4.106c)
$$

$$
\begin{aligned}
D_{\rho\rho a} =& \frac{a^{\frac{n-2}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(\frac{q^{n-1} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right)}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}} - \frac{n-2}{n-1} \frac{q^{n-1} a}{\rho^{\frac{3n-5}{n-1}}} - \frac{n-1}{2\ell} \frac{a q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}} \right) \bar{h}_{\rho a} \epsilon \\
&+ \frac{a^{\frac{1}{n-1}} q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \bar{h}_{\rho a} \epsilon + \mathcal{O}(\epsilon^2), \qquad (4.106d)
\end{aligned}
$$

$$
\begin{aligned}
D_{\rho ab} =& \frac{1}{2(n-1)} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a^{\frac{2-n}{n-1}} \rho^{\frac{2}{n-1}} \partial_\rho a + 2 a^{\frac{1}{n-1}} \rho^{\frac{3-n}{n-1}} \right) \\
&+ \frac{q^{n-2} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}}{2(n-1)} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \Bigg[- \frac{1}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} - (n-1) \bar{h}_{ab} \right) \\
&\times \left(\frac{\partial_\rho a}{a} - \frac{(n-1)(n-2)}{q\ell} + \frac{2}{\rho} \right) - \partial_\rho a \bar{h}_{\rho\rho} + a \partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \\
&- (n-1) \partial_\rho \bar{h}_{ab} \Bigg] \epsilon + \mathcal{O}(\epsilon^2), \qquad (4.106e)
\end{aligned}
$$

$$
\begin{aligned}
D_{\rho\rho\rho} =& - \frac{n-2}{2(n-1)} \frac{a^{\frac{2-n}{n-1}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(\frac{\partial_\rho a}{a} + \frac{2}{\rho} \right) \\
&+ \frac{n-2}{2(n-1)^2} \frac{q^{n-2} a^{\frac{2-n}{n-1}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(\frac{\partial_\rho a}{a} + \frac{2}{\rho} + \frac{n-1}{q\ell} \right) \left((2-n) a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&- \frac{q^{n-2}}{2(n-1)} \frac{a^{\frac{2-n}{n-1}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[(2-n) \partial_\rho (a \bar{h}_{\rho\rho}) + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \right] \epsilon \\
&+ \mathcal{O}(\epsilon^2). \qquad (4.106f)
\end{aligned}
$$

Note that the boundary behaviour of $D_{\rho\rho a}$ and $D_{a\rho b}$ is purely governed by a small perturbation, i.e. the purely AdS$_n$ part of these components of the auxiliary variable vanishes, while the boundary behaviour of the other variables is governed by the purely AdS$_n$ part $\hat{D}_{ijk}$ of the auxiliary variable plus some small perturbation $\epsilon h_{D_{ijk}}$.

Conclusively, we have calculated how the 33 auxiliary variables, as defined in Eq. (2.63), have to behave near the boundary $q = 0$. Again, these auxiliary variables were introduced in order to rewrite the second-order $CCZ4$ system as a first-order system and have no physical meaning.

4.5.6 The extrinsic curvature $K_{\mu\nu}$

Furthermore, we will need to derive the boundary behaviour of the extrinsic curvature K_{ij} that is a measure of how the hypersurfaces Σ_t are embedded into the spacetime manifold M. The second fundamental form K_{ij} is an evolution variable of the originally derived ADM evolution system but got replaced by the trace of the extrinsic curvature K and the conformally decomposed traceless part of the extrinsic curvature in the derivation of the CCZ4 system.

Before we can calculate the boundary behaviour of the trace of the extrinsic curvature K, we need to derive the behaviour of the extrinsic curvature K_{ij} first. To do so, we first need to find an expression of the extrinsic curvature K_{ij} in terms of the auxiliary variables from above. Therefore, let us rewrite Eq. (2.21a) to

$$K_{ij} = -\frac{1}{2\alpha}\left(\partial_t - \mathcal{L}_\beta\right)\gamma_{ij}, \tag{4.107}$$

where

$$\begin{aligned}
\mathcal{L}_\beta\gamma_{ij} &= \partial_j\beta_i + \partial_i\beta_j - 2\Gamma^k{}_{ij}\beta_k \\
&= \partial_j\left(\gamma_{il}\beta^l\right) + \partial_i\left(\gamma_{jl}\beta^l\right) - 2\frac{1}{2}\gamma^{kl}\left(\partial_i\gamma_{lj} + \partial_j\gamma_{il} - \partial_l\gamma_{ij}\right)\beta_k \\
&= \cancel{\beta^l\partial_j\gamma_{il}} + \gamma_{il}\partial_j\beta^l + \cancel{\beta^l\partial_i\gamma_{jl}} + \gamma_{jl}\partial_i\beta^l \cancel{- \beta^l\partial_i\gamma_{lj}} \cancel{- \beta^l\partial_j\gamma_{il}} + \beta^l\partial_l\gamma_{ij} \\
&= \gamma_{il}\partial_j\beta^l + \gamma_{jl}\partial_i\beta^l + \beta^l\partial_l\gamma_{ij} \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + \beta^l\partial_l\gamma_{ij} \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + \beta^l\partial_l\left(\phi^2\gamma_{ij}\phi^{-2}\right) \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + \phi^{-2}\beta^l\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{ij}\beta^l\partial_l\phi^{-2} \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + 2\phi^{-2}\beta^l D_{lij} + \gamma_{ij}\beta^l\phi^2\partial_l\phi^{-2} \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + 2\phi^{-2}\beta^l D_{lij} - \gamma_{ij}\beta^l\phi^{-2}\partial_l\phi^2 \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + 2\phi^{-2}\beta^l D_{lij} - 2\gamma_{ij}\beta^l\partial_l\ln\phi \\
&= \gamma_{il}B^l_j + \gamma_{jl}B^l_i + 2\phi^{-2}\beta^l D_{lij} - 2\gamma_{ij}\beta^l P_l
\end{aligned} \tag{4.108}$$

is the Lie derivative of the spatial metric γ_{ij} in the direction of the shift vector β. Plugging this formula into Eq. (4.107), we receive

$$K_{ij} = -\frac{1}{2\alpha}\left(\partial_t\gamma_{ij} - \gamma_{ik}B^k_j - \gamma_{jk}B^k_i - 2\phi^{-2}\beta^l D_{kij} + 2\gamma_{ij}\beta^k P_k\right), \tag{4.109}$$

an expresion of the extrinsic curvature in terms of the auxiliary variable. Then, by using the boundary behaviour of the auxiliary variables, we can write the boundary behaviour of the extrinsic curvature as

$$\begin{aligned}
K_{ij} &= -\frac{1}{2\alpha}\left(\partial_t\gamma_{ij} - \gamma_{ik}B^k_j - \gamma_{jk}B^k_i - 2\phi^{-2}\beta^k D_{kij} + 2\gamma_{ij}\beta^k P_k\right) \\
&= -\frac{1}{2\hat{\alpha}(1+\epsilon h_\alpha + \mathcal{O}(\epsilon^2))}\Big[\partial_t\Big(\hat{\gamma}_{ij} + \epsilon h_{ij}\Big) - \left(\hat{\gamma}_{ik} + \epsilon h_{ik}\right)\epsilon h_{B^k_j} - \left(\hat{\gamma}_{jk} + \epsilon h_{jk}\right)\epsilon h_{B^k_i} \\
&\quad - 2\hat{\phi}^{-2}\left(1 + \epsilon h_{\phi^{-2}}\right)\epsilon h_{\beta^k}\Big(\hat{D}_{kij} + \epsilon h_{D_{kij}}\Big) + 2\left(\hat{\gamma}_{ij} + \epsilon h_{ij}\right)h_{\beta^k}\left(\hat{P}_k + \epsilon h_{P_k}\right) + \mathcal{O}(\epsilon^2)\Big] \\
&= -\frac{1}{2\hat{\alpha}}(1 - \epsilon h_\alpha)\Big[\partial_t h_{ij} - \hat{\gamma}_{ik}h_{B^k_j} - \hat{\gamma}_{jk}h_{B^k_i} - 2\hat{\phi}^{-2}h_{\beta^k}\hat{D}_{kij} + 2\hat{\gamma}_{ij}\hat{P}_k h_{\beta^k}\Big]\epsilon + \mathcal{O}(\epsilon^2) \\
&= -\frac{1}{2\hat{\alpha}}\Big[\partial_t h_{ij} - \hat{\gamma}_{ik}h_{B^k_j} - \hat{\gamma}_{jk}h_{B^k_i} - 2\hat{\phi}^{-2}h_{\beta^k}\hat{D}_{kij} + 2\hat{\gamma}_{ij}\hat{P}_k h_{\beta^k}\Big]\epsilon + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{4.110}$$

where we have used the boundary behaviour of the spatial metric, as given in Eq. (4.58), the boundary behaviour of the shift vector β as in Eq. (4.71), the boundary behaviour of the lapse function α as in Eq. (4.75), the boundary behaviour of the auxiliary variable P_i as in Eq. (4.91), the boundary behaviour of the auxiliary variable B_k^l as in Eq. (4.99), the fact that B_k^l and β^k are purely perturbative and the boundary behaviour of the D_{ijk} auxiliary variable as in Eq. (4.102). Furthermore, we have simplified the equations by keeping only the lowest order in ϵ. In general, the boundary behaviour of the extrinsic curvature can be written as

$$K_{ij} = \hat{K}_{ij} + \epsilon h_{K_{ij}} + \mathcal{O}(\epsilon^2), \tag{4.111}$$

where $\hat{K}_{ij} = 0$ for all combinations and $h_{K_{ij}}$ is given by

$$h_{K_{ij}} = -\frac{1}{2\hat{\alpha}}\Big[\partial_t h_{ij} - \hat{\gamma}_{ik} h_{B_j^k} - \hat{\gamma}_{jk} h_{B_i^k} - 2\hat{\phi}^{-2} h_{\beta^k}\hat{D}_{kij} + 2\hat{\gamma}_{ij}\hat{P}_k h_{\beta^k}\Big]. \tag{4.112}$$

Now, the components of the extrinsic curvature K_{ij}, as calculated in Appendix E.5, are explicitly given by

$$K_{\rho\rho} = -\frac{q^{n-3}}{2\sqrt{a}}\Big[\partial_t \bar{h}_{\rho\rho} - 2q\partial_\rho \bar{h}_{t\rho} + \left(\frac{2(n-1)}{l} - \frac{2\rho}{aL^2}\right)\bar{h}_{t\rho}\Big]\epsilon + \mathcal{O}(\epsilon^2), \tag{4.113a}$$

$$\begin{aligned} K_{ab} = -\frac{q^{n-3}}{2\sqrt{a}}\Big[&\rho^2 g_{ab,S^{n-2}}\partial_t \bar{h}_{ab} - 2g_{(ac,S^{n-2}}\partial_{b)} g^{dc}_{S^{n-2}}\bar{h}_{td} - 2\partial_{(a}\bar{h}_{tb)} \\ &- g^{dc}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}\bar{h}_{td} - 2\rho g_{ab,S^{n-2}} a\bar{h}_{t\rho}\Big]\epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.113b}$$

$$K_{\rho a} = -\frac{q^{n-3}}{2\sqrt{a}}\left[q\partial_t \bar{h}_{\rho a} - q\partial_a \bar{h}_{t\rho} + \left(\frac{n-2}{q\ell} + \frac{2}{\rho}\right)\bar{h}_{ta} - \partial_\rho \bar{h}_{ta}\right]\epsilon + \mathcal{O}(\epsilon^2). \tag{4.113c}$$

Near the boundary, the behaviour of the components of the extrinsic curvature K_{ij} is governed by the perturbation, while they vanish at the boundary $q = 0$.

Let us now derive the boundary behaviour of the trace of the extrinsic curvature K. Then,

$$\begin{aligned} K &= \gamma^{ij}K_{ij} = \gamma^{\rho\rho}K_{\rho\rho} + 2\gamma^{\rho a}K_{\rho a} + \gamma^{ab}K_{ab} \\ &= \left(\hat{\gamma}^{\rho\rho} - \epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{\rho\rho}q^{n-4}\bar{h}_{\rho\rho}\right)\left(\hat{K}_{\rho\rho} + \epsilon h_{K_{\rho\rho}}\right) + 2\left(\hat{\gamma}^{\rho a} - \epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{ab}q^{n-3}\bar{h}_{\rho b}\right)\left(\hat{K}_{\rho a} + \epsilon h_{K_{\rho a}}\right) \\ &\qquad + \left(\hat{\gamma}^{ab} - \epsilon q^{n-4}\rho^2 g_{ab,S^{n-2}}\hat{\gamma}^{ac}\hat{\gamma}^{bd}\bar{h}_{cd}\right)\left(\hat{K}_{ab} + \epsilon h_{K_{ab}}\right) + \mathcal{O}(\epsilon^2) \\ &= \hat{\gamma}^{\rho\rho}h_{K_{\rho\rho}}\epsilon + \hat{\gamma}^{ab}h_{K_{ab}}\epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.114}$$

where we have used the formula for the inverse of a perturbated metric, i.e.

$$\gamma^{ij} = \hat{\gamma}^{ij} - \epsilon\hat{\gamma}^{il}\hat{\gamma}^{jk}h_{kl} + \mathcal{O}(\epsilon^2),$$

the boundary behaviour of the extrinsic curvature K_{ij}, as given in Eq. (4.111), the fact that the extrinsic curvature K_{ij} is purely perturbative and the latter equations for $K_{\rho\rho}$, K_{ab}. By defining

$$h_K = \hat{\gamma}^{\rho\rho}h_{K_{\rho\rho}} + \hat{\gamma}^{ab}h_{K_{ab}}, \tag{4.115}$$

the boundary behaviour of the trace of the extrinsic curvature K is given by

$$K = \hat{K} + \epsilon h_K + \mathcal{O}(\epsilon^2), \tag{4.116}$$

where $\hat{K} = 0$. After some calculations, that can be found in Appendix C, the boundary behaviour of the trace of the extrinsic curvature K can be explicitly summarized by

$$\begin{aligned} K = -\frac{q^{n-1}}{2\sqrt{a}} \Bigg[& a\partial_t \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} - 2qa\partial_\rho \bar{h}_{t\rho} + \Bigg(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} \\ & - \frac{2(n-2)q^2}{\rho} \Bigg) \bar{h}_{t\rho} - \frac{2}{\rho^2} g^{ab}_{S^{n-2}} \Big(\partial_a \bar{h}_{tb} + \frac{1}{2} g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} \Big) \Bigg] \epsilon \\ & + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.117}$$

Note that the boundary behaviour of the trace of the extrinsic curvature K is governed by the small perturbation ϵh_K, while the purely AdS$_n$ part vanishes.

4.5.7 The conformally decomposed traceless part of the extrinsic curature $\tilde{A}_{ij}$

The conformally decomposed traceless part $\tilde{A}_{ij}$ of the extrinsic curvature, i.e. Eq. (2.34), was first introduced for the derivation of the CCZ4 system. Together with the trace of the extrinsic curvature, it replaces the evolution equation of the extrinsic curvature.

Before we can calculate the boundary behaviour of the conformally decomposed traceless part of the extrinsic curvature, we need to derive the boundary condition for the traceless part of the extrinsic curvature A_{ij}, as defined in Eq. (2.30). Using the boundary behaviour of the extrinsic curvature K_{ij} and spatial metric γ_{ij}, we can calculate the boundary behaviour of the conformally decomposed traceless part of the extrinsic curvature to

$$\begin{aligned} A_{ij} = K_{ij} - \frac{1}{n-1} K\gamma_{ij} &= \Big(\hat{K}_{ij} + \epsilon h_{K_{ij}} \Big) - \frac{1}{n-1} \Big(\hat{K} + \epsilon h_K \Big) (\hat{\gamma}_{ij} + \epsilon h_{ij}) + \mathcal{O}(\epsilon^2) \\ &= \hat{K}_{ij} + \epsilon h_{K_{ij}} - \frac{1}{n-1} \Big(\hat{K}\hat{\gamma}_{ij} + \epsilon \hat{K} h_{ij} + \epsilon \hat{\gamma}_{ij} h_K \Big) + \mathcal{O}(\epsilon^2) \\ &= \hat{K}_{ij} - \frac{1}{n-1} \hat{K}\hat{\gamma}_{ij} + \Big(h_{K_{ij}} - \frac{1}{n-1} \hat{\gamma}_{ij} h_K \Big) \epsilon + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.118}$$

where we have used the boundary behaviour of the extrinsic curvature K_{ij}, as given in Eq. (4.111), and the boundary behaviour of the spatial metric γ_{ij}. By defining

$$\hat{A}_{ij} = \hat{K}_{ij} - \frac{1}{n-1} \hat{K}\hat{\gamma}_{ij} \tag{4.119}$$

and

$$h_{A_{ij}} = h_{K_{ij}} - \frac{1}{n-1} \hat{\gamma}_{ij} h_K, \tag{4.120}$$

we can write A_{ij} in terms of the perturbation as

$$A_{ij} = \hat{A}_{ij} + h_{A_{ij}}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.121}$$

Note that as $\hat{K}_{ij} = 0$, $\hat{K}$ vanishes as well. Therefore, the boundary behaviour of the traceless part of the extrinsic curvature A_{ij} is governed by the small perturbation $\epsilon h_{A_{ij}}$.

The components of the traceless part of the extrinsic curvature, as calculated in Appendix E.5 by using Eq. (4.118), are given then by

$$\begin{aligned}
A_{\rho\rho} = &-\frac{q^{n-3}}{2\sqrt{a}}\Bigg[\frac{n-2}{n-1}\partial_t\bar{h}_{\rho\rho} - \frac{1}{(n-1)a}\sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} - \frac{2(n-2)q}{n-1}\partial_\rho\bar{h}_{t\rho} + \frac{2g^{ab}_{S^{n-2}}\partial_b\bar{h}_{ta}}{(n-1)\rho^2 a} \\
&+\left(\frac{2(n-2)}{\ell} + \frac{2(n-2)q^2}{(n-1)a\rho}\right)\bar{h}_{t\rho} + \frac{g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}}{(n-1)\rho^2 a}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}\Bigg]\epsilon \\
&+\mathcal{O}(\epsilon^2), && \text{(4.122a)}\\
A_{\rho a} = &-\frac{q^{n-3}}{2\sqrt{a}}\left[q\partial_t\bar{h}_{\rho a} - q\partial_a\bar{h}_{t\rho} + \left(\frac{n-2}{q\ell} + \frac{2}{\rho}\right)\bar{h}_{ta} - \partial_\rho\bar{h}_{ta}\right]\epsilon + \mathcal{O}(\epsilon^2), && \text{(4.122b)}\\
A_{ab} = &-\frac{q^{n-3}}{2\sqrt{a}}\Bigg[\frac{\rho^2 g_{ab,S^{n-2}}}{n-1}\left(-a\partial_t\bar{h}_{\rho\rho} - \sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} + (n-1)\partial_t\bar{h}_{ab}\right) - 2\partial_{(a}\bar{h}_{tb)} \\
&- 2g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td} - g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc} - 2\rho g_{ab,S^{n-2}}a\bar{h}_{t\rho} \\
&+\frac{g_{ab,S^{n-2}}\rho^2}{n-1}\Bigg(2qa\partial_\rho\bar{h}_{t\rho} - \left(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho}\right)\bar{h}_{t\rho} \\
&+\frac{2}{\rho^2}g^{cd}_{S^{n-2}}\partial_d\bar{h}_{tc} + \frac{1}{\rho^2}g^{ef}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ef,S^{n-2}}\bar{h}_{tc}\Bigg)\Bigg]\epsilon + \mathcal{O}(\epsilon^2). && \text{(4.122c)}
\end{aligned}$$

Now, we are able to derive the boundary behaviour for the conformally decomposed traceless part of the extrinsic curvature. Then, using Eq. (2.58d), we get

$$\begin{aligned}
\tilde{A}_{ij} &= \phi^2 A_{ij} = \hat{\phi}^2\left(1+\epsilon h_{\phi^2}\right)\left(\hat{A}_{ij} + \epsilon h_{A_{ij}}\right) + \mathcal{O}(\epsilon^2) \\
&= \hat{\phi}^2\left(\hat{A}_{ij} + \epsilon h_{A_{ij}} + \epsilon\hat{A}_{ij}h_{\phi^2}\right) + \mathcal{O}(\epsilon^2) \\
&= \hat{\phi}^2\hat{A}_{ij} + \hat{\phi}^2\left(h_{A_{ij}} + \hat{A}_{ij}h_{\phi^2}\right)\epsilon + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{4.123}$$

where we have used the boundary behaviour of the conformal factor ϕ, as defined in Eq. (4.50), the boundary behaviour of the traceless part of the extrinsic curvature A_{ij}, as given in Eq. (4.118) and kept the lowest order in the auxiliary variable ϵ. As the traceless part of the extrinsic curvature is purely perturbative, i.e. $\hat{A}_{ij} = 0$, the latter equation simplifies to

$$\tilde{A}_{ij} = \hat{\phi}^2 h_{A_{ij}}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.124}$$

By defining

$$h_{\tilde{A}_{ij}} = \hat{\phi}^2 h_{A_{ij}}, \tag{4.125}$$

we can write the boundary behaviour of $\tilde{A}_{ij}$ as

$$\tilde{A}_{ij} = \hat{\tilde{A}}_{ij} + \epsilon h_{\tilde{A}_{ij}} + \mathcal{O}(\epsilon^2), \tag{4.126}$$

where $\hat{\tilde{A}}_{ij}$ vanishes. Therefore, the boundary behaviour of the conformally decomposed tracless-part of the extrinsic curvature is purely perturbative. By using Eq. (4.48) and Eq. (4.127a)-(4.127c), we can explicitly write down the boundary behaviour of the components of the conformally decomposed traceless part as

$$\begin{aligned}
\tilde{A}_{\rho\rho} =& - \frac{a^{\frac{3-n}{2(n-1)}} q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[\frac{n-2}{n-1}\partial_t \bar{h}_{\rho\rho} - \frac{1}{(n-1)a} \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} - \frac{2(n-2)q}{n-1} \partial_\rho \bar{h}_{t\rho} \right. \\
& \left. + \frac{2g^{ab}_{S^{n-2}} \partial_b \bar{h}_{ta}}{(n-1)\rho^2 a} + \left(\frac{2(n-2)}{\ell} + \frac{2(n-2)q^2}{(n-1)a\rho} \right) \bar{h}_{t\rho} + \frac{g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}}}{(n-1)\rho^2 a} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} \right] \epsilon \\
& + \mathcal{O}(\epsilon^2), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.127a) \\
\tilde{A}_{\rho a} =& - \frac{a^{\frac{3-n}{2(n-1)}} q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[q\partial_t \bar{h}_{\rho a} - q\partial_a \bar{h}_{t\rho} + \left(\frac{n-2}{q\ell} + \frac{2}{\rho} \right) \bar{h}_{ta} - \partial_\rho \bar{h}_{ta} \right] \epsilon \\
& + \mathcal{O}(\epsilon^2), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.127b) \\
\tilde{A}_{ab} =& - \frac{a^{\frac{3-n}{2(n-1)}} q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[\frac{\rho^2 g_{ab,S^{n-2}}}{n-1} \left(-a\partial_t \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} + (n-1)\partial_t \bar{h}_{ab} \right) \right. \\
& - 2\partial_{(a} \bar{h}_{tb)} - 2g_{(ac,S^{n-2}} \partial_{b)} g^{dc}_{S^{n-2}} \bar{h}_{td} - g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} - 2\rho g_{ab,S^{n-2}} a \bar{h}_{t\rho} \\
& - \frac{g_{ab,S^{n-2}}\rho^2}{n-1} \left(-2qa\partial_\rho \bar{h}_{t\rho} + \left(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho} \right) \bar{h}_{t\rho} \right. \\
& \left.\left. - \frac{2}{\rho^2} g^{cd}_{S^{n-2}} \partial_d \bar{h}_{tc} - \frac{1}{\rho^2} g^{ef}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{ef,S^{n-2}} \bar{h}_{tc} \right) \right] \epsilon + \mathcal{O}(\epsilon^2). \qquad (4.127c)
\end{aligned}$$

As caculated in Appendix E.6, these equations verify that the trace of the conformally decomposed traceless part of the extrinsic curvature $\tilde{A}_{ij}$ with respect to the conformal spatial metric $\tilde{\gamma}_{ij}$ should vanish, i.e.

$$\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0. \tag{4.128}$$

This gives us a hint that the calculated equations for the conformally decomposed traceless part variables $\tilde{A}_{ij}$ of the state vector $\mathbf{Q}$ are correct. As already noted, these variables are governed purely by the small perturbation $\epsilon h_{\tilde{A}_{ij}}$, while the purely AdS$_n$ part vanishes.

4.5.8 The $\tilde{\Gamma}^i$ and $\hat{\Gamma}^i$ variables

Another fundamental variable of the state vector $\mathbf{Q}$ that was first introducced for the derivation of the $CCZ4$ system is the $\hat{\Gamma}^i$ variable, as defined in Eq. (2.58d). Before we can write down the boundary behaviour of $\hat{\Gamma}^i$, we need to calculate the boundary behaviour of the $\tilde{\Gamma}^i$ variable.

The $\tilde{\Gamma}^i$ variable

The auxiliary variable $\tilde{\Gamma}^i$ was first introduced to reduce the Ricci tensor to a Laplacian-like operator for the derivation of the BSSNOK system and is defined as

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}. \tag{4.129}$$

However, the idea is to allow for any value of the coordinate choice $\tilde{\Gamma}^i$ [27]. To be able to write down the boundary behaviour, let us first rewrite this variable by some fundamental variables of the state vector $\mathbf{Q}$. Thus,

$$\begin{aligned}
\tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = \tilde{\gamma}^{jk}\frac{1}{2}\tilde{\gamma}^{il}\left(\partial_j\tilde{\gamma}_{kl} + \partial_k\tilde{\gamma}_{jl} - \partial_l\tilde{\gamma}_{jk}\right) \\
&= \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_j\tilde{\gamma}_{kl} + \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_k\tilde{\gamma}_{jl} - \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_l\tilde{\gamma}_{jk} \\
&= \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_k\tilde{\gamma}_{jl} + \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_k\tilde{\gamma}_{jl} - \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_l\tilde{\gamma}_{jk} \\
&= \tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_k\tilde{\gamma}_{jl} - \frac{1}{2}\tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_l\tilde{\gamma}_{jk} \\
&= \tilde{\gamma}^{jk}\tilde{\gamma}^{il}\partial_k\tilde{\gamma}_{jl} \\
&= 2\tilde{\gamma}^{ij}\tilde{\gamma}^{kl}D_{ljk},
\end{aligned} \tag{4.130}$$

where we have switched, due to symmetry reason, $j \leftrightarrow k$ in the first term in the second line, and used

$$-\frac{1}{2}\tilde{\gamma}^{il}\tilde{\gamma}^{jk}\partial_l\tilde{\gamma}_{jk} = -\tilde{\gamma}^{il}\frac{1}{2}\operatorname{tr}(\partial_l\tilde{\gamma}_{jk}) = -\tilde{\gamma}^{il}\partial_l\sqrt{\det(\tilde{\gamma})} = 0, \tag{4.131}$$

as $\det(\tilde{\gamma}) = n$ is constant. Using Eq. (4.66) and the boundary behaviour for the auxiliary variable D_{ijk}, we can derive

$$\begin{aligned}
\tilde{\Gamma}^i &= 2\left(\hat{\gamma}^{ij} + \epsilon h_{\tilde{\gamma}^{ij}}\right)\left(\hat{\gamma}^{kl} + \epsilon h_{\tilde{\gamma}^{kl}}\right)\left(\hat{D}_{ljk} + \epsilon h_{D_{ljk}}\right) \\
&= 2\left(\hat{\gamma}^{ij}\hat{\gamma}^{kl} + \epsilon h_{\tilde{\gamma}^{ij}}\hat{\gamma}^{kl} + \epsilon h_{\tilde{\gamma}^{kl}}\hat{\gamma}^{ij}\right)\left(\hat{D}_{ljk} + \epsilon h_{D_{ljk}}\right) + \mathcal{O}(\epsilon^2) \\
&= 2\left(\hat{\gamma}^{ij}\hat{\gamma}^{kl}\hat{D}_{ljk} + \epsilon h_{\tilde{\gamma}^{ij}}\hat{\gamma}^{kl}\hat{D}_{ljk} + \epsilon h_{\tilde{\gamma}^{kl}}\hat{\gamma}^{ij}\hat{D}_{ljk} + \epsilon\hat{\gamma}^{ij}\hat{\gamma}^{kl}h_{D_{ljk}}\right) + \mathcal{O}(\epsilon^2) \\
&= 2\hat{\gamma}^{ij}\hat{\gamma}^{kl}\hat{D}_{ljk} + 2\left(h_{\tilde{\gamma}^{ij}}\hat{\gamma}^{kl}\hat{D}_{ljk} + h_{\tilde{\gamma}^{kl}}\hat{\gamma}^{ij}\hat{D}_{ljk} + \hat{\gamma}^{ij}\hat{\gamma}^{kl}h_{D_{ljk}}\right)\epsilon + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{4.132}$$

By defining the purely AdS$_n$ part as

$$\hat{\tilde{\Gamma}}^i = 2\hat{\gamma}^{ij}\hat{\gamma}^{kl}\hat{D}_{ljk} \tag{4.133}$$

and the perturbation as

$$h_{\tilde{\Gamma}^i} = 2\left(h_{\tilde{\gamma}^{ij}}\hat{\gamma}^{kl}\hat{D}_{ljk} + h_{\tilde{\gamma}^{kl}}\hat{\gamma}^{ij}\hat{D}_{ljk} + \hat{\gamma}^{ij}\hat{\gamma}^{kl}h_{D_{ljk}}\right), \tag{4.134}$$

we can write the boundary behaviour of $\tilde{\Gamma}^i$ as

$$\tilde{\Gamma}^i = \hat{\tilde{\Gamma}}^i + h_{\tilde{\Gamma}^i}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.135}$$

After some calculations that can be found in Appendix E.7, we can write down the explicit boundary behaviour of the components of the $\tilde{\Gamma}^i$ variable

$$
\begin{aligned}
\tilde{\Gamma}^\rho =& -\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{1}{1-n}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\partial_\rho a-\frac{2(n-2)}{n-1}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\\
&+\frac{n-2}{(n-1)^2}q^{n-2}\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{2(n-2)}{(n-1)^2}q^{n-2}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-\frac{n-2}{(n-1)\ell}q^{n-3}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{1}{n-1}\frac{q^{n-2}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{1-n}}}\left((n-2)\left(\partial_\rho a\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}\right)-\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}g^{ab}_{S^{n-2}}\partial_a\bar{h}_{\rho b}\epsilon+\frac{q^{n-1}}{n-1}\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\frac{g^{bc}_{S^{n-2}}\partial_c\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}g^{bc}_{S^{n-2}}g^{ad}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}q^{n-1}\bar{h}_{\rho d}\epsilon+\mathcal{O}(\epsilon^2),
\end{aligned}
\tag{4.136a}
$$

$$
\begin{aligned}
\tilde{\Gamma}^a =& -\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{1}{n-1}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\partial_\rho\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\left(\frac{2}{n-1}\frac{1}{\rho}+\frac{n-1}{q\ell}-\frac{n-2}{n-1}\frac{\partial_\rho a}{a}\right)\bar{h}_{\rho b}\epsilon\\
&+\frac{q^{n-2}a^{\frac{1}{1-n}}\rho^{\frac{2}{1-n}}}{(n-1)^2}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\epsilon\\
&-\frac{q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\left(a\partial_d\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i}-(n-1)\partial_d\bar{h}_{bc}\right)\epsilon\\
&+\frac{2}{n-1}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ae}_{S^{n-2}}g^{bf}_{S^{n-2}}g_{ef,S^{n-2}}\bar{h}_{ef}\epsilon\\
&-\frac{2}{(n-1)^2}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\left[a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right]\frac{g^{ab}_{S^{n-2}}\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}\epsilon\\
&+\mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.136b}
$$

Note that we can not just sum over $g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}$ in Eq. (4.136a) as the $\bar{h}_{\mu\nu}$ functions have the same indicies and we need to sum over all of them. Furthermore, the AdS$_n$ part of the boundary behaviour of $\tilde{\Gamma}^i$ is given by

$$
\begin{aligned}
\hat{\tilde{\Gamma}}^\rho =& -\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{1}{1-n}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\partial_\rho a\\
&-\frac{2(n-2)}{n-1}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}
\end{aligned}
\tag{4.137a}
$$

$$\hat{\tilde{\Gamma}}^a = -\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{1}{n-1}\frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} g^{ab}_{S^{n-2}}, \tag{4.137b}$$

while the rest is the small perturbation $\epsilon h_{\tilde{\Gamma}^i}$.

The fundamental $\hat{\Gamma}^i$ variable

Before we are able to state the asymptotic behaviour of the fundamental $\hat{\Gamma}$-variable, as defined in Eq. (2.58d), we need to derive the boundary behaviour of the algebraic constraint vector Z^μ. As the boundary behaviour of Z^μ should match those of the generators ξ^μ as in Eq. (12) in [5], their behaviour is given by

$$Z^t = \xi^t + \mathcal{O}(q), \quad Z^a = \xi^a + \mathcal{O}(q) \quad \text{and} \quad Z^\rho = -\frac{\rho^3}{qL}\left[\xi^\rho + \mathcal{O}(q)\right]. \tag{4.138}$$

As derived in Appendix E.8, the algebraic constraint vector Z_μ can be decomposed into a purely AdS$_n$ part of the algebraic vector plus a small perturbation, i.e.

$$Z_\mu = \hat{Z}_\mu + \epsilon h_{Z_\mu} + \mathcal{O}(\epsilon^2). \tag{4.139}$$

Therefore, the boundary behaviour of $\hat{\Gamma}^i$ is given by

$$\begin{aligned}
\hat{\Gamma}^i &= \tilde{\Gamma}^i + 2\tilde{\gamma}^{ij}Z_j \\
&= \hat{\tilde{\Gamma}}^i + \epsilon h_{\tilde{\Gamma}^i} + 2\left(\hat{\tilde{\gamma}}^{ij} + \epsilon h_{\tilde{\gamma}^{ij}}\right)\left(\hat{Z}_j + \epsilon h_{Z_j}\right) + \mathcal{O}(\epsilon^2) \\
&= \hat{\tilde{\Gamma}}^i + 2\hat{\tilde{\gamma}}^{ij}\hat{Z}_j + \epsilon h_{\tilde{\Gamma}^i} + 2\epsilon\hat{\tilde{\gamma}}^{ij}h_{Z_j} + 2\epsilon h_{\tilde{\gamma}^{ij}}\hat{Z}_j + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{4.140}$$

where we have used the boundary behaviour of the $\tilde{\Gamma}^i$ variable as in Eq. (4.135), the boundary behaviour of the algebraic constraint four-vector Z_μ as in E.8 and the boundary behaviour of the conformally decomposed spatial metric $\tilde{\gamma}_{ij}$ as defined in Eq. (4.62). By defining

$$\hat{\hat{\Gamma}}^i = \hat{\tilde{\Gamma}}^i + 2\hat{\tilde{\gamma}}^{ij}\hat{Z}_j, \tag{4.141}$$

and

$$h_{\hat{\Gamma}^i} = h_{\tilde{\Gamma}^i} + 2\hat{\tilde{\gamma}}^{ij}h_{Z_j} + 2\hat{\tilde{\gamma}}^{ij}h_{\tilde{\gamma}^{ij}}\hat{Z}_j, \tag{4.142}$$

we can decompose the boundary behaviour into a purely AdS$_n$ part plus some perturbation

$$\hat{\Gamma}^i = \hat{\hat{\Gamma}}^i + \epsilon h_{\hat{\Gamma}^i} + \mathcal{O}(\epsilon^2), \tag{4.143}$$

Now, we are ready to state the boundary behaviour of the $\hat{\Gamma}^i$ variables. The calculation can be found in Appendix E.9, while the components can be summarized as

$$\begin{aligned}
\hat{\Gamma}^\rho = \tilde{\Gamma}^\rho &+ \frac{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}Z^\rho + 2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-3}\bar{h}_{t\rho}Z^t\epsilon \\
&+ \frac{2q^{n-4}}{n-1}\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)Z^\rho\epsilon + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{4.144a}$$

$$\hat{\Gamma}^a = \tilde{\Gamma}^a + 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}\frac{\rho^{\frac{2(n-2)}{n-1}}}{q^2}Z^a + 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}q^{n-4}\bar{h}_{tb}Z^t\epsilon$$
$$+ \frac{2q^{n-4}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)Z^a\epsilon + \mathcal{O}(\epsilon^2), \qquad (4.144b)$$

where the first terms without an $\bar{h}_{ij}$ term represents the purely AdS part, while the terms containing an ϵ term represent the perturbation. We can see that the boundary behaviour of non of these three variables is purely perturbative, but has at least one purely AdS term.

4.5.9 The t-component of the algebraic constraint

The algebraic constraint Z vector, as defined in Eq. (2.49), was first introduced in Section 2.4 to replace the constraint equations with evolution equations.

The boundary behaviour of the t-component of the algebraic constraint, i.e. θ, can be derived by inserting the boundary behaviour of the lapse function α, as defined in Eq. (4.50), into Eq. (2.49). Then,

$$\theta = \alpha Z^t = \hat{\alpha}\left(1 + \epsilon h_\alpha\right)Z^t + \mathcal{O}(\epsilon^2) = \frac{\sqrt{a}}{q}\left(1 - \frac{q^{n-2}}{2a}\bar{h}_{tt}\epsilon\right)Z^t + \mathcal{O}(\epsilon^2)$$
$$= \frac{\sqrt{a}}{q}Z^t - \frac{q^{n-3}}{2\sqrt{a}}\bar{h}_{tt}Z^t\epsilon + \mathcal{O}(\epsilon^2). \qquad (4.145)$$

By defining

$$\hat{\theta} = \frac{\sqrt{a}}{q}Z^t \qquad (4.146)$$

and

$$h_\theta = -\frac{q^{n-3}}{2\sqrt{a}}\bar{h}_{tt}Z^t, \qquad (4.147)$$

we can write the boundary behaviour of the t-component of the algebraic constraint as

$$\theta = \hat{\theta} + \epsilon h_\theta + \mathcal{O}(\epsilon^2). \qquad (4.148)$$

We see that the purely AdS$_n$ part diverges at the boundary $q = 0$, while the small perturbation ϵh_θ vanishes.

4.5.10 The auxiliary field b^i

The auxiliary field b^i was introduced in [27] to rewrite the second-order hyperbolic PDE for the shift vector β as a first-order system. It is, therefore, part of the standard Gamma-driver shift-condition-choice for choosing spatial coordinates.

Substituting the boundary condition of the shift vector β, as defined in Eq. (4.71), and the boundary behaviour of the auxiliary variable B^l_k, as defined in Eq. (4.99) into Eq. (2.88a) for the Gamma-driver, we obtain the boundary behaviour of the auxiliary field. Thus,

$$b^i = \frac{1}{k}\left[\partial_t\beta^i - \beta^k B^i_k\right]$$

$$
\begin{aligned}
&= \frac{1}{k}\left[\partial_t\left(\hat{\beta}^i + \epsilon h_{\beta^i}\right) - \left(\hat{\beta}^k + \epsilon h_{\beta^k}\right)\left(\hat{B}^i_k + \epsilon h_{B^i_k}\right)\right] + \mathcal{O}(\epsilon^2)\\
&= \frac{1}{k}\left[\partial_t\hat{\beta}^i + \epsilon\partial_t h_{\beta^i} - \hat{\beta}^k\hat{B}^i_k - \epsilon h_{\beta^k}\hat{B}^i_k - \epsilon\hat{\beta}^k h_{B^i_k}\right] + \mathcal{O}(\epsilon^2)\\
&= \frac{1}{k}\left[\partial_t\hat{\beta}^i - \hat{\beta}^k\hat{B}^i_k\right] + \frac{1}{k}\left[\partial_t h_{\beta^i} - h_{\beta^k}\hat{B}^i_k - \hat{\beta}^k h_{B^i_k}\right]\epsilon + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.149}
$$

By defining

$$
\hat{b}^i = \frac{1}{k}\left[\partial_t\hat{\beta}^i - \hat{\beta}^k\hat{B}^i_k\right] \tag{4.150}
$$

and

$$
h_{b^i} = \frac{1}{k}\left[\partial_t h_{\beta^i} - h_{\beta^k}\hat{B}^i_k - \hat{\beta}^k h_{B^i_k}\right], \tag{4.151}
$$

we can decompose the boundary behaviour of b^i into

$$
b^i = \hat{b}^i + \epsilon h_{b^i} + \mathcal{O}(\epsilon^2), \tag{4.152}
$$

where $\hat{b}^i = 0$. Whereas a full derivation can be found in Appendix E.10, the boundary behaviour of the components are summarized as

$$
b^\rho = \frac{q^{n-1}a}{k}\partial_t\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2), \tag{4.153a}
$$

$$
b^a = \frac{q^{n-2}}{k\rho^2} g^{ab}_{S^{n-2}}\partial_t\bar{h}_{tb}\epsilon + \mathcal{O}(\epsilon^2). \tag{4.153b}
$$

We note that the boundary behaviour of the auxiliary field b^i is governed by the small perturbation ϵh_{b^i}, while the AdS$_n$ part vanishes.

Finally, we have derived for all variables of the state vector $\mathbf{Q}$ the boundary behaviour near $q = 0$. We can summarize that we can decompose the boundary behaviour of each term of the state vector into a purely AdS$_n$ part of this very variable plus a small perturbation. This can be written, if u is a variable of $\mathbf{Q}$, as

$$
u = \hat{u} + \epsilon h_u + \mathcal{O}(\epsilon^2). \tag{4.154}
$$

By using the explicit formulas for the variables of the state vector $\mathbf{Q}$, we gained some intuition on how these fields behave near the boundary.

Furthermore, we have written a Mathematica script [36] that double-checks the boundary condition equations for all variables of the state vector $\mathbf{Q}$ in three and four dimensions. This gives us a good hint that the generalized boundary equations for the variables of the state vector $\mathbf{Q}$ for an asymptotically AdS$_n$ spacetime are as well correct.

To obtain a well-posed and deterministic initial value problem for the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b) for an asymptotically Anti-de Sitter spacetime parametrized by the coordinates $(t, \rho, \theta_{n-2}, \dots, \theta_1) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi)^{n-3} \times (0, 2\pi)$ and with time-symmetric initial data from the PDE (3.25), we need to explicitly set the derived boundary conditions (4.67a)-(4.67c) for $\tilde{\gamma}_{ij}$, (4.76) for $\ln\alpha$, (4.70a)-(4.70b) for β^i, (4.51) for $\log\phi$, (4.127a)-(4.127c) for $\tilde{A}_{ij}$, (4.117) for K, (4.145) for θ, (4.144a)-(4.144b) for $\hat{\Gamma}^i$, (4.153a)-(4.153b) for b^i, (4.83a)-(4.83a) for A_k, (4.100a)-(4.100d) for

B^i_k, (E.17a)-(4.106f) for D_{ijk} and (4.92a)-(4.92b) for P_k at spacelike infinity in the ExaHyPE code.

As we aim to numerically solve the latter time evolution system for a $n = 3$ and $n = 4$ AdS spacetime, let us explicitly write out the boundary conditions for an AdS_3 and AdS_4 space in the following section.

Chapter 5

Boundary Conditions for an AdS_3 and AdS_4 spacetime

"The Cauchy problem for an evolution equation is the problem of determining this evolution from the knowledge of its initial value." [28]
T. Gowers
Mathematician

In the following section, we state, respectively, the boundary conditions at $\rho = \ell$ or $q = 0$ for the 30 and 58 variables of the state vector $\mathbf{Q}$ to obtain a well-posed and deterministic initial value problem for the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b) for an asymptotically Anti-de Sitter spacetime parametrized by the coordinates

$$(t, \rho, \theta_{n-2}, \ldots, \theta_1) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi)^{n-3} \times (0, 2\pi) \tag{5.1}$$

and with initial data (3.25) that still needs to be solved for.

5.1 Boundary conditions for an AdS_3 spacetime

We will set $\ell = 1$ and $L = 1$ in the code, thereby fixing the cosmological constant to $\Lambda_3 = -1$ and keep, for the sake of simplicity, the following quantities

$$a(\rho) = q^2 + \frac{\rho^2}{L^2} \qquad \text{and} \qquad q(\rho) = 1 - \frac{\rho}{\ell}. \tag{5.2}$$

Note that $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}(t, \rho, \chi)$, depending on the spherical coordinates (t, ρ, χ), satisfies a Dirichlet boundary problem, i.e.

$$\begin{aligned}
\bar{h}_{tt}\Big|_{\rho=\ell} &= 0 \\
\bar{h}_{t\rho}\Big|_{\rho=\ell} &= 0 \\
\bar{h}_{t\chi}\Big|_{\rho=\ell} &= 0 \\
\bar{h}_{\rho\rho}\Big|_{\rho=\ell} &= 0
\end{aligned}$$

$$\bar{h}_{\rho\chi}\Big|_{\rho=\ell} = 0$$
$$\bar{h}_{\chi\chi}\Big|_{\rho=\ell} = 0, \tag{5.3}$$

while the origin and axis regularity conditions, taken from [5], are given, respectively, by

$$\begin{aligned} \partial_\rho \bar{h}_{tt}\Big|_{\rho=0} &= 0 \\ \bar{h}_{t\rho}\Big|_{\rho=0} &= 0 \\ \partial_\rho \bar{h}_{t\chi}\Big|_{\rho=0} &= 0 \\ \partial_\rho \bar{h}_{\rho\rho}\Big|_{\rho=0} &= 0 \\ \bar{h}_{\rho\chi}\Big|_{\rho=0} &= 0 \\ \partial_\rho \bar{h}_{\chi\chi}\Big|_{\rho=0} &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \partial_\chi \bar{h}_{tt}\Big|_{\chi=0,\pi} &= 0 \\ \partial_\chi \bar{h}_{t\rho}\Big|_{\chi=0,\pi} &= 0 \\ \bar{h}_{t\chi}\Big|_{\chi=0,\pi} &= 0 \\ \partial_\chi \bar{h}_{\rho\rho}\Big|_{\chi=0,\pi} &= 0 \\ \bar{h}_{\rho\chi}\Big|_{\chi=0,\pi} &= 0 \\ \partial_\chi \bar{h}_{\chi\chi}\Big|_{\chi=0,\pi} &= 0, \end{aligned} \tag{5.4}$$

Now, the 30 boundary equations have to satisfy the following equations.

The boundary behaviour of the $\tilde{\gamma}_{ij}$ variable

is given by

$$\tilde{\gamma}_{\rho\rho} = \frac{1}{\sqrt{a}\rho}\left(1 + \frac{q}{2}\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right)\right) + \mathcal{O}(h^2), \tag{5.5a}$$
$$\tilde{\gamma}_{\rho\chi} = \frac{q^2\sqrt{a}}{\rho}\bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.5b}$$
$$\tilde{\gamma}_{\chi\chi} = \sqrt{a}\rho\left(1 + \frac{q}{2}\left(-a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi}\right)\right) + \mathcal{O}(h^2), \tag{5.5c}$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the logarithm of the lapse function $\ln\alpha$

is given by

$$\ln\alpha = \ln(1-q) + \ln\frac{\ell}{L} + \frac{1}{2}\ln\left(1 + \frac{q^2L^2}{\rho^2}\right) - \ln q - \frac{q}{2a}\bar{h}_{tt} + \mathcal{O}(h^2), \tag{5.6}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

The boundary behaviour of the shift vector β

is given by

$$\beta^\rho = q^2 a\bar{h}_{t\rho} + \mathcal{O}(h^2), \tag{5.7a}$$
$$\beta^\chi = \frac{q}{\rho^2}\bar{h}_{t\chi} + \mathcal{O}(h^2), \tag{5.7b}$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the logarithm of the conformal factor $\ln\phi$

is given by

$$\ln\phi = \ln q + \frac{\ln a}{4} - \frac{\ln\rho}{2} - \frac{q}{4}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \tag{5.8}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

The boundary behaviour of the conformally decomposed traceless-part of the extrinsic curvature $\tilde{A}_{ij}$

is given by

$$\begin{aligned}
\tilde{A}_{\rho\rho} =& -\frac{q^2}{2\rho}\left[\frac{1}{2}\partial_t\bar{h}_{\rho\rho} - \frac{1}{2a}\partial_t\bar{h}_{\chi\chi} - q\partial_\rho\bar{h}_{t\rho} + \frac{1}{\rho^2 a}\partial_\chi\bar{h}_{t\chi} + \left(\frac{2}{\ell} + \frac{q^2}{a\rho}\right)\bar{h}_{t\rho}\right] \\
&+ \mathcal{O}(h^2), && (5.9a) \\
\tilde{A}_{\rho\chi} =& -\frac{q^2}{2\rho}\left[q\partial_t\bar{h}_{\rho\chi} - q\partial_\chi\bar{h}_{t\rho} + \left(\frac{1}{q\ell} + \frac{2}{\rho}\right)\bar{h}_{t\chi} - \partial_\rho\bar{h}_{t\chi}\right] + \mathcal{O}(h^2), && (5.9b) \\
\tilde{A}_{\chi\chi} =& -\frac{q^2\rho}{4}\left[-a\partial_t\bar{h}_{\rho\rho} + \partial_t\bar{h}_{\chi\chi} - \frac{2}{\rho^2}\partial_\chi\bar{h}_{t\chi} + 2qa\partial_\rho\bar{h}_{t\rho} - \left(\frac{4a}{l} + \frac{2q^2}{\rho}\right)\bar{h}_{t\rho}\right] \\
&+ \mathcal{O}(h^2), && (5.9c)
\end{aligned}$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the trace of the extrinsic curvature K

is given by

$$\begin{aligned}
K = -\frac{q^2}{2\sqrt{a}}\Bigg(& a\partial_t\bar{h}_{\rho\rho} + \partial_t\bar{h}_{\chi\chi} - 2qa\partial_\rho\bar{h}_{t\rho} \\
&+ \left(\frac{4a}{\ell} - \frac{2q^2}{\rho} - \frac{4\rho}{L^2}\right)\bar{h}_{t\rho} - \frac{2}{\rho^2}\partial_\chi\bar{h}_{t\chi}\Bigg) + \mathcal{O}(h^2),
\end{aligned} \tag{5.10}$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the θ **variable**

is given by

$$\theta = \frac{\sqrt{a}}{q}Z^t - \frac{1}{2\sqrt{a}}\bar{h}_{tt}Z^t + \mathcal{O}(h^2), \tag{5.11}$$

and diverges at the boundary $q = 0$.

The boundary behaviour of the $\hat{\Gamma}^i$ variable

is given by

$$\begin{aligned}
\hat{\Gamma}^\rho =& -\frac{1}{2}\frac{\rho}{\sqrt{a}}\partial_\rho a - \sqrt{a} + \left(\frac{1}{4}\frac{q\rho}{\sqrt{a}}\partial_\rho a + \frac{1}{2}q\sqrt{a} - \frac{1}{2\ell}\sqrt{a}\rho\right)\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right) \\
&+ \frac{1}{2}\sqrt{a}q\rho\left(\partial_\rho a\bar{h}_{\rho\rho} + a\partial_\rho\bar{h}_{\rho\rho} - \partial_\rho\bar{h}_{\chi\chi}\right) + \frac{q^2\sqrt{a}}{\rho}\partial_\chi\bar{h}_{\rho\chi} \\
&+ \frac{2\rho}{\sqrt{a}q^2}Z^\rho + 2\rho\sqrt{a}\bar{h}_{t\rho}Z^t + \frac{\rho}{q\sqrt{a}}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi}\right)Z^\rho + \mathcal{O}(h^2), \qquad (5.12a)\\
\hat{\Gamma}^\chi =& \frac{q^2\sqrt{a}}{\rho}\partial_\rho\bar{h}_{\rho\chi} - \frac{q^2\sqrt{a}}{\rho}\left(\frac{1}{\rho} + \frac{2}{q\ell} - \frac{1}{2}\frac{\partial_\rho a}{a}\right)\bar{h}_{\rho\chi} - \frac{1}{2}\frac{q}{\rho\sqrt{a}}\left(a\partial_\chi\bar{h}_{\rho\rho} - \partial_\chi\bar{h}_{\chi\chi}\right) \\
&+ \frac{2\rho}{\sqrt{a}q^2}Z^\chi + \frac{2}{\sqrt{a}q\rho}\bar{h}_{t\chi}Z^t + \frac{\rho}{q\sqrt{a}}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi}\right)Z^\chi + \mathcal{O}(h^2), \qquad (5.12b)
\end{aligned}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

The boundary behaviour of the auxiliary field b^i

is given by

$$b^\rho = \frac{q^2 a}{k}\partial_t\bar{h}_{t\rho} + \mathcal{O}(h^2), \qquad (5.13a)$$

$$b^\chi = \frac{q}{k\rho^2}\partial_t\bar{h}_{t\chi} + \mathcal{O}(h^2), \qquad (5.13b)$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the auxiliary variable A_i

is given by

$$A_\rho = \frac{\rho}{L^2 aq} + \left(\frac{q\rho}{a^2L^2} - \frac{q^2}{a^2\ell} + \frac{1}{2a\ell}\right)\bar{h}_{tt} - \frac{q}{2a}\partial_\rho\bar{h}_{tt} + \mathcal{O}(h^2), \qquad (5.14a)$$

$$A_\chi = -\frac{q}{2a}\partial_\chi\bar{h}_{tt} + \mathcal{O}(h^2), \qquad (5.14b)$$

and diverges at the boundary $q = 0$.

The boundary behaviour of the auxiliary variable B^i_k

is given by

$$B^\rho_\rho = \left(-\frac{2qa}{\ell} - q^2\left(\frac{2q}{\ell} - \frac{2\rho}{L^2}\right)\right)\bar{h}_{t\rho} + q^2 a\partial_\rho\bar{h}_{t\rho} + \mathcal{O}(h^2), \qquad (5.15a)$$

$$B^\rho_\chi = q^2 a\partial_\chi\bar{h}_{t\rho} + \mathcal{O}(h^2), \qquad (5.15b)$$

$$B^\chi_\rho = -\left(\frac{1}{\rho^2\ell} + \frac{2q}{\rho^3}\right)\bar{h}_{t\chi} + \frac{q}{\rho^2}\partial_\rho\bar{h}_{t\chi} + \mathcal{O}(h^2), \qquad (5.15c)$$

$$B^\chi_\chi = \frac{q}{\rho^2}\partial_\chi\bar{h}_{t\chi} + \mathcal{O}(h^2), \qquad (5.15d)$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the auxiliary variable D_{ijk}

is given by

$$D_{\chi\chi\chi} = -\frac{1}{4}\sqrt{a}\rho q\left(a\partial_\chi \bar{h}_{\rho\rho} - \partial_\chi \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \tag{5.16a}$$

$$D_{\chi\rho\rho} = \frac{q}{4\sqrt{a}\rho}\left(a\partial_\chi \bar{h}_{\rho\rho} - \partial_\chi \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \tag{5.16b}$$

$$D_{\chi\rho\chi} = \frac{\sqrt{a}q^2}{2\rho}\partial_\chi \bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.16c}$$

$$\begin{aligned} D_{\rho\rho\rho} = &-\left(\frac{1}{2L^2 a^{3/2}} - \frac{q}{2\ell\rho a^{3/2}} + \frac{1}{2\sqrt{a}\rho^2}\right)\left(1 + \frac{q}{2}\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right)\right) \\ &+ \frac{q}{4\rho\sqrt{a}}\left(\partial_\rho a\bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} - \partial_\rho \bar{h}_{\chi\chi}\right) - \frac{1}{4\rho\sqrt{a}\ell}\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.16d}$$

$$D_{\rho\rho\chi} = \left(\frac{q^2}{2L^2\sqrt{a}} - \frac{q^3}{2\ell\rho\sqrt{a}} - \frac{\sqrt{a}q}{\rho\ell} - \frac{\sqrt{a}q^2}{2\rho^2}\right)\bar{h}_{\rho\chi} + \frac{\sqrt{a}q^2}{2\rho}\partial_\rho \bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.16e}$$

$$\begin{aligned} D_{\rho\chi\chi} = &\frac{\rho^2}{2L^2\sqrt{a}} - \frac{\rho q}{2\ell\sqrt{a}} + \frac{\sqrt{a}}{2} - \left(\frac{q\rho\partial_\rho a}{8\sqrt{a}} + \frac{q\sqrt{a}}{4} - \frac{\sqrt{a}\rho}{2\ell}\right)\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right) \\ &- \frac{\sqrt{a}\rho}{4\ell}\left(a\bar{h}_{\rho\rho} - \bar{h}_{\chi\chi}\right) - \frac{\sqrt{a}\rho q}{4}\left(\partial_\rho a\bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} - \partial_\rho \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.16f}$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the auxiliary variable P_i

is given by

$$\begin{aligned} P_\rho = &-\frac{1}{ql} + \frac{\rho}{2aL^2} - \frac{q}{2a\ell} - \frac{1}{2\rho} + \frac{1}{4\ell}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi}\right) \\ &- \frac{q}{4}\left(\left(-\frac{2q}{\ell} + \frac{2\rho}{L^2}\right)\bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} + \partial_\rho \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.17a}$$

$$P_\chi = -\frac{q}{4}\left(a\partial_\chi \bar{h}_{\rho\rho} + \partial_\chi \bar{h}_{\chi\chi}\right) + \mathcal{O}(h^2), \tag{5.17b}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

5.2 Boundary conditions for an AdS$_4$ spacetime

We will set $\ell = 1$ and $L = 1$ in the code, thereby fixing the cosmological constant to $\Lambda_4 = -3$ and keep, for the sake of simplicity, the following quantities

$$a(\rho) = q^2 + \frac{\rho^2}{L^2} \qquad \text{and} \qquad q(\rho) = 1 - \frac{\rho}{\ell}. \tag{5.18}$$

Note that $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}(t, \rho, \chi, \theta)$, depending on the spherical coordinates (t, ρ, χ, θ), satisfies a Dirichlet boundary problem, i.e.

$$\bar{h}_{tt}\Big|_{\rho=\ell} = 0$$

$$
\begin{aligned}
\bar{h}_{t\rho}\Big|_{\rho=\ell} &= 0\\
\bar{h}_{t\chi}\Big|_{\rho=\ell} &= 0\\
\bar{h}_{\rho\rho}\Big|_{\rho=\ell} &= 0\\
\bar{h}_{\rho\chi}\Big|_{\rho=\ell} &= 0\\
\bar{h}_{\chi\chi}\Big|_{\rho=\ell} &= 0\\
\bar{h}_{\theta\theta}\Big|_{\rho=\ell} &= 0,
\end{aligned} \tag{5.19}
$$

while the origin and axis regularity conditions, taken from [5], are given, respectively, by

$$
\begin{aligned}
\partial_\rho\bar{h}_{tt}\Big|_{\rho=0} &= 0 & \qquad & \partial_\chi\bar{h}_{tt}\Big|_{\chi=0,\pi} &= 0\\
\bar{h}_{t\rho}\Big|_{\rho=0} &= 0 & & \partial_\chi\bar{h}_{t\rho}\Big|_{\chi=0,\pi} &= 0\\
\partial_\rho\bar{h}_{t\chi}\Big|_{\rho=0} &= 0 & & \bar{h}_{t\chi}\Big|_{\chi=0,\pi} &= 0\\
\partial_\rho\bar{h}_{\rho\rho}\Big|_{\rho=0} &= 0 & \text{and} & \partial_\chi\bar{h}_{\rho\rho}\Big|_{\chi=0,\pi} &= 0\\
\bar{h}_{\rho\chi}\Big|_{\rho=0} &= 0 & & \bar{h}_{\rho\chi}\Big|_{\chi=0,\pi} &= 0\\
\partial_\rho\bar{h}_{\chi\chi}\Big|_{\rho=0} &= 0 & & \partial_\chi\bar{h}_{\chi\chi}\Big|_{\chi=0,\pi} &= 0\\
\partial_\rho\bar{h}_{\theta\theta}\Big|_{\rho=0} &= 0 & & \partial_\chi\bar{h}_{\theta\theta}\Big|_{\chi=0,\pi} &= 0.
\end{aligned} \tag{5.20}
$$

Now, the 58 boundary equations have to satisfy the following equations.

The boundary behaviour of the $\tilde{\gamma}_{ij}$ variable

is given by

$$\tilde{\gamma}_{\rho\rho} = \frac{1}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi} - \frac{q^2}{3a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(-2a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} + \bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.21a}$$

$$\tilde{\gamma}_{\rho\chi} = \frac{a^{1/3}q^3}{\rho^{4/3}\sin^{2/3}\chi}\bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.21b}$$

$$\tilde{\gamma}_{\rho\theta} = 0, \tag{5.21c}$$

$$\tilde{\gamma}_{\chi\chi} = \frac{a^{1/3}\rho^{2/3}}{\sin^{2/3}\chi} - \frac{q^2a^{1/3}\rho^{2/3}}{3\sin^{2/3}\chi}\left(a\bar{h}_{\rho\rho} - 2\bar{h}_{\chi\chi} + \bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.21d}$$

$$\tilde{\gamma}_{\chi\theta} = 0, \tag{5.21e}$$

$$\tilde{\gamma}_{\theta\theta} = a^{1/3}\rho^{2/3}\sin^{4/3}\chi - \frac{q^2a^{1/3}\rho^{2/3}\sin^{4/3}\chi}{3}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} - 2\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.21f}$$

and diverges at the origin $\rho = 0$ and for the angular coordinates $\chi = \{0, \pi\}$. The (ρ, θ) and (χ, θ) components vanish due to the symmetry reason discussed in Section 4.5.

The boundary behaviour of the logarithm of the lapse function $\ln\alpha$

is given by

$$\ln\alpha = \ln(1-q) + \ln\frac{\ell}{L} + \frac{1}{2}\ln\left(1+\frac{q^2L^2}{\rho^2}\right) - \ln q - \frac{q^2}{2a}h_{tt} + \mathcal{O}(h^2), \tag{5.22}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

The boundary behaviour of the shift vector β

is given by

$$\beta^\rho = q^3 a\bar{h}_{t\rho} + \mathcal{O}(h^2), \tag{5.23a}$$
$$\beta^\chi = \frac{q^2}{\rho^2}\bar{h}_{t\chi} + \mathcal{O}(h^2), \tag{5.23b}$$
$$\beta^\theta = 0, \tag{5.23c}$$

diverges at the origin $\rho = 0$ and for the angular coordinates $\chi = 0, \pi$. The θ-component vanishes due to symmetry reasons.

The boundary behaviour of the logarithm of the conformal factor $\ln\phi$

is given by

$$\ln\phi = \ln q + \frac{1}{6}\ln a - \frac{2}{3}\ln\rho - \frac{1}{3}\ln\sin\chi - \frac{q^2}{6}\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} + \bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.24}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

The boundary behaviour of the conformally decomposed traceless part of the extrinsic curvature $\tilde{A}_{ij}$

is given by

$$\begin{aligned}\tilde{A}_{\rho\rho} = -\frac{q^3}{2a^{1/6}\rho^{4/3}\sin^{2/3}\chi}\Bigg(&\frac{2}{3}\partial_t\bar{h}_{\rho\rho} - \frac{1}{3a}\partial_t\bar{h}_{\theta\theta} - \frac{1}{3a}\partial_t\bar{h}_{\chi\chi} - \frac{4q}{3}\partial_\rho\bar{h}_{t\rho}\\ &+\left(\frac{4}{\ell} + \frac{4q^2}{3a\rho}\right)\bar{h}_{t\rho} + \frac{2}{3a\rho^2}\left(\partial_\chi\bar{h}_{t\chi} + \cot\chi\bar{h}_{t\chi}\right)\Bigg) + \mathcal{O}(h^2),\end{aligned} \tag{5.25a}$$

$$\begin{aligned}\tilde{A}_{\chi\chi} = -\frac{q^3\rho^{2/3}}{2a^{1/6}\sin^{2/3}\chi}\Bigg(&\frac{2}{3}\partial_t\bar{h}_{\chi\chi} - \frac{a}{3}\partial_t\bar{h}_{\rho\rho} - \frac{1}{3}\partial_t\bar{h}_{\theta\theta} - \frac{4}{3\rho^2}\partial_\chi\bar{h}_{t\chi} + \frac{2qa}{3}\partial_\rho\bar{h}_{t\rho}\\ &-\left(\frac{2a}{\ell} + \frac{2q^2}{3\rho}\right)\bar{h}_{t\rho} + \frac{2}{3\rho^2}\cot\chi\bar{h}_{t\chi}\Bigg) + \mathcal{O}(h^2),\end{aligned} \tag{5.25b}$$

$$\tilde{A}_{\theta\theta} = -\frac{q^3\rho^{2/3}\sin^{4/3}\chi}{2a^{1/6}}\Bigg(-\frac{a}{3}\partial_t\bar{h}_{\rho\rho} - \frac{1}{3}\partial_t\bar{h}_{\chi\chi} + \frac{2}{3}\partial_t\bar{h}_{\theta\theta} + \frac{2qa}{3}\partial_\rho\bar{h}_{t\rho}$$

$$- \left(\frac{2a}{\ell} + \frac{2q^2}{3\rho} \right) \bar{h}_{t\rho} - \frac{4 \cot \chi}{3\rho^2} \bar{h}_{t\chi} + \frac{2}{3\rho^2} \partial_\chi \bar{h}_{t\chi} \Bigg) + \mathcal{O}(h^2), \quad (5.25c)$$

$$\tilde{A}_{\rho\chi} = -\frac{q^3}{2a^{1/6}\rho^{4/3}\sin^{2/3}\chi} \Bigg(q\partial_t \bar{h}_{\rho\chi} - \partial_\rho \bar{h}_{t\chi} + \left(\frac{2}{q\ell} + \frac{2}{\rho} \right) \bar{h}_{t\chi} - q\partial_\chi \bar{h}_{t\rho} \Bigg) + \mathcal{O}(h^2), \quad (5.25d)$$

$$\tilde{A}_{\rho\theta} = \frac{q^4}{2a^{1/6}\rho^{4/3}\sin^{2/3}\chi} \partial_\theta \bar{h}_{t\rho} + \mathcal{O}(h^2), \quad (5.25e)$$

$$\tilde{A}_{\theta\chi} = \frac{q^3}{2a^{1/6}\rho^{4/3}\sin^{2/3}\chi} \partial_\theta \bar{h}_{t\chi} + \mathcal{O}(h^2), \quad (5.25f)$$

and diverges at the origin $\rho = 0$ and for the angular coordinates $\chi = \{0, \pi\}$.

The boundary behaviour of the trace of the extrinsic curvature K

is given by

$$K = -\frac{q^3}{2\sqrt{a}} \Bigg(a\partial_t \bar{h}_{\rho\rho} + \partial_t \bar{h}_{\theta\theta} + \partial_t \bar{h}_{\chi\chi} - 2qa\partial_\rho \bar{h}_{t\rho} + \left(\frac{6a}{\ell} - \frac{6\rho}{L^2} - \frac{4q^2}{\rho} \right) \bar{h}_{t\rho} - \frac{2}{\rho^2} \left(\partial_\chi \bar{h}_{t\chi} + \cot \chi \bar{h}_{t\chi} \right) \Bigg) + \mathcal{O}(h^2), \quad (5.26)$$

and diverges at the origin $\rho = 0$.

The boundary behaviour of the θ variable

is given by

$$\theta = \frac{\sqrt{a}}{q} Z^t - \frac{q}{2\sqrt{a}} \bar{h}_{tt} Z^t + \mathcal{O}(h^2), \quad (5.27)$$

diverges at the boundary $q = 0$.

The boundary behaviour of the $\hat{\Gamma}^i$ variable,

given by

$$\begin{aligned}
\hat{\Gamma}^\rho = & -\frac{2}{3} \frac{\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right) - \frac{4}{3} \rho^{1/3} a^{2/3} \sin^{2/3}\chi + \frac{2\rho^{4/3}\sin^{2/3}\chi}{q^2 a^{1/3}} Z^\rho \\
& - \frac{2}{9} q^2 a^{2/3} \rho^{4/3} \sin^{2/3}\chi \left(\frac{1}{a} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right) + \frac{2}{\rho} - \frac{3}{q\ell} \right) \left(-2a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} + \bar{h}_{\theta\theta} \right) \\
& - \frac{1}{3} q^2 a^{2/3} \rho^{4/3} \sin^{2/3}\chi \left(-2\partial_\rho a \bar{h}_{\rho\rho} - 2a\partial_\rho \bar{h}_{\rho\rho} + \partial_\rho \bar{h}_{\chi\chi} + \partial_\rho \bar{h}_{\theta\theta} \right) \\
& + \frac{q^3 a^{2/3} \sin^{2/3}\chi}{\rho^{2/3}} \partial_\chi \bar{h}_{\rho\chi} + \frac{2}{3} \frac{q^3 a^{2/3} \sin^{2/3}\chi \cot\chi}{\rho^{2/3}} \bar{h}_{\rho\chi} + 2qa^{2/3}\rho^{4/3}\sin^{2/3}\chi \bar{h}_{t\rho} Z^t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{3}\frac{\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}}\left(a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)Z^{\rho}+\mathcal{O}(h^2), &(5.28a)\\
\hat{\Gamma}^{\chi} =& -\frac{2}{3}\frac{\sin^{2/3}\chi\cot\chi}{\rho^{2/3}a^{1/3}}+\frac{2\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}q^2}Z^{\chi}+\frac{q^3a^{2/3}\sin^{2/3}\chi}{\rho^{2/3}}\partial_\rho\bar{h}_{\rho\chi}\\
&-\frac{q^3a^{2/3}\sin^{2/3}\chi}{\rho^{2/3}}\left(\frac{2}{3\rho}+\frac{3}{q\ell}-\frac{4\rho}{3aL^2}+\frac{4q}{3a\ell}\right)\bar{h}_{\rho\chi}\\
&-\frac{2q^2\sin^{2/3}\chi\cot\chi}{9a^{1/3}\rho^{2/3}}\left(a\bar{h}_{\rho\rho}-2\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)-\frac{q^2\sin^{2/3}\chi}{3a^{1/3}\rho^{2/3}}\left(a\partial_\chi\bar{h}_{\rho\rho}-2\partial_\chi\bar{h}_{\chi\chi}+\partial_\chi\bar{h}_{\theta\theta}\right)\\
&+\frac{2}{3}\frac{\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}}\left(a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)Z^{\chi}+\frac{2\sin^{2/3}\chi}{a^{1/3}\rho^{2/3}}\bar{h}_{t\chi}Z^t+\mathcal{O}(h^2), &(5.28b)\\
\hat{\Gamma}^{\theta} =& \frac{2\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}q^2}Z^{\theta}-\frac{q^2}{3\rho^{2/3}\sin^{4/3}\chi a^{1/3}}\left(a\partial_\theta\bar{h}_{\rho\rho}+\partial_\theta\bar{h}_{\chi\chi}-2\partial_\theta\bar{h}_{\theta\theta}\right)\\
&+\frac{2}{3}\frac{\rho^{4/3}\sin^{2/3}\chi}{a^{1/3}}\left(a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)Z^{\theta}+\mathcal{O}(h^2), &(5.28c)
\end{aligned}
$$

and diverges at the origin $\rho = 0$, at the boundary $q = 0$ and for the angular components $\chi = \{0, \pi\}$.

The boundary behaviour of the auxiliary field b^i

is given by

$$b^{\rho} = \frac{q^3 a}{k}\partial_t\bar{h}_{t\rho}+\mathcal{O}(h^2), \tag{5.29a}$$

$$b^{\chi} = \frac{q^2}{k\rho^2}\partial_t\bar{h}_{t\chi}+\mathcal{O}(h^2), \tag{5.29b}$$

$$b^{\theta} = 0, \tag{5.29c}$$

and diverges at the origin $\rho = 0$. Again, the θ component vanishes due to the SO(3) symmetry as discussed in 4.5.

The boundary behaviour of the auxiliary variable A_i

is given by

$$A_{\rho} = \frac{\rho}{L^2aq}+\left(\frac{q^2\rho}{a^2L^2}-\frac{q^3}{a^2\ell}+\frac{q}{a\ell}\right)\bar{h}_{tt}-\frac{q^2}{2a}\partial_\rho\bar{h}_{tt}+\mathcal{O}(h^2), \tag{5.30a}$$

$$A_{\chi} = -\frac{q^2}{2a}\partial_\chi\bar{h}_{tt}+\mathcal{O}(h^2), \tag{5.30b}$$

$$A_{\theta} = -\frac{q^2}{2a}\partial_\theta\bar{h}_{tt}+\mathcal{O}(h^2), \tag{5.30c}$$

and diverges at the boundary $q = 0$.

The boundary behaviour of the auxiliary variable B_k^i

is given by

$$B_{\rho}^{\rho} = q^3a\partial_\rho\bar{h}_{t\rho}-\left(\frac{3q^2a}{\ell}+q^3\left(\frac{2q}{\ell}-\frac{2\rho}{L^2}\right)\right)\bar{h}_{t\rho}+\mathcal{O}(h^2), \tag{5.31a}$$

$$B_\rho^\chi = \frac{q^2}{\rho^2}\partial_\rho \bar{h}_{t\chi} - \left(\frac{2q}{\rho^2\ell} + \frac{2q^2}{\rho^3}\right)\bar{h}_{t\chi} + \mathcal{O}(h^2), \tag{5.31b}$$

$$B_\rho^\theta = 0, \tag{5.31c}$$

$$B_\chi^\rho = q^3 a \partial_\chi \bar{h}_{t\rho} + \mathcal{O}(h^2), \tag{5.31d}$$

$$B_\chi^\chi = \frac{q^2}{\rho^2}\partial_\chi \bar{h}_{t\chi} + \mathcal{O}(h^2), \tag{5.31e}$$

$$B_\chi^\theta = 0, \tag{5.31f}$$

$$B_\theta^\rho = q^3 a \partial_\theta \bar{h}_{t\rho} + \mathcal{O}(h^2), \tag{5.31g}$$

$$B_\theta^\chi = \frac{q^2}{\rho^2}\partial_\theta \bar{h}_{t\chi} + \mathcal{O}(h^2), \tag{5.31h}$$

$$B_\theta^\theta = 0, \tag{5.31i}$$

and diverges at the origin $\rho = 0$, while the zero B_i^θ components vanish due to the SO(3) symmetry.

The boundary behaviour of the auxiliary variable D_{ijk}

is given by

$$\begin{aligned} D_{\chi\theta\theta} =& \frac{2}{3}\rho^{2/3}a^{1/3}\sin^{1/3}\chi\cos\chi - \frac{2}{9}q^2a^{1/3}\rho^{2/3}\sin^{1/3}\chi\cos\chi\left(a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} - 2\bar{h}_{\theta\theta}\right) \\ &- \frac{1}{6}q^2a^{1/3}\rho^{2/3}\sin^{4/3}\chi\left(a\partial_\chi\bar{h}_{\rho\rho} + \partial_\chi\bar{h}_{\chi\chi} - 2\partial_\chi\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.32a}$$

$$D_{\theta\theta\theta} = -\frac{1}{6}q^2\rho^{2/3}\sin^{4/3}\chi a^{1/3}\left(a\partial_\theta\bar{h}_{\rho\rho} + \partial_\theta\bar{h}_{\chi\chi} - 2\partial_\theta\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.32b}$$

$$D_{\theta\chi\chi} = -\frac{1}{6}\frac{q^2a^{1/3}\rho^{2/3}}{\sin^{2/3}\chi}\left(a\partial_\theta\bar{h}_{\rho\rho} - 2\partial_\theta\bar{h}_{\chi\chi} + \partial_\theta\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.32c}$$

$$\begin{aligned} D_{\chi\chi\chi} =& -\frac{1}{3}\frac{\rho^{2/3}a^{1/3}\cos\chi}{\sin^{5/3}\chi} + \frac{1}{9}\frac{q^2a^{1/3}\rho^{2/3}\cos\chi}{\sin^{5/3}\chi}\left(a\bar{h}_{\rho\rho} - 2\bar{h}_{\chi\chi} + \bar{h}_{\theta\theta}\right) \\ &- \frac{1}{6}\frac{q^2a^{1/3}\rho^{2/3}}{\sin^{2/3}\chi}\left(a\partial_\chi\bar{h}_{\rho\rho} - 2\partial_\chi\bar{h}_{\chi\chi} + \partial_\chi\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.32d}$$

$$\begin{aligned} D_{\chi\rho\rho} =& -\frac{1}{3}\frac{\cos\chi}{a^{2/3}\rho^{4/3}\sin^{5/3}\chi} + \frac{1}{9}\frac{q^2\cos\chi}{a^{2/3}\rho^{4/3}\sin^{5/3}\chi}\left(-2a\bar{h}_{\rho\rho} + \bar{h}_{\chi\chi} + \bar{h}_{\theta\theta}\right) \\ &- \frac{1}{6}\frac{q^2}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(-2a\partial_\chi\bar{h}_{\rho\rho} + \partial_\chi\bar{h}_{\chi\chi} + \partial_\chi\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \end{aligned} \tag{5.32e}$$

$$D_{\theta\rho\rho} = -\frac{1}{6}\frac{q^2}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(-2a\partial_\theta\bar{h}_{\rho\rho} + \partial_\theta\bar{h}_{\chi\chi} + \partial_\theta\bar{h}_{\theta\theta}\right) + \mathcal{O}(h^2), \tag{5.32f}$$

$$D_{\chi\rho\chi} = -\frac{1}{3}\frac{q^3a^{1/3}}{\rho^{4/3}\sin^{2/3}\chi}\cot\chi\bar{h}_{\rho\chi} + \frac{1}{2}\frac{q^3a^{1/3}}{\rho^{4/3}\sin^{2/3}\chi}\partial_\chi\bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.32g}$$

$$D_{\chi\rho\theta} = 0, \tag{5.32h}$$

$$D_{\theta\rho\chi} = \frac{1}{2}\frac{q^3a^{1/3}}{\rho^{4/3}\sin^{2/3}\chi}\partial_\theta\bar{h}_{\rho\chi} + \mathcal{O}(h^2), \tag{5.32i}$$

$$D_{\theta\rho\theta} = 0, \tag{5.32j}$$

$$D_{\chi\chi\theta} = 0, \tag{5.32k}$$
$$D_{\theta\chi\theta} = 0, \tag{5.32l}$$
$$D_{\rho\chi\theta} = 0, \tag{5.32m}$$
$$\begin{aligned}D_{\rho\rho\rho} =& -\frac{1}{3}\frac{1}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(\frac{2\rho}{aL^2}-\frac{2q}{a\ell}+\frac{2}{\rho}\right)\\ &+\frac{q^2}{9}\frac{1}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(\frac{2\rho}{aL^2}-\frac{2q}{a\ell}+\frac{2}{\rho}\right)\left(-2a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)\\ &+\frac{q^2a^{-2/3}\rho^{-4/3}}{3\sin^{2/3}\chi}\left[\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}-\frac{1}{2}\partial_\rho\bar{h}_{\chi\chi}-\frac{1}{2}\partial_\rho\bar{h}_{\theta\theta}\right]\\ &-\frac{1}{3\ell}\frac{q}{a^{2/3}\rho^{4/3}\sin^{2/3}\chi}\left(-2a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2),\end{aligned} \tag{5.32n}$$
$$D_{\rho\rho\chi} =\frac{q^3a^{1/3}}{\rho^{4/3}\sin^{2/3}\chi}\left[\left(\frac{\frac{2\rho}{L^2}-\frac{2q}{\ell}}{6a}-\frac{2}{3\rho}-\frac{3}{2q\ell}\right)\bar{h}_{\rho\chi}+\frac{1}{2}\partial_\rho\bar{h}_{\rho\chi}\right]+\mathcal{O}(h^2), \tag{5.32o}$$
$$D_{\rho\rho\theta} = 0, \tag{5.32p}$$
$$\begin{aligned}D_{\rho\chi\chi} =&\frac{\rho^{2/3}}{6a^{2/3}\sin^{2/3}\chi}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)+\frac{a^{1/3}}{3\rho^{1/3}\sin^{2/3}\chi}-\left(a\bar{h}_{\rho\rho}-2\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)\\ &\times\left(\frac{\rho^{2/3}q^2}{9a^{2/3}\sin^{2/3}\chi}\left(\frac{\rho}{L^2}-\frac{q}{\ell}\right)+\frac{a^{1/3}q^2}{9\rho^{1/3}\sin^{2/3}\chi}-\frac{a^{1/3}\rho^{2/3}q}{3\ell\sin^{2/3}\chi}\right)\\ &-\frac{\rho^{2/3}q^2a^{1/3}}{6\sin^{2/3}\chi}\left(a\partial_\rho\bar{h}_{\rho\rho}+\bar{h}_{\rho\rho}\partial_\rho a-2\partial_\rho\bar{h}_{\chi\chi}+\partial_\rho\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2),\end{aligned} \tag{5.32q}$$
$$\begin{aligned}D_{\rho\theta\theta} =&\frac{\rho^{2/3}\sin^{4/3}\chi}{6a^{2/3}}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)+\frac{a^{1/3}\sin^{4/3}\chi}{3\rho^{1/3}}-\left(a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}-2\bar{h}_{\theta\theta}\right)\\ &\times\left(\frac{\rho^{2/3}q^2\sin^{4/3}\chi}{9a^{2/3}}\left(\frac{\rho}{L^2}-\frac{q}{\ell}\right)+\frac{a^{1/3}q^2\sin^{4/3}\chi}{9\rho^{1/3}}-\frac{a^{1/3}\rho^{2/3}q\sin^{4/3}\chi}{3\ell}\right)\\ &-\frac{\rho^{2/3}\sin^{4/3}\chi q^2a^{1/3}}{6}\left(a\partial_\rho\bar{h}_{\rho\rho}+\bar{h}_{\rho\rho}\partial_\rho a+\partial_\rho\bar{h}_{\chi\chi}-2\partial_\rho\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2),\end{aligned} \tag{5.32r}$$

and diverges at the origin $\rho = 0$ and for the angular components $\chi = \{0, \pi\}$. Again, some components vanish due to the SO(3) symmetry.

The boundary behaviour of the auxiliary variable P_i

is given by

$$\begin{aligned}P_\rho =& -\frac{1}{ql}-\frac{q}{3a\ell}+\frac{\rho}{3aL^2}-\frac{2}{3\rho}+\frac{q}{3\ell}\left(a\bar{h}_{\rho\rho}+\bar{h}_{\chi\chi}+\bar{h}_{\theta\theta}\right)\\ &-\frac{q^2}{6}\left(\left(-\frac{2q}{\ell}+\frac{2\rho}{L^2}\right)\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}+\partial_\rho\bar{h}_{\chi\chi}+\partial_\rho\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2),\end{aligned} \tag{5.33a}$$
$$P_\chi = -\frac{\cot\chi}{3}-\frac{q^2}{6}\left(a\partial_\chi\bar{h}_{\rho\rho}+\partial_\chi\bar{h}_{\chi\chi}+\partial_\chi\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2), \tag{5.33b}$$
$$P_\theta = -\frac{q^2}{6}\left(a\partial_\theta\bar{h}_{\rho\rho}+\partial_\theta\bar{h}_{\chi\chi}+\partial_\theta\bar{h}_{\theta\theta}\right)+\mathcal{O}(h^2), \tag{5.33c}$$

and diverges at the origin $\rho = 0$ and at the boundary $q = 0$.

Now, we need to implement these boundary function in terms of the used coordinates in the ExaHyPE-code. However, as we do not know the explicit behaviour of the $\bar{h}$-functions, this turns out to be a two-step problem. First, we need to numerically solve the Dirichlet boundary problem of the $\bar{h}$-functions and then insert the results back into the boundary conditions of the variables of the state vector $\mathbf{Q}$. As this would go beyong the scope of this thesis, we argued to numerically solve only the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b) for an Anti-de Sitter spacetime parametrized by the coordinates $(t, \rho, \theta_{n-2}, \dots, \theta_1) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi)^{n-3} \times (0, 2\pi)$ and with time-symmetric, matter-free initial data (3.21) and with the AdS metric $\hat{\gamma}_{ij}$, without any $\bar{h}$-functions, as boundary values. Since we start at $t = 0$ with an AdS metric as initial data, the $\bar{h}$-functions that were derived for asymptotically AdS spacetimes with matter fields within the bulk, should vanish everywhere. Therefore, for the sake of simplicity, we want to numerically solve the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b) for an Anti-de Sitter spacetime in three dimensions parametrized by the coordinates $(t, \rho, \chi) \in (-\infty, \infty) \times (0, 1) \times (0, 2\pi)$, with matter-free AdS initial data (3.25) and the AdS metric $\hat{\gamma}_{ij}$ as boundary values withhin the ExaHyPE framework. Let us therefore introduce the PDE engine in the next section.

Chapter 6

Introduction into ExaHyPE

"Due to the robustness and shock-capturing abilities of ExaHyPE's numerical methods, users of the engine can simulate linear and non-linear hyperbolic PDEs with very high accuracy." [58]

We are aiming, even beyond the scope of this thesis, to eventually solve the initial value problem for the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b), time-symmetric initial data from the PDE (3.25) and the boundary conditions (4.67a)-(4.67c) for $\tilde{\gamma}_{ij}$, (4.76) for $\ln \alpha$, (4.70a)-(4.70b) for β^i, (4.51) for $\log \phi$, (4.127a)-(4.127c) for $\tilde{A}_{ij}$, (4.117) for K, (4.145) for θ, (4.144a)-(4.144b) for $\hat{\Gamma}^i$, (4.153a)-(4.153b) for b^i, (4.83a)-(4.83a) for A_k, (4.100a)-(4.100d) for B_k^i, (E.17a)-(4.106f) for D_{ijk} and (4.92a)-(4.92b) for P_k for an three- and four-dimensional asymptotically Anti-de Sitter spacetime parametrized by the spherical coordinates

$$(t, \rho, \theta_{n-2}, \dots, \theta_1) \in (-\infty, \infty) \times (0, \ell) \times (0, \pi)^{n-3} \times (0, 2\pi) \tag{6.1}$$

at spacelike infinity with the ExaHyPE software. Let us therefore state the main properties, taken from the ExaHyPE guidebook [18], in the following chapter.

6.1 Setup and installation

We will assume that a Linux-based operating system is used, though ExaHyPE should be made configurable to work with Windows or Mac. We will run the commands in a terminal window and have the following dependencies installed.

- A C++ compiler, as ExaHyPE source code is written in C++
- A Fortran compiler if we wish to write our code in Fortran
- GNU Make
- Python 3 as some development environments rely on python 3 dependencies
- Peano, as ExaHyPE is built on top of the AMR framework Peano
- TBB or OpenMP, as they can be used for parallel programming. Both work with GCC and Intel compilers

- MPI, as it can be used for working on distributed memory clusters.

There are different ways to set up ExaHyPE. One of them is to clone the ExaHyPE repository via

```
git clone https://gitlab.lrz.de/exahype/ExaHyPE-Engine.git
git clone https://gitlab.lrz.de/exahype/ExaHyPE-Astrophysics.git
git clone https://gitlab.lrz.de/exahype/ExaHyPE-Documentation.git
```

into a specific folder. But note that one needs an LRZ GitLab account to clone the Astrophysics repository. Then, as all python 3 dependecies and Peano are registered as git submodules, they can be obtained (if not installed and used in different projects) by running

```
./updateSubmodules.sh
```

in the `./ExaHyPE-Enginge/Submodules`-folder. Furthermore, by running

```
./toolkit.sh -h
```

in the `./ExaHyPE-Enginge/Toolkit`-folder we should get a description of the various toolkit options telling us that the installation is complete. Furthermore, we need to register the Astrophysics repository by

```
cd ExaHyPE-Astrophysics && ./link-to-exahype.sh && cd ..
```

Finally, we are ready to start programming.

6.2 The ExaHyPE workflow

Let us first summarise the overall workflow before getting into more detail.

To solve a specific physical PDE problem using ExaHyPE, one must write a plain specification text file that holds all required data. This file has to be written completely by the ExaHyPE user. It is then handed over to the ExaHyPE toolkit that will create glue code and empty files. These empty files have to be filled by the ExaHyPE user with the actual PDE, the PDE related flux functions, eigenvalues and so forth. The simple `make` command will create the ExaHyPE executable file, and we can run it by handing the specification file over to the executable file. Let us now go through the particular steps by running, for example, the gauge-wave benchmark ExaHyPE project that can be found in the ExaHyPE repository.

6.2.1 The specification file

First, we need to write a specification file.

```
exahype-project CCZ4

  peano-kernel-path const      = ./Peano
  exahype-path const           = ./ExaHyPE
```

```
output-directory const      = ./AstroApplications/CCZ4
architecture const          = snb
plotter-subdirectory const  = Writers

computational-domain
  dimension const = 3
  width           = 1.0, 1.0, 1.0
  offset          = 0.0, 0.0, 0.0
  end-time        = 0.5
end computational-domain

shared-memory
  identifier      = dummy
  configure       = {}
  cores           = 2
  properties-file = sharedmemory.properties
end shared-memory

distributed-memory
  identifier      = static_load_balancing
  configure       = {hotspot,fair}
  buffer-size     = 64
  timeout         = 60
end distributed-memory

global-optimisation
  spawn-predictor-as-background-thread = off
  spawn-amr-background-threads = off
  disable-vertex-exchange-in-time-steps = off
  time-step-batch-factor          = 0.0
  disable-metadata-exchange-in-batched-time-steps = off
  double-compression = 0.0
  spawn-double-compression-as-background-thread = on
end global-optimisation

solver ADER-DG CCZ4Solver_ADERDG
  variables const       = G:6,K:6,theta:1,Z:3,lapse:1,shift:3,b:3,
                          dLapse:3,dxShift:3,dyShift:3,dzShift:3,
                          dxG:6,dyG:6,dzG:6,traceK:1,phi:1,P:3,K0:1,
                          domain:1,pos:3
  order const           = 3
  maximum-mesh-size     = 0.33339
  maximum-mesh-depth    = 0
  time-stepping         = global
  type const            = nonlinear
  terms const           = ncp,source
  optimisation const    = generic
  language const        = C
```

```
    constants         = mexa/ref:/exahype/solvers/solver/constants,
                        mexa/style:adapted,
                        mexa/encoding:quotedprintable,
                        initialdata/name:GaugeWave,
                        ccz4/k1:0.0,
                        ccz4/k2:0.0,
                        ccz4/k3:0.0,
                        ccz4/eta:0.0,
                        ccz4/itau:0.0,
                        ccz4/f:0.0,
                        ccz4/g:0.0,
                        ccz4/xi:0.0,
                        ccz4/e:2.0,
                        ccz4/c:0.0,
                        ccz4/mu:0.0,
                        ccz4/ds:1.0,
                        ccz4/sk:0.0,
                        ccz4/bs:0.0,
                        ccz4/lapsetype:0,
                        limiter/criterion:geometric_sphere,
                        limiter/radius:0.5,
                        refinement/criterion:geometric_sphere,
                        refinement/radius:0.5

    plot vtk::Cartesian::cells::ascii ConservedWriter
      variables const = 59
      time     = 0.0
      repeat   = 0.01
      output   = ./vtk-output
    end plot
  end solver
end exahype-project
```

This .exahype file is the centrepiece of solving the equations numerically. The first block defines all the mandatory paths to the Peano- and ExaHyPE-folders, where the output is saved, specifies the microarchitecture of our processor and specifies where the used plotters get generated.

The second block defines the computational domain, where we can choose between two- and three-dimensional setup. The parameters width and offset have to be adopted, respectively. Furthermore, we specify the number of timesteps.

The third block defines shared-memory parallelisation through Intel's Threading Building Blocks (TBB) or OpenMP. Whenever we add this block, the default value will be a shared-memory using TBB or OpenMP, depending on whether we have specified the SHAREDMEM environment variable with

```
export SHAREDMEM=TBB
or
export SHAREDMEM=OpenMP.
```

Note that as all arguments within the shared-memory block are not marked with `const.`, they will be read at runtime and can be changed without rerunning the toolkit. Using OpenMP, we have to set the environment variables TBB_INC and TBB_SHLIB[1]. If we want at any time, after successfully running the toolkit, to not use shared-memory or use a different shared-memory method, we can redefine the SHAREDMEM environment variable via

```
export SHAREDMEM=None
```

or as noted above. If we have redefined the SHAREDMEM environment variable, we do not have to rerun the toolkit but the makefile at this point. For more information on shared-memory parallelisation, see the respective chapter in [18].

We can add the distributed-memory block to the specification file if we want to use MPI with our ExaHyPE project, and switch MPI on/off through the environment variable DISTRIBUTEDMEM. If we have added the distributed-memory block, the default value will be DISTRIBUTEDMEM=mpi and by running

```
export DISTRIBUTEDMEM=None
```

we can switch it off. If we have not specified the distributed-memory block, then the default value will be DISTRIBUTEDMEM=None. Then, by adding this block, we can either rerun the toolkit and recompile or

```
export DISTRIBUTEDMEM=mpi
```

and recompile. Make sure that an MPI compiler is installed, and note that we can reconfigure it by rewriting the `EXAHYPE_CC` flag.

The fifth block defines some optimisations that can be switched on or off on runtime and can be changed without rerunning the toolkit as they are not marked as `const`. See the respective chapter in [18] for further information for these parameters.

The last environment, reading

```
solver [solver] [name_of_the_solver]
...
end solver,
```

sets the main properties. First, we need to choose a specific numerical PDE solver. Here, we can use either use

- ADER-DG
- Finite-Volumes
- or Limiting-ADER-DG.

Then, we give the solver a name. In the latter gauge-wave benchmark, we used `ADER-DG` as a numerical scheme and gave the solver environment the name `CCZ4Solver_ADERDG`. Before we characterise the parameters within this solver block, let us first

[1] TBB_INC=-I/mypath/include and TBB_SHLIB="-L/mypath/lib64/intel64/gcc4.4 -ltbb"

give an overview of ExaHyPE's main solver concepts. By using ExaHyPE, we can solve equations of the following form

$$\underbrace{\mathbf{P}}_{\texttt{materialmatrix}} \frac{\partial}{\partial t}\mathbf{Q} + \boldsymbol{\nabla} \cdot \underbrace{\mathbf{F}}_{\texttt{fluxes}}(\mathbf{Q}) + \underbrace{\sum_{i=1}^{d} \mathbf{B}_i(\mathbf{Q}) \frac{\partial \mathbf{Q}}{\partial x_i}}_{\texttt{ncp}} = \underbrace{\mathbf{S(Q)}}_{\texttt{sources}} + \underbrace{\sum \delta}_{\texttt{pointsources}} \quad . \tag{6.2}$$

Depending on the mathematical problem, not all specified terms are required, and therefore, we need to specify in our specification file which terms we want to use. This can be done by specifying the `terms` parameters, where the following values, taken from [18], are supported

`materialparameters`	If this PDE term is present, ExaHyPE allows the user to specify a spd matrix P. If it is not present, ExaHyPE uses $P = id$.
`fluxes`	Informs ExaHyPE that we want to use a standard, conservative first-order flux formulation.
`ncp`	Informs ExaHyPE that we plan to use non-conservative formulations.
`sources`	Informs ExaHyPE that we plan to use algebraic sources, i.e. right-hand side terms.
`pointSources`	Informs ExaHyPE that we plan to use point sources, i.e. Dirac distribution right-hand side terms.

Table 6.1: Supported solver PDE terms

Furthermore, we need to specify which type of solver and integration scheme we want to use. Again, the supported values are take from [18]

ADER-DG	
`linear`	Is a shortcut for `linear, Legendre`.
`nonlinear`	Is a shortcut for `nonlinear, Legendre`.
`linear, Legendre`	Kernel for linear PDEs solved on Gauss-Legendre-Nodes
`nonlinear, Legendre`	Kernel for nonlinear PDEs, solved on Gauss-Legendre-Nodes
`linear, Lobatto`	Kernel for linear PDEs solved on Gauss-Lobatto-Nodes
`nonlinear, Lobatto`	Kernel for nonlinear PDEs solved on Gauss-Lobatto-Nodes
Finite Volumes	
`musclhancock`	Use a MUSCL-Hancock Riemann solver.
`robustmusclhancock`	Use a slightly more robust version of the MUSCL-Hancock Riemann solver.
`godunov`	Use a standard Godunov Riemann solver.

Table 6.2: Solver types.

Within the `global optimisation` block, we can choose a particular optimisation method. For further reading about all the used parameters, take a look at the respective chapter in [18].

In ExaHyPE, there are two possible ways of telling our program about all the unknown variables of the state vector $\mathbf{Q}$. Here, we work with symbolic identifiers but refer the reader to the respective chapter in [18] for more information.

The global `time-stepping` parameter characterises that all cells have the same time step size.

The `constants` environment sets some constants that are used in the PDE, specifies which initial data should be used and sets some further parameters that we do not have to worry about.

Furthermore, we add the respective

```
plot [plotter_type] [plotted_data]
...
end plot
```

environment within the solver environment to create output. The different types and data that could be plotted can be found in [18], while the parameters within this environment are self-explanatory. If we want to change the type or the number of our output data, then add the respective block to the specification file, rerun the toolkit and rerun our code.

6.2.2 The ExaHyPE toolkit and make

After writing the specification file, the next step will be to hand the specification file over to the toolkit. This can be done by running the following command

```
./Toolkit/toolkit.sh [path_to_the_exahype_file]
```

in ./ExaHyPE-Enginge. Depending on our system, we might have to change some environment variables, taken from [18],

```
export COMPILER=Intel          Select Intel compiler (default)
export COMPILER=GNU            Select GNU compiler

export MODE=Debug            Build debug version of our code
export MODE=Asserts            Build release version of our code that is
                           augmented with assertions
export MODE=Profile            Build release version of our code that
                           produces profiling information
export MODE=Release            Build release version of our code (default)

export SHAREDMEM=TBB           Uses TBB
export SHAREDMEM=OMP           Uses OpenMP
export SHAREDMEM=None          Do not use TBB or OpenMP (default)

export TBB_INC=-I/mypath/include
export TBB_SHLIB=-L/mypath/lib64/intel64/gcc4.4 -ltbb
```

```
export DISTRIBUTEDMEM=MPI     Uses MPI
export DISTRIBUTEDMEM=None    Do not use MPI (default)
```

Afterwards, we need to navigate to our application's folder and run

```
make.
```

This step will create a makefile, a README-file, a lot of glue code and an executable file in our directory.

6.2.3 Running the code

Once we have successfully run the toolkit and the make command, our terminal will prompt

```
....
=================
An ExaHyPE solver
```

Now, we can start running the code by handing the specification file over to the executable

```
./ExaHyPE-CCZ4 [path_to_the_exahype_file]
```

And at last, after successfully running our code, we can open our output with, e.g. Paraview [2].

[2]https://www.paraview.org/

Chapter 7

Numerical Output

"With the aid of computer, it is possible to tackle
these highly nonlinear equations numerically
in oder to examine these scenarios in detail" [7]
Thomas W. Baumgarte
Physicist

The first example of numerical output is the evolution the generalized FO-CCZ4 system of the Einstein field equations (2.69a)-(2.69k) with gauge fixing (2.82), (2.88a)-(2.88b), with matter-free initial data (3.25) and the AdS metric $\hat{\gamma}_{ij}$ as boundary values for a static Anti-de Sitter spacetime in three dimensions parametrized by the coordinates $(t, \rho, \chi) \in (-\infty, \infty) \times (0, 1) \times (0, 2\pi)$ using the ExaHyPE software engine. This means that we want to numerically evolve the AdS_3 vacuum spacetime metric while we set the values of the state vector at the boundary to the purely AdS_3 part. The evolution of a static AdS spacetime is a trivial example and nothing should happen, but it will give us a good hint that the numerical program works. The initial data for a static AdS_3 spacetime is given by

$$\gamma_{ij} = \hat{\gamma}_{ij}, \quad p_i = 0, \quad E = 0 \quad \text{and} \quad K_{ij} = 0, \tag{7.1}$$

where the spatial metric on the initial slice is given by the vacuum AdS_3 metric $\hat{\gamma}_{ij}$, while the energy and matter distribution vanishes, i.e. $p_i = 0$ and $E = 0$. Furthermore, we have specified time-symmetric initial data, which means setting the extrinsic curvature to zero for the initial slice Σ_0. i.e. $K_{ij} = 0$. Furthermore, we will set the variables of the state vector at the boundary exactly to the purely AdS_3 part of these variables to ensure the well-posedness of the initial value problem of the generalized FO-CCZ4 system. Now, let us note where we have modified the ExaHyPE code.

Initial Data

First we modified the InitialData.f90 file. Here, we added the following case to the InitParameters(STRLEN,PARSETUP) subroutine

```
case('AdS3')
    ! We use the same parameters as for the Gauge wave
    EQN%CCZ4k1 = 0.0
    EQN%CCZ4k2 = 0.0
```

```
        EQN%CCZ4k3 = 0.0
        EQN%CCZ4eta = 0.0
        EQN%CCZ4f  = 0.0
        EQN%CCZ4g  = 0.0
        EQN%CCZ4xi = 0.0
        EQN%CCZ4e  = 2.0
        EQN%CCZ4c  = 0.0
        EQN%CCZ4mu = 0.0
        EQN%CCZ4ds = 1.0
        EQN%CCZ4sk = 0.0
        EQN%CCZ4bs = 0.0
        EQN%CCZ4LapseType = 0 ! harmonic lapse
        isAdS3 = LOGICAL( .TRUE., KIND=C_BOOL ),
```

and the following case to the InitialData(xGP, tGp, Q) subroutine

```
case('AdS3')
    ! Let us first define the following:
    ! The compactified radial coordiante rho = r/(1+r)

    AdSrho = xGP(1)/(1+xGP(1))

    ! The scalar field q = 1-rho
    AdSq = 1-AdSrho

    ! The auxiliary variable a = q^2 + rho^2
    AdSa = AdSq**2 + AdSrho**2

    ! The derivative of a
    DAdSa = -2 + 4*AdSrho

        V0(:) = 0.0

        ! The conformally decomposed spatial metric
        V0(1)  = 1/(SQRT(AdSa)*AdSrho)
        V0(4)  = SQRT(AdSa)*AdSrho

        ! The conformally decomposed traceless part of the extrinsic
           curvature

        ! Theta

        ! Gamma-hat
        V0(14) = -0.5*AdSrho/SQRT(AdSa)*DAdSa - SQRT(AdSa)

        ! Logarithm of the lapse function
        V0(17) = LOG(1-AdSq) + 0.5*LOG(1+AdSq**2/AdSrho**2) - LOG(AdSq)

        ! The shift vector
```

```
! Gamma driver condition

! A_i
V0(24) = AdSrho/(AdSa*AdSq)

! B_i^j

! D_ijk
V0(36) = -1/(2*AdSa**(1.5)) + AdSq/(2*AdSrho*AdSa**(1.5)) -
    1/(2*SQRT(AdSa)*AdSrho**2)

V0(39) = AdSrho**2/(2*SQRT(AdSa)) - AdSrho*AdSq / (2*SQRT(AdSa)) +
    0.5*SQRT(AdSa)

! The trace K of the extrinsic curvature K_ij

! The logarithm of the conformal factor phi
V0(55) = LOG(AdSq) + 0.25*LOG(AdSa) - 0.5*LOG(AdSrho)

! P_i
V0(56) = -1/AdSq + AdSrho/(2*AdSa) - AdSq/(2*AdSa) - 1/(2*AdSrho)

! Trace K of the extrinsic curvature K_ij at time t=0
```

The first case sets some parameters that can be found in the FO-CCZ4 (2.69a)-(2.69k), while the second case sets initial data. First, we defined some AdS specific variables, then set the variables of the state vector at $t = 0$ to $V0(:) = 0.0$ and overwrote the nonvanishing variables afterwards.

Boundary Values

Then, we modified the FOCCZ4::FOCCZ4Solver::boundaryValues function in the FOCCZ4Solver.cpp file for setting boundary values to the finite volume solver. Here, we added the following if(AdS3_) statement.

```
void FOCCZ4::FOCCZ4Solver::boundaryValues(
    const double* const x,
    const double t,const double dt,
    const int faceIndex,
    const int direction,
    const double* const stateInside,
    double* const stateOutside) {
    const int nVars = FOCCZ4::FOCCZ4Solver::NumberOfVariables;
   double Qgp[nVars];

   double ti = t + 0.5 * dt;
   // Compute the outer state according to the initial condition
  double x_3[3];
  x_3[2]=0;
```

```
std::copy_n(&x[0],DIMENSIONS,&x_3[0]);

if (isAdS3_){
// rho = r/(1+r)
double rho = AdSrho(x[0]);
// q=1-rho
double q = AdSq(rho);
// a=q^2 + rho^2
double a = AdSa(AdSq(rho), AdSrho(x[0]));
double Da = DAdSa(AdSrho(x[0]));

// Setting all statevariables which are zero to -stateInside
for(int m=0; m < nVars; m++){
 stateOutside[m]=-stateInside[m];
}

// Overwriting the components which are not zero
// conformally decomposed spatial metric
 stateOutside[0] = 1/(sqrt(a)*rho);
 stateOutside[3] = sqrt(a)*rho;

// conformally traceless part of the extrinsic curvature

// Theta

// Gamma hat
 stateOutside[13] = - 0.5*rho*Da/sqrt(a) - sqrt(a);

// lapse function
 stateOutside[16] = log(1-q) + 0.5*log(1+q*q/(rho*rho)) - log(q);

// shift vector

// Gamma driver condition

// Auxiliary variable A_i
 stateOutside[23] = rho/(a*q);

// Auxiliary variable B_i^j

// Auxiliary variable D_ijk
 stateOutside[35] = -1/(2*pow(a,3/2)) + q/(2*rho*pow(a,3/2)) -
     1/(2*sqrt(a)*rho*rho);
 stateOutside[38] = rho*rho/(2*sqrt(a)) - q*rho/(2*sqrt(a)) +
     0.5*sqrt(a);

// Trace of the extrinsic curvature

// conformal factor
```

```
   stateOutside[54] = log(q) + 0.25*log(a) - 0.5*log(rho);

  // Auxiliary variable P_i
   stateOutside[55] = -1/q + rho/(2*a) - q/(2*a) - 1/(2*rho);

  // Trace of the extrinsic curvature at t=0
  }
  else {
    initialdata_(x_3, &ti, Qgp);
    for(int m=0; m < nVars; m++) {
          stateOutside[m] = Qgp[m];
    }
  }
}
```

Furthermore, we have implemented the following header file specified for some AdS variables in the FOCCZ4Solver.cpp file

```
#pragma once

double AdSrho(double a){
   double b;
   b = a/(1+a);
   return b;
   }

double AdSq(double a){
   double q;
   q = 1-a;
   return q;
   }

double AdSa( double b, double c){
   double a;
   a = b*b + c*c;
   return a;
   }

double DAdSa(double a){
   double b;
   b = -2 + 4*a;
   return b;
   }
```

PDE

And finally, we had to add the non-trivial cosmological constant to the partial differential equation system. As we set $L = 1$, the cosmological constant will be given for a four-dimensional manifold by $\Lambda_4 = -3$. Therefore, we have to add

$+9alpha$ and $+6$, respectively, in the conditional preprocessing block

```
#if defined(CCZ4EINSTEIN) || defined(CCZ4GRHD) || defined(CCZ4GRMHD) ||
    defined(CCZ4GRGPR)
```

in the PDE.f90 file to the evolution equation of the trace of the extrinsic curvature

```
dtTraceK = - nablanablaalphaNCP - nablanablaalphaSrc + alpha*(
    RPlusTwoNablaZNCP + RPlusTwoNablaZSrc + traceK**2 - 2*Theta*traceK )
    - 3*alpha*k1*(1+k2)*Theta + SUM(beta(:)*dtraceK(:)) + 9*alpha,
```

and to the evolution equation of the theta algebraic constraint

```
dtTheta = 0.5*alpha*e**2*(RplusTwoNablaZNCP + RplusTwoNablaZSrc) +
    beta(1)*dTheta(1) + beta(2)*dTheta(2) + beta(3)*dTheta(3) &  !
    temporal Z
            + 0.5*alpha*e**2*( - Aupdown + 2./3.*traceK**2 + 6 ) -
               alpha*Theta*traceK - SUM(Zup*alpha*AA) -
               alpha*k1*(2+k2)*Theta
```

Note, we have used the value for the four-dimensional cosmological constant because ExaHyPE still uses the equations for a four-dimensional spacetime but plots a 2-dimensional slice of the three-dimensional space by setting the z-component to zero. If we want to solve for a "real" AdS_3 spacetime numerically, we need to change some of the prefactors, depending on the dimension of the spacetime, in the FO-CCZ4 system (2.69a)-(2.69k).

Unfortunately, the ExaHyPE software for solving the simplified Cauchy-problem of a static vacuum three-dimensional AdS spacetime with time-symmetric initial data, i.e. $(E = 0, p_i = 0, K_{ij} = 0, \hat{\gamma}_{ij})$, and the anti-de Sitter metric $\hat{\gamma}_{ij}$ as boundary condition broke up after several timesteps. Within the time framework of this thesis, we could not debug this error message and had to leave it open for further investigation.

Chapter 8

Conclusion

"I became interested in this question of whether
you can build wormholes for interstellar travel.
I realized that if you had a wormhole,
the theory of general relativity by itself
would permit you to go backward in time." [34]
Kip Thorne
Physicist

By reading the quote above, we realize how fantastic general relativity is. However, as a theory combining such beautiful math of differential geometry with such complicated answers of the universe, it still cannot accurately describe compact objects such as neutron stars and black holes in the strong-field regime. Therefore, much work has been put into solving Einstein's field equations for compact objects numerically.

This thesis presented a derivation for the well-posed initial value problem for the strongly hyperbolic FO-CCZ4 system for an asymptotically Anti-de Sitter space in n spacetime dimensions. First, the FO-CCZ4 system for a general n-dimensional manifold with a non-trivial cosmological constant was derived by Grosvenor in [37]. Then, we argued why this augmented system of PDE remained for a three- and four-dimensional manifold strongly hyperbolic and wrote it in a compact, non-conservative matrix form similar as in Dumser et al. [20]. However, we still need to show strong hyperbolicity of the FO-CCZ4 system for a $n \neq 4$-dimensional manifold.

As the Anti-de Sitter spacetime is not globally hyperbolic, we had to derive conditions for the $n^3/2 + n^2 + 5n/2$ unknown variables of the FO-CCZ4 system at the boundary. First, the AdS space was compactified to some finite value ρ, then, the metric near the boundary was written as a first-order perturbation series, i.e. $g_{\mu\nu} = \hat{g}_{\mu\nu} + \epsilon h_{\mu\nu}$, and finally all conditions for the $n^3/2+n^2+5n/2$ unknown variables were derived from this. If we do not specify some boundary conditions at timelike infinity, we could only evolve forward in time some given initial data on a spacelike hypersphere at time $t = 0$ for a small environment in the causal future. Moreover, these boundary conditions were verified for $n = 3$ and $n = 4$ by a Mathematica script [36].

As initial data on the spacelike hypersurface Σ_0 must satisfy the conservation of energy and momenta, the set-up is non-trivial. For the sake of simplicity, we restricted ourselves to time-symmetric initial data and used a scalar field ζ as a non-trivial deviation from the purely matter-free AdS spacetime. In order to obtain physically relevant initial data on the Cauchy surface Σ_0, a time-independent, elliptic, second-order initial value problem was derived with the help of a conformal decomposition. For a time-symmetric, matter-free distribution, i.e. $(E = 0, p_i = 0, K_{ij} = 0)$, we found a solution to the initial value problem and reconstructed the spatial AdS spacetime metric $\hat{\gamma}_{ij}$ as last part of the initial data. Nevertheless, further work in the area of inital data needs to be done. First, we need to find a solution to the less-trivial second-order, elliptic initial value problem on $\Sigma_0 = \mathbb{R}^{n-1}$ with scalar field ζ as matter (3.25) by using, e.g., spectral methods. Then, for choosing the topology on the initial leave as $\Sigma_0 = \mathbb{R}^{n-1} \backslash B_\rho(0)$ or as $\Sigma_0 = \mathbb{R}^{n-1} \backslash O$, we need to set boundary conditions for the conformal factor $\zeta \in \mathcal{C}^\infty(M, \mathbb{R})$, respectively, on the sphere or at the puncture O.

Further work in the field of numerically solving the FO-CCZ4 system for an asymptotically AdS_3 and AdS_4 spacetime with ExaHyPE is needed. As we want to numerically evolve some initial data on a compactified asymptotically Anti-de Sitter spacetime, we need to explicitly set some boundary conditions near $\rho = 1$. As the boundary behaviour of each variable of the state vector $\mathbf{Q}$ is given by the purely AdS_n part plus a small deviation $\epsilon h_{statevariable}$, we need to tell ExaHyPE how these fields behave near the boundary. While the boundary conditions (4.18) give us the power of the radial coordinate r with which the deviation falls off, we do not know the explicit behaviour concerning time and angle coordinates. Therefore, solving the FO-CCZ4 system for an asymptotically AdS spacetime is twofold: First, we need to numerically solve the FO-CCZ4 initial value problem for the $\bar{h}$-functions, and then insert the numerical output into the boundary behaviour of each state variable. After finding the exact behaviour of the fall-off conditions near the boundary, we can use the set-up of initial data in the paper [22] as a benchmark for the time-evolution of an asymptotically AdS_3 spacetime. Then, after successfully evolving the AdS-benchmark, we can try to evolve some less-trivial initial data for an asymptotically AdS_4 spacetime.

To address the complication to explicitly set the $\bar{h}$-function, we tried to solve the simplified Cauchy-problem of a static vacuum three-dimensional AdS spacetime with time-symmetric initial data, i.e. $(E = 0, p_i = 0, K_{ij} = 0, \hat{\gamma}_{ij})$, and the anti-de Sitter metric $\hat{\gamma}_{ij}$ as a boundary condition within the ExaHyPE framework. This, however, could not be solved within the time framework of this thesis, as the numerics of the ExaHyPE code broke up after some timesteps.

Appendix A

The mathematically rigorous Definition of the Space and Time Decomposition

"Numerical relativity picks up where post-Newtonian theory and general relativistic perturbation theory leave off." [6]
Thomas W. Baumgarte
Physicist

The basis of numerical relativity is the notion of a hypersurface Σ of a spacetime M. Since we could not find a rigorous introduction to numerical relativity in physics textbooks, we have devoted much effort to defining the main objects from differential geometry in a mathematically rigorous manner for recasting Einstein's equations as a time-evolution problem. Therefore, the first chapter of the appendix, independent of the Einstein field equations, is entirely devoted to differential geometry, and we will introduce physical quantities only in Chapter 2. While we define the main objects for working with the Einstein field equations in the first section, we state in a mathematically rigorous manner the essential objects that are necessary for rewriting Einstein's field equations as an initial value problem in the sections that follow.

A.1 The basis of differential geometry

As the spacetime M will be the framework on which we will build the theory of numerical relativity, let us state the properties that it carries. A general relativistic n-dimensional spacetime is a pair (M, g), where M is a real smooth n-dimensional manifold and g a Lorentzian metric on M. [44] However, we need to ask ourselves which condition the spacetime M must satisfy such that it admits a Lorentzian metric. Well, a smooth manifold M admits a Lorentzian metric if there exists a non-vanishing vector field X on M. [45] As we want to define a global time direction, we assume that the spacetime (M, g) is time-orientable. Furthermore, as we want the entire manifold to be determined by some specified initial data on the initial Cauchy surface Σ_t at $t = 0$, the spacetime must be globally hyperbolic. As the definitions of time-orientability and global hyperbolicity would go beyond the scope of this chapter, we have summarized the main aspects in Appendix B.

We denote by T_pM the tangent space and T_p^*M the cotangent space of the manifold M at the point $p \in M$. The set of all disjoint tangent spaces will form a smooth manifold, the tangent bundle TM of M, i.e.

$$TM = \coprod_{p \in M} T_pM. \tag{A.1}$$

As every tangent vector $v_p \in TM$ has a base point $p \in M$, there exists a canonical projection

$$\mathrm{pr}\colon TM \ni v_p \mapsto p \in M. \tag{A.2}$$

The space for l-fold contravariant and k-fold covariant tensor fields on M will be given by the set of smooth sections $\Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$ of the generalized tangent bundel $TM^{\otimes l} \otimes T^*M^{\otimes k}$. A smooth section $S \in \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$ is a smooth map

$$S\colon M \to TM^{\otimes l} \otimes T^*M^{\otimes k} \tag{A.3}$$

such that

$$\mathrm{pr}_{TM^{\otimes l} \otimes T^*M^{\otimes k}} \circ S = \mathrm{id}_M, \tag{A.4}$$

where

$$\mathrm{pr}_{TM^{\otimes l} \otimes T^*M^{\otimes k}}\colon TM^{\otimes l} \otimes T^*M^{\otimes k} \ni S_p \mapsto p \in M \tag{A.5}$$

is the projection of the tensor S_p to its base point $p \in M$ as defined in [64]. This way, a vector field $X \in \Gamma^\infty(TM)$ is a 1-fold contravariant tensor field.

Furthermore, a local frame for the tangent bundle $TM\big|_U$ and the cotangent bundle $T^*M\big|_U$ restricted to some open neighbourhood $U \subseteq M$ of $p \in M$, respectively, is a set of smooth sections

$$\frac{\partial}{\partial x^0}, \ldots, \frac{\partial}{\partial x^{n-1}} \in \Gamma^\infty(TM\big|_U) \quad \text{and} \quad dx^0, \ldots, dx^{n-1} \in \Gamma^\infty(T^*M\big|_U) \tag{A.6}$$

such that

$$\frac{\partial}{\partial x^0}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n-1}}\bigg|_p \in T_pM \quad \forall p \in U \quad \text{and} \quad dx^0_p, \ldots, dx^{n-1}_p \in T_p^*M \quad \forall p \in U \tag{A.7}$$

form a vector basis of T_pM and a dual basis of T_p^*M. This way we have

$$dx^\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \delta^\alpha{}_\beta \tag{A.8}$$

on an open neighbourhood U of $p \in M$, and any ℓ-fold contravariant and k-fold covariant tensor $S \in \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k})$ on M can be written in a local chart (U, x) for M as in [64] by

$$S\big|_U = S^{\mu_1 \ldots \mu_\ell}{}_{\nu_1 \ldots \nu_k} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_\ell}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_k}, \tag{A.9}$$

where $S^{\mu_1 \ldots \mu_\ell}{}_{\nu_1 \ldots \nu_k} \in \mathcal{C}^\infty(U, \mathbb{R})$.

Now, a Lorentzian metric is a pseudo-Riemannian metric that is a smooth symmetric 2-fold covariant tensor field

$$\begin{aligned} g\colon M &\to T^*M \otimes T^*M \\ p &\mapsto g\Big|_p := g_p \end{aligned} \tag{A.10}$$

that is non-degenerate at each point $p \in M$ with the signature $(-,+,\cdots,+)$ [45], and whose value g_p at each $p \in M$ is an inner product on T_pM defined by

$$\begin{aligned} g_p\colon T_pM \times T_pM &\to \mathbb{R} \\ (u_p, v_p) &\mapsto g_p(u_p, v_p). \end{aligned} \tag{A.11}$$

Locally, the metric can be written as

$$g\Big|_U = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{A.12}$$

where $g_{\mu\nu} \in \mathcal{C}^\infty(U, \mathbb{R})$, $dx^\mu \in \Gamma^\infty(T^*M)$ and $g_{\mu\nu}(p) = g_p\left(\partial_\mu\Big|_p, \partial_\nu\Big|_p\right)$. Furthermore, we need the notion of a connection on our manifold M that is a bilinear map

$$\nabla\colon \Gamma^\infty(TM) \times \Gamma^\infty(TM) \ni (X, Y) \mapsto \nabla_X Y \in \Gamma^\infty(TM) \tag{A.13}$$

satisfying for $f \in \mathcal{C}^\infty(M)$ the following two properties [45]

$$\nabla_{fX} Y = f \nabla_X Y \tag{A.14a}$$
$$\nabla_X(fY) = (\mathcal{L}_X f)Y + f\nabla_X Y, \tag{A.14b}$$

where $\mathcal{L}_X f$ is the Lie-derivative of f in the direction of the vector field $X \in \Gamma^\infty(TM)$. See Appendix A for some properties of the Lie-derivative.

Now, we say that $\nabla_X Y$ is the covariant derivative of the vector field Y in the direction of the vector field X. Locally, the covariant derivative of the coordinate vector fields $\frac{\partial}{\partial x^\alpha} \in \Gamma^\infty(TM)$ can be written as

$$\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta} \frac{\partial}{\partial x^\mu}, \tag{A.15}$$

where $\Gamma^\mu{}_{\alpha\beta}\colon U \to \mathbb{R}$ are called the connection coefficients of ∇. The connection ∇ is completely determined in a neighbourhood U by the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$, and ∇ uniqueley determines another connection ∇ on the tensor tangent bundle $TM^{\otimes l} \otimes T^*M^{\otimes k}$ [45] that we will denote by the same symbol. The covariant derivative of any ℓ-fold contravariant and k-fold covariant tensor field $S \in \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$, where S is expressed locally by (A.9), is given in [45] locally by

$$\begin{aligned} \nabla_X S\Big|_U = &\left(\mathcal{L}_X S^{\mu_1\ldots\mu_\ell}{}_{\nu_1\ldots\nu_k} + \sum_{s=1}^{\ell} X^\rho S^{\mu_1\ldots\sigma\ldots\mu_\ell}{}_{\nu_1\ldots\nu_k} \Gamma^{\mu_s}{}_{\rho\sigma} - \sum_{s=1}^{k} X^\rho S^{\mu_1\ldots\mu_\ell}{}_{\nu_1\ldots\sigma\ldots\nu_k} \Gamma^\sigma{}_{\rho\nu_s}\right) \\ &\times \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_\ell}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_k}, \end{aligned} \tag{A.16}$$

and we define for the sake of simplicity

$$\nabla_{\partial_\gamma} S^{\alpha_1\ldots\alpha_\ell}_{\beta_1\ldots\beta_k} := \partial_\gamma S^{\alpha_1\ldots\alpha_\ell}_{\beta_1\ldots\beta_k} + \sum_{s=1}^{\ell} S^{\alpha_1\ldots\mu\ldots\alpha_\ell}_{\beta_1\ldots\beta_k}\Gamma^{\alpha_s}_{\gamma\mu} - \sum_{s=1}^{k} S^{\alpha_1\ldots\alpha_\ell}_{\beta_1\ldots\mu\ldots\beta_k}\Gamma^{\mu}_{\gamma\beta_s}. \tag{A.17}$$

To choose a particular connection on our Lorentzian manifold that reflects the properties of the Lorentzian metric, it has to have the following two properties:

- compatibility with the metric: $\nabla_X\langle Y,Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ [1]
- symmetric: $\nabla_X Y - \nabla_Y X \equiv [X,Y]$

for all vector fields $X,Y,Z \in \Gamma^\infty(TM)$. These two conditions are enough to uniquely determine the connection associated to the Lorentzian metric [55], and it will be called the Levi-Civita connection after Tullio Levi-Civita. Furthermore, the connection coefficients associated to the Levi-Civita connection are given locally by

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\mu}\left(\frac{\partial g_{\beta\mu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right). \tag{A.18}$$

The covariant derivative of a tensor $S \in \Gamma^\infty(TM^{\otimes\ell}\otimes T^*M^{\otimes k})$ of type (ℓ,k) on M in all directions, given by the map

$$\nabla S\colon \Gamma^\infty(T^*M)\times\cdots\times\Gamma^\infty(T^*M)\times\Gamma^\infty(TM)\times\cdots\times\Gamma^\infty(TM)\to\mathcal{C}^\infty(M), \tag{A.19}$$

defines by

$$(\nabla S)\left(\omega^1,\ldots,\omega^k,Y_1,\ldots,Y_l,X\right) = (\nabla_X S)\left(\omega^1,\ldots,\omega^k,Y_1,\ldots,Y_l\right), \tag{A.20}$$

where $\omega^1,\ldots,\omega^k \in \Gamma^\infty(T^*M)$ and $Y_1,\ldots,Y_l,X \in \Gamma^\infty(TM)$, a single ℓ-fold contravariant and $k{+}1$-fold covariant tensor tensor field that is called the total covariant derivative [45].

Thanks to the canonical isomorphism [64]

$$\Gamma^\infty(T^*M^{\otimes 3}\otimes TM) \simeq \mathrm{Hom}_{\mathcal{C}^\infty(M,\mathbb{R})}(\Gamma^\infty(TM),\Gamma^\infty(TM),\Gamma^\infty(TM);\Gamma^\infty(TM)) \tag{A.21}$$

there exists for the multilinear map Riem over $\mathcal{C}^\infty(M)$

$$\begin{aligned}\mathrm{Riem}\colon \Gamma^\infty(TM)\times\Gamma^\infty(TM)\times\Gamma^\infty(TM) &\to \Gamma^\infty(TM)\\ (X,Y,Z) &\mapsto [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z\end{aligned} \tag{A.22}$$

a uniquely determined $(1,3)$ tensor field $\mathrm{Riem} \in \Gamma^\infty(TM\otimes T^*M^{\otimes 3})$ that we will call the Riemannian curvature tensor. This tensor can be written locally as

$$\mathrm{Riem}\Big|_U = R^\sigma_{\mu\nu\rho}dx^\mu\otimes dx^\nu\otimes dx^\rho\otimes\frac{\partial}{\partial x^\sigma}, \tag{A.23}$$

[1]To say that the connection ∇ is compatible with the metric g is equivalent to that the metric g is parallel with respect to the connection ∇: $\nabla g \equiv 0$.

where the components of the tensor are given by

$$R^{\delta}_{\alpha\beta\gamma} = \partial_\alpha \Gamma^{\delta}_{\beta\gamma} - \partial_\beta \Gamma^{\delta}_{\alpha\gamma} + \Gamma^{\mu}_{\beta\gamma}\Gamma^{\delta}_{\alpha\mu} - \Gamma^{\mu}_{\alpha\gamma}\Gamma^{\delta}_{\beta\mu}. \tag{A.24}$$

By taking the trace of the Riemannian curvature tensor we can define the Ricci tensor $\mathrm{Ric} \in \Gamma^\infty(T^*M \otimes T^*M)$ of ∇ globally by

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\,(Z \mapsto R(Z, X)Y)\,, \tag{A.25}$$

where $X, Y \in \Gamma^\infty(TM)$, and locally by

$$\mathrm{Ric}\Big|_U = \mathrm{R}_{\mu\nu}\, dx^\mu \otimes dx^\nu, \tag{A.26}$$

where $\mathrm{R}_{\alpha\beta} = \mathrm{R}^{\mu}_{\mu\alpha\beta}$. By taking the trace of the Ricci tensor we get the Ricci scalar curvature

$$\mathrm{R} = \mathrm{tr}\,\mathrm{Ric} = R^{\mu}{}_{\mu}. \tag{A.27}$$

We have now defined all differential geometric objects that we need to decompose our spacetime manifold M into space and time. Let us, therefore, begin by defining a hypersurface and the induced spatial metric.

A.2 Definition of the hypersurface Σ_t and the spatial metric γ

In this section, we will make the notion of a hypersurface Σ. As well as in the latter chapter, (M, g) denotes an n-dimensional Lorentzian manifold.

A spacelike $(n-1)$-dimensional submanifold Σ_t of the ambient spacetime M is called a hypersurface and can be defined by the smooth embedding [3], i.e. an injective immersion and a topological embedding,

$$\begin{aligned} \Phi\colon \hat{\Sigma}_t &\to M \\ q &\mapsto \Phi(q) = p = (t, q). \end{aligned} \tag{A.28}$$

We can write $p = (t, q) \in M$ globally as a product, as we have assumed that M is globally hyperpolic. A topological embedding is injective and a homeomorphism onto its non-intersecting-self-image $\Sigma_t := \Phi(\hat{\Sigma}_t)$. [64] Locally, a hypersurface can be defined as a scalar field $\hat{t} \in \mathcal{C}^\infty(M, \mathbb{R})$ by

$$\Sigma_t = \{p \in M | \hat{t}(p) = t\}, \tag{A.29}$$

where $t \in \mathbb{R}$. [27] The smooth map Φ induces a linear map for all $q \in \hat{\Sigma}_t$

$$T_q\Phi\colon T_q\hat{\Sigma}_t \to T_{\Phi(q)}M, \tag{A.30}$$

where any $v_q = v^i_q \partial_i\Big|_q \in T_q\hat{\Sigma}_t$ will be mapped to $T_q\Phi(v_q) = (t, v_q) \in T_{\Phi(q)}M$ such that $\Phi(q) = p$. This is sometimes also called the push-forward or tangent map of Φ. Furthermore, the embedding function induces another important mapping, called the pull-back. In order to define the pull-back $\Phi^* S$ of an ℓ-fold contravariant and k-fold covariant tensor field $S \in \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$ by

$$\Phi^* S_q(\alpha_1, \dots, \alpha_\ell, v_1, \dots, v_k) = S_{\Phi(q)}(\alpha_1 \circ (T_q\Phi)^{-1}, \dots, \alpha_\ell \circ (T_q\Phi)^{-1}, T_q\Phi(v_1), \dots, T_q\Phi(v_k)), \tag{A.31}$$

where $q \in \hat{\Sigma}_t$, $\alpha_1, \dots, \alpha_\ell \in T_q^*\hat{\Sigma}_t$ and $v_1, \dots, v_k \in T_q\hat{\Sigma}_t$ [64], we need to assume furthermore that Φ is at least locally a diffeomorphism. The first and most important pull-back for us, for which Φ does not need to be locally diffeomorphic but only smooth, will be the induced metric on $\hat{\Sigma}_t$

$$\gamma := \Phi^* g \in \Gamma^\infty(T^*\hat{\Sigma}_t^{\otimes 2}). \tag{A.32}$$

For the sake of simplicity, we will identify $\hat{\Sigma}_t$ with $\Sigma_t = \Phi(\hat{\Sigma}_t) \subset M$ and write for any vector $v_q \in T_q\Sigma_t$ just $v_p = T_q\Phi(v_q) \in T_p\Sigma_t$. The induced metric γ is related to the ambient metric g by

$$\gamma_q(v_q, w_q) = g_{\Phi(q)}(T_q\Phi(v_q), T_q\Phi(w_q)) = g_p(v_p, w_p) \tag{A.33}$$

for any tangent vectors $v_q, w_q \in T_q\Sigma$.

In this way, we have defined a spatial hypersurface Σ_t of the spacetime M and the spatial metric γ in a rigorous manner. Now, we can see if there exists a unique connection D defined on the submanifold Σ_t.

A.3 Definition of the Levi-Civita connection D and extrinsic curvature K

As Φ is a smooth embedding, i.e. an injective immersion that is also a topological embedding, and Σ_t is spacelike, we can split the tangent space T_pM into the direct sum

$$T_pM = T_p\Sigma_t \oplus N_p\Sigma_t, \tag{A.34}$$

where we have defined $(T_p\Sigma_t)^\perp := N_p\Sigma_t$ to be the 1-dimensional orthogonal complement generated by the unit normal vector N_p that will be defined in Section A.45. The Gauss formula says that if $X, Y \in \Gamma^\infty(TM)$ are extended arbitrarily to smooth vector fields on an open neighbourhood of Σ_c in M, then we can decompose the ambient covariant derivative with respect to the Levi-Civita connection by

$$\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp, \tag{A.35}$$

where $^\top$ and $^\perp$ represents the tangential and normal projection, respectively. [55] Furthermore, we know that the tangential projection $(\nabla_X Y)^\top$ is just the covariant derivative with respect to the Levi-Civita connection of the induced metric on Σ_c,

$$(\nabla_X Y)^\top = D_X Y, \tag{A.36}$$

where D is the pull-back of ∇ by the embedding map

$$D := \Phi^* \nabla. \tag{A.37}$$

Again, the fundamental theorem of Riemannian geometry says that there exists a unique connection D on $T\Sigma$ that is compatible with γ and symmetric, and we will call it the Levi-Civita connection of γ.

The normal projection $(\nabla_X Y)^\perp$ can be described either by the second fundamental form of Σ_c or by the scalar-valued second fundamental form, which is defined by the normal projection of the covariant derivative of tangent vector fields and by the tangent projection of the covariant derivative of a normal vector field, respectively.

The second fundamental form

We can define the second fundamental form of Σ_c by the map

$$K\colon \Gamma^\infty(T\Sigma) \times \Gamma^\infty(T\Sigma) \to \Gamma^\infty(N\Sigma), \tag{A.38}$$

such that $K(X,Y) := (\nabla_X Y)^\perp$. Thus, we can write Eq. (A.35) as

$$\nabla_X Y = D_X Y + K(X,Y). \tag{A.39}$$

The scalar-valued second fundamental form

Here, we define for each normal vector field $N \in \Gamma^\infty(N\Sigma)$ the scalar-valued second fundamental form by the map

$$K_N\colon \Gamma^\infty(T\Sigma) \times \Gamma^\infty(T\Sigma) \to \mathcal{C}^\infty(M,\mathbb{R}), \tag{A.40}$$

such that $K_N(X,Y) := \langle N, K(X,Y)\rangle = \langle X, W_N(Y)\rangle$, where we have used the self-adjoint linear Weingarten map

$$W_N\colon \Gamma^\infty(T\Sigma) \to \Gamma^\infty(T\Sigma). \tag{A.41}$$

For every $X \in \Gamma^\infty(T\Sigma)$ and $N \in \Gamma^\infty(NM)$ the Weingarten map is given by $W_N(X) = -(\nabla_X N)^\top$, where N is an arbitrary extension to an open subset of M. Thus, we can write Eq. (A.35) explicitely as

$$\begin{aligned}\nabla_X Y &= D_X Y + K_N(X,Y)N \\ &= D_X Y - \langle X, (\nabla_X N)^\top\rangle N.\end{aligned} \tag{A.42}$$

Let us give some remarks at the end: The Levi-Civita connection D on $T\Sigma$ is called the intrinsic curvature of (Σ,γ) and there exists as well a Riemannian curvature tensor, Ricci tensor and Ricci scalar on Σ with respect to D. The scalar-valued second fundamental form $K_N \in \Gamma^\infty(T^*\Sigma^{\otimes 2})$ is also called the extrinsic curvature tensor and tells us how the hypersurface is embedded into the spacetime manifold. From Eq. (A.42), we can interpret it as a measure of the difference between the intrinsic Levi-Civita connection on $T\Sigma$ and the ambient Levi-Civita connection on TM. Another interpretation is given in [27]: the extrinsic curvature tensor K measures the failure of a geodesic of (Σ,γ) to be a geodesic of (M,g), and the two notions only coincide in the case where $K = 0$.

Next, as we want to define a global direction of time, define a timelike normal vector field N in the next section.

A.4 Characterisation of the normal vector field N

In the latter chapter, we have defined (Σ_c, γ) as an embedded spacelike $(n-1)$-dimensional submanifold of an n-dimensional ambient Lorentzian manifold (M, g). In this case, there are at each point $q \in \Sigma_c$ two unit normal vectors. However, it is generally not clear that a smooth unit normal vector field exists on all of Σ_c. However, there will always be a unit normal vector field on some open neighbourhood of $q \in \Sigma_c$. [45]

Let $t \in \mathcal{C}^\infty(M, \mathbb{R})$ be a smooth scalar field such that $\Sigma_c = t^{-1}(c)$. Then we can define the one-form $dt_p \in T_p^*M$ at the point $p \in M$ by

$$dt_p(v_p) = v_p(t), \tag{A.43}$$

where $v_p \in T_pM$. Let now $v_p \in T_p\Sigma_c$, then the one-form and the gradient of the scalar field $t \in \mathcal{C}^\infty(M, \mathbb{R})$ are normal to Σ_c, since

$$g_p(\nabla t\Big|_p, v_p) = dt_p(v_p) = v_p(t\Big|_\Sigma) = 0, \tag{A.44}$$

because the scalar field is constant on Σ_c. [55] Since Σ_t is a spacelike hypersurface, we can normalise the timelike tangent vector $\nabla t\Big|_p \in T_pM$ as in [27] by

$$N_p := -\frac{1}{\sqrt{-g_p(\nabla t\Big|_p, \nabla t\Big|_p)}} \nabla t\Big|_p \in T_pM, \tag{A.45}$$

where the minus sign is chosen so that the vector N_p is future-oriented if the scalar field $t \in \mathcal{C}^\infty(M, \mathbb{R})$ is increasing towards the future. By calculating the scalar product of the normal vector, i.e.,

$$g_p(N_p, N_p) = -1, \tag{A.46}$$

we see that the normal vector is timelike, and we have found an unit timelike normal vector field $N \in \Gamma^\infty(TM)$ on some open neighbourhood of $q \in \Sigma_c$.

The definition of the lapse function α

If we define the normalization factor of Eq. (A.45) as in [27] by

$$\alpha_p := \frac{1}{\sqrt{-g_p(\nabla t\Big|_p, \nabla t\Big|_p)}}, \tag{A.47}$$

then $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$ is a scalar field on M called the lapse function.

The definition of the normal evolution vector field M

We can define the normal evolution vector as well as in [27] by setting

$$M_p := \alpha_p N_p. \tag{A.48}$$

This way, $M \in \Gamma^\infty(TM)$ will be a timelike vector field on some open neighbourhood of $q \in \Sigma_c$ with the properties

$$g_p(M_p, M_p) = -\alpha_p^2 \quad \text{and} \quad \nabla_M t\Big|_p = \mathcal{L}_M t\Big|_p = M_p(t) = g_p(\nabla_p t, M_p) = 1. \tag{A.49}$$

From the very definition of the Lie-derivative and $\mathcal{L}_M t\Big|_p = 1$, we can see that $t(p') = t(p) + \delta t$, where $p' = p + \delta t M_p \in \Sigma_{t+\delta t}$. Geometrically, this means that the vector $\delta t M_p$ carries the hypersurface Σ_t to the neighbouring one $\Sigma_{t+\delta t}$. [27] If you are not familiar with the notion of a Lie derivative, you may consult Appendix B.

The definition of the shift vector field β

Let us take $(t, x_1, \ldots, x_{n-1})$ as natural basis of M. Then, $\partial_t \in \Gamma^\infty(TM\Big|_U)$ and $dt \in \Gamma^\infty(T^*M\Big|_U)$ will be smooth sections on the tangent and cotangent bundle satisfying $dt\,(\partial_t) = 1$. Then, as

$$dt_p\,(M_p) = M_p(t) = 1 \;\forall p \in M, \tag{A.50}$$

we can see that the smooth time section $\partial_t \in \Gamma^\infty(TM)$ has the same properties as the normal evolution vector field $M \in \Gamma^\infty(TM)$. In general, these two sections differ by the shift vector field $\beta \in \Gamma^\infty(TM)$ by

$$\partial_t =: M + \beta. \tag{A.51}$$

By calculating

$$dt_p\,(\beta_p) = dt_p\left(\partial_t\Big|_p\right) - dt_p\,(M_p) = 1 - 1 = 0 \tag{A.52}$$

we can see that the shift vector β_p at some point $p \in M$ is tangent to the hypersurfaces Σ_t. Furthermore, by using the definition of the normal evolution vector field, we can view

$$\partial_t = \alpha N + \beta \tag{A.53}$$

as the decomposition of the time vector section $\partial_t \in \Gamma^\infty(TM)$ into time and space. [27] Locally, we can write the shift vector field $\beta \in \Gamma^\infty(TM)$ as

$$\beta\Big|_U = \beta^i \partial_i \quad \text{and} \quad \beta^\flat\Big|_U = \beta_i dx^i. \tag{A.54}$$

Now, we are ready to foliate our spacetime M into spacelike hypersurfaces in such a way that we can define a global direction of time.

The characterization of Eulerian observer

Since $N \in \Gamma^\infty(TM)$ denotes a unit timelike vector field, we can view it as the four-velocity vector field of some observer. As time goes by, the observer moves along the vector flow Φ of N. We call it the wordline of the Eulerian observer, since the vector flow traces out the time history of the observer. This means as well that the locally defined hypersurface Σ_t is the set of events that happen at the same time from the point of view of the Eulerian observer. The proper time τ of two close events, p and $p' = p + \delta t M_p$, measured by the Eulerian observer is given by [27]

$$\delta\tau = \sqrt{-g_p(\delta t M_p, \delta t M_p)} = \sqrt{-g_p(M_p, M_p)}\delta t = \alpha_p \delta t, \tag{A.55}$$

and the four-acceleration vector field $A \in \Gamma^\infty(TM)$ of the Eulerian observer is defined via

$$A = \nabla_N N, \tag{A.56}$$

and with

$$\begin{aligned} \langle A, N\rangle\Big|_p = \langle N, \nabla_N N\rangle\Big|_p &= \frac{1}{2}\nabla_N\langle N, N\rangle\Big|_p = \frac{1}{2}\mathcal{L}_N\langle N, N\rangle\Big|_p \\ &= \frac{1}{2}N_p(\langle N, N\rangle) = 0, \end{aligned} \tag{A.57}$$

since $\langle N, N\rangle \in \mathcal{C}^\infty(M, \mathbb{R})$ is constant on Σ_t, it follows that $A_p = \nabla_N N\Big|_p \in T_p\Sigma_t$.

A.5 The foliation of the spacetime M

The principle idea to decompose the spacetime M into "time" and "space" dates back to the beginning of the Hamiltonian formulation of general relativity. [59] It is commonly used in modern numerical relativity to rewrite the Einstein equations as a "time" evolution problem suitable for numerical implementation.

A foliation of an n-dimensional manifold M is a partition of M into injectively immersed $(n-1)$-dimensional submanifolds Σ_t, i.e.,

$$M = \bigcup_{t\in\mathbb{R}} \Sigma_t \quad \text{with} \quad \Sigma_t \cap \Sigma_{t'} = \emptyset \text{ for } t \neq t', \tag{A.58}$$

in such a way that for each $p \in M$ there is a coordinate chart (U, x) that maps the intersection $\Sigma_t \cap U$ to a subset in $\mathbb{R}^{n-1} \times \{t\} \subset \mathbb{R}^{n-1} \times \mathbb{R}^1 = \mathbb{R}^n$. [17] In general, we could have foliated the manifold M by any $(n-k)$-dimensional submanifolds, but since we want to split "time" and "space" this seems to be the right choice. By the very definition, the hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ do not intersect themself and others and by the assumption that M is globally hyperbolic, we can write the spacetime as a global product

$$M = \Sigma_t \times \mathbb{R}, \tag{A.59}$$

where the topology is given by the product topology of the leave-topology and the standard topology of $\mathbb{R}$.

As we want to project our objects living on our spacetime M onto the spatial hypersurface Σ_t and along the normal vector field N, we need to define the notion of the projection onto the submanifolds in the next section.

A.6 The definition of the projection operator γ^μ_ν

In the latter chapter, we have denoted the orthogonal projection of some section of the tangent bundle of M by $^\top$. From now on, we will denote it by γ and define it by

$$\begin{aligned}\gamma\colon \Gamma^\infty(TM) &\to \Gamma^\infty(T\Sigma)\\ V &\mapsto V + \langle N, V\rangle N,\end{aligned}\tag{A.60}$$

where $N \in \Gamma^\infty(N\Sigma)$ and $V \in \Gamma^\infty(TM)$ [27]. Evaluating the right-hand side at a point p, we see that

$$\gamma(N)\Big|_p = 0 \quad \text{and} \quad \gamma(V)\Big|_p = V_p \text{ for all } V_p \in T_p\Sigma. \tag{A.61}$$

Locally, we can write

$$\begin{aligned}\gamma(V)\Big|_p &= V_p^\mu \frac{\partial}{\partial x^\mu}\Big|_p + N_p^\mu V_p^\nu \left\langle \frac{\partial}{\partial x^\mu}\Big|_p, \frac{\partial}{\partial x^\nu}\Big|_p \right\rangle N_p^\rho \frac{\partial}{\partial x^\rho}\Big|_p\\ &= V_p^\mu \frac{\partial}{\partial x^\mu}\Big|_p + N_p^\mu V_p^\nu g_{\mu\nu}(p) N_p^\rho \frac{\partial}{\partial x^\rho}\Big|_p\\ &= \left(V_p^\mu + N_{p,\nu} V_p^\nu N_p^\mu\right) \frac{\partial}{\partial x^\mu}\Big|_p\\ &= \left(\delta^\mu{}_\nu + N_p^\mu N_{p,\nu}\right) V_p^\nu \frac{\partial}{\partial x^\mu}\Big|_p\\ &= \gamma^\mu{}_\nu(p) V_p^\nu \frac{\partial}{\partial x^\mu}\Big|_p,\end{aligned}\tag{A.62}$$

where we have defined the components of γ with respect to the basis $\frac{\partial}{\partial x^\alpha}\Big|_p$ of T_pM by

$$\gamma^\mu{}_\nu = \delta^\mu{}_\nu + N^\mu N_\nu. \tag{A.63}$$

Furthermore, we can define a mapping to extend any k-fold covariant tensor on $T^*\Sigma^{\otimes k}$ to a k-fold covariant tensor on $T^*M^{\otimes k}$ by

$$\gamma_M^*\colon \Gamma^\infty(T^*\Sigma^{\otimes k}) \to \Gamma^\infty(T^*M^{\otimes k}), \tag{A.64}$$

such that

$$\gamma_M^*(A)(p)(V_1, \ldots, V_k) := A(p)(\gamma(V_1)\Big|_p, \ldots, \gamma(V_k)\Big|_p). \tag{A.65}$$

We will be interested in the extension of the spatial metric γ and the extrinsic curvature K_N to all vectors. These are given by

$$\gamma_M^*(\gamma)(p)(V_1, V_2) := \gamma(p)(\gamma(V_1)\Big|_p, \gamma(V_2)\Big|_p) \tag{A.66}$$

and

$$\gamma_M^*(K_N)(p)(V_1, V_2) := K_N(p)(\gamma(V_1)\Big|_p, \gamma(V_2)\Big|_p). \tag{A.67}$$

The extended spatial metric $\gamma_M^*(\gamma)$ can be expressed globally by

$$\gamma_M^*(\gamma) = g + N^\flat \otimes N^\flat, \tag{A.68}$$

and locally by

$$\gamma_M^*(\gamma)\Big|_U = (\gamma_M^*(\gamma))_{ij}\, dx^i \otimes dx^j = (g_{ij} + N_i N_j) dx^i \otimes dx^j. \tag{A.69}$$

The extended extrinsic curvature $\gamma_M^*(K_N)$ can be expressed globally by

$$\gamma_M^*(K_N) = -\nabla N^\flat - A^\flat \otimes N^\flat, \tag{A.70}$$

since for all $U, V \in \Gamma^\infty(TM)$

$$\begin{aligned}
\gamma_M^*(K_N)(U,V) &= K_N(\gamma(U),\gamma(V)) = -\langle \gamma(U), \left(\nabla_{\gamma(V)} N\right)^\top \rangle \\
&= -\langle U + \langle N,U\rangle N, \left(\nabla_{V+\langle N,V\rangle N} N\right)^\top \rangle \\
&= -\langle U + \langle N,U\rangle N, \nabla_V N + \langle N,V\rangle \nabla_N N\rangle \\
&= -\langle U, \nabla_V N\rangle - \langle N,V\rangle\langle U,\nabla_N N\rangle - \langle N,U\rangle\langle N,\nabla_V N\rangle \\
&\quad - \langle N,U\rangle\langle N,V\rangle\langle N,\nabla_N N\rangle \\
&= -\langle \nabla_V N, U\rangle - \langle \nabla_N N, U\rangle\langle N,V\rangle \\
&= -\left(\nabla_V N\right)^\flat(U) - A^\flat(U) N^\flat(V) \\
&= -\nabla_V N^\flat(U) - A^\flat(U) N^\flat(V) \\
&= -\nabla N^\flat(U,V) - A^\flat(U) N^\flat(V),
\end{aligned} \tag{A.71}$$

where $\nabla N^\flat$ is the total derivative of $N^\flat \in \Gamma^\infty(T^*M)$ given by the map A.19. Locally, it can be expressed by [27]

$$\gamma_M^*(K_N)\Big|_U = -\left(N_{\mu;\nu} + A_\mu N_\nu\right) dx^\mu \otimes dx^\nu, \tag{A.72}$$

where $N_{\alpha;\beta} = \frac{\partial N_\alpha}{\partial x^\beta} - N_\mu \Gamma^\mu_{\beta\alpha}$. To close this section, we will introduce another map to project any tensor of M to the hypersurface $\Sigma \subset M$. This can be done by extending the above defined projection operator γ to any ℓ-fold contravariant and k-fold covariant tensor $S \in \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$ by the map

$$\gamma_M^* \colon \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k}) \to \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k}) \tag{A.73}$$

such that the components of $\gamma^* S$ are projected to the hypersurface Σ_t by

$$(\gamma_M^* S)^{\alpha_1,\dots,\alpha_\ell}{}_{\beta_1,\dots,\beta_k} = \gamma^{\alpha_1}{}_{\mu_1} \dots \gamma^{\alpha_\ell}{}_{\mu_\ell} \gamma^{\nu_1}{}_{\beta_1} \dots \gamma^{\nu_k}{}_{\beta_k} S^{\mu_1,\dots,\mu_\ell}{}_{\nu_1,\dots,\nu_k}. \tag{A.74}$$

Thus, we can write locally

$$\gamma_M^* S\Big|_U = \gamma^{\sigma_1}{}_{\mu_1} \dots \gamma^{\sigma_\ell}{}_{\mu_\ell} \gamma^{\nu_1}{}_{\rho_1} \dots \gamma^{\nu_k}{}_{\rho_k} \; S^{\mu_1,\dots,\mu_\ell}{}_{\nu_1,\dots,\nu_k} \; dx^{\rho_1} \otimes \dots \otimes dx^{\rho_k} \otimes \partial_{\sigma_1} \otimes \dots \otimes \partial_{\sigma_\ell}. \tag{A.75}$$

Using the projection map γ^*, we can link the total covariant derivative of a tensor

$$S \in \Gamma^\infty(T\Sigma^{\otimes l} \otimes T^*\Sigma^{\otimes k})$$

on Σ with the total covariant derivative of some extension of S on M by

$$\gamma_M^*(DS) = \gamma^* \left[\nabla \left(\gamma_M^* S\right)\right]. \tag{A.76}$$

For the sake of simplicity, we will denote the extensions always by the same symbol

$$DS := \gamma_M^*(DS), \quad S := \gamma_M^* S \quad , \gamma := \gamma_M^*(\gamma) \quad \text{and} \quad K_N := \gamma_M^*(K_N). \tag{A.77}$$

This way we can write Eq. (A.68), Eq. (A.70) and Eq. (A.76) as

$$\gamma = g + N^\flat \otimes N^\flat \tag{A.78}$$

with $\gamma_{\alpha\beta} = g_{\alpha\beta} + N_\alpha N_\beta$,

$$K_N = -\nabla N^\flat - A^\flat \otimes N^\flat \tag{A.79}$$

with $K_{N,\alpha\beta} = N_{\alpha;\beta} + A_\alpha N_\beta$ and

$$DS = \gamma^* \nabla S \tag{A.80}$$

with $D_\gamma S^{\alpha_1,\ldots,\alpha_\ell}{}_{\beta_1,\ldots,\beta_k} = \gamma^{\alpha_1}{}_{\mu_1} \ldots \gamma^{\alpha_\ell}{}_{\mu_\ell} \gamma^{\nu_1}{}_{\beta_1} \ldots \gamma^{\nu_k}{}_{\beta_k} \gamma^\rho{}_\gamma \nabla_\rho S^{\mu_1,\ldots,\mu_\ell}{}_{\nu_1,\ldots,\nu_k}$.

Using the projection operators of this section, we can split our full metric tensor g into its space and time components in the following section.

A.7 The $(n-1)+1$-decomposition of the metric

Locally, we can write the spatial metric γ and the Lorentzian metric g, respectively, as

$$\gamma\Big|_U = \gamma_{ij} dx^i \otimes dx^j \tag{A.81}$$

and

$$g\Big|_U = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{A.82}$$

By using

$$g_{\alpha\beta} = g\left(\partial_\alpha, \partial_\beta\right), \tag{A.83}$$

we can calculate

$$g_{tt} = g\left(\partial_t, \partial_t\right) = -\alpha^2 + g\left(\beta, \beta\right) = -\alpha^2 + \beta_i \beta^i \tag{A.84a}$$

$$g_{ti} = g\left(\partial_t, \partial_i\right) = g\left(M + \beta, \partial_i\right) = g\left(\beta, \partial_i\right) = \beta_i \tag{A.84b}$$

$$g_{ij} = g\left(\partial_i, \partial_j\right) = \gamma\left(\partial_i, \partial_j\right) = \gamma_{ij}. \tag{A.84c}$$

Therefore, we can write the metric components in terms of the $(n-1)+1$ split as

$$g_{\alpha\beta} = \begin{pmatrix} g_{tt} & g_{tj} \\ g_{it} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{A.85}$$

and the inverse metric by

$$g^{\alpha\beta} = \begin{pmatrix} g^{tt} & g^{tj} \\ g^{it} & g^{ij} \end{pmatrix} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}. \tag{A.86}$$

We can easily see that these matrices satisfy

$$g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha{}_\beta \tag{A.87}$$

and we can write the $(n-1)+1$ decomposition of the Lorentzian metric g in terms of the line element as

$$g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right). \tag{A.88}$$

So far, we have decomposed the metric tensor g and the smooth section ∂_t into space and time. We can do this for all objects and the next one will be the Riemannian curvature tensor $R^\gamma{}_{\mu\alpha\beta}$. From the decomposition of this object, we can derive the space and time decomposition of the Ricci tensor $R_{\mu\nu}$. This will be needed for decomposing the Einstein field equations into space and time.

A.8 Derivation of the Gauss-, Codazzi- and Ricci-relation

The Gauß-Peterson-Mainardi-Codazzi relations are the most important equations for deriving of the $(n-1)+1$ decomposition of the spacetime since they link objects living on the hypersurface Σ with the projection of objects living on the spacetime M. We will not derive any of these equations but instead, refer the reader to the respective section in [27].

From here on, we will use the local component notation with respect to a local frame $(\partial_\alpha)_{\alpha\in I} \in \Gamma^\infty(TM\big|_U)$ and $(dx^\alpha)_{\alpha\in I} \in \Gamma^\infty(T^*M\big|_U)$ where $I = \{0, \dots, n-1\}$, and label all objects that live in the n-dimensional manifold M with a superscript $^{(n)}$.

The Gauss-Relation

We can derive the Gauss-equation from the extended Ricci identity

$$(D_\alpha D_\beta - D_\beta D_\alpha)\, v^\gamma = R^\gamma{}_{\mu\alpha\beta} v^\mu, \tag{A.89}$$

where $v_p = v_p^\mu \partial_\mu\big|_p \in T_pM$ is a tangent vector to Σ, D_α the components of the extended covariant derivative and $R^\gamma{}_{\delta\alpha\beta}$ the components of the extended Riemannian curvature tensor. By using the relation between the total derivative on Σ and M, i.e. Eq. (A.80), and a lot of index gymnastic, we can derive the **Gauss relation**

$$\gamma^\mu{}_\alpha \gamma^\nu{}_\beta \gamma^\gamma{}_\rho \gamma^\sigma{}_\delta {}^{(n)}R^\rho{}_{\sigma\mu\nu} = R^\gamma{}_{\delta\alpha\beta} + K^\gamma{}_\alpha K_{\delta\beta} - K^\gamma{}_\beta K_{\alpha\delta}. \tag{A.90}$$

By contracting the Gauss equation on the indices γ and α, we obtain the **contracted Gauss relation**

$$\gamma^\mu{}_\alpha \gamma^\nu{}_\beta {}^{(n)}R_{\mu\nu} + \gamma_{\alpha\mu} N^\nu \gamma^\rho{}_\beta N^{\sigma\,(n)} R^\mu{}_{\nu\rho\sigma} = R_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\mu} K^\mu{}_\beta, \tag{A.91}$$

and by taking the trace with respect to γ, the **scalar Gauss relation**

$$^{(n)}R + 2\,^{(n)}R_{\mu\nu} N^\mu N^\nu = R + K^2 - K_{ij} K^{ij}. \tag{A.92}$$

The scalar Gauss relation is a generalization of the Theorema Egregium by Gauss and relates the intrinsic curvature of Σ represented by R with the extrinsic curvature represented by $K^2 - K_{ij}K^{ij}$.

The Codazzi-Relation

We can get this relation by projecting the Ricci idendity applied locally to some normal vector field, i.e.

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, N^\gamma = {}^{(n)}R^\gamma{}_{\mu\alpha\beta} N^\mu, \tag{A.93}$$

three times onto the hypersurface Σ_t, we get the **Peterson-Mainardi-Codazzi relation**

$$\gamma^\gamma{}_\rho N^\rho \gamma^\mu{}_\alpha \gamma^\nu{}_\beta {}^{(n)}R^\rho{}_{\sigma\mu\nu} = D_\beta K^\gamma{}_\alpha - D_\alpha K^\gamma{}_\beta. \tag{A.94}$$

Contracting this equation on the indices α and γ yields the **contracted Peterson-Mainardi-Codazzi relation**

$$\gamma^\mu{}_\alpha N^{\nu\,(n)} R_{\mu\nu} = D_\alpha K - D_\mu K^\mu{}_\alpha. \tag{A.95}$$

The Ricci-Relation

We can get the Ricci-relation by projecting the Ricci identity applied to some normal vector field, i.e.

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) N^\gamma = {}^{(n)}R^\gamma{}_{\mu\alpha\beta} N^\mu, \tag{A.96}$$

twice onto the hypersurface Σ_t and once along the normal vector filed N. After a lot of index gymnastic that can be found in [27], we get the so-called **Ricci equation**

$$\gamma_{\alpha\mu} \gamma^\nu{}_\beta N^\rho N^{\sigma\,(n)} R^\mu{}_{\rho\nu\sigma} = \frac{1}{\alpha} \mathcal{L}_m K_{\alpha\beta} + \frac{1}{\alpha} D_\alpha D_\beta \alpha + K_{\alpha\mu} K^\mu{}_\beta. \tag{A.97}$$

We have projected the Riemannian curvature tensor

- fully onto Σ_t
- three times onto Σ_t and once along the normal vector field N
- and twice onto Σ_t and twice along the normal vector field N.

The other possibility to project the Riemannian tensor three times along the normal vector field N will vanish identically, since

$$ {}^{(n)}\mathrm{Riem}(N^\flat, N, N, \cdot) = 0 \quad \text{and} \quad {}^{(n)}\mathrm{Riem}(\cdot, N, N, N) = 0 \tag{A.98}$$

due to the partial antisymmetry. [27] Thus, the Gauss-relation (A.90), the Codazzi-relation (A.94) and the Ricci equation (A.97) state the full decomposition of the n-dimensional Riemannian curvature tensor.

There are three more important equations that we will state here and refer the attentive reader to the book [27] by Éric Gourgoulhon. By plugging the Ricci-equation (A.97) into the contracted Gauss-relation (A.91) we get an equation holding only the spacetime Ricci tensor

$$\gamma^\mu{}_\alpha \gamma^\nu{}_\beta \, {}^{(n)}\mathrm{R}_{\mu\nu} = -\frac{1}{\alpha} \mathcal{L}_m K_{\alpha\beta} - \frac{1}{\alpha} D_\alpha D_\beta \alpha + R_{\alpha\beta} + K K_{\alpha\beta} - 2 K_{\alpha\mu} K^\mu{}_\beta. \tag{A.99}$$

By taking the trace of the latter equation with respect to the spatial metric γ, we get

$$ {}^{(n)}R + {}^{(n)}R_{\mu\nu} N^\mu N^\nu = R + K^2 - \frac{1}{\alpha} \mathcal{L}_m K - \frac{1}{\alpha} D_i D^i \alpha, \tag{A.100}$$

and by combining the latter equation with the scalar Gauss relation (A.92), we get an equation involving only the spacetime scalar curvature ${}^{(n)}R$

$$ {}^{(n)}R = R + K^2 + K_{ij} K^{ij} - \frac{2}{\alpha} \mathcal{L}_m K - \frac{2}{\alpha} D_i D^i \alpha. \tag{A.101}$$

Appendix B

The Lie Derivative

In this section, we will shortly summarize the main aspects of Lie derivatives of tensor fields in the direction of a vector field as this will be used for the derivation of the FO-CCZ4 system. All definitions, theorems and formulas are taken from [64].

B.1 The Lie Derivative of a Tensor Field

B.1.1 The idea of a Lie Derivative

Let us assume that we have a field of Ⓐ on a manifold M, and we want to quantify how it changes along a smooth vector field $X \in \Gamma^\infty(TM)$. To see how the field of Ⓐ changes along the vector field X, we could move forward along the integral curve ϕ_t^X of X and compare the field of Ⓐ at the point p and $\phi_t^X(p) \in M$. However, as we want to quantify the change of the field at the point p, it is not a good idea to move away from p. But, we could pull back and evaluate the field of Ⓐ at the point p. Then, by dividing by the stepsize and taking the limit gives as the idea of the Lie derivative:

$$\mathcal{L}_X \text{Ⓐ}\Big|_p := \lim_{t \to 0} \frac{\mathit{backdragged}\ \text{Ⓐ}(p) - \text{Ⓐ}(p)}{t}. \tag{B.1}$$

Before we can make the idea of a Lie derivative precise, we will need the notion of an integral curve and how to pull back fields on a manifold.

B.1.2 Integral Curve and Pull-Backs

We can view a vector field $X \in \Gamma^\infty(TM)$ as the map

$$\begin{aligned} X : \mathcal{C}^\infty(M, \mathbb{R}) &\to \mathcal{C}^\infty(M, \mathbb{R}) \\ f &\mapsto X(f) \end{aligned} \tag{B.2}$$

defined by

$$X(f)_{|_p} := X_p(f), \tag{B.3}$$

where we identify the tangent vector $X_p \in T_pM$ as a $\mathbb{R}$-linear derivation on germs of functions

$$X_p : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}. \tag{B.4}$$

and we can write locally on (U, x)

$$X(f)_{|p} := X_p(f) = X_p^\mu \frac{\partial}{\partial x^\mu}\bigg|_p f = X_p^\mu \frac{\partial (f \circ x^{-1})}{\partial x^\mu}\bigg|_{x(p)} \tag{B.5}$$

Let $X \in \Gamma^\infty(TM)$ be a vector field on M. Then for each tangent vector $X_p \in T_pM$, it is easy to find a curve γ through p having X_p as tangent vector by the equivalent definition of a tangent vector by a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$

$$\dot\gamma(0) : \mathcal{C}_p^\infty(M, \mathbb{R}) \ni f \mapsto \dot\gamma(0)(f) := \frac{d(f \circ \gamma)}{dt}\bigg|_{t=0} \in \mathbb{R} \tag{B.6}$$

But if this should hold for each point $p \in \mathrm{im}(\gamma)$, we get the notion of an integral curve.

Definition B.1.1 (Integral Curve)**.** Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M. Then a smooth curve $\gamma : I \to M$ with some open interval $I \subseteq \mathbb{R}$ is called an integral curve of X if for all $t \in I$

$$\dot\gamma(t) = X_{\gamma(t)}. \tag{B.7}$$

We can indeed compare both sides: $\dot\gamma(t)$ is a tangent vector of γ and $X_{\gamma(t)}$ is a tangent vector of the vector field X at the point $\gamma(t)$. As we want to move points $p \in M$ along their integral curve, we need the notion of the flow of a vector field.

Definition B.1.2 (Flow of a Vector Field)**.** Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M and $I_p \subseteq \mathbb{R}$ be the maximal interval on which the integral curve γ of X with $\gamma(0) = p$ is defined. Then

$$\mathcal{U} = \bigcup_{p \in M} I_p \times \{p\} \subseteq \mathbb{R} \times M \tag{B.8}$$

is an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$ and we call the smooth map

$$\Phi \colon \mathcal{U} \to M \tag{B.9}$$

$$(t, p) \mapsto \Phi(t, p) := \gamma(t) \tag{B.10}$$

satisfying

$$\Phi(0, p) = \gamma(0) = p \tag{B.11}$$

$$\Phi(t, \Phi(s, p)) = \Phi(t + s, p) \tag{B.12}$$

for all $p \in M$, $t, s \in \mathbb{R}$ the flow of the vector field X.

Furthermore, we can write for all $(t, p) \in \mathcal{U}$

$$\frac{d}{dt}\Phi(t, p) = X_{\Phi(t,p)} = X_{\gamma(t)}, \tag{B.13}$$

and restrict the flow of a complete vector field, i.e $\mathcal{U} = \mathbb{R} \times M$, for each $t \in \mathbb{R}$ to the map

$$\Phi_t \colon M \to M$$

$$p \mapsto \Phi_t(p) := \Phi(t,p) = \gamma(t)$$

satisfying

$$\Phi_0 = \mathrm{id}_M \quad \text{and} \quad \Phi_t \circ \Phi_s = \Phi_{t+s} \tag{B.14}$$

for all $t, s \in \mathbb{R}$ by the flow property. The procedure of moving points along the integral curve γ_p of a vector field X is called *Lie dragging.* We can Lie drag points, functions and tensor fields forward from a point p to q or backwards from q to p. Lie dragging forward is called the push-forward $\Phi_{t,*}$ and Lie dragging backwards is called the pull-back Φ_t^*. If Φ_t is a diffeomorphism, then both are related by

$$\Phi_{t,*} = (\Phi_t^{-1})^* = \Phi_{-t}^*.$$

Now, let us define the pull-back of a function $f \in \mathcal{C}^\infty(M,\mathbb{R})$ and a ℓ-fold contravariant and k-fold covariant tensor field $S \in \Gamma^\infty(TM^{\otimes l} \otimes T^*M^{\otimes k})$ by the restricted flow map phi_t.

Definition B.1.3 (Pull-back of a function)**.** Let $\Phi_t \colon M \to M$ be a map on the manifold M and $f \colon M \to \mathbb{R}$ be a function. Then the pull-back $\Phi_t^* f$ of f by Φ_t is defined by

$$\Phi_t^* f = f \circ \Phi_t \colon M \to \mathbb{R}. \tag{B.15}$$

Definition B.1.4 (Pull-back of a tensor field)**.** Let $\Phi_t \colon M \to M$ be a local diffeomorphism, $k, \ell \in \mathbb{N}_0$ and $S \in \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k})$ be a ℓ-fold contravariant and k-fold covariant tensor field on M. Then, for $p \in M$ the pull-back $\Phi_t^* S$ of S by Φ_t is defined by

$$\begin{aligned}(\Phi_t^* S)_p&(\alpha_1, \ldots, \alpha_\ell, v_1, \ldots, v_k) \\ &= S_{\Phi_t(p)}(\alpha_1 \circ (T_p\Phi_t)^{-1}, \ldots, \alpha_\ell \circ (T_p\Phi_t)^{-1}, T_p\Phi_t(v_1), \ldots, T_p\Phi_t(v_k)),\end{aligned} \tag{B.16}$$

where $\alpha_1, \ldots, \alpha_\ell \in T_p^* M$ and $v_1, \ldots, v_k \in T_p M$.

This definition can be simplified to only covariant or contravariant tensor fields on M by setting respectively $\ell = 0$ or $k = 0$. An alternative way to write a contravariant tensor field $S \in \Gamma^\infty(TM^{\otimes \ell})$ is given by

$$(\Phi^* S)(p) = ((T_p\Phi)^{-1} \otimes \cdots \otimes (T_p\Phi)^{-1})(S(\Phi(p))) \tag{B.17}$$

for all $p \in M$.

B.1.3 The Lie Derivative

Theorem B.1.1 (Lie derivative of a function)**.** *Let $X \in \Gamma^\infty(TM)$ be a smooth vector field with complete flow $\Phi \colon \mathbb{R} \times M \to M$. Then for every $f \in \mathcal{C}^\infty(M,\mathbb{R})$ we have*

$$\mathcal{L}_X f = \lim_{t \to 0} \frac{\Phi_t^* f - f}{t} = \frac{d}{dt}\Big|_{t=0} \Phi_t^* f = X(f). \tag{B.18}$$

Proof. Let $\gamma_p\colon \mathbb{R} \to M$ be an integral curve of M with $\gamma_p(0) = p$. Then we can calculate

$$\begin{aligned}\mathcal{L}_X f\Big|_p &:= \lim_{t\to 0} \frac{\Phi_t^* f(p) - f(p)}{t} = \lim_{t\to 0} \frac{(f\circ\Phi_t)(p) - f(p)}{t} = \lim_{t\to 0} \frac{(f\circ\gamma_p)(t) - (f\circ\gamma_p)(0)}{t}\\ &= \frac{d}{dt}\Big|_{t=0} (f\circ\gamma_p)(t) = \frac{d}{dt}\Big|_{t=0} (f\circ\Phi)(t,p) = \frac{d}{dt}\Big|_{t=0} (\Phi_t^* f)(p)\\ &= \frac{d}{dt}\Big|_{t=0} (f\circ\gamma_p)(t) = \dot\gamma_p(0)(f) = (X\circ\gamma_p)(0)(f) = X_p(f) = X(f)\Big|_p\end{aligned}$$

□

Let us give some remarks for the Lie derivative of functions.

Remark B.1.1. If the vector field flow would not be complete, then there would not exist a map $\Phi_t\colon M \to M$ for all $t \neq 0$. However, around every point $p \in M$ we find a neighbourhood $U \subseteq M$ and an $\epsilon > 0$ such that for all $t \in (-\epsilon, \epsilon)$ the restrictions $\Phi_t\colon U \to M$ are defined and diffeomorphism onto their images and we can consider the local pull-back $(\Phi_t\Big|_U)^* S$ for some tensor fields S on M yielding a tensor field on U for all $t \in (-\epsilon, \epsilon)$. Since it is enough for the Lie derivative to be computed locally around p and as the size of ϵ is irrelevant, we can define the Lie derivative for a non-complete flow as well. Bearing this in mind, we can now state the definition of the Lie derivative of a vector and tensor field.

Theorem B.1.2 (Lie derivative of a vector field)**.** *Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M with flow Φ. For a vector field $Y \in \Gamma^\infty(TM)$ one defines the Lie derivative of Y in the direction of X by*

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{\Phi_t^* Y - Y}{t} = \frac{d}{dt}\Big|_{t=0} \Phi_t^* Y = [X, Y], \tag{B.19}$$

where $[\cdot,\cdot]$ denotes the Lie bracket.

Theorem B.1.3 (Lie derivative of a tensor field)**.** *Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M with flow Φ. For a tensor field $S \in \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k})$ one defines the Lie derivative of S in the direction of X by*

$$\mathcal{L}_X S = \lim_{t\to 0} \frac{\Phi_t^* S - S}{t} = \frac{d}{dt}\Big|_{t=0} \Phi_t^* S. \tag{B.20}$$

Remark B.1.2. Let us notice that the Lie derivative is a linear map

$$\mathcal{L}_X\colon \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k}) \to \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k}) \tag{B.21}$$

and a differential operator of order one, but for practical reasons, these definitions are not very suitable as it requires to compute the flow of the vector field X, which amounts to solving a differential equation. Let us therefore state in the next section some local formulas of the Lie derivative.

B.1.4 Local Formulas

Theorem B.1.4 (Local formulas)**.** *Let (U, x) be a local chart for M. Then we have the following local Lie derivatives*

i.) *For coordinate vector field one has for all $\mu, \nu = 0, \ldots, n-1$*

$$\mathcal{L}_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = \left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right] = 0 \tag{B.22}$$

ii.) *For coordinate one-forms one has for all $\mu, \nu = 0, \ldots, n-1$*

$$\mathcal{L}_{\frac{\partial}{\partial x^\mu}} dx^\nu = 0 \tag{B.23}$$

iii.) *The Lie derivative $\mathcal{L}_X$ is only $\mathcal{C}^\infty(M, \mathbb{R})$-linear in the argument X for functions $g \in \mathcal{C}^\infty(M, \mathbb{R})$*

$$\mathcal{L}_{fX} g = f \mathcal{L}_X g. \tag{B.24}$$

For vector fields we get

$$\mathcal{L}_{fX} Y = [fX, Y] = f \mathcal{L}_X Y + Y(f) X. \tag{B.25}$$

iv.) *Let $S \in \Gamma^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k})$ and $X \in \Gamma^\infty(TM)$ be a tensor and vector field, respectively, on M with the local forms*

$$S\Big|_U = S^{\mu_1 \ldots \mu_\ell}_{\nu_1 \ldots \nu_k} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_\ell}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_k} \tag{B.26}$$

and

$$X\Big|_U = X^\mu \frac{\partial}{\partial x^\mu} \tag{B.27}$$

we can write the Lie derivative $\mathcal{L}_X S$ locally by

$$\begin{aligned} \mathcal{L}_X S\Big|_U = &\left(X^\rho \frac{\partial S^{\mu_1 \ldots \mu_\ell}_{\nu_1 \ldots \nu_k}}{\partial x^\rho} - \sum_{r=1}^{\ell} S^{\mu_1 \ldots \mu_{r-1} \sigma \mu_{r+1} \ldots \mu_\ell}_{\nu_1 \ldots \nu_k} \frac{\partial X^{\mu_r}}{\partial x^\sigma} + \sum_{s=1}^{k} S^{\mu_1 \ldots \mu_\ell}_{\nu_1 \ldots \nu_{s-1} \sigma \nu_{s+1} \ldots \nu_k} \frac{\partial X^\sigma}{\partial x^{\nu_s}} \right) \\ &\frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_\ell}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_k}. \end{aligned} \tag{B.28}$$

v.) *Notice that the local formula B.28 can be written by any connection without torsion, such as the Levi-Civita connection ∇ associated with the metric $\mathbf{g}$ by*

$$\begin{aligned} \mathcal{L}_X S\Big|_U = &\left(X^\rho \nabla_\rho S^{\mu_1 \ldots \mu_\ell}_{\nu_1 \ldots \nu_k} - \sum_{r=1}^{\ell} S^{\mu_1 \ldots \mu_{r-1} \sigma \mu_{r+1} \ldots \mu_\ell}_{\nu_1 \ldots \nu_k} \nabla_\sigma X^{\mu_r} + \sum_{s=1}^{k} S^{\mu_1 \ldots \mu_\ell}_{\nu_1 \ldots \nu_{s-1} \sigma \nu_{s+1} \ldots \nu_k} \nabla_{\nu_s} X^\sigma \right) \\ &\frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_\ell}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_k}. \end{aligned} \tag{B.29}$$

Appendix C

Properties of a relativistic Spacetime

In this section, we will shortly summarize the main properties the spacetime manifold M has to satisfy. First, we will recall what a differentiable manifold is and when it admits a Lorentzian metric. Then, as we assume that the spacetime M will be time-orientable and globally hyperbolic, we will recall the definitions of the latter and give some remarks.

C.1 Manifold Properties

Definition C.1.1 (Topological manifold)**.** Let $(M, \mathcal{M})$ be a topological space. Then $(M, \mathcal{M})$ is called a **topological manifold** of dimension $n \in \mathbb{N}_0$ if it allows for an n-dimensional topological atals, satisfies the second countability axiom and is Hausdorff.

Since we can use local coordinates around every point $p \in M$, we can use the familiar concepts of differentiation in $\mathbb{R}^n$ and transfer them to topological manifolds. This will need the additional assumption that for any two local charts (U_α, x_α) and (U_β, x_β) of the atlas $\mathfrak{U} = \{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ either $U_\alpha \cap U_\beta = \emptyset$ or $x_\alpha \circ x_\beta^{-1}\Big|_{x_\beta(U_\alpha \cap U_\beta)} \in \mathcal{C}^k$ and we will call an atlas with such a structure a $\mathcal{C}^k$-atlas. Therefore,

Definition C.1.2. A $\mathcal{C}^k$-manfiold is a topological manifold together with a maximal $\mathcal{C}^k$-atlas.

Definition C.1.3 (Spacetime)**.** The pair (M, g) of a differentiable manifold M and a Lorentz metric $g \in \Gamma^\infty(T^*M^{\otimes 2})$ is called a **general relativistic spacetime**.

But how do we know that there exists for any spacetime M a Lorentzian metric g? Well, [55] states:

Proposition C.1.1. *For a smooth manifold M the following statements are equivalent*

(1) There exists a Lorentz metric $g \in \Gamma^\infty(T^*M^{\otimes 2})$ on M

(2) There exists a time-orientable Lorentz metric $g \in \Gamma^\infty(T^*M^{\otimes 2})$ on M

(3) There exists a non-vanishing vector field $X \in \Gamma^\infty(TM)$ *on* M

(4) Either M *is non-compact, or* M *is compact and has Euler characterisic zero*

C.2 Time-orientability

Furthermore, as we want the spacetime to have a global direction of time, we need to assume that (M, g) will be **time-orientable**. But what does this mean? Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M, then a tangent vector $X_p \in T_pM$ at a point $p \in M$ can be

- timelike if $g_p(X_p, X_p) < 0$
- null or lightlike if $g_p(X_p, X_p) = 0$
- spacelike if $g_p(X_p, X_p) > 0$,

where we have used the Lorentzian signature $(-, +, +, +)$ of the metric g. Furthermore, a tangent vector X_p is called *non-spacelike* if $g_p(X_p, X_p) \leq 0$, so if it is time- or lightlike. Besides, every timelike vector $X_p \in T_pM$ at a point $p \in M$ can be divided into two equivalence classes - the "future" and the "past" direction.

To do so, we first recall the concepts of orientation for any smooth manifold: at each $p \in M$, we partition all ordered bases in T_pM into exactly two equivalence classes depending on if the change of basis from one (ordered) basis to another basis has positive or negative determinant. The determinant of the change of basis can't be zero since the change of basis is bijective. We can now orient M by simply picking one equivalence class at each T_pM and say that M is smoothly positively oriented if at each point we can find a smooth local frame $E_i \in \Gamma^\infty(TM\big|_U)$ of TM such that $(E_0\big|_p, \cdots, E_{n-1}\big|_p)$ is a positively oriented basis for T_pM at each point $p \in U$ [44]. The concept of time orientation works the same way. Instead of ordered bases at each T_pM we have time-cones: namely, for any timelike vector $X_p \in T_pM$ we define the time-cone of T_pM containing X_p to be

$$C(X_p) :=\{Y_p \text{ any timelike vector in } T_pM : g_p(X_p, Y_p) < 0\}.$$

Two timelike vectors X_p, Y_p are in the same time-cone at T_pM if and only if $g_p(X_p, Y_p) < 0$. So just as with orientation, we can time-orient our manifold by picking one of the two time-cones from each tangent space T_pM [55]. We can now arbitrarily call one equivalence class *future-directed* and the other one *past-directed.* If we can do this continuously for the entire manifold, i.e. $\forall p \in M$, then we say that the Lorentzian manifold M is time-orientable.

Furthermore, the following lemma is taken from [54]

Lemma C.2.1. *The following statements are equivalent:*

- *There exists a continuous non-vanishing time-like vector field* $X \in \Gamma^\infty(TM)$ *on* M

- *Any means of transporting a timelike vector around an arbitrarily closed loop in spacetime that is continuous and keeps the vector timelike does not result in time inversion of the vector when it returns to the starting point*
- *Parallel transport of a timelike vector around an arbitrarily closed loop in the spacetime does not result in time inversion*

Lemma C.2.2. *For any (M, g) if M is simply connected, then (M, g) is time orientable.*

Lemma C.2.3. *If (M, g) is not temporally orientable, then time orientability can be achieved by passing to a covering spacetime.*

C.3 Global Hyperbolicity

The following lemma and definition about strong causality is taken from [50]

Lemma C.3.1. *For any event p of a spacetime (M,g), the following sentences are equivalent:*

- *Given any neighborhood U of p there exists a neighborhood $V \subset U$, $p \in V$ (which can be chosen globally hyperbolic), such that V is causally convex in M and thus in U.*
- *Given any neighborhood U of p there exists a neighborhood $V \subset U$, $p \in V$, such that any future-directed (and hence also any past-directed) causal curve $\gamma \colon I \to M$ with endpoints at V is entirely contained in U.*

Definition C.3.1 (Strongly Causal)**.** A spacetime (M, g) is called strongly causal at $p \in M$ if it satisfies one of the equivalent properties in Lemma C.3.1. A spacetime is strongly causal if it is strongly causal $\forall p \in M$.

Definition C.3.2. A spacetime (M, g) is globally hyperbolic if and only if M is strongly causal and $\forall p, q \in M$ such that $p << q$, $C^-(q) \cap C^+(p)$ is compact.

The set $C^-(p)$ and $C^+(p)$, respectivley, are the causal past and causal future, and $p << q$ means that p chronologically preceeds to q.

Lemma C.3.2. *The spacetime (M, g) is globally hyperbolic if and only if it admits a Cauchy surface, i.e. a spacelike hypersurface which meets every maximally extended timelike curve exactly once.*

Appendix D

Characterization of Killing Vector Fields

In this short appendix, we will state the definition of a Killing vector field, as they can be used to characterise symmetries of spacetimes. And exactly those spacetimes with maximal linear independent Killing vector fields are of interest and we call them maximally symmetric spacetimes.

Definition D.0.1. Let (M, g) and (N, h) be n-dimensional pseudo-Riemannian manifolds with Lorentzian metrics. Then, a diffeomorphism

$$\begin{aligned} \phi \colon M &\to N \\ p &\mapsto \phi(p) = q \end{aligned} \tag{D.1}$$

is called an *isometry* of a pseudo-Riemannian manfiold (M, g) if it preserves the metric tensor [55]

$$h_{\phi(p)} \left(T_p\phi(v_p), T_p\phi(w_p)\right) = g_p \left(v_p, w_p\right), \tag{D.2}$$

for all $p \in M$ and $v_p, w_p \in T_pM$. The set of all isometries of (M, g) is denoted by $\mathrm{Iso}(M, g)$.

As the metrics are 2-fold covariant tensor fields, we can write the definition of an isometry equivalently as

$$\phi^* h = g, \tag{D.3}$$

where ϕ^* is the pull-back map as defined in (A.31). From the properties of the pull-back map, we can easily see that $\mathrm{Iso}(M, g)$ forms a group, i.e.

- the identity map id is an isometry
- the composition of isometries is an isometry,
- and the inverse map of an isometry is an isometry.

Isometries are of importance, as they leave the metric and all other objects expressed in terms of the metric invariant. Let us therefore see how the metric components $g_{\alpha\beta}(p)$ at the point $p \in M$ can be related to the metric components $g_{\mu\nu}(\phi(p))$ at the point $\phi(p) \in N$.

Proposition D.0.1. *Let (U, x) and (V, y), respectively, be local charts of M around p and N around $\phi(p) = q$, and ϕ an isometry of M. Then,*

$$g_{\alpha\beta} = \frac{\partial\left(y^\mu \circ \phi\right)}{\partial x^\alpha} \frac{\partial\left(y^\nu \circ \phi\right)}{\partial x^\beta} h_{\mu\nu} \circ \phi. \tag{D.4}$$

Proof. Let us calculate how the metric components are related via the isometry ϕ

$$\begin{aligned} g_{\alpha\beta}(p) &= g\Big|_p \left(\frac{\partial}{\partial x^\alpha}\Big|_p, \frac{\partial}{\partial x^\beta}\Big|_p\right) = \left(\phi^* h\right)\Big|_p \left(\frac{\partial}{\partial x^\alpha}\Big|_p, \frac{\partial}{\partial x^\beta}\Big|_p\right) \\ &= h\Big|_{\phi(p)} \left(T_p\phi\left(\frac{\partial}{\partial x^\alpha}\Big|_p\right), T_p\phi\left(\frac{\partial}{\partial x^\beta}\Big|_p\right)\right) \\ &= \frac{\partial\left(y^\mu \circ \phi \circ x^{-1}\right)}{\partial x^\alpha}\Big|_{x(p)} \frac{\partial\left(y^\nu \circ \phi \circ x^{-1}\right)}{\partial x^\beta}\Big|_{x(p)} h\Big|_{\phi(p)} \left(\frac{\partial}{\partial y^\mu}\Big|_q, \frac{\partial}{\partial y^\nu}\Big|_q\right) \\ &= \frac{\partial\left(y^\mu \circ \phi \circ x^{-1}\right)}{\partial x^\alpha}\Big|_{x(p)} \frac{\partial\left(y^\nu \circ \phi \circ x^{-1}\right)}{\partial x^\beta}\Big|_{x(p)} h_{\mu\nu}(\phi(p)) \\ &= \frac{\partial}{\partial x^\alpha}\Big|_p \left(y^\mu \circ \phi\right) \frac{\partial}{\partial x^\beta}\Big|_p \left(y^\nu \circ \phi\right) h_{\mu\nu}(\phi(p)) \ \forall p \in M. \end{aligned} \tag{D.5}$$

Thus,

$$g_{\alpha\beta} = \frac{\partial\left(y^\mu \circ \phi\right)}{\partial x^\alpha} \frac{\partial\left(y^\nu \circ \phi\right)}{\partial x^\beta} h_{\mu\nu} \circ \phi. \tag{D.6}$$

If we want to figure out, how the metric $g \in \Gamma^\infty(T^*M^{\otimes 2})$ behaves under coordinate transformation, we can take the identity $\phi = \mathrm{id}$ as isometry and simplify

$$\begin{aligned} g_{\alpha\beta}(p) &= \frac{\partial\left(y^\mu \circ \phi \circ x^{-1}\right)}{\partial x^\alpha}\Big|_{x(p)} \frac{\partial\left(y^\nu \circ \phi \circ x^{-1}\right)}{\partial x^\beta}\Big|_{x(p)} h_{\mu\nu}(\phi(p)) \\ &= \frac{\partial\left(y^\mu \circ x^{-1}\right)}{\partial x^\alpha}\Big|_{x(p)} \frac{\partial\left(y^\nu \circ x^{-1}\right)}{\partial x^\beta}\Big|_{x(p)} h_{\mu\nu}(\phi(p)) \\ &= \frac{\partial}{\partial x^\alpha}\Big|_p y^\mu \frac{\partial}{\partial x^\beta}\Big|_p y^\nu \ h_{\mu\nu}(p) \ \forall p \in M. \end{aligned} \tag{D.7}$$

Thus,

$$g_{\alpha\beta} = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} \ h_{\mu\nu} \in \mathcal{C}^\infty(M, \mathbb{R}). \tag{D.8}$$

□

Let $\phi \colon \mathcal{U} \to M$ be the flow of the vector field $X \in \Gamma^\infty(TM)$, then we can restrict this map to the smooth map

$$\phi_t \colon U \to M \tag{D.9}$$

for all $t \in I \subset \mathbb{R}$ and $U \subset M$. If X is a complete vector field, then we can restrict the flow to the diffeomorphism

$$\phi_t \colon M \to M \tag{D.10}$$

for all $t \in \mathbb{R}$. If ϕ_t is a (local) isometry, i.e. preserving the metric, then

$$\phi_t^* g = g \tag{D.11}$$

and can we motivate the infinitesimal couterpart to (D.11) by taking the Lie derivative:

$$\begin{aligned}
\mathcal{L}_X g\Big|_p (v_p, w_p) &= \left(\frac{d}{dt}\Big|_{t=0} \phi_t^* g \right) (p) (v_p, w_p) \\
&= \frac{d}{dt}\Big|_{t=0} (\phi_t^* g)_p (v_p, w_p) \\
&= \frac{d}{dt}\Big|_{t=0} g_p (v_p, w_p) \\
&= 0
\end{aligned} \tag{D.12}$$

for all $p \in M$ and $v_p, w_p \in T_pM$. This gives us the following definition

Definition D.0.2. Let (M, g) be a pseudo-Riemannian manifold. Then, a vector field $X \in \Gamma^\infty(TM)$ is called a Killing vector field with respect to the metric g if

$$\mathcal{L}_X g = 0. \tag{D.13}$$

The set of Killing vector fields is a Lie subalgebra denoted by $\mathfrak{iso}(M, g) \subset \Gamma^\infty(TM)$.

Thus, by Eq. (B.1.3), the metric does not change under the flow ϕ_t^* of the vector field $X \in \Gamma^\infty(TM)$ and we can view a Killing vector field as an "infinitesimal isometry" [55]. Furthermore, it follows from Lemma 28 in [55], that the Lie subalgebra $\mathfrak{iso}(M, g)$ for a connected pseudo-Riemannian manifold M has dimension at most $n(n + 1)/2$.

For further characterisation of Killing vector fields, see the respective chapters in [55] and references within.

Appendix E

Calculations for the Boundary Behaviour

In this appendix, you can find all the cumbersome derivation of the condition of the $n^3/2 + n^2 + 5/2n$ unknown variables of the state vector $\mathbf{Q}$ at the boundary $\rho = \ell$, $q = 0$ or $r \to \infty$. To gain some intuition on how these fields behave near the boundary, we have perturbated the AdS metric $\hat{g}_{\mu\nu}$ with a small deviation $\epsilon h_{\mu\nu}$.

E.1 The conformally-decomposed spatial metric

Let us first prove that the spatial metric γ_{ij} and its inverse γ^{ij} satisfy the condition

$$\gamma^{ij}\gamma_{ij} = n - 1. \tag{E.1}$$

Then,

$$\begin{aligned}
\gamma^{ij}\gamma_{ij} &= \gamma^{\rho\rho}\gamma_{\rho\rho} + 2\gamma^{\rho a}\gamma_{\rho a} + \gamma^{ab}\gamma_{ab} \\
&= \left(\hat{\gamma}_{\rho\rho} + q^{n-4}\bar{h}_{\rho\rho}\epsilon\right)\left(\hat{\gamma}^{\rho\rho} - \hat{\gamma}^{\rho\rho}\hat{\gamma}^{\rho\rho}q^{n-4}\bar{h}_{\rho\rho}\epsilon\right) - 2q^{n-3}\bar{h}_{\rho a}\epsilon\hat{\gamma}^{\rho\rho}\hat{\gamma}^{ab}q^{n-3}\bar{h}_{\rho b}\epsilon \\
&\qquad + \left(\hat{\gamma}_{ab} + q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab}\epsilon\right)\left(\hat{\gamma}^{ab} - \hat{\gamma}^{ac}\hat{\gamma}^{bd}q^{n-4}\rho^2 g_{cd,S^{n-2}}\bar{h}_{cd}\epsilon\right) \\
&= \hat{\gamma}_{\rho\rho}\hat{\gamma}^{\rho\rho} + \hat{\gamma}_{ab}\hat{\gamma}^{ab} + \mathcal{O}(\epsilon^2) \\
&= n - 1 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{E.2}$$

Furthermore, we can show that

$$\begin{aligned}
\phi^2 \times \phi^{-2} &= \hat{\phi}^2\left(1 + \epsilon h_{\phi^2} + \mathcal{O}(\epsilon^2)\right) \times \hat{\phi}^{-2}\left(1 + \epsilon h_{\phi^{-2}} + \mathcal{O}(\epsilon^2)\right) \\
&= \left(1 - \frac{q^2}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2)\right) \\
&\qquad \times \left(1 + \frac{q^2}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2)\right) \\
&= 1 - \frac{q^2}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon + \frac{q^2}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon + \mathcal{O}(\epsilon^2) \\
&= 1 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{E.3}$$

Now, we can calculate the boundary behaviour of the conformally decomposed spatial metric via

$$\tilde{\gamma}_{ij} = \hat{\tilde{\gamma}}_{ij} + \epsilon h_{\tilde{\gamma}_{ij}} + \mathcal{O}(\epsilon^2) \qquad \text{and} \qquad \tilde{\gamma}^{ij} = \hat{\tilde{\gamma}}^{ij} + \epsilon h_{\tilde{\gamma}^{ij}} + \mathcal{O}(\epsilon^2), \tag{E.4}$$

where the purely AdS$_n$ part is given by

$$\hat{\tilde{\gamma}}_{ij} = \hat{\phi}^2 \tilde{\gamma}_{ij} \quad \text{and} \quad \hat{\tilde{\gamma}}^{ij} = \hat{\phi}^{-2} \tilde{\gamma}^{ij} \tag{E.5}$$

while the deviation is given by

$$h_{\tilde{\gamma}_{ij}} = \hat{\phi}^2 \left(h_{ij} + h_{\phi^2} \hat{\gamma}_{ij} \right), \quad \text{and} \quad h_{\tilde{\gamma}^{ij}} = \hat{\phi}^{-2} \left(h^{ij} + h_{\phi^{-2}} \hat{\gamma}^{ij} \right), \tag{E.6}$$

Then,

$$\begin{aligned}
\tilde{\gamma}_{\rho\rho} =& \hat{\tilde{\gamma}}_{\rho\rho} + \hat{\phi}^2 \left(q^{n-4} \bar{h}_{\rho\rho} + h_{\phi^2} \hat{\gamma}_{\rho\rho} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{a^{\frac{n-2}{1-n}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} + \frac{1}{n-1} \frac{q^{n-2} a^{\frac{n-2}{1-n}} \rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left((n-2) a \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{1}{a^{\frac{n-2}{n-1}} \rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[1 + \frac{q^{n-2}}{n-1} \left((n-2) a \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \right] \\
& + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{E.7a}$$

$$\tilde{\gamma}_{\rho b} = \hat{\phi}^2 q^{n-3} \bar{h}_{\rho b} \epsilon + \mathcal{O}(\epsilon^2) = \frac{a^{\frac{1}{n-1}} q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \bar{h}_{\rho b} \epsilon + \mathcal{O}(\epsilon^2) \tag{E.7b}$$

$$\begin{aligned}
\tilde{\gamma}_{ab} =& \hat{\tilde{\gamma}}_{ab} + \hat{\phi}^2 \left(q^{n-4} \rho^2 g_{ab,S^{n-2}} \bar{h}_{ab} + h_{\phi^2} \hat{\gamma}_{ab} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \frac{\rho^2}{q^2} g_{ab,S^{n-2}} + \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[q^{n-4} \rho^2 g_{ab,S^{n-2}} \bar{h}_{ab} \right. \\
& \left. - \frac{q^{n-2}}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \frac{\rho^2}{q^2} g_{ab,S^{n-2}} \right] \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \frac{\rho^2}{q^2} g_{ab,S^{n-2}} + \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \frac{\rho^2}{q^2} g_{ab,S^{n-2}} \\
& \times \left[q^{n-2} \bar{h}_{ab} - \frac{q^{n-2}}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \right] \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left[1 - \frac{q^{n-2}}{n-1} \left(a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} - (n-1) \bar{h}_{ab} \right) \epsilon \right] \\
& + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{E.7c}$$

and

$$\begin{aligned}
\tilde{\gamma}^{\rho\rho} =& \hat{\tilde{\gamma}}^{\rho\rho} + \hat{\phi}^{-2} \left(-\hat{\gamma}^{\rho\rho} \hat{\gamma}^{\rho\rho} q^{n-4} \bar{h}_{\rho\rho} + h_{\phi^{-2}} \hat{\gamma}^{\rho\rho} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} q^2} q^2 a + \frac{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} q^2} \Big[- q^2 a q^2 a q^{n-4} \bar{h}_{\rho\rho}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) q^2 a\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \\
&= a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} + a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\Bigg[-aq^{n-2}\bar{h}_{\rho\rho} \\
&+ \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{1-n}}}\left[1 + \frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right] + \mathcal{O}(\epsilon^2) \qquad \text{(E.8a)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\gamma}^{\rho a} &= -\hat{\phi}^{-2}\hat{\gamma}^{\rho\rho}\hat{\gamma}^{ba}q^{n-3}\bar{h}_{\rho b}\epsilon + \mathcal{O}(\epsilon^2) \\
&= -\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2} q^2 a \frac{q^2}{\rho^2} g^{ba}_{S^{n-2}} q^{n-3}\bar{h}_{\rho b}\epsilon + \mathcal{O}(\epsilon^2) \\
&= -\frac{a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}} g^{ba}_{S^{n-2}} q^{n-1}\bar{h}_{\rho b}\epsilon + \mathcal{O}(\epsilon^2) \qquad \text{(E.8b)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\gamma}^{ab} &= \hat{\tilde{\gamma}}^{ab} + \hat{\phi}^{-2}\left(-\hat{\gamma}^{ac}\hat{\gamma}^{bd}q^{n-4}\rho^2 g_{cd,S^{n-2}}\bar{h}_{cd} + h_{\phi^{-2}}\hat{\gamma}^{ab}\right)\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}\frac{q^2}{\rho^2}g^{ab}_{S^{n-2}} - \frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2} \\
&\times\left[\frac{q^2}{\rho^2}g^{ac}_{S^{n-2}}\frac{q^2}{\rho^2}g^{bd}_{S^{n-2}}q^{n-4}\rho^2 g_{cd,S^{n-2}}\bar{h}_{cd} - \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{q^2}{\rho^2}g^{ab}_{S^{n-2}}\right]\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}a^{\frac{1}{n-1}}}\Bigg[g^{ab}_{S^{n-2}} - q^{n-2}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd} \\
&+ \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \qquad \text{(E.8c)}
\end{aligned}
$$

Finally, we can calculate the trace of the conformally decomposed spatial metric to prove the correctness of these equations. Therefore,

$$
\begin{aligned}
\tilde{\gamma}^{ij}\tilde{\gamma}_{ij} &= \tilde{\gamma}^{\rho\rho}\tilde{\gamma}_{\rho\rho} + \tilde{\gamma}^{ab}\tilde{\gamma}_{ab} \\
&= a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left[1 + \frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right] \\
&\quad\times\frac{1}{a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left[1 + \frac{q^{n-2}}{n-1}\left((n-2)a\bar{h}_{\rho\rho} - \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right] \\
&\quad+ \frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}a^{\frac{1}{n-1}}}\left[g^{ab}_{S^{n-2}} - q^{n-2}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd}\epsilon + \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\epsilon\right] \\
&\quad\times\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left[1 - \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab}\right)\epsilon\right] + \mathcal{O}(h^2) \\
&= \left(1 - \frac{q^{n-2}}{n-1}\left((n-2)a\bar{h}_{\rho\rho} - \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left(1+\frac{q^{n-2}}{n-1}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)\\
&\quad +\left(g^{ab}_{S^{n-2}}-q^{n-2}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd}\epsilon+\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\epsilon\right)\\
&\quad \times \left(g_{ab,S^{n-2}}-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{ab}\right)g_{ab,S^{n-2}}\epsilon\right)+\mathcal{O}(\epsilon^2)\\
&=1+g^{ab}_{S^{n-2}}g_{ab,S^{n-2}}-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{ab}\right)g^{ab,S^{n-2}}g_{ab,S^{n-2}}\epsilon\\
&\quad -q^{n-2}g_{ab,S^{n-2}}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd}\epsilon\\
&\quad +q^{n-2}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{ab}\right)g_{ab,S^{n-2}}\epsilon\\
&\quad +\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}g_{ab,S^{n-2}}\epsilon\\
&\quad -\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{ab}\right)g_{ab,S^{n-2}}\epsilon\\
&=1+n-2-\frac{n-2}{n-1}q^{n-2}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon+q^{n-2}g^{ab}_{S^{n-2}}g_{ab,S^{n-2}}\bar{h}_{ab}\epsilon\\
&\quad -q^{n-2}g^{ab}_{S^{n-2}}g_{ab,S^{n-2}}\bar{h}_{ab}\epsilon+\frac{n-2}{n-1}q^{n-2}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon+\mathcal{O}(\epsilon^2)\\
&=n-1+\mathcal{O}(\epsilon^2).
\end{aligned}
\tag{E.9}
$$

This gives a good hint that our calculated equations for the conformally decomposed spatial metric are correct. This will be verified by a Mathematica script for an $n=3$ and $n=4$ dimensional manifold M.

E.2 The auxiliary variable A_i

The individual derivatives of

$$A_\rho=\partial_\rho\ln\alpha=\partial_\rho\ln\hat{\alpha}+\partial_\rho h_\alpha+\mathcal{O}(h_\alpha^2) \tag{E.10}$$

are given, using Eq. (4.73) and Eq. (4.74) respectively, by

$$\partial_\rho\ln\hat{\alpha}=\partial_\rho\ln\left(\frac{1}{q}\sqrt{q^2+\frac{\rho^2}{L^2}}\right)=\partial_\rho\ln\left(\sqrt{1+\frac{\rho^2}{q^2L^2}}\right)=\frac{\rho}{L^2qa}, \tag{E.11}$$

and

$$
\begin{aligned}
\partial_\rho h_\alpha&=-\partial_\rho\left(\frac{q^{n-2}}{2a}\right)\bar{h}_{tt}-\frac{q^{n-2}}{2a}\partial_\rho\bar{h}_{tt}+\mathcal{O}(h^2)\\
&=\left(\frac{q^{n-2}\left(\frac{2\rho}{L^2}-\frac{2q}{l}\right)}{2a^2}+\frac{(n-2)q^{n-3}}{2la}\right)\bar{h}_{tt}-\frac{q^{n-2}}{2a}\partial_\rho\bar{h}_{tt}+\mathcal{O}(h^2).
\end{aligned}
\tag{E.12}
$$

E.3 The auxiliary variable P_i

The individual derivatives of

$$P_\rho = \partial_\rho \ln \hat{\phi} + \partial_\rho h_\phi + \mathcal{O}(h_\phi^2), \tag{E.13}$$

are given, using Eq. (4.52) and Eq. (4.53) respectively, by

$$\begin{aligned}\partial_\rho \ln \hat{\phi} &= \partial_\rho \left(\ln q + \frac{\ln a}{2(n-1)} - \frac{\ln\left(\rho^{2(n-2)} \det(g_{ab,S^{n-2}})\right)}{2(n-1)} \right) \\ &= -\frac{1}{ql} + \frac{1}{2(n-1)} \frac{\partial_\rho a}{a} - \frac{n-2}{(n-1)} \frac{1}{\rho},\end{aligned} \tag{E.14}$$

and

$$\begin{aligned}\partial_\rho h_\phi &= -\partial_\rho \left(\frac{q^{n-2}}{2(n-1)} \right) \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) - \frac{q^{n-2}}{2(n-1)} \partial_\rho \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \\ &= \frac{(n-2)}{2(n-1)\ell} q^{n-3} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \\ &\quad - \frac{q^{n-2}}{2(n-1)} \left(\partial_\rho a \bar{h}_{\rho\rho} + a \partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \right) + \mathcal{O}(h^2).\end{aligned} \tag{E.15}$$

E.4 The auxiliary variables D_{ijk}

By using

$$\begin{aligned}\partial_\rho \hat{\phi}^2 &= \partial_\rho \left[\frac{a q^{2(n-1)}}{\rho^{2(n-2)} \det(g_{ab,S^{n-2}})} \right]^{\frac{1}{n-1}} = \frac{1}{(n-1)\det(g_{ab,S^{n-2}})} \left[\frac{a q^{2(n-1)}}{\rho^{2(n-2)} \det(g_{ab,S^{n-2}})} \right]^{\frac{2-n}{n-1}} \\ &\times \left(\frac{q^{2(n-1)} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right)}{\rho^{2(n-2)}} - \frac{2(n-1) q^{2n-3} a}{\rho^{2(n-2)} \ell} - \frac{2(n-2) q^{2(n-1)} a}{\rho^{2n-3}} \right)\end{aligned} \tag{E.16a}$$

$$\partial_a \hat{\phi}^2 = \partial_a \sqrt[n-1]{\frac{a q^{2(n-1)}}{\rho^{2(n-2)} \det(g_{ab,S^{n-2}})}} = -\frac{1}{n-1} \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}}} \frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} \tag{E.16b}$$

$$\begin{aligned}\partial_\rho \hat{\hat{\gamma}}_{\rho\rho} &= \partial_\rho \left(\sqrt[n-1]{\frac{a q^{2(n-1)}}{\rho^{2(n-2)} \det(g_{ab,S^{n-2}})}} \frac{1}{q^2 a} \right) = \partial_\rho \left(\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \right) \\ &= \frac{\partial_\rho a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} + \frac{a^{\frac{2-n}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \rho^{\frac{-2(n-2)}{n-1}} \\ &= \frac{2-n}{n-1} \frac{a^{\frac{3-2n}{n-1}} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right)}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} - \frac{2(n-2)}{n-1} \frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\end{aligned} \tag{E.16c}$$

$$\partial_b \hat{\hat{\gamma}}_{\rho\rho} = \partial_b \left(\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \right) = \frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}} \partial_b \det(g_{ab,S^{n-2}})^{\frac{-1}{n-1}}$$

$$= -\frac{1}{n-1}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_b \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} \tag{E.16d}$$

$$\begin{aligned}\partial_\rho h_{\phi^2} &= \partial_\rho\left(-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) = \frac{(n-2)q^{n-3}}{(n-1)\ell}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\\ &\quad - \frac{q^{n-2}}{n-1}\left(\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)\end{aligned} \tag{E.16e}$$

$$\partial_b h_{\phi^2} = \partial_b\left(-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) = -\frac{q^{n-2}}{n-1}\left(a\partial_b\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_b\bar{h}_{\theta_i\theta_i}\right) \tag{E.16f}$$

$$\begin{aligned}\partial_\rho\hat{\bar{\gamma}}_{ab} &= \partial_\rho\left(\sqrt[n-1]{\frac{aq^{2(n-1)}}{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}}\frac{\rho^2}{q^2}g_{ab,S^{n-2}}\right) = \partial_\rho\left(\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}g_{ab,S^{n-2}}\right)\\ &= \frac{\partial_\rho a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}g_{ab,S^{n-2}} + \frac{a^{\frac{1}{n-1}}\partial_\rho\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}g_{ab,S^{n-2}}\\ &= \frac{1}{n-1}\frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left(a^{\frac{2-n}{n-1}}\rho^{\frac{2}{n-1}}\partial_\rho a + 2a^{\frac{1}{n-1}}\rho^{\frac{3-n}{n-1}}\right)\end{aligned} \tag{E.16g}$$

$$\begin{aligned}\partial_c\hat{\bar{\gamma}}_{ab} &= \partial_c\left(\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}g_{ab,S^{n-2}}\right)\\ &= a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})^{\frac{-1}{n-1}} + \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_c g_{ab,S^{n-2}}\\ &= -\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}g_{ab,S^{n-2}}\partial_a\det(g_{ab,S^{n-2}})}{(n-1)\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} + \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_c g_{ab,S^{n-2}}\end{aligned} \tag{E.16h}$$

we can calculate the D_{ijk} auxiliary varibles.

$$\begin{aligned}D_{cab} &= \frac{1}{2}\partial_c\hat{\bar{\gamma}}_{ab} + \frac{1}{2}\left(h_{ab}\partial_c\hat{\phi}^2 + h_{\phi^2}\hat{\gamma}_{ab}\partial_c\hat{\phi}^2 + \hat{\phi}^2\partial_c h_{ab} + \hat{\phi}^2\hat{\gamma}_{ab}\partial_c h_{\phi^2} + \hat{\phi}^2 h_{\phi^2}\partial_c\hat{\gamma}_{ab}\right)\epsilon + \mathcal{O}(\epsilon^2)\\ &= -\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}g_{ab,S^{n-2}}\partial_a\det(g_{ab,S^{n-2}})}{2(n-1)\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} + \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_c g_{ab,S^{n-2}}\\ &\quad - \frac{1}{2}q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab}\frac{1}{n-1}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\ &\quad + \frac{1}{2}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{\rho^2}{q^2}g_{ab,S^{n-2}}\frac{1}{n-1}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\ &\quad + \frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}q^{n-4}\rho^2\left(\partial_c g_{ab,S^{n-2}}\bar{h}_{ab} + g_{ab,S^{n-2}}\partial_c\bar{h}_{ab}\right)\epsilon\\ &\quad - \frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\frac{\rho^2}{q^2}g_{ab,S^{n-2}}\frac{q^{n-2}}{n-1}\left(a\partial_b\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_b\bar{h}_{\theta_i\theta_i}\right)\epsilon\\ &\quad - \frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{\rho^2}{q^2}\partial_c g_{ab,S^{n-2}}\epsilon + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}\left(\frac{1}{n-1}\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\quad - \frac{1}{2(n-1)}q^{n-2}\rho^{\frac{2}{n-1}}a^{\frac{1}{n-1}}\frac{g_{ab,S^{n-2}}\bar{h}_{ab}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\
&\quad + \frac{1}{2(n-1)^2}q^{n-2}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\
&\quad + \frac{1}{2}a^{\frac{1}{n-1}}q^{n-2}\rho^{\frac{2}{n-1}}\left(\frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\bar{h}_{ab} + \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_c\bar{h}_{ab}\right)\epsilon\\
&\quad - \frac{1}{2(n-1)}\frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^{n-2}\left(a\partial_c\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_c\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&\quad - \frac{1}{2(n-1)}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^{n-2}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\epsilon + \mathcal{O}(\epsilon^2)\\
&= -\frac{1}{2}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}\left(\frac{1}{n-1}\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\quad + \frac{q^{n-2}a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2(n-1)^2}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab}\right)\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\
&\quad - \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^{n-2}}{2(n-1)}\frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left(a\partial_c\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_c\bar{h}_{\theta_i\theta_i} - (n-1)\partial_c\bar{h}_{ab}\right)\epsilon\\
&\quad - \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^{n-2}}{2(n-1)}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab}\right)\frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\epsilon\\
&\quad + \mathcal{O}(\epsilon^2) \qquad \text{(E.17a)}
\end{aligned}
$$

$$
\begin{aligned}
D_{a\rho\rho} &= \frac{1}{2}\partial_a\hat{\tilde{\gamma}}_{\rho\rho} + \frac{1}{2}\left(h_{\rho\rho}\partial_a\hat{\phi}^2 + h_{\phi^2}\hat{\gamma}_{\rho\rho}\partial_a\hat{\phi}^2 + \hat{\phi}^2\partial_a h_{\rho\rho} + \hat{\phi}^2\hat{\gamma}_{\rho\rho}\partial_a h_{\phi^2} + \hat{\phi}^2 h_{\phi^2}\partial_a\hat{\gamma}_{\rho\rho}\right)\epsilon + \mathcal{O}(\epsilon^2)\\
&= \frac{1}{2}\left(-\frac{1}{n-1}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\right)\\
&\quad + \frac{1}{2}q^{n-4}\bar{h}_{\rho\rho}\left(-\frac{1}{n-1}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\right)\epsilon\\
&\quad + \frac{1}{2}\left(-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)\frac{1}{q^2 a}\left(-\frac{1}{n-1}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\right)\\
&\quad + \frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_a q^{n-4}\bar{h}_{\rho\rho}\epsilon\\
&\quad + \frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\frac{1}{q^2 a}\left(-\frac{q^{n-2}}{n-1}\left(a\partial_a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_a\bar{h}_{\theta_i\theta_i}\right)\epsilon\right) + \mathcal{O}(\epsilon^2)\\
&= -\frac{1}{2(n-1)}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}} - \frac{q^{n-2}}{2(n-1)}\bar{h}_{\rho\rho}\frac{a^{\frac{1}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2(n-1)^2}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{q^{n-2}}{a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\epsilon\\
&+\frac{1}{2}\frac{a^{\frac{1}{n-1}}q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_a\bar{h}_{\rho\rho}\epsilon\\
&-\frac{1}{2(n-1)}\frac{q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left(a\partial_a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_a\bar{h}_{\theta_i\theta_i}\right)\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{1}{2(n-1)}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\\
&+\frac{1}{2(n-1)^2}\frac{q^{n-2}}{a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-\frac{1}{2(n-1)}\frac{q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left((2-n)a\partial_a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_a\bar{h}_{\theta_i\theta_i}\right)\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{1}{2(n-1)}\frac{\rho^{\frac{2(n-2)}{1-n}}}{a^{\frac{n-2}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\left[1-\frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right]\\
&-\frac{q^{n-2}}{2(n-1)}\frac{a^{\frac{n-2}{1-n}}\rho^{\frac{2(n-2)}{1-n}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left[(2-n)a\partial_a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_a\bar{h}_{\theta_i\theta_i}\right]\epsilon+\mathcal{O}(\epsilon^2)\quad\text{(E.17b)}\\
D_{a\rho b}=&\frac{1}{2}h_{\rho b}\partial_a\hat{\phi}^2\epsilon+\frac{1}{2}\hat{\phi}^2\partial_a h_{\rho b}\epsilon+\mathcal{O}(\epsilon^2)\\
=&\frac{1}{2}q^{n-3}\bar{h}_{\rho b}\left(-\frac{1}{n-1}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\right)\epsilon\\
&\quad+\frac{1}{2}\frac{a^{\frac{1}{n-1}}q^2}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}q^{n-3}\partial_a\bar{h}_{\rho b}\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{1}{2(n-1)}q^{n-1}\frac{a^{\frac{1}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\bar{h}_{\rho b}\epsilon\\
&\quad+\frac{1}{2}\frac{a^{\frac{1}{n-1}}q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\partial_a\bar{h}_{\rho b}\epsilon+\mathcal{O}(\epsilon^2)\\
=&\frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}}\left[\frac{\partial_a\bar{h}_{\rho b}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}-\frac{1}{n-1}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}\bar{h}_{\rho b}\right]\epsilon+\mathcal{O}(\epsilon^2)\quad\text{(E.17c)}\\
D_{\rho\rho a}=&\frac{1}{2}h_{\rho a}\partial_\rho\hat{\phi}^2\epsilon+\frac{1}{2}\hat{\phi}^2\partial_\rho h_{\rho a}\epsilon+\mathcal{O}(\epsilon^2)\\
=&\frac{1}{2}q^{n-3}\bar{h}_{\rho a}\frac{1}{(n-1)\det(g_{ab,S^{n-2}})}\left(\frac{aq^{2(n-1)}}{\rho^{2(n-2)}\det(g_{ab,S^{n-2}})}\right)^{\frac{2-n}{n-1}}\\
\times&\left(\frac{q^{2(n-1)}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)}{\rho^{2(n-2)}}-\frac{2(n-1)q^{2n-3}a}{\rho^{2(n-2)}\ell}-\frac{2(n-2)q^{2(n-1)}a}{\rho^{2n-3}}\right)\epsilon
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{n-3}{2\ell} \frac{a^{\frac{1}{n-1}} q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \bar{h}_{\rho a}\epsilon + \frac{1}{2} \frac{a^{\frac{1}{n-1}} q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \bar{h}_{\rho a}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{2(n-1)} q^{1-n} \frac{1}{\det(g_{ab,S^{n-2}})} \left(\frac{a}{\rho^{2(n-2)} \det(g_{ab,S^{n-2}})} \right)^{\frac{2-n}{n-1}} \\
&\quad \times \left(\frac{q^{2(n-1)} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right)}{\rho^{2(n-2)}} - \frac{2(n-1)q^{2n-3}a}{\rho^{2(n-2)}\ell} - \frac{2(n-2)q^{2(n-1)}a}{\rho^{2n-3}} \right) \bar{h}_{\rho a}\epsilon \\
&- \frac{n-3}{2\ell} \frac{a^{\frac{1}{n-1}} q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \bar{h}_{\rho a}\epsilon + \frac{1}{2} \frac{a^{\frac{1}{n-1}} q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \bar{h}_{\rho a}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \frac{1}{a^{\frac{n-2}{n-1}}} \left(\frac{1}{2(n-1)} \frac{q^{n-1}\left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}} - \frac{q^{n-2}a}{\rho^{\frac{2(n-2)}{n-1}}\ell} - \frac{n-2}{n-1} \frac{q^{n-1}a}{\rho^{\frac{3n-5}{n-1}}} \right. \\
&\left. - \frac{n-3}{2\ell} \frac{a q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}} \right) \bar{h}_{\rho a}\epsilon + \frac{1}{2} \frac{a^{\frac{1}{n-1}} q^{n-1}}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \bar{h}_{\rho a}\epsilon + \mathcal{O}(\epsilon^2) \qquad \text{(E.17d)}
\end{aligned}
$$

$$
\begin{aligned}
D_{\rho ab} &= \frac{1}{2}\partial_\rho \hat{\hat{\gamma}}_{ab} + \frac{1}{2}\left(h_{ab}\partial_\rho \hat{\phi}^2 + h_{\phi^2}\hat{\gamma}_{ab}\partial_\rho \hat{\phi}^2 + \hat{\phi}^2 \partial_\rho h_{ab} + \hat{\phi}^2 \hat{\gamma}_{ab}\partial_\rho h_{\phi^2} + \hat{\phi}^2 h_{\phi^2}\partial_\rho \hat{\gamma}_{ab} \right)\epsilon + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{2}\frac{1}{n-1} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a^{\frac{2-n}{n-1}} \rho^{\frac{2}{n-1}} \partial_\rho a + 2a^{\frac{1}{n-1}} \rho^{\frac{3-n}{n-1}} \right) \\
&+ \frac{1}{2} q^{n-4}\rho^2 g_{ab,S^{n-2}} \bar{h}_{ab} \frac{1}{(n-1)} \frac{\rho^{\frac{2(n-2)^2}{n-1}}}{a^{\frac{n-2}{n-1}} q^{2(n-2)} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \\
&\quad \times \left(\frac{q^{2(n-1)} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right)}{\rho^{2(n-2)}} - \frac{2(n-1)q^{2n-3}a}{\rho^{2(n-2)}\ell} - \frac{2(n-2)q^{2(n-1)}a}{\rho^{2n-3}} \right)\epsilon \\
&+ \frac{1}{2}\left(-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right)\epsilon \right) \frac{\rho^2}{q^2} g_{ab,S^{n-2}} \frac{q^{2(2-n)}}{(n-1)} \frac{\rho^{\frac{2(n-2)^2}{n-1}}}{a^{\frac{n-2}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \\
&\quad \times \left(\frac{q^{2(n-1)} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right)}{\rho^{2(n-2)}} - \frac{2(n-1)q^{2n-3}a}{\rho^{2(n-2)}\ell} - \frac{2(n-2)q^{2(n-1)}a}{\rho^{2n-3}} \right) \\
&+ \frac{1}{2} \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \partial_\rho \left(q^{n-4}\rho^2 g_{ab,S^{n-2}} \bar{h}_{ab} \right) \\
&+ \frac{1}{2} \frac{a^{\frac{1}{n-1}} q^2}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \frac{\rho^2}{q^2} g_{ab,S^{n-2}} \left[\frac{(n-2)q^{n-3}}{(n-1)\ell} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \right. \\
&\left. - \frac{q^{n-2}}{n-1} \left(\left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right) \bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \right) \right]\epsilon \\
&+ \frac{1}{2} \frac{a^{\frac{1}{n-1}} \rho^{\frac{2(n-2)}{1-n}} q^2}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(-\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right)\epsilon \right) \partial_\rho \left(\frac{\rho^2}{q^2} g_{ab,S^{n-2}} \right) + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{2}\frac{1}{n-1} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a^{\frac{2-n}{n-1}} \rho^{\frac{2}{n-1}} \partial_\rho a + 2a^{\frac{1}{n-1}} \rho^{\frac{3-n}{n-1}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2(n-1)} q^{n-2} \frac{\rho^{\frac{2}{n-1}}}{a^{\frac{n-2}{n-1}}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \bar{h}_{ab} \left(\partial_\rho a - \frac{2(n-1)a}{q\ell} - \frac{2(n-2)a}{\rho} \right) \epsilon \\
&- \frac{1}{2(n-1)^2} q^{n-2} \frac{\rho^{\frac{2}{n-1}}}{a^{\frac{n-2}{n-1}}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&\quad \times \left(\partial_\rho a - \frac{2(n-1)a}{q\ell} - \frac{2(n-2)a}{\rho} \right) \\
&- \frac{n-4}{2\ell} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-3} \bar{h}_{ab} \epsilon + \frac{a^{\frac{1}{n-1}}}{\rho^{\frac{n-3}{n-1}}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-2} \bar{h}_{ab} \epsilon \\
&+ \frac{1}{2} \rho^{\frac{2}{n-1}} a^{\frac{1}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-2} \partial_\rho \bar{h}_{ab} \epsilon \\
&+ \frac{(n-2)}{2(n-1)\ell} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-3} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&- \frac{1}{2(n-1)} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-2} \left(\partial_\rho a \bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&- \frac{1}{(n-1)\ell} a^{\frac{1}{n-1}} q^{n-3} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon \\
&- \frac{1}{(n-1)} \frac{a^{\frac{1}{n-1}} q^{n-2}}{\rho^{\frac{n-3}{n-1}}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{1}{2} \frac{1}{n-1} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a^{\frac{2-n}{n-1}} \rho^{\frac{2}{n-1}} \partial_\rho a + 2a^{\frac{1}{n-1}} \rho^{\frac{3-n}{n-1}} \right) \\
&- \frac{q^{n-2}}{2(n-1)^2} \frac{\rho^{\frac{2}{n-1}}}{a^{\frac{n-2}{n-1}}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab} \right) \epsilon \\
&\quad \times \left(\partial_\rho a - \frac{2(n-1)a}{q\ell} + \frac{2a}{\rho} \right) \\
&+ \frac{n-4}{2(n-1)\ell} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} q^{n-3} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{ab} \right) \epsilon \\
&- \frac{q^{n-2}}{2(n-1)} a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}} \frac{g_{ab,S^{n-2}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \Bigg(\partial_\rho a \bar{h}_{\rho\rho} + a\partial_\rho \bar{h}_{\rho\rho} \\
&\qquad\qquad + \sum_{i=1}^{n-2} \partial_\rho \bar{h}_{\theta_i\theta_i} - (n-1)\partial_\rho \bar{h}_{ab} \Bigg) \epsilon + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.17e}
$$

$$
\begin{aligned}
D_{\rho\rho\rho} =& \frac{1}{2} \partial_\rho \hat{\tilde{\gamma}}_{\rho\rho} \left(1 + h_{\phi^2}\epsilon + q^{n-2} a\bar{h}_{\rho\rho}\epsilon + \mathcal{O}(\epsilon^2) \right) + \frac{1}{2} \hat{\tilde{\gamma}}_{\rho\rho} \Bigg(\partial_\rho h_{\phi^2} - \frac{(n-2)}{\ell} q^{n-3} a\bar{h}_{\rho\rho} \\
&+ q^{n-2} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right) \bar{h}_{\rho\rho} + q^{n-2} a\partial_\rho \bar{h}_{\rho\rho} \Bigg) \epsilon + \mathcal{O}(\epsilon^2) \\
=& \frac{1}{2} \left(\frac{2-n}{n-1} \frac{a^{\frac{3-2n}{n-1}} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell} \right)}{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} - \frac{2(n-2)}{n-1} \frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(1-\frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)\\
&+\frac{1}{2}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left(\frac{(n-2)q^{n-3}}{(n-1)\ell}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right.\\
&-\frac{q^{n-2}}{n-1}\left(\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)-\frac{(n-2)}{\ell}q^{n-3}a\bar{h}_{\rho\rho}\\
&\left.+q^{n-2}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)\bar{h}_{\rho\rho}+q^{n-2}a\partial_\rho\bar{h}_{\rho\rho}\right)\epsilon+\mathcal{O}(\epsilon^2)\\
=&\frac{1}{2}\left(\frac{2-n}{n-1}\frac{a^{\frac{3-2n}{n-1}}\partial_\rho a}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}-\frac{2(n-2)}{n-1}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\times\left(1-\frac{q^{n-2}}{n-1}\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\right)\\
&-\frac{1}{2(n-1)}\frac{a^{\frac{2-n}{n-1}}\rho^{\frac{2(n-2)}{1-n}}q^{n-2}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left((2-n)\partial_\rho a\bar{h}_{\rho\rho}+(2-n)a\partial_\rho\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{n-2}{2(n-1)\ell}\frac{a^{\frac{2-n}{n-1}}\rho^{\frac{2(n-2)}{1-n}}q^{n-3}}{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.17f}
$$

This was the cumbersome calculation for the boundary behaviour of the auxiliary variables D_{ijk}.

E.5 The extrinsic curvature K_i

Let us now calculate the boundary behaviour of the extrinsic curvature

$$
\begin{aligned}
K_{\rho\rho}&=-\frac{1}{2\hat{\alpha}}\left[\partial_t h_{\rho\rho}-2\hat{\gamma}_{\rho k}h_{B^k_\rho}-2\hat{\phi}^{-2}h_{\beta^k}\hat{D}_{k\rho\rho}+2\hat{\gamma}_{\rho\rho}\hat{P}_k h_{\beta^k}\right]\epsilon+\mathcal{O}(\epsilon^2)\\
&=-\frac{q}{2\sqrt{a}}\left[q^{n-4}\partial_t\bar{h}_{\rho\rho}-2\hat{\gamma}_{\rho\rho}h_{B^\rho_\rho}-2\hat{\gamma}_{\rho a}h_{B^a_\rho}-2\hat{\phi}^{-2}h_{\beta^\rho}\hat{D}_{\rho\rho\rho}\right.\\
&\quad\left.-2\hat{\phi}^{-2}h_{\beta^a}\hat{D}_{a\rho\rho}+2\hat{\gamma}_{\rho\rho}\hat{P}_\rho h_{\beta^\rho}+2\hat{\gamma}_{\rho\rho}\hat{P}_a h_{\beta^a}\right]+\mathcal{O}(h^2)\\
&=-\frac{q}{2\sqrt{a}}\left[q^{n-4}\partial_t\bar{h}_{\rho\rho}-\frac{2}{q^2a}\left(-\frac{(n-1)q^{n-2}}{\ell}a\bar{h}_{t\rho}+q^{n-1}\partial_\rho a\bar{h}_{t\rho}+q^{n-1}a\partial_\rho\bar{h}_{t\rho}\right)\right.\\
&\quad-2\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}q^{n-1}a\bar{h}_{t\rho}\\
&\quad\times\frac{1}{2}\left(\frac{2-n}{n-1}\frac{a^{\frac{3-2n}{n-1}}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}-\frac{2(n-2)}{n-1}\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\quad+2\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}q^{n-4}\frac{q^2}{\rho^2}g^{ba}_{S^{n-2}}\bar{h}_{tb}\frac{a^{\frac{2-n}{n-1}}}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})^{\frac{n}{n-1}}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{q^2 a}\left(-\frac{1}{ql}+\frac{\frac{2\rho}{L^2}-\frac{2q}{l}}{2(n-1)a}-\frac{n-2}{(n-1)\rho}\right)q^{n-1}a\bar{h}_{t\rho}\\
&-\frac{2}{q^2 a}\frac{1}{2(n-1)}\frac{\partial_a \det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}q^{n-4}\frac{q^2}{\rho^2}g^{ba}_{S^{n-2}}\bar{h}_{tb}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\partial_t\bar{h}_{\rho\rho}+\frac{2(n-1)q^{n-4}}{\ell}\bar{h}_{t\rho}-\frac{2}{a}q^{n-3}\partial_\rho a\bar{h}_{t\rho}-2q^{n-3}\partial_\rho\bar{h}_{t\rho}\\
&-q^{n-3}\bar{h}_{t\rho}\left(\frac{2-n}{n-1}\frac{\partial_\rho a}{a}-\frac{2(n-2)}{n-1}\frac{1}{\rho}\right)+\frac{1}{n-1}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}\frac{q^{n-4}}{a\rho^2}g^{ba}_{S^{n-2}}\bar{h}_{tb}\\
&+2\left(-\frac{1}{ql}+\frac{\partial_\rho a}{2(n-1)a}-\frac{n-2}{(n-1)\rho}\right)q^{n-3}\bar{h}_{t\rho}\\
&-\frac{1}{n-1}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}\frac{q^{n-4}}{a\rho^2}g^{ba}_{S^{n-2}}\bar{h}_{tb}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\partial_t\bar{h}_{\rho\rho}+\frac{2(n-1)q^{n-4}}{\ell}\bar{h}_{t\rho}-\frac{2}{a}q^{n-3}\partial_\rho a\bar{h}_{t\rho}-2q^{n-3}\partial_\rho\bar{h}_{t\rho}\\
&\cancel{+\frac{1}{n-1}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}\frac{q^{n-4}}{a\rho^2}}g^{ba}_{S^{n-2}}\bar{h}_{tb}+2\left(-\frac{1}{ql}+\frac{\partial_\rho a}{2a}\right)q^{n-3}\bar{h}_{t\rho}\\
&\cancel{-\frac{1}{(n-1)}\frac{\partial_a\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}\frac{q^{n-4}}{a\rho^2}}g^{ba}_{S^{n-2}}\bar{h}_{tb}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\partial_t\bar{h}_{\rho\rho}-2q^{n-3}\partial_\rho\bar{h}_{t\rho}-\left(-\frac{2(n-2)}{l}+\frac{2\rho q}{aL^2}-\frac{2q^2}{a\ell}\right)q^{n-4}\bar{h}_{t\rho}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\partial_t\bar{h}_{\rho\rho}-2q^{n-3}\partial_\rho\bar{h}_{t\rho}-\left(-\frac{2(n-2)}{l}-\frac{2}{\ell}+\frac{2\rho}{aL^2}\right)q^{n-4}\bar{h}_{t\rho}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\partial_t\bar{h}_{\rho\rho}-2q^{n-3}\partial_\rho\bar{h}_{t\rho}+\left(\frac{2(n-1)}{l}-\frac{2\rho}{aL^2}\right)q^{n-4}\bar{h}_{t\rho}\Big]\epsilon\\
&+\mathcal{O}(\epsilon^2) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(E.18)}\\
K_{ab}=&-\frac{1}{2\hat{\alpha}}\Big[\partial_t h_{ab}-\hat{\gamma}_{ak}h_{B^k_b}-\hat{\gamma}_{bk}h_{B^k_a}-2\hat{\phi}^{-2}h_{\beta^k}\hat{D}_{kab}+2\hat{\gamma}_{ab}\hat{P}_k h_{\beta^k}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab}-\rho^2q^{n-6}g_{ac,S^{n-2}}\left(\frac{q^2}{\rho^2}\partial_b g^{dc}_{S^{n-2}}\bar{h}_{td}+\frac{q^2}{\rho^2}g^{dc}_{S^{n-2}}\partial_b\bar{h}_{td}\right)\\
&-\rho^2q^{n-6}g_{bc,S^{n-2}}\left(\frac{q^2}{\rho^2}\partial_a g^{dc}_{S^{n-2}}\bar{h}_{td}+\frac{q^2}{\rho^2}g^{dc}_{S^{n-2}}\partial_a\bar{h}_{td}\right)\\
&-\frac{g_{ab,S^{n-2}}}{n-1}q^{n-3}\bar{h}_{t\rho}\left(\rho^2\partial_\rho a+2a\rho\right)+q^{n-4}g^{dc}_{S^{n-2}}\bar{h}_{td}\left(\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{(n-1)\det(g_{ab,S^{n-2}})}-\partial_c g_{ab,S^{n-2}}\right)\\
&+2\rho^2 g_{ab,S^{n-2}}\left(-\frac{1}{ql}+\frac{\partial_\rho a}{2(n-1)a}-\frac{n-2}{(n-1)\rho}\right)q^{n-3}a\bar{h}_{t\rho}\\
&-\frac{1}{n-1}\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{\det(g_{ab,S^{n-2}})}q^{n-4}g^{dc}_{S^{n-2}}\bar{h}_{td}\Big]\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{q}{2\sqrt{a}}\Big[q^{n-4}\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab}-q^{n-4}g_{ac,S^{n-2}}\left(\partial_b g^{dc}_{S^{n-2}}\bar{h}_{td}+g^{dc}_{S^{n-2}}\partial_b\bar{h}_{td}\right)\\
&-q^{n-4}g_{bc,S^{n-2}}\left(\partial_a g^{dc}_{S^{n-2}}\bar{h}_{td}+g^{dc}_{S^{n-2}}\partial_a\bar{h}_{td}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cancel{-\frac{g_{ab,S^{n-2}}}{n-1}q^{n-3}\bar{h}_{t\rho}\rho^2\partial_\rho a} - \frac{g_{ab,S^{n-2}}}{n-1}q^{n-3}\bar{h}_{t\rho}2a\rho \\
&\cancel{+q^{n-4}g^{dc}_{S^{n-2}}\bar{h}_{td}\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{(n-1)\det(g_{ab,S^{n-2}})}} - q^{n-4}g^{dc}_{S^{n-2}}\bar{h}_{td}\partial_c g_{ab,S^{n-2}} \\
&- 2\rho^2 g_{ab,S^{n-2}}\left(\frac{1}{ql}+\frac{n-2}{(n-1)\rho}\right)q^{n-3}a\bar{h}_{t\rho} + \cancel{\rho^2 g_{ab,S^{n-2}}\frac{\partial_\rho a}{n-1}q^{n-3}\bar{h}_{t\rho}} \\
&\cancel{-\frac{g_{ab,S^{n-2}}\partial_c\det(g_{ab,S^{n-2}})}{(n-1)\det(g_{ab,S^{n-2}})}q^{n-4}g^{dc}_{S^{n-2}}\bar{h}_{td}}\Big]\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -\frac{q^{n-3}}{2\sqrt{a}}\Big[\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab} - 2g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td} - 2\partial_{(a}\bar{h}_{tb)} \\
&- g^{dc}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}\bar{h}_{td} - 2\rho g_{ab,S^{n-2}}a\bar{h}_{t\rho}\Big]\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.19}
$$

$$
\begin{aligned}
K_{\rho a} ={}& -\frac{1}{2\hat{\alpha}}\left[\partial_t h_{\rho a} - \hat{\gamma}_{\rho k}h_{B^k_a} - \hat{\gamma}_{ak}h_{B^k_\rho}\right]\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -\frac{q}{2\sqrt{a}}\left[\partial_t h_{\rho a} - \hat{\gamma}_{\rho\rho}h_{B^\rho_a} - \hat{\gamma}_{\rho b}h_{B^b_a} - \hat{\gamma}_{a\rho}h_{B^\rho_\rho} - \hat{\gamma}_{ab}h_{B^b_\rho}\right]\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -\frac{q}{2\sqrt{a}}\Bigg[q^{n-3}\partial_t\bar{h}_{\rho a} - \frac{1}{q^2 a}q^{n-1}a\partial_a\bar{h}_{t\rho} \\
&- \frac{\rho^2}{q^2}g_{ab,S^{n-2}}g^{cb}_{S^{n-2}}\left(-\frac{(n-2)q^{n-5}}{\ell}\frac{q^2}{\rho^2}\bar{h}_{tc} - q^{n-4}\frac{2q^2}{\rho^3}\bar{h}_{tc} + q^{n-4}\frac{q^2}{\rho^2}\partial_\rho\bar{h}_{tc}\right)\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -\frac{q}{2\sqrt{a}}\left[q^{n-3}\partial_t\bar{h}_{\rho a} - q^{n-3}\partial_a\bar{h}_{t\rho} + q^{n-4}\left(\frac{n-2}{q\ell}+\frac{2}{\rho}\right)\bar{h}_{ta} - q^{n-4}\partial_\rho\bar{h}_{ta}\right]\epsilon \\
&+ \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.20}
$$

Now, we are able to calculate the trace of the extrinsic curvature

$$
\begin{aligned}
K ={}& \gamma^{ij}K_{ij} = \gamma^{\rho\rho}K_{\rho\rho} + 2\gamma^{\rho a}K_{\rho a} + \gamma^{ab}K_{ab} \\
={}& \left(\hat{\gamma}^{\rho\rho} - \hat{\gamma}^{\rho\rho}\hat{\gamma}^{\rho\rho}q^{n-4}\bar{h}_{\rho\rho}\epsilon\right)\left(\hat{K}_{\rho\rho} + h_{K_{\rho\rho}}\epsilon\right) + 2\left(\hat{\gamma}^{\rho a} - \hat{\gamma}^{\rho\rho}\hat{\gamma}^{ab}q^{n-3}\bar{h}_{\rho b}\epsilon\right)\left(\hat{K}_{\rho a} + h_{K_{\rho a}}\epsilon\right) \\
&+ \left(\hat{\gamma}^{ab} - q^{n-4}\rho^2 g_{ab,S^{n-2}}\hat{\gamma}^{ac}\hat{\gamma}^{bd}\bar{h}_{cd}\epsilon\right)\left(\hat{K}_{ab} + h_{K_{ab}}\epsilon\right) + \mathcal{O}(\epsilon^2) \\
={}& \left(\hat{\gamma}^{\rho\rho}h_{K_{\rho\rho}} + \hat{\gamma}^{ab}h_{K_{ab}}\right)\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -q^2 a\frac{q^{n-3}}{2\sqrt{a}}\left[\partial_t\bar{h}_{\rho\rho} - 2q\partial_\rho\bar{h}_{t\rho} + \left(\frac{2(n-1)}{l} - \frac{2\rho}{aL^2}\right)\bar{h}_{t\rho}\right]\epsilon + \mathcal{O}(\epsilon^2) \\
&- \frac{q^2}{\rho^2}g^{ab}_{S^{n-2}}\frac{q^{n-3}}{2\sqrt{a}}\Big[\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab} - 2g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td} - 2\partial_{(a}\bar{h}_{tb)} \\
&\qquad\qquad - g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc} - 2\rho a g_{ab,S^{n-2}}\bar{h}_{t\rho}\Big]\epsilon + \mathcal{O}(\epsilon^2) \\
={}& -\frac{q^{n-1}}{2\sqrt{a}}\Bigg[a\partial_t\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} - 2qa\partial_\rho\bar{h}_{t\rho} + \left(\frac{2(n-1)a}{l} - \frac{2\rho}{L^2} - \frac{2(n-2)a}{\rho}\right)\bar{h}_{t\rho} \\
&- \frac{2}{\rho^2}\underbrace{g^{ab}_{S^{n-2}}\left(g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td}\right.}_{=0} + \partial_{(a}\bar{h}_{tb)} + \left.\frac{1}{2}g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}\right)\Bigg]\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$= -\frac{q^{n-1}}{2\sqrt{a}}\Bigg[a\partial_t\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} - 2qa\partial_\rho\bar{h}_{t\rho} + \left(\frac{2(n-1)a}{l} - \frac{2\rho}{L^2} - \frac{2(n-2)a}{\rho}\right)\bar{h}_{t\rho}$$
$$- \frac{2}{\rho^2}g^{ab}_{S^{n-2}}\left(\partial_a\bar{h}_{tb} + \frac{1}{2}g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}\right)\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \tag{E.21}$$

E.6 The traceless-part of the extrinsic curvature

Let us calculate the traceless-part of the extrinsic curvature. Then,

$$A_{\rho\rho} = -\frac{q^{n-3}}{2\sqrt{a}}\Bigg(\partial_t\bar{h}_{\rho\rho} - 2q\partial_\rho\bar{h}_{t\rho} + \left(\frac{2(n-1)}{l} - \frac{2\rho}{aL^2}\right)\bar{h}_{t\rho}$$
$$- \frac{1}{n-1}\partial_t\bar{h}_{\rho\rho} - \frac{1}{(n-1)a}\sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} + \frac{2q}{n-1}\partial_\rho\bar{h}_{t\rho}$$
$$- \left(\frac{2}{l} - \frac{2\rho}{aL^2} - \frac{2(n-2)q^2}{(n-1)a\rho}\right)\bar{h}_{t\rho}$$
$$+ \frac{2}{(n-1)\rho^2 a}g^{ab}_{S^{n-2}}\partial_b\bar{h}_{ta} + \frac{1}{(n-1)\rho^2 a}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}\Bigg)\epsilon + \mathcal{O}(\epsilon^2)$$
$$= -\frac{q^{n-3}}{2\sqrt{a}}\Bigg[\frac{n-2}{n-1}\partial_t\bar{h}_{\rho\rho} - \frac{1}{(n-1)a}\sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} - \frac{2(n-2)q}{n-1}\partial_\rho\bar{h}_{t\rho}$$
$$+ \left(\frac{2(n-2)}{\ell} + \frac{2(n-2)q^2}{(n-1)a\rho}\right)\bar{h}_{t\rho} + \frac{2}{(n-1)\rho^2 a}g^{ab}_{S^{n-2}}\partial_b\bar{h}_{ta}$$
$$+ \frac{1}{(n-1)\rho^2 a}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \tag{E.22a}$$
$$A_{\rho a} = -\frac{q^{n-3}}{2\sqrt{a}}\left[q\partial_t\bar{h}_{\rho a} - q\partial_a\bar{h}_{t\rho} + \left(\frac{n-2}{q\ell} + \frac{2}{\rho}\right)\bar{h}_{ta} - \partial_\rho\bar{h}_{ta}\right]\epsilon + \mathcal{O}(\epsilon^2) \tag{E.22b}$$
$$A_{ab} = -\frac{q^{n-3}}{2\sqrt{a}}\Big[\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab} - 2g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td} - 2\partial_{(a}\bar{h}_{tb)}$$
$$- g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc} - 2\rho g_{ab,S^{n-2}}a\bar{h}_{t\rho}\Big]\epsilon + \frac{\rho^2 g_{ab,S^{n-2}}}{n-1}\frac{q^{n-3}}{2\sqrt{a}}\Bigg(a\partial_t\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} - 2qa\partial_\rho\bar{h}_{t\rho}$$
$$+ \left(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho}\right)\bar{h}_{t\rho} - \frac{2}{\rho^2}g^{cd}_{S^{n-2}}\partial_d\bar{h}_{tc}$$
$$- \frac{1}{\rho^2}g^{ef}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ef,S^{n-2}}\bar{h}_{tc}\Bigg)\epsilon + \mathcal{O}(\epsilon^2)$$
$$= -\frac{q^{n-3}}{2\sqrt{a}}\Bigg[\rho^2 g_{ab,S^{n-2}}\partial_t\bar{h}_{ab} - 2g_{(ac,S^{n-2}}\partial_{b)}g^{dc}_{S^{n-2}}\bar{h}_{td} - 2\partial_{(a}\bar{h}_{tb)} - g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}$$
$$- 2\rho g_{ab,S^{n-2}}a\bar{h}_{t\rho} - \frac{\rho^2 g_{ab,S^{n-2}}a}{n-1}\partial_t\bar{h}_{\rho\rho} - \frac{\rho^2 g_{ab,S^{n-2}}}{n-1}\sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} + \frac{2qa\rho^2 g_{ab,S^{n-2}}}{n-1}\partial_\rho\bar{h}_{t\rho}$$

$$
\begin{aligned}
&- \rho^2 g_{ab,S^{n-2}} \left(\frac{2a}{l} - \frac{2\rho}{L^2} - \frac{2(n-2)q^2}{(n-1)\rho} \right) \bar{h}_{t\rho} + \frac{2 g_{ab,S^{n-2}}}{n-1} g^{cd}_{S^{n-2}} \partial_d \bar{h}_{tc} \\
&+ \frac{g^{ab}_{S^{n-2}} g^{ef}_{S^{n-2}} g^{cd}_{S^{n-2}}}{n-1} \partial_d g_{ef,S^{n-2}} \bar{h}_{tc} \Bigg) \Bigg] \epsilon + \mathcal{O}(\epsilon^2) \\
= &- \frac{q^{n-3}}{2\sqrt{a}} \Bigg[\frac{\rho^2 g_{ab,S^{n-2}}}{n-1} \left(-a \partial_t \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} + (n-1) \partial_t \bar{h}_{ab} \right) - 2 g_{(ac,S^{n-2}} \partial_{b)} g^{dc}_{S^{n-2}} \bar{h}_{td} \\
&- 2 \partial_{(a} \bar{h}_{tb)} - g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} - 2\rho g_{ab,S^{n-2}} a \bar{h}_{t\rho} \\
&- \frac{g_{ab,S^{n-2}} \rho^2}{n-1} \Bigg(- 2qa \partial_\rho \bar{h}_{t\rho} + \left(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho} \right) \bar{h}_{t\rho} \\
&- \frac{2}{\rho^2} g^{cd}_{S^{n-2}} \partial_d \bar{h}_{tc} - \frac{1}{\rho^2} g^{ef}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{ef,S^{n-2}} \bar{h}_{tc} \Bigg) \Bigg] \epsilon + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{E.22c}
$$

From theory, we know that

$$
\gamma^{ij} A_{ij} = 0. \tag{E.23}
$$

Therefore, let us see if our derived equations satisfy this condition.

$$
\begin{aligned}
\gamma^{ij} \left(\hat{A}_{ij} + h_{A_{ij}} \epsilon + \mathcal{O}(\epsilon^2) \right) &= \gamma^{ij} h_{A_{ij}} \epsilon + \mathcal{O}(\epsilon^2) = \hat{\gamma}^{jj} \left(1 + h_{\hat{\gamma}^{jj}} \epsilon + \mathcal{O}(\epsilon^2) \right) h_{A_{jj}} \epsilon + \mathcal{O}(\epsilon^2) = \hat{\gamma}^{jj} h_{A_{jj}} \epsilon + \\
&= \left(\hat{\gamma}^{\rho\rho} h_{A_{\rho\rho}} + \hat{\gamma}^{ab} h_{A_{ab}} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
&= \left(q^2 a h_{A_{\rho\rho}} + \frac{q^2}{\rho^2} g^{ab}_{S^{n-2}} h_{A_{ab}} \right) \epsilon + \mathcal{O}(\epsilon^2) \\
&= -q^2 a \frac{q^{n-3}}{2\sqrt{a}} \Bigg[\frac{n-2}{n-1} \partial_t \bar{h}_{\rho\rho} - \frac{1}{(n-1)a} \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} - \frac{2(n-2)q}{n-1} \partial_\rho \bar{h}_{t\rho} + \frac{2 g^{ab}_{S^{n-2}} \partial_b \bar{h}_{ta}}{(n-1)\rho^2 a} \\
&\quad + \left(\frac{2(n-2)}{\ell} + \frac{2(n-2)q^2}{(n-1)a\rho} \right) \bar{h}_{t\rho} + \frac{1}{(n-1)\rho^2 a} g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} \Bigg] \epsilon \\
&\quad - \frac{q^2}{\rho^2} g^{ab}_{S^{n-2}} \frac{q^{n-3}}{2\sqrt{a}} \Bigg\{ \frac{\rho^2 g_{ab,S^{n-2}}}{n-1} \left[-a \partial_t \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} + (n-1) \partial_t \bar{h}_{ab} \right] \\
&\quad - 2 g_{(ac,S^{n-2}} \partial_{b)} g^{dc}_{S^{n-2}} \bar{h}_{td} - 2 \partial_{(a} \bar{h}_{tb)} - g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} - 2\rho g_{ab,S^{n-2}} a \bar{h}_{t\rho} \\
&\quad - \frac{g_{ab,S^{n-2}} \rho^2}{n-1} \Bigg(- 2qa \partial_\rho \bar{h}_{t\rho} + \left(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho} \right) \bar{h}_{t\rho} \\
&\quad - \frac{2}{\rho^2} g^{cd}_{S^{n-2}} \partial_d \bar{h}_{tc} - \frac{1}{\rho^2} g^{ef}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{ef,S^{n-2}} \bar{h}_{tc} \Bigg) \Bigg\} \epsilon + \mathcal{O}(\epsilon^2) \\
&= -\frac{q^{n-1}}{2\sqrt{a}} \Bigg[\frac{n-2}{n-1} a \partial_t \bar{h}_{\rho\rho} - \frac{1}{n-1} \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} - \frac{2(n-2)qa}{n-1} \partial_\rho \bar{h}_{t\rho} + \frac{2 g^{ab}_{S^{n-2}} \partial_b \bar{h}_{ta}}{(n-1)\rho^2} \\
&\quad + \left(\frac{2(n-2)a}{\ell} + \frac{2(n-2)q^2}{(n-1)\rho} \right) \bar{h}_{t\rho} + \frac{1}{(n-1)\rho^2} g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{ab,S^{n-2}} \bar{h}_{tc} \\
&\quad + g^{ab}_{S^{n-2}} \Bigg\{ \frac{g_{ab,S^{n-2}}}{n-1} \left(-a \partial_t \bar{h}_{\rho\rho} - \sum_{i=1}^{n-2} \partial_t \bar{h}_{\theta_i\theta_i} + (n-1) \partial_t \bar{h}_{ab} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{\rho^2} g_{(ac,S^{n-2}}\partial_{b)} g^{dc}_{S^{n-2}} \bar{h}_{td} - \frac{2}{\rho^2}\partial_{(a}\bar{h}_{tb)} - \frac{1}{\rho^2} g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc} - \frac{2 g_{ab,S^{n-2}} a}{\rho}\bar{h}_{t\rho} \\
&- \frac{g_{ab,S^{n-2}}}{n-1}\Bigg(-2qa\partial_\rho \bar{h}_{t\rho} + \Big(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho}\Big)\bar{h}_{t\rho} \\
&- \frac{2}{\rho^2} g^{cd}_{S^{n-2}}\partial_d \bar{h}_{tc} - \frac{1}{\rho^2} g^{ef}_{S^{n-2}} g^{cd}_{S^{n-2}}\partial_d g_{ef,S^{n-2}}\bar{h}_{tc}\Bigg)\Bigg\}\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \\
= & -\frac{q^{n-1}}{2\sqrt{a}}\Bigg[\cancel{\frac{n-2}{n-1}a\partial_t\bar{h}_{\rho\rho}} - \cancel{\frac{1}{n-1}\sum_{i=1}^{n-2}\partial_t \bar{h}_{\theta_i\theta_i}} - \cancel{\frac{2(n-2)qa}{n-1}\partial_\rho\bar{h}_{t\rho}} + \cancel{\frac{2}{(n-1)\rho^2} g^{ab}_{S^{n-2}}\partial_b \bar{h}_{ta}} \\
&+ \cancel{\Big(\frac{2(n-2)a}{\ell} + \frac{2(n-2)q^2}{(n-1)\rho}\Big)}\bar{h}_{t\rho} + \cancel{\frac{1}{(n-1)\rho^2} g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}} \\
&- \cancel{\frac{n-2}{n-1}a\partial_t\bar{h}_{\rho\rho}} - \cancel{\frac{n-2}{n-1}\sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i} + \sum_{i=1}^{n-2}\partial_t\bar{h}_{\theta_i\theta_i}} \underbrace{- \frac{2}{\rho^2} g^{ab}_{S^{n-2}} g_{(ac,S^{n-2}}\partial_{b)} g^{dc}_{S^{n-2}}\bar{h}_{td}}_{=0} \\
&- \cancel{\frac{2}{\rho^2}g^{ab}_{S^{n-2}}\partial_{(a}\bar{h}_{tb)}} - \cancel{\frac{1}{\rho^2}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ab,S^{n-2}}\bar{h}_{tc}} - \cancel{\frac{2(n-2)a}{\rho}\bar{h}_{t\rho}} \\
&+ \cancel{\frac{n-2}{n-1}2qa\partial_\rho\bar{h}_{t\rho}} - \cancel{\frac{n-2}{n-1}\Big(\frac{2(n-1)a}{l} - \frac{2(n-1)\rho}{L^2} - \frac{2(n-2)q^2}{\rho}\Big)}\bar{h}_{t\rho} \\
&+ \cancel{\frac{n-2}{n-1}\frac{2}{\rho^2}g^{cd}_{S^{n-2}}\partial_d\bar{h}_{tc}} + \cancel{\frac{n-2}{n-1}\frac{1}{\rho^2}g^{ef}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{ef,S^{n-2}}\bar{h}_{tc}}\Bigg)\Bigg\}\Bigg]\epsilon + \mathcal{O}(\epsilon^2) \\
= & \, 0 + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.24}
$$

We can verify if the $\tilde{A}$-equations Eq. (E.22a) - Eq. (E.22c) are correct by calculating

$$
\begin{aligned}
\tilde{\gamma}^{ij}\tilde{A}_{ij} &= \tilde{\gamma}^{ij} h_{\tilde{A}_{ij}}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \hat{\tilde{\gamma}}^{\rho\rho} h_{\tilde{A}_{\rho\rho}}\epsilon + \hat{\tilde{\gamma}}^{ab} h_{\tilde{A}_{ab}}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \hat{\gamma}^{\rho\rho} h_{A_{\rho\rho}}\epsilon + \hat{\gamma}^{ab} h_{A_{ab}}\epsilon + \mathcal{O}(\epsilon^2) \\
&= \gamma^{ij} A_{ij} \\
&= 0 + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{E.25}
$$

where we have used the fact $\phi^2 \times \phi^{-2} = 1$ and $\gamma^{ij}A_{ij} = 0 + \mathcal{O}(\epsilon^2)$.

E.7 The $\tilde{\Gamma}$ equations

Let us calculate the $\hat{\Gamma}$-variables.

$$
\begin{aligned}
\tilde{\Gamma}^{\rho} &= 2\tilde{\gamma}^{\rho\rho}\Big(\tilde{\gamma}^{\rho\rho}D_{\rho\rho\rho} + \tilde{\gamma}^{\rho a}D_{a\rho\rho} + \tilde{\gamma}^{a\rho}D_{\rho\rho a} + \tilde{\gamma}^{ab}D_{a\rho b}\Big) \\
&\quad + 2\tilde{\gamma}^{\rho a}\Big(\tilde{\gamma}^{\rho\rho}D_{\rho a\rho} + \tilde{\gamma}^{\rho b}D_{ba\rho} + \tilde{\gamma}^{b\rho}D_{\rho ab} + \tilde{\gamma}^{bc}D_{cab}\Big) \\
&= 2\tilde{\gamma}^{\rho\rho}\Big\{ \Big(\hat{\tilde{\gamma}}^{\rho\rho} + \epsilon h_{\tilde{\gamma}^{\rho\rho}}\Big)\Big(\hat{D}_{\rho\rho\rho} + \epsilon h_{D_{\rho\rho\rho}}\Big) + \epsilon h_{\tilde{\gamma}^{\rho a}}\Big(\hat{D}_{a\rho\rho} + \epsilon h_{D_{a\rho\rho}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \epsilon h_{\tilde{\gamma}^{a\rho}} \epsilon h_{D_{\rho\rho a}} + \left(\hat{\tilde{\gamma}}^{ab} + \epsilon h_{\hat{\tilde{\gamma}}^{ab}}\right) \epsilon h_{D_{a\rho b}} \Big\} \\
&+ 2\tilde{\gamma}^{\rho a} \Big\{ \left(\hat{\tilde{\gamma}}^{\rho\rho} + \epsilon h_{\hat{\tilde{\gamma}}^{\rho\rho}}\right) \epsilon h_{D_{\rho a\rho}} + \epsilon h_{\tilde{\gamma}^{\rho b}} \epsilon h_{D_{ba\rho}} + \epsilon h_{\tilde{\gamma}^{b\rho}} \left(\hat{D}_{\rho ab} + \epsilon h_{D_{\rho ab}}\right) \\
&+ \left(\tilde{\gamma}^{bc} + \epsilon h_{\tilde{\gamma}^{bc}}\right) \left(\hat{D}_{cab} + \epsilon h_{D_{cab}}\right) \Big\} + \mathcal{O}(\epsilon^2) \\
={}& 2\tilde{\gamma}^{\rho\rho} \left(\hat{\tilde{\gamma}}^{\rho\rho} \hat{D}_{\rho\rho\rho} + \epsilon \hat{D}_{\rho\rho\rho} h_{\tilde{\gamma}^{\rho\rho}} + \epsilon \hat{\tilde{\gamma}}^{\rho\rho} h_{D_{\rho\rho\rho}} + \epsilon h_{\tilde{\gamma}^{\rho a}} \hat{D}_{a\rho\rho} + \epsilon \hat{\tilde{\gamma}}^{ab} h_{D_{a\rho b}} \right) \\
&+ 2\tilde{\gamma}^{\rho a} \left(\epsilon \hat{\tilde{\gamma}}^{\rho\rho} h_{D_{\rho a\rho}} + \epsilon h_{\tilde{\gamma}^{b\rho}} \hat{D}_{\rho ab} + \hat{\tilde{\gamma}}^{bc} \hat{D}_{cab} + \epsilon \hat{\tilde{\gamma}}^{bc} h_{D_{cab}} + \epsilon \hat{D}_{cab} h_{\tilde{\gamma}^{bc}} \right) + \mathcal{O}(\epsilon^2) \\
={}& 2 \left(\hat{\tilde{\gamma}}^{\rho\rho} + \epsilon h_{\tilde{\gamma}^{\rho\rho}}\right) \left(\hat{\tilde{\gamma}}^{\rho\rho} \hat{D}_{\rho\rho\rho} + \epsilon \hat{D}_{\rho\rho\rho} h_{\tilde{\gamma}^{\rho\rho}} + \epsilon \hat{\tilde{\gamma}}^{\rho\rho} h_{D_{\rho\rho\rho}} + \epsilon h_{\tilde{\gamma}^{\rho a}} \hat{D}_{a\rho\rho} + \epsilon \hat{\tilde{\gamma}}^{ab} h_{D_{a\rho b}} \right) \\
&+ 2\epsilon h_{\tilde{\gamma}^{\rho a}} \left(\epsilon \hat{\tilde{\gamma}}^{\rho\rho} h_{D_{\rho a\rho}} + \epsilon h_{\tilde{\gamma}^{b\rho}} \hat{D}_{\rho ab} + \hat{\tilde{\gamma}}^{bc} \hat{D}_{cab} + \epsilon \hat{\tilde{\gamma}}^{bc} h_{D_{cab}} + \epsilon \hat{D}_{cab} h_{\tilde{\gamma}^{bc}} \right) + \mathcal{O}(\epsilon^2) \\
={}& 2\hat{\tilde{\gamma}}^{\rho\rho} \hat{\tilde{\gamma}}^{\rho\rho} \hat{D}_{\rho\rho\rho} + \Big(4\hat{\tilde{\gamma}}^{\rho\rho} \hat{D}_{\rho\rho\rho} h_{\tilde{\gamma}^{\rho\rho}} + 2\hat{\tilde{\gamma}}^{\rho\rho} \hat{\tilde{\gamma}}^{\rho\rho} h_{D_{\rho\rho\rho}} + 2\hat{\tilde{\gamma}}^{\rho\rho} h_{\tilde{\gamma}^{\rho a}} \hat{D}_{a\rho\rho} \\
&+ 2\hat{\tilde{\gamma}}^{\rho\rho} \hat{\tilde{\gamma}}^{ab} h_{D_{a\rho b}} + 2h_{\tilde{\gamma}^{\rho a}} \hat{\tilde{\gamma}}^{bc} \hat{D}_{cab} \Big) \epsilon + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{E.26}
$$

Let us now calculate the individual terms by using

$$\phi^{-2} = \frac{\rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} q^2} \tag{E.27}$$

$$\hat{\tilde{\gamma}}^{\rho\rho} = \rho^{\frac{2(n-2)}{n-1}} \det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} a^{\frac{n-2}{n-1}} \tag{E.28}$$

$$\hat{\tilde{\gamma}}^{ab} = \frac{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} \tag{E.29}$$

$$h_{\tilde{\gamma}^{\rho\rho}} = \frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{-\frac{n-2}{n-1}} \rho^{-\frac{2(n-2)}{n-1}}} \frac{q^{n-2}}{n-1} \left((2-n) a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) + \mathcal{O}(h^2) \tag{E.30}$$

$$h_{\tilde{\gamma}^{\rho a}} = -\frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}} q^{n-1} g^{ab}_{S^{n-2}} \bar{h}_{\rho b} + \mathcal{O}(h^2) \tag{E.31}$$

$$\hat{D}_{\rho\rho\rho} = -\frac{1}{2\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}} \frac{n-2}{n-1} \left(\frac{a^{\frac{3-2n}{n-1}} \partial_\rho a}{\rho^{\frac{2(n-2)}{n-1}}} + 2 \frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}} \right) \tag{E.32}$$

$$\hat{D}_{a\rho\rho} = -\frac{a^{\frac{2-n}{n-1}}}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}} \frac{\partial_a \det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}} \tag{E.33}$$

$$\hat{D}_{cab} = -\frac{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}}{2} \left(\frac{g_{ab,S^{n-2}}}{n-1} \frac{\partial_c \det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}} \right) \tag{E.34}$$

$$
\begin{aligned}
h_{D_{\rho\rho\rho}} ={}& \frac{q^{n-2}}{2(n-1)} \frac{n-2}{n-1} \frac{1}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}} \left(\frac{a^{\frac{3-2n}{n-1}} \left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}} \right) \\
&\times \left((2-n) a \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{a^{\frac{2-n}{n-1}}\rho^{\frac{2(n-2)}{1-n}}q^{n-2}}{2(n-1)\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\Bigg[(2-n)\partial_\rho a\bar{h}_{\rho\rho} + (2-n)a\partial_\rho\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i} \\
&- \frac{n-2}{q\ell}\left((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\Bigg] + \mathcal{O}(h^2) \qquad \text{(E.35)}
\end{aligned}
$$

$$
h_{D_{a\rho b}} = -\frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}}\left(\frac{\partial_a\det(g_{cd,S^{n-2}})\bar{h}_{\rho b}}{(n-1)\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_a\bar{h}_{\rho b}}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\right) \qquad \text{(E.36)}
$$

Then,

$$
\begin{aligned}
2\hat{\bar{\gamma}}^{\rho\rho}\hat{\bar{\gamma}}^{\rho\rho}\hat{D}_{\rho\rho\rho} &= -\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{1}{1-n}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left(\frac{2\rho}{L^2} - \frac{2q}{\ell}\right) \\
&\quad - \frac{2(n-2)}{n-1}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} \qquad \text{(E.37a)} \\
4\hat{\bar{\gamma}}^{\rho\rho}\hat{D}_{\rho\rho\rho}h_{\bar{\gamma}^{\rho\rho}} &= 4\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}} \\
&\quad \times -\frac{1}{2\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\frac{n-2}{n-1}\left(\frac{a^{\frac{3-2n}{n-1}}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}}\right) \\
&\quad \times -\frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{-\frac{n-2}{n-1}}\rho^{-\frac{2(n-2)}{n-1}}}\left(aq^{n-2}\bar{h}_{\rho\rho} - \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) \\
&= 2\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{n-2}{n-1}}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}} \\
&\quad \times \left(\frac{a^{\frac{3-2n}{n-1}}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}}\right)\left(aq^{n-2}\bar{h}_{\rho\rho} - \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\right) \\
&= \frac{2(n-2)}{(n-1)^2}q^{n-2}\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho} - \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) \\
&\quad + \frac{4(n-2)}{(n-1)^2}\frac{q^{n-2}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}}{\det(g_{ab,S^{n-2}})^{\frac{1}{1-n}}}\left((n-2)a\bar{h}_{\rho\rho} - \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) \qquad \text{(E.37b)} \\
2\hat{\bar{\gamma}}^{\rho\rho}\hat{\bar{\gamma}}^{\rho\rho}h_{D_{\rho\rho\rho}} &= 2\rho^{\frac{4(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{2}{n-1}}a^{\frac{2(n-2)}{n-1}} \\
&\quad \times \Bigg[\frac{q^{n-2}}{2(n-1)}\frac{n-2}{n-1}\frac{1}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\left(\frac{a^{\frac{3-2n}{n-1}}\left(\frac{2\rho}{L^2}-\frac{2q}{\ell}\right)}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}}\right) \\
&\quad \times \left((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) \\
&\quad - \frac{a^{\frac{2-n}{n-1}}q^{n-2}}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\Big\{(2-n)\partial_\rho a\bar{h}_{\rho\rho} + (2-n)a\partial_\rho\bar{h}_{\rho\rho} \\
&\quad + \sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i} - \frac{n-2}{q\ell}\Big((2-n)a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\Big)\Big\}\Bigg] \\
&= \frac{n-2}{(n-1)^2}q^{n-2}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left(\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a + 2\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left((2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\\
&-\frac{1}{n-1}a^{\frac{n-2}{n-1}}q^{n-2}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\Bigg\{(2-n)\partial_\rho a\bar{h}_{\rho\rho}+(2-n)a\partial_\rho\bar{h}_{\rho\rho}\\
&+\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}-\frac{n-2}{q\ell}\left[(2-n)a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right]\Bigg\}
\end{aligned}
\tag{E.37c}
$$

$$
\begin{aligned}
2\hat{\tilde{\gamma}}^{\rho\rho}h_{\tilde{\gamma}^{\rho a}}\hat{D}_{a\rho\rho}&=2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}\times\left(-\frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}q^{n-1}g^{ab}_{S^{n-2}}\bar{h}_{\rho b}\right)\\
&\quad\times-\frac{a^{\frac{2-n}{n-1}}}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}}\frac{\partial_a\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}}\\
&=\cancel{\frac{1}{n-1}\frac{\partial_a\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}\frac{g^{ab}_{S^{n-2}}a^{\frac{n-2}{n-1}}q^{n-1}}{\rho^{\frac{2}{n-1}}}}\bar{h}_{\rho b}
\end{aligned}
\tag{E.37d}
$$

$$
\begin{aligned}
2\hat{\tilde{\gamma}}^{\rho\rho}\hat{\tilde{\gamma}}^{ab}h_{D_{a\rho b}}&=2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}\frac{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\\
&\quad\times-\frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}}\left(\frac{\partial_a\det(g_{cd,S^{n-2}})\bar{h}_{\rho b}}{(n-1)\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}}-\frac{\partial_a\bar{h}_{\rho b}}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&=\cancel{-\frac{1}{n-1}\frac{\partial_a\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}\frac{g^{ab}_{S^{n-2}}a^{\frac{n-2}{n-1}}q^{n-1}}{\rho^{\frac{2}{n-1}}}}\bar{h}_{\rho b}\\
&\quad+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}g^{ab}_{S^{n-2}}\partial_a\bar{h}_{\rho b}
\end{aligned}
\tag{E.37e}
$$

$$
\begin{aligned}
2h_{\tilde{\gamma}^{\rho a}}\hat{\tilde{\gamma}}^{bc}\hat{D}_{cab}&=-2\frac{\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}q^{n-1}g^{ad}_{S^{n-2}}\bar{h}_{\rho d}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{bc}_{S^{n-2}}\\
&\quad\times\left(-\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}g_{ab,S^{n-2}}\partial_c\det(g_{cd,S^{n-2}})}{2(n-1)\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}}+\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}\partial_c g_{ab,S^{n-2}}}{2\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&=-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{ef,S^{n-2}})^{\frac{2}{n-1}}g^{bc}_{S^{n-2}}g^{ad}_{S^{n-2}}q^{n-1}\bar{h}_{\rho d}\\
&\quad\times\left(-\frac{g_{ab,S^{n-2}}\partial_c\det(g_{cd,S^{n-2}})}{(n-1)\det(g_{cd,S^{n-2}})^{\frac{n}{n-1}}}+\frac{\partial_c g_{ab,S^{n-2}}}{\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&=\frac{1}{n-1}\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\frac{\partial_c\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}g^{bc}_{S^{n-2}}q^{n-1}\bar{h}_{\rho b}\\
&\quad-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}g^{bc}_{S^{n-2}}g^{ad}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}q^{n-1}\bar{h}_{\rho d}
\end{aligned}
\tag{E.37f}
$$

and

$$
\tilde{\Gamma}^\rho=-\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{1}{1-n}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\partial_\rho a-\frac{2(n-2)}{n-1}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}
$$

$$
\begin{aligned}
&+2q^{n-2}\frac{n-2}{(n-1)^2}\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+4q^{n-2}\frac{n-2}{(n-1)^2}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-q^{n-2}\frac{n-2}{(n-1)^2}\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-2q^{n-2}\frac{n-2}{(n-1)^2}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{1}{n-1}a^{\frac{n-2}{n-1}}q^{n-2}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\\
&\quad\times\left((n-2)\partial_\rho a\bar{h}_{\rho\rho}+(n-2)a\partial_\rho\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-\frac{n-2}{(n-1)\ell}q^{n-3}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}g^{ab}_{S^{n-2}}\partial_a\bar{h}_{\rho b}\epsilon+\frac{1}{n-1}\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\frac{\partial_c\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}g^{bc}_{S^{n-2}}q^{n-1}\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}g^{bc}_{S^{n-2}}g^{ad}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}q^{n-1}\bar{h}_{\rho d}\epsilon+\mathcal{O}(\epsilon^2)\\
=&-\frac{n-2}{n-1}\rho^{\frac{2(n-2)}{n-1}}a^{\frac{1}{1-n}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\partial_\rho a-\frac{2(n-2)}{n-1}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\\
&+q^{n-2}\frac{n-2}{(n-1)^2}\frac{\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\partial_\rho a\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+2q^{n-2}\frac{n-2}{(n-1)^2}\rho^{\frac{n-3}{n-1}}a^{\frac{n-2}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&-\frac{n-2}{(n-1)\ell}q^{n-3}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\left((n-2)a\bar{h}_{\rho\rho}-\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{1}{n-1}a^{\frac{n-2}{n-1}}q^{n-2}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}\\
&\quad\times\left((n-2)\left(\partial_\rho a\bar{h}_{\rho\rho}+a\partial_\rho\bar{h}_{\rho\rho}\right)-\sum_{i=1}^{n-2}\partial_\rho\bar{h}_{\theta_i\theta_i}\right)\epsilon\\
&+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}g^{ab}_{S^{n-2}}\partial_a\bar{h}_{\rho b}\epsilon+\frac{1}{n-1}\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\frac{\partial_c\det(g_{cd,S^{n-2}})}{\det(g_{cd,S^{n-2}})^{\frac{n-2}{n-1}}}g^{bc}_{S^{n-2}}q^{n-1}\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}g^{bc}_{S^{n-2}}g^{ad}_{S^{n-2}}\partial_c g_{ab,S^{n-2}}q^{n-1}\bar{h}_{\rho d}\epsilon+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.38}
$$

Furthermore,

$$
\tilde{\Gamma}^a=2\tilde{\gamma}^{a\rho}\left(\tilde{\gamma}^{\rho\rho}D_{\rho\rho\rho}+\tilde{\gamma}^{\rho b}D_{b\rho\rho}+\tilde{\gamma}^{b\rho}D_{\rho\rho b}+\tilde{\gamma}^{bc}D_{c\rho b}\right)
$$

$$
\begin{aligned}
&+ 2\tilde{\gamma}^{ab}\Big(\tilde{\gamma}^{\rho\rho}D_{\rho b\rho} + \tilde{\gamma}^{\rho c}D_{cb\rho} + \tilde{\gamma}^{c\rho}D_{\rho bc} + \tilde{\gamma}^{cd}D_{dbc}\Big) \\
=& 2\epsilon h_{\tilde{\gamma}^{a\rho}}\left[\hat{\tilde{\gamma}}^{\rho\rho}\hat{D}_{\rho\rho\rho} + \epsilon h_{\tilde{\gamma}^{\rho\rho}}\hat{D}_{\rho\rho\rho} + \epsilon\hat{\tilde{\gamma}}^{\rho\rho}h_{D_{\rho\rho\rho}} + \epsilon h_{\tilde{\gamma}^{\rho b}}\hat{D}_{b\rho\rho} + \epsilon\hat{\tilde{\gamma}}^{bc}h_{D_{c\rho b}}\right] \\
&+ 2\left(\hat{\tilde{\gamma}}^{ab} + \epsilon h_{\tilde{\gamma}^{ab}}\right)\left[\epsilon\hat{\tilde{\gamma}}^{\rho\rho}h_{D_{\rho\rho b}} + \epsilon h_{\tilde{\gamma}^{c\rho}}\hat{D}_{\rho bc} + \hat{\tilde{\gamma}}^{cd}\hat{D}_{dbc} + \epsilon h_{\tilde{\gamma}^{cd}}\hat{D}_{dbc} + \epsilon\hat{\tilde{\gamma}}^{cd}h_{D_{dbc}}\right] + \mathcal{O}(\epsilon^2) \\
=& 2\hat{\tilde{\gamma}}^{ab}\hat{\tilde{\gamma}}^{cd}\hat{D}_{dbc} + \Big(2h_{\tilde{\gamma}^{a\rho}}\hat{\tilde{\gamma}}^{\rho\rho}\hat{D}_{\rho\rho\rho} + 2\hat{\tilde{\gamma}}^{ab}\hat{\tilde{\gamma}}^{\rho\rho}h_{D_{\rho\rho b}} + 2\hat{\tilde{\gamma}}^{ab}h_{\tilde{\gamma}^{c\rho}}\hat{D}_{\rho bc} \\
&+ 2\hat{\tilde{\gamma}}^{ab}h_{\tilde{\gamma}^{cd}}\hat{D}_{dbc} + 2\hat{\tilde{\gamma}}^{ab}\hat{\tilde{\gamma}}^{cd}h_{D_{dbc}} + 2h_{\tilde{\gamma}^{ab}}\hat{\tilde{\gamma}}^{cd}\hat{D}_{dbc}\Big)\epsilon + \mathcal{O}(\epsilon^2). \qquad \text{(E.39)}
\end{aligned}
$$

Let us now calculate the individual terms of the latter equation by using the following

$$\hat{\tilde{\gamma}}^{\rho\rho} = \rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}} \qquad \text{(E.40)}$$

$$\hat{\tilde{\gamma}}^{cd} = \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{cd}_{S^{n-2}} \qquad \text{(E.41)}$$

$$h_{\tilde{\gamma}^{a\rho}} = -\frac{a^{\frac{n-2}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}}q^{n-1}g^{ab}_{S^{n-2}}\bar{h}_{\rho b} \qquad \text{(E.42)}$$

$$
\begin{aligned}
h_{\tilde{\gamma}^{ab}} =& \hat{\phi}^{-2}\left(h^{ab} + h_{\phi^{-2}}\hat{\gamma}^{ab}\right) \\
=& \frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}\Bigg(-\frac{q^4}{\rho^4}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}q^{n-4}\rho^2 g_{cd,S^{n-2}}\bar{h}_{cd} \\
&+ \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\frac{q^2}{\rho^2}g^{ab}_{S^{n-2}}\Bigg) \\
=& \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\Bigg(-q^{n-2}g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd} \\
&+ \frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\Bigg) \\
=& -\frac{q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\Bigg((n-1)g^{ac}_{S^{n-2}}g^{bd}_{S^{n-2}}g_{cd,S^{n-2}}\bar{h}_{cd} \\
&- \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\Bigg) \qquad \text{(E.43)}
\end{aligned}
$$

$$\hat{D}_{\rho\rho\rho} = -\frac{1}{2\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\frac{n-2}{n-1}\left(\frac{a^{\frac{3-2n}{n-1}}\partial_\rho a}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}}\right) \qquad \text{(E.44)}$$

$$\hat{D}_{\rho bc} = \frac{1}{2(n-1)}\frac{g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\left(a^{\frac{2-n}{n-1}}\rho^{\frac{2}{n-1}}\partial_\rho a + 2a^{\frac{1}{n-1}}\rho^{\frac{3-n}{n-1}}\right) \qquad \text{(E.45)}$$

$$\hat{D}_{dbc} = -\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2}\left(\frac{g_{bc,S^{n-2}}}{n-1}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right) \qquad \text{(E.46)}$$

$$h_{D_{\rho\rho b}} = \Bigg(\frac{a^{\frac{2-n}{n-1}}q^{n-1}\partial_\rho a}{2(n-1)\rho^{\frac{2(n-2)}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}} - \frac{a^{\frac{1}{n-1}}q^{n-2}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}\ell}$$

$$
\begin{aligned}
&- \frac{(n-2)a^{\frac{1}{n-1}}q^{n-1}}{(n-1)\rho^{\frac{3n-5}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}} - \frac{(n-3)a^{\frac{1}{n-1}}q^{n-2}}{2\ell\rho^{\frac{2(n-2)}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\Bigg)\bar{h}_{\rho b}\\
&+ \frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{cd,S^{n-2}})^{\frac{1}{n-1}}}\partial_\rho \bar{h}_{\rho b} + \mathcal{O}(h^2)\\
=& \frac{q^{n-2}a^{\frac{1}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\left(\frac{q}{2(n-1)a}\partial_\rho a - \frac{n-2}{n-1}\frac{q}{\rho} - \frac{n-1}{2\ell}\right)\bar{h}_{\rho b}\\
&+ \frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\partial_\rho \bar{h}_{\rho b} + \mathcal{O}(h^2)
\end{aligned}
\tag{E.47}
$$

$$
\begin{aligned}
h_{D_{dbc}} =& \frac{q^{n-2}}{n-1}\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2}\left(\frac{g_{bc,S^{n-2}}}{n-1}\frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\times\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i} - (n-1)\bar{h}_{bc}\right)\\
&- \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^{n-2}g_{bc,S^{n-2}}}{2(n-1)\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\left(a\partial_d\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i} - (n-1)\partial_d\bar{h}_{bc}\right) + \mathcal{O}(h^2)
\end{aligned}
\tag{E.48}
$$

Then,

$$
\begin{aligned}
2h_{\hat{\gamma}^{a\rho}}\hat{\hat{\gamma}}^{\rho\rho}\hat{D}_{\rho\rho\rho} =& -2\frac{a^{\frac{n-2}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}}q^{n-1}g^{ab}_{S^{n-2}}\bar{h}_{\rho b}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}\\
&\times -\frac{1}{2\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\frac{n-2}{n-1}\left(\frac{a^{\frac{3-2n}{n-1}}\partial_\rho a}{\rho^{\frac{2(n-2)}{n-1}}} + 2\frac{a^{\frac{2-n}{n-1}}}{\rho^{\frac{3n-5}{n-1}}}\right)\\
=&\frac{n-2}{n-1}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}g^{ab}_{S^{n-2}}\left(\frac{\partial_\rho a}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} + 2\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{n+1}{n-1}}}\right)\bar{h}_{\rho b}
\end{aligned}
\tag{E.49}
$$

$$
\begin{aligned}
2\hat{\hat{\gamma}}^{ab}\hat{\hat{\gamma}}^{\rho\rho}h_{D_{\rho\rho b}} =&2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}\\
&\times\Bigg\{\frac{q^{n-2}a^{\frac{1}{n-1}}}{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\left(\frac{q}{2(n-1)a}\partial_\rho a - \frac{n-2}{n-1}\frac{q}{\rho} - \frac{n-1}{2\ell}\right)\bar{h}_{\rho b}\\
&+ \frac{a^{\frac{1}{n-1}}q^{n-1}}{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\partial_\rho\bar{h}_{\rho b}\Bigg\}\\
=&\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-2}\left(\frac{q}{(n-1)a}\partial_\rho a - \frac{2(n-2)}{n-1}\frac{q}{\rho} - \frac{n-1}{\ell}\right)\bar{h}_{\rho b}\\
&+ \frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\partial_\rho\bar{h}_{\rho b}
\end{aligned}
\tag{E.50}
$$

$$
2\hat{\hat{\gamma}}^{ab}h_{\hat{\gamma}^{c\rho}}\hat{D}_{\rho bc} = -2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\frac{a^{\frac{n-2}{n-1}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{\rho^{\frac{2}{n-1}}}q^{n-1}g^{cd}_{S^{n-2}}\bar{h}_{\rho d}
$$

$$\times \frac{1}{2(n-1)} \frac{g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}} \left(a^{\frac{2-n}{n-1}} \rho^{\frac{2}{n-1}} \partial_\rho a + 2 a^{\frac{1}{n-1}} \rho^{\frac{3-n}{n-1}}\right)$$

$$= -\frac{q^{n-1}}{n-1} \det(g_{ef,S^{n-2}})^{\frac{1}{n-1}} g^{ab}_{S^{n-2}} \left(\frac{\partial_\rho a}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} + \frac{2a^{\frac{n-2}{n-1}}}{\rho^{\frac{n+1}{n-1}}}\right) \bar{h}_{\rho b} \tag{E.51}$$

$$2\hat{\tilde{\gamma}}^{ab}\hat{\tilde{\gamma}}^{cd}\hat{D}_{dbc} = 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}}$$

$$\times -\frac{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}}{2} \left(\frac{g_{bc,S^{n-2}}}{n-1} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right)$$

$$= -\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} \left(\frac{1}{n-1} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})} - g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}}\right)$$

$$= -\frac{1}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} \frac{1}{n-1} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} g^{ab}_{S^{n-2}} + \underbrace{\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}}}_{=0}$$

$$= -\frac{1}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} \frac{1}{n-1} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} g^{ab}_{S^{n-2}}$$

$$2\hat{\tilde{\gamma}}^{ab} h_{\tilde{\gamma}^{cd}} \hat{D}_{dbc} = 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}}$$

$$\times \left(-q^{n-2} g^{ce}_{S^{n-2}} g^{df}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} + \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right) g^{cd}_{S^{n-2}}\right)$$

$$\times -\frac{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}}{2} \left(\frac{g_{bc,S^{n-2}}}{n-1} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right)$$

$$= -\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}}$$

$$\times \left(-q^{n-2} g^{ce}_{S^{n-2}} g^{df}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} + \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right) g^{cd}_{S^{n-2}}\right)$$

$$\times \left(\frac{g_{bc,S^{n-2}}}{n-1} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})} - \partial_d g_{bc,S^{n-2}}\right)$$

$$= -\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}} \rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}}$$

$$\times \Bigg\{ -\frac{1}{n-1} q^{n-2} g_{bc,S^{n-2}} g^{ce}_{S^{n-2}} g^{df}_{S^{n-2}} g_{ef,S^{n-2}} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})} \bar{h}_{ef}$$

$$+ \frac{q^{n-2}}{(n-1)^2} g^{cd}_{S^{n-2}} g_{bc,S^{n-2}} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right)$$

$$+ q^{n-2} g^{ce}_{S^{n-2}} g^{df}_{S^{n-2}} g_{ef,S^{n-2}} \partial_d g_{bc,S^{n-2}} \bar{h}_{ef}$$

$$- \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i}\right) g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}} \Bigg\}$$

$$
\begin{aligned}
=&\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{1}{n-1}q^{n-2}g^{ab}_{S^{n-2}}g^{df}_{S^{n-2}}g_{bf,S^{n-2}}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}\bar{h}_{bf}\\
&-\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{q^{n-2}}{(n-1)^2}g^{ab}_{S^{n-2}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\\
&-\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}q^{n-2}g^{ab}_{S^{n-2}}g^{ce}_{S^{n-2}}g^{df}_{S^{n-2}}g_{ef,S^{n-2}}\partial_d g_{bc,S^{n-2}}\bar{h}_{ef}\\
&+\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}}
\end{aligned}
\tag{E.52}
$$

$$
\begin{aligned}
2\hat{\bar{\gamma}}^{ab}\hat{\bar{\gamma}}^{cd}h_{D_{dbc}}=&2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{cd}_{S^{n-2}}\\
&\times\left\{\frac{q^{n-2}}{n-1}\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2}\left(\frac{g_{bc,S^{n-2}}}{n-1}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}}-\frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right)\right.\\
&\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}+(n-1)\bar{h}_{bc}\right)-\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^2g_{bc,S^{n-2}}}{2(n-1)\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\\
&\left.\times\left(a\partial_d\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i}-(n-1)\partial_d\bar{h}_{bc}\right)\right\}\\
=&2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{cd}_{S^{n-2}}\\
&\times\frac{q^{n-2}}{n-1}\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2}\left(\frac{g_{bc,S^{n-2}}}{n-1}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}}-\frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\right)\\
&\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\\
&-2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{cd}_{S^{n-2}}\frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}q^2g_{bc,S^{n-2}}}{2(n-1)\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}\\
&\times\left(a\partial_d\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i}-(n-1)\partial_d\bar{h}_{bc}\right)\\
=&\frac{1}{(n-1)^2}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\\
&\qquad\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\\
&-\frac{q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}}\\
&\qquad\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \frac{q^{n-2}}{n-1} g^{ab}_{S^{n-2}} g^{cd}_{S^{n-2}} g_{bc,S^{n-2}} \\
&\qquad\qquad\qquad \times \left(a\partial_d \bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \partial_d \bar{h}_{\theta_i\theta_i} - (n-1)\partial_d \bar{h}_{bc} \right)
\end{aligned} \tag{E.53}
$$

$$
\begin{aligned}
2h_{\tilde{\gamma}^{ab}} \hat{\tilde{\gamma}}^{cd} \hat{D}_{dbc} =& \frac{2\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}} q^{2-n}} \left[-g^{ac}_{S^{n-2}} g^{bd}_{S^{n-2}} g_{cd,S^{n-2}} \bar{h}_{cd} + \frac{1}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \right] \\
& \times \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \\
& \times - \frac{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}{2} \left(\frac{g_{bc,S^{n-2}}}{n-1} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n}{n-1}}} - \frac{\partial_d g_{bc,S^{n-2}}}{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}} \right) \\
=& \left(q^{n-2} g^{ae}_{S^{n-2}} g^{bf}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} - \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \right) \\
& \times \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \\
& \times \left(\frac{g_{bc,S^{n-2}}}{n-1} \frac{\partial_d \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})} - \partial_d g_{bc,S^{n-2}} \right) \\
=& \left(q^{n-2} g^{ae}_{S^{n-2}} g^{bf}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} - \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \right) \\
& \times \left(\frac{1}{n-1} \frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} - \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}} \right) \\
=& q^{n-2} g^{ae}_{S^{n-2}} g^{bf}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} \\
& \times \left(\frac{1}{n-1} \frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} - \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}} \right) \\
& - \frac{q^{n-2}}{n-1} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \\
& \times \left(\frac{1}{n-1} \frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} - \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}} \right) \\
=& \frac{1}{n-1} \frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}} g^{ae}_{S^{n-2}} g^{bf}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} \\
& - q^{n-2} \frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{cd}_{S^{n-2}} \partial_d g_{bc,S^{n-2}} g^{ae}_{S^{n-2}} g^{bf}_{S^{n-2}} g_{ef,S^{n-2}} \bar{h}_{ef} \\
& - \frac{q^{n-2}}{(n-1)^2} \frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} \left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2} \bar{h}_{\theta_i\theta_i} \right) g^{ab}_{S^{n-2}} \frac{\partial_b \det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}
\end{aligned}
$$

$$+\frac{q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}} \tag{E.54}$$

and

$$\begin{aligned}
\tilde{\Gamma}^a =& -\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{1}{n-1}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\partial_\rho\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\left(\frac{2}{n-1}\frac{1}{\rho}+\frac{n-1}{q\ell}-\frac{n-2}{n-1}\frac{\partial_\rho a}{a}\right)\bar{h}_{\rho b}\epsilon\\
&-\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}q^{n-2}\overbrace{g^{ab}_{S^{n-2}}g^{ce}_{S^{n-2}}g^{df}_{S^{n-2}}g_{ef,S^{n-2}}\partial_d g_{bc,S^{n-2}}\bar{h}_{ef}}^{=0}\epsilon\\
&+\frac{1}{(n-1)^2}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\\
&\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\epsilon\\
&-\frac{q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\overbrace{g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}}}^{=0}\\
&\times\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\epsilon\\
&-\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{q^{n-2}}{n-1}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\\
&\times\left(a\partial_d\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i}-(n-1)\partial_d\bar{h}_{bc}\right)\epsilon\\
&+\frac{2}{n-1}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ae}_{S^{n-2}}g^{bf}_{S^{n-2}}g_{ef,S^{n-2}}\bar{h}_{ef}\epsilon\\
&-q^{n-2}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\overbrace{g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}}g^{ae}_{S^{n-2}}g^{bf}_{S^{n-2}}g_{ef,S^{n-2}}\bar{h}_{ef}\epsilon}^{=0}\\
&-\frac{2q^{n-2}}{(n-1)^2}\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}\epsilon\\
&+\frac{2q^{n-2}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)\overbrace{g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}\partial_d g_{bc,S^{n-2}}\epsilon}^{=0}+\mathcal{O}(\epsilon^2)\\
=& -\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{1}{n-1}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}+\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\partial_\rho\bar{h}_{\rho b}\epsilon\\
&-\frac{a^{\frac{n-2}{n-1}}}{\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}q^{n-1}\left(\frac{2}{n-1}\frac{1}{\rho}+\frac{n-1}{q\ell}-\frac{n-2}{n-1}\frac{\partial_\rho a}{a}\right)\bar{h}_{\rho b}\epsilon
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(n-1)^2}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_d\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}-(n-1)\bar{h}_{bc}\right)\epsilon\\
&-\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{q^{n-2}}{n-1}g^{ab}_{S^{n-2}}g^{cd}_{S^{n-2}}g_{bc,S^{n-2}}\left(a\partial_d\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\partial_d\bar{h}_{\theta_i\theta_i}-(n-1)\partial_d\bar{h}_{bc}\right)\epsilon\\
&+\frac{2}{n-1}\frac{q^{n-2}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}g^{ae}_{S^{n-2}}g^{bf}_{S^{n-2}}g_{ef,S^{n-2}}\bar{h}_{ef}\epsilon\\
&-\frac{2q^{n-2}}{(n-1)^2}\frac{1}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}\left(a\bar{h}_{\rho\rho}+\sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)g^{ab}_{S^{n-2}}\frac{\partial_b\det(g_{ef,S^{n-2}})}{\det(g_{ef,S^{n-2}})^{\frac{n-2}{n-1}}}\epsilon+\mathcal{O}(\epsilon^2)
\end{aligned}
\tag{E.55}
$$

E.8 The algebraic constraint vector Z^μ and Z_μ

The covariant components of the algebraic constraint Z_μ can be split into purely AdS and a perturbated part by using

$$Z_\mu = g_{\nu\mu}Z^\nu. \tag{E.56}$$

Then,

$$
\begin{aligned}
Z_t &= \hat{g}_{tt}Z^t + q^{n-4}\bar{h}_{tt}Z^t\epsilon + q^{n-3}\bar{h}_{t\rho}Z^\rho\epsilon + q^{n-4}\bar{h}_{ta}Z^a\epsilon + \mathcal{O}(\epsilon^2),\\
Z_\rho &= q^{n-3}\bar{h}_{t\rho}Z^t\epsilon + \hat{g}_{\rho\rho}Z^\rho + q^{n-4}\bar{h}_{\rho\rho}Z^\rho\epsilon + q^{n-3}\bar{h}_{\rho a}Z^a\epsilon + \mathcal{O}(\epsilon^2),\\
Z_a &= q^{n-4}\bar{h}_{ta}Z^t\epsilon + q^{n-3}\bar{h}_{\rho a}Z^\rho\epsilon + \hat{g}_{ab}Z^b + q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab}Z^b + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{E.57}
$$

By defining the purely AdS part

$$\hat{Z}_t = \hat{g}_{tt}Z^t, \quad \hat{Z}_\rho = \hat{g}_{\rho\rho}Z^\rho \quad \text{and} \quad \hat{Z}_a = \hat{g}_{ab}Z^b \tag{E.58}$$

and the deviation

$$
\begin{aligned}
h_{Z_t} &= q^{n-4}\bar{h}_{tt}Z^t + q^{n-3}\bar{h}_{t\rho}Z^\rho + q^{n-4}\bar{h}_{ta}Z^a\\
h_{Z_\rho} &= q^{n-3}\bar{h}_{t\rho}Z^t + q^{n-4}\bar{h}_{\rho\rho}Z^\rho + q^{n-3}\bar{h}_{\rho a}Z^a\\
h_{Z_a} &= q^{n-4}\bar{h}_{ta}Z^t + q^{n-3}\bar{h}_{\rho a}Z^\rho + q^{n-4}\rho^2 g_{ab,S^{n-2}}\bar{h}_{ab}Z^b,
\end{aligned}
\tag{E.59}
$$

we can decompose the boundary behaviour of the algebraic constraint vector Z_μ into

$$Z_\mu = \hat{Z}_\mu + h_{Z_\mu}\epsilon + \mathcal{O}(\epsilon^2). \tag{E.60}$$

Here we can see that the algebraic constraint behaves near the boundary as the purely AdS_n part plus a small perturbation ϵh_{Z_μ}.

E.9 The $\hat{\Gamma}$ variables

The ρ-component of the $\hat{\Gamma}$-variable is given by

$$\hat{\Gamma}^\rho = \tilde{\Gamma}^\rho + 2\tilde{\gamma}^{\rho\rho}Z_\rho + 2\tilde{\gamma}^{\rho a}Z_a$$

$$
\begin{aligned}
&= \tilde{\Gamma}^\rho + 2\left(\hat{\tilde{\gamma}}^{\rho\rho} + \epsilon h_{\tilde{\gamma}^{\rho\rho}}\right)\left(\hat{Z}_\rho + \epsilon h_{Z_\rho}\right) + 2\epsilon h_{\tilde{\gamma}^{\rho a}}\hat{Z}_a + \mathcal{O}(\epsilon^2) \\
&= \tilde{\Gamma}^\rho + 2\hat{\tilde{\gamma}}^{\rho\rho}\hat{Z}_\rho + 2\hat{\tilde{\gamma}}^{\rho\rho}h_{Z_\rho}\epsilon + 2h_{\tilde{\gamma}^{\rho\rho}}\hat{Z}_\rho\epsilon + 2h_{\tilde{\gamma}^{\rho a}}\hat{Z}_a\epsilon + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{E.61}
$$

Now, let us calculate the terms of the latter equations individually

$$
2\hat{\tilde{\gamma}}^{\rho\rho}\hat{Z}_\rho = \frac{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}Z^\rho \tag{E.62a}
$$

$$
\begin{aligned}
2\hat{\tilde{\gamma}}^{\rho\rho}h_{Z_\rho} &= 2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-3}\bar{h}_{t\rho}Z^t \\
&\quad \cancel{+2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-4}\bar{h}_{\rho\rho}Z^\rho} \\
&\quad \cancel{+2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-3}\bar{h}_{\rho a}Z^a}
\end{aligned} \tag{E.62b}
$$

$$
\begin{aligned}
2h_{\tilde{\gamma}^{\rho\rho}}\hat{Z}_\rho &= \cancel{-2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-4}\bar{h}_{\rho\rho}Z^\rho} \\
&\quad + 2\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}\frac{q^{n-2}}{n-1}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)Z^\rho
\end{aligned} \tag{E.62c}
$$

$$
2h_{\tilde{\gamma}^{\rho a}}\hat{Z}_a = \cancel{-2\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}\rho^{\frac{2(n-2)}{n-1}}q^{n-3}\bar{h}_{\rho a}Z^a} \tag{E.62d}
$$

Therefore,

$$
\begin{aligned}
\hat{\Gamma}^\rho &= \tilde{\Gamma}^\rho + \frac{2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}q^2}Z^\rho + 2\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}a^{\frac{n-2}{n-1}}q^{n-3}\bar{h}_{t\rho}Z^t\epsilon \\
&\quad + \frac{2q^{n-4}}{n-1}\frac{\rho^{\frac{2(n-2)}{n-1}}\det(g_{ab,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right)Z^\rho\epsilon + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{E.63}
$$

Now, the angle components are given by

$$
\begin{aligned}
\hat{\Gamma}^a &= \tilde{\Gamma}^a + 2\tilde{\gamma}^{a\rho}Z_\rho + 2\tilde{\gamma}^{ab}Z_b \\
&= \tilde{\Gamma}^a + 2\epsilon h_{\tilde{\gamma}^{a\rho}}\hat{Z}_\rho + 2\left(\hat{\tilde{\gamma}}^{ab} + \epsilon h_{\tilde{\gamma}^{ab}}\right)\left(\hat{Z}_b + \epsilon h_{Z_b}\right) + \mathcal{O}(\epsilon^2) \\
&= \tilde{\Gamma}^a + 2\epsilon h_{\tilde{\gamma}^{a\rho}}\hat{Z}_\rho + 2\hat{\tilde{\gamma}}^{ab}\hat{Z}_b + 2\epsilon\hat{\tilde{\gamma}}^{ab}h_{Z_b} + 2\epsilon h_{\tilde{\gamma}^{ab}}\hat{Z}_b + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{E.64}
$$

where the individual terms are given by

$$
2h_{\tilde{\gamma}^{a\rho}}\hat{Z}_\rho = \cancel{-2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}q^{n-3}g^{ab}_{S^{n-2}}\bar{h}_{\rho b}Z^\rho} \tag{E.65a}
$$

$$
2\hat{\tilde{\gamma}}^{ab}\hat{Z}_b = 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}\frac{\rho^{\frac{2(n-2)}{n-1}}}{q^2}Z^a \tag{E.65b}
$$

$$
\begin{aligned}
2\hat{\tilde{\gamma}}^{ab}h_{Z_b} &= 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}q^{n-4}\bar{h}_{tb}Z^t \\
&\quad \cancel{+2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}}g^{ab}_{S^{n-2}}q^{n-3}\bar{h}_{\rho b}Z^\rho}
\end{aligned}
$$

$$
\begin{aligned}
&\cancel{+ 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}} g^{ab}_{S^{n-2}} q^{n-4} g_{bc,S^{n-2}} \bar{h}_{bc} Z^c} && \text{(E.65c)}\\
2h_{\tilde{\gamma}^{ab}}\hat{Z}_b &= \cancel{-2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}} q^{n-4} g^{ac}_{S^{n-2}} g_{cd,S^{n-2}} \bar{h}_{cd} Z^d}\\
&= \frac{2q^{n-4}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) Z^a && \text{(E.65d)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\Gamma}^a = \tilde{\Gamma}^a &+ 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}}\frac{\rho^{\frac{2(n-2)}{n-1}}}{q^2} Z^a + 2\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}\rho^{\frac{2}{n-1}}} g^{ab}_{S^{n-2}} q^{n-4}\bar{h}_{tb} Z^t \epsilon\\
&+ \frac{2q^{n-4}}{n-1}\frac{\det(g_{ef,S^{n-2}})^{\frac{1}{n-1}}\rho^{\frac{2(n-2)}{n-1}}}{a^{\frac{1}{n-1}}}\left(a\bar{h}_{\rho\rho} + \sum_{i=1}^{n-2}\bar{h}_{\theta_i\theta_i}\right) Z^a\epsilon + \mathcal{O}(\epsilon^2). \qquad \text{(E.66)}
\end{aligned}
$$

E.10 Auxiliary Field for the Gamma-driver shift condition

Let us Eq. (2.88a), i.e.

$$
b^i = \frac{1}{k}\left[\partial_t\beta^i - \beta^k B_k{}^i\right], \qquad \text{(E.67)}
$$

to calculate the asymptotic behaviour of the auxiliary field b^i. The individual terms of the latter are given by

$$
\begin{aligned}
\partial_t\beta^\rho &= q^{n-1}a\partial_t\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2), && \text{(E.68a)}\\
\partial_t\beta^a &= q^{n-4}\hat{\gamma}^{aa}\partial_t\bar{h}_{ta}\epsilon + \mathcal{O}(\epsilon^2), && \text{(E.68b)}\\
\beta^\rho B_\rho{}^\rho &= \left(q^{n-1}a\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2)\right) q^{n-2}\left(-\tfrac{(n-1)}{\ell}a\bar{h}_{t\rho}\epsilon + q\partial_\rho a\bar{h}_{t\rho}\epsilon + qa\partial_\rho\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2)\right)\\
&= 0 + \mathcal{O}(\epsilon^2), && \text{(E.68c)}\\
\beta^a B_a{}^\rho &= \left(q^{n-4}\hat{\gamma}^{aa}\bar{h}_{ta}\epsilon + \mathcal{O}(\epsilon^2)\right)\left(q^{n-1}a\partial_a\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2)\right)\\
&= 0 + \mathcal{O}(\epsilon^2), && \text{(E.68d)}\\
\beta^\rho B_\rho{}^a &= \left(q^{n-1}a\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2)\right) q^{n-5}\left(-\tfrac{(n-4)}{\ell}\hat{\gamma}^{aa}\bar{h}_{ta}\epsilon + q\partial_\rho\hat{\gamma}^{aa}\bar{h}_{ta}\epsilon + q\hat{\gamma}^{aa}\partial_\rho\bar{h}_{ta}\epsilon + \mathcal{O}(\epsilon^2)\right)\\
&= 0 + \mathcal{O}(\epsilon^2), && \text{(E.68e)}\\
\beta^a B_a{}^b &= \left(q^{n-4}\hat{\gamma}^{aa}\bar{h}_{ta}\epsilon + \mathcal{O}(\epsilon^2)\right)\left(q^{n-4}\partial_a\hat{\gamma}^{bb}\bar{h}_{tb}\epsilon + q^{n-4}\hat{\gamma}^{bb}\partial_a\bar{h}_{tb}\epsilon + \mathcal{O}(\epsilon^2)\right)\\
&= 0 + \mathcal{O}(\epsilon^2). && \text{(E.68f)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
b^\rho &= \frac{1}{k}\left[\partial_t\beta^\rho - \beta^\rho B_\rho{}^\rho - \beta^a B_a{}^\rho\right]\\
&= \frac{1}{k}\partial_t h_{\beta^\rho}\epsilon + \mathcal{O}(\epsilon^2) = \frac{q^{n-1}a}{k}\partial_t\bar{h}_{t\rho}\epsilon + \mathcal{O}(\epsilon^2), && \text{(E.69a)}
\end{aligned}
$$

$$\begin{aligned} b^a &= \frac{1}{k}\left[\partial_t \beta^a - \beta^\rho B_\rho{}^a - \beta^b B_b{}^a\right] \\ &= \frac{1}{k}\partial_t h_{\beta^a}\epsilon + \mathcal{O}(\epsilon^2) = \frac{q^{n-4}\hat{\gamma}^{aa}}{k}\partial_t \bar{h}_{ta}\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{E.69b}$$

Appendix F

Jupyter Notebook for the Initial Data Calculation

This will be a simple jupyter notebook to write down the Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$, the Riemannian tensor $R^{\mu}_{\nu\sigma\rho}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, as well as the spatial Christoffel symbols $\hat{\Gamma}^{k}_{lm}$, the spatial Riemannian tensor $\hat{R}^{k}_{lmn}$, the spatial Ricci tensor $\hat{R}_{ij}$ and the spatial Ricci scalar $\hat{R}$ for the AdS$_3$ spacetime. We will need the spatial Ricci scalar $\hat{R}$ and some of the spatial Christoffel symbols $\hat{\Gamma}^{k}_{lm}$ to derive the initial value problem.

F.1 AdS$_3$ spacetime

[116]:
```
from gravipy.tensorial import *
from sympy import init_printing
init_printing()
```

The AdS$_3$ metric in globale coordinates

$$x^{\mu} = (t, r, \chi) \tag{F.1}$$

is given by

$$g_{\mu\nu} = \begin{pmatrix} -g(r) & 0 & 0 \\ 0 & \frac{1}{g(r)} & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \tag{F.2}$$

where $g(r) = 1 + r^2/L^2$. By compactifying the "radial" coordiante r to a finite value ρ by using the transformation

$$r = \frac{\rho}{1 - \rho/\ell}, \tag{F.3}$$

we get the metric

$$g_{\mu\nu} = \begin{pmatrix} -\frac{f(\rho)}{(1-\rho/\ell)^2} & 0 & 0 \\ 0 & \frac{1}{f(\rho)(1-\rho/\ell)^2} & 0 \\ 0 & 0 & \frac{\rho^2}{(1-\rho/\ell)^2} \end{pmatrix}, \tag{F.4}$$

where $f(\rho) = (1 - \rho/\ell)^2 + \rho^2/L^2$.

[117]:
```
t, rho, chi, L, l = symbols('t, rho, chi, L, l')
f = Function('f')(rho)
# create a coordinate four-vector object instantiating
# the Coordinates class
x = Coordinates('x', [t, rho, chi])
# define a matrix of a metric tensor components
#Metric = diag((1-2*M/r), -1/(1-2*M/r), -r**2,␣
↪-r**2*sin(theta)**2)
Metric = diag( - (1 + rho**2 / ((1-rho/l)**2*L**2) ), 1 / ( (1-rho/
↪l)**4 + (1-rho/l)**2 * rho**2 / L**2), rho**2 / (1-rho/l)**2 )
# create a metric tensor object instantiating the MetricTensor␣
↪class
g = MetricTensor('g', x, Metric)
```

The metric $g_{\mu\nu}$ is given by

[118]:
```
g(All, All)
```

[118]:

$$\begin{bmatrix} -1-\frac{\rho^2}{L^2\left(1-\frac{\rho}{l}\right)^2} & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{\rho}{l}\right)^4+\frac{\rho^2\left(1-\frac{\rho}{l}\right)^2}{L^2}} & 0 \\ 0 & 0 & \frac{\rho^2}{\left(1-\frac{\rho}{l}\right)^2} \end{bmatrix}$$

while the inverse metric $g^{\mu\nu}$ is given by g(-All,-All). Let us next calculate the Christoffel symbols defined by

$$\Gamma_{\mu\nu\sigma} = g_{\rho\sigma}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{F.5}$$

This can be done in jupyter by

[119]:
```
Ga = Christoffel('Ga',g)
Ga(All,All,All)
```

[119]:

$$\left[\begin{bmatrix} 0 & -\frac{l^3\rho}{L^2(l-\rho)^3} & 0 \\ -\frac{l^3\rho}{L^2(l-\rho)^3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{l^3\rho}{L^2(l-\rho)^3} & 0 & 0 \\ 0 & \frac{L^2l^4\left(2L^2(l-\rho)^2+l^2\rho^2-l^2\rho(l-\rho)\right)}{(l-\rho)^3\left(L^2(l-\rho)^2+l^2\rho^2\right)^2} & 0 \\ 0 & 0 & -\frac{l^3\rho}{(l-\rho)^3} \end{bmatrix} \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{l^3\rho}{(l-\rho)^3} \\ 0 & \frac{l^3\rho}{(l-\rho)^3} & 0 \end{bmatrix} \right] \tag{F.6}$$

The Riemannian tensor of the AdS$_3$ metric can be, respectively, calculated and displayed in jupyter by

$$R_{\mu\nu\rho\sigma} = \partial_\rho\Gamma_{\mu\nu\sigma} - \partial_\sigma\Gamma_{\mu\nu\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma_{\mu\sigma\alpha} - \Gamma^{\alpha}_{\nu\rho}\Gamma_{\mu\sigma\alpha} - \partial_\rho g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} + \partial_\sigma g_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho} \tag{F.7}$$

and

[120]:
```
Rm = Riemann('Rm',g)
```

The Ricci tensor $R_{\mu\nu}$ can be, respectively, calculated and displayed in jupyter by

$$R_{\alpha\beta} = R^{\mu}_{\mu\alpha\beta} \tag{F.8}$$

and

[121]:
```
Ri = Ricci('Ri', g)
```

and the Ricci scalar R can be, respectively, calculated and displayed in jupyter by

$$R = R^{\mu}_{\mu} \tag{F.9}$$

and

[122]:
```
Ri.scalar()
```

[122]: $-\frac{6}{L^2}$

Let us calculate the spatial Ricci scalar:

$$\hat{R} = \gamma^{ij}\hat{R}_{ij} \tag{F.10}$$

For this, we need to redefine the spatial γ matrix and calculate the same objects as above.

[123]:
```
rho, chi, L, l = symbols('rho, chi, L, l')
g = Function('g')(rho)
# create a coordinate four-vector object instantiating
# the Coordinates class
x = Coordinates('x', [rho, chi])
# define a matrix of a metric tensor components
#Metric = diag((1-2*M/r), -1/(1-2*M/r), -r**2,␣
 ↪-r**2*sin(theta)**2)
Metric = diag( 1 / ( (1-rho/l)**4 + (1-rho/l)**2 * rho**2 / L**2),␣
 ↪rho**2 / (1-rho/l)**2 )
# create a metric tensor object instantiating the MetricTensor␣
 ↪class
h = MetricTensor('h', x, Metric)
```

[124]:
```
h(All, All)
```

[124]:
$$\begin{bmatrix} \frac{1}{\left(1-\frac{\rho}{l}\right)^4+\frac{\rho^2\left(1-\frac{\rho}{l}\right)^2}{L^2}} & 0 \\ 0 & \frac{\rho^2}{\left(1-\frac{\rho}{l}\right)^2} \end{bmatrix}$$

[125]:
```
Ga_spatial = Christoffel('Ga_spatial',h)
Rm_spatial = Riemann('Rm_spatial',h)
Ri_spatial = Ricci('Ri_spatial', h)
```

The Christoffel symbols can be calculated as above, where

$$Ga(k,l,m) = \hat{\Gamma}_{klm} \tag{F.11}$$

$$Ga(-k,l,m) = \hat{\Gamma}^{k}_{lm} \tag{F.12}$$

[126]:
```
Ga_spatial(-1,All,All)
```

[126]:

$$\begin{bmatrix}\frac{2L^2(l-\rho)^2+l^2\rho^2-l^2\rho(l-\rho)}{(l-\rho)\left(L^2(l-\rho)^2+l^2\rho^2\right)} & 0\\ 0 & -\frac{\rho\left(L^2(l-\rho)^2+l^2\rho^2\right)}{L^2l(l-\rho)}\end{bmatrix}$$

In this way, $\hat{\Gamma}^{\rho}_{\rho\rho}$ and $\hat{\Gamma}^{\chi}\chi\chi$ can be written in terms of $q = 1-\rho/\ell$ and $a = q^2 + \frac{\rho^2}{L^2}$ by

$$\hat{\Gamma}^{\rho}_{\rho\rho} = \frac{q}{a\ell} + \frac{1}{q\ell} - \frac{\rho}{L^2 a} \tag{F.13}$$

$$\hat{\Gamma}^{\rho}_{\chi\chi} = -\frac{\rho a}{q} \tag{F.14}$$

and, thus

$$\hat{\gamma}^{\rho\rho}\hat{\Gamma}^{\rho}_{\rho\rho} = \frac{q^3}{\ell} + \frac{qa}{\ell} - \frac{q^2\rho}{L^2} \tag{F.15}$$

$$\hat{\gamma}^{\chi\chi}\hat{\Gamma}^{\rho}_{\chi\chi} = -\frac{qa}{\rho}. \tag{F.16}$$

[127]:
```
Ga_spatial(-2,All,All)
```

[127]:

$$\begin{bmatrix}0 & \frac{l}{\rho(l-\rho)}\\ \frac{l}{\rho(l-\rho)} & 0\end{bmatrix}$$

The spatial Ricci scalar $\hat{R}$ is given by

[128]:
```
Ri_spatial.scalar()
```

[128]:

$$-\frac{2}{L^2}$$

F.2 AdS$_4$ spacetime

[129]:
```
from gravipy.tensorial import *
from sympy import init_printing
init_printing()
```

The AdS_4 metric in globale coordinates

$$x^\mu = (t, r, \chi, \theta) \tag{F.17}$$

is given by

$$g_{\mu\nu} = \begin{pmatrix}-g(r) & 0 & 0 & 0\\ 0 & \frac{1}{g(r)} & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\chi\end{pmatrix}, \tag{F.18}$$

where $g(r) = 1 + r^2/L^2$. By compactifying the “radial” coordiante r to a finite value ρ by using the transformation

$$r = \frac{\rho}{1-\rho/\ell}, \tag{F.19}$$

we get the metric

$$g_{\mu\nu} = \begin{pmatrix} -\frac{f(\rho)}{(1-\rho/\ell)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{f(\rho)(1-\rho/\ell)^2} & 0 & 0 \\ 0 & 0 & \frac{\rho^2}{(1-\rho/\ell)^2} & 0 \\ 0 & 0 & 0 & \frac{\rho^2 \sin^2\chi}{(1-\rho/\ell)^2} \end{pmatrix}, \tag{F.20}$$

where $f(\rho) = (1-\rho/\ell)^2 + \rho^2/L^2$.

[130]:
```
t, rho, chi, theta, L, l = symbols('t, rho, chi, theta, L, l')
f = Function('f')(rho)
# create a coordinate four-vector object instantiating
# the Coordinates class
x = Coordinates('x', [t, rho, chi, theta])
# define a matrix of a metric tensor components
#Metric = diag((1-2*M/r), -1/(1-2*M/r), -r**2,␣
↪-r**2*sin(theta)**2)
Metric = diag( - (1 + rho**2 / ((1-rho/l)**2*L**2) ), 1 / ( (1-rho/
↪l)**4 + (1-rho/l)**2 * rho**2 / L**2), rho**2 / (1-rho/l)**2,␣
↪rho**2*sin(chi)**2/ (1-rho/l)**2 )
# create a metric tensor object instantiating the MetricTensor␣
↪class
g = MetricTensor('g', x, Metric)
```

The metric $g_{\mu\nu}$ is given by

[131]:
```
g(All, All)
```

[131]:

$$\begin{bmatrix} -1-\frac{\rho^2}{L^2\left(1-\frac{\rho}{l}\right)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{\rho}{l}\right)^4+\frac{\rho^2\left(1-\frac{\rho}{l}\right)^2}{L^2}} & 0 & 0 \\ 0 & 0 & \frac{\rho^2}{\left(1-\frac{\rho}{l}\right)^2} & 0 \\ 0 & 0 & 0 & \frac{\rho^2\sin^2(\chi)}{\left(1-\frac{\rho}{l}\right)^2} \end{bmatrix}$$

Let us next calculate the Christoffel symbols defined by

$$\Gamma_{\mu\nu\sigma} = g_{\rho\sigma}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{F.21}$$

They can be displayed in jupyter by

[132]:
```
Ga = Christoffel('Ga',g)
Ga(1,All,All),Ga(2,All,All)
```

[132]:

$$\begin{bmatrix} 0 & -\frac{l^3\rho}{L^2(l-\rho)^3} & 0 & 0 \\ -\frac{l^3\rho}{L^2(l-\rho)^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{l^3\rho}{L^2(l-\rho)^3} & 0 & 0 & 0 \\ 0 & \frac{L^2l^4\left(2L^2(l-\rho)^2+l^2\rho^2-l^2\rho(l-\rho)\right)}{(l-\rho)^3\left(L^2(l-\rho)^2+l^2\rho^2\right)^2} & 0 & 0 \\ 0 & 0 & -\frac{l^3\rho}{(l-\rho)^3} & 0 \\ 0 & 0 & 0 & -\frac{l^3\rho\sin^2(\chi)}{(l-\rho)^3} \end{bmatrix}$$

[133]:
```
Ga(3,All,All),Ga(4,All,All)
```

[133]:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l^3\rho}{(l-\rho)^3} & 0 \\ 0 & \frac{l^3\rho}{(l-\rho)^3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{l^2\rho^2\sin(2\chi)}{2(l-\rho)^2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{l^3\rho\sin^2(\chi)}{(l-\rho)^3} \\ 0 & 0 & 0 & \frac{l^2\rho^2\sin(2\chi)}{2(l-\rho)^2} \\ 0 & \frac{l^3\rho\sin^2(\chi)}{(l-\rho)^3} & \frac{l^2\rho^2\sin(2\chi)}{2(l-\rho)^2} & 0 \end{bmatrix}$$

The Riemannian tensor of the AdS_4 metric can be, respectively, calculated and displayed in jupyter by

$$R_{\mu\nu\rho\sigma} = \partial_\rho\Gamma_{\mu\nu\sigma} - \partial_\sigma\Gamma_{\mu\nu\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma_{\mu\rho\alpha} - \Gamma^{\alpha}_{\nu\rho}\Gamma_{\mu\sigma\alpha} - \partial_\rho g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} + \partial_\sigma g_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho} \tag{F.22}$$

and

[134]:
```
Rm = Riemann('Rm',g)
```

The Ricci tensor $R_{\mu\nu}$ can be, respectively, calculated and displayed in jupyter by

$$R_{\alpha\beta} = R^{\mu}_{\mu\alpha\beta} \tag{F.23}$$

and

[135]:
```
Ri = Ricci('Ri', g)
Ri(All, All)
```

[135]:

$$\begin{bmatrix} \frac{3\left(L^2l^2-2L^2l\rho+L^2\rho^2+l^2\rho^2\right)}{L^4\left(l^2-2l\rho+\rho^2\right)} & 0 & 0 & 0 \\ 0 & -\frac{3l^4}{L^2l^4-4L^2l^3\rho+6L^2l^2\rho^2-4L^2l\rho^3+L^2\rho^4+l^4\rho^2-2l^3\rho^3+l^2\rho^4} & 0 & 0 \\ 0 & 0 & -\frac{3l^2\rho^2}{L^2\left(l^2-2l\rho+\rho^2\right)} & 0 \\ 0 & 0 & 0 & -\frac{3l^2\rho^2\sin^2(\chi)}{L^2\left(l^2-2l\rho+\rho^2\right)} \end{bmatrix}$$

while the Ricci scalar R can be, respectively, calculated and displayed in jupyter by

$$R = R^{\mu}_{\mu} \tag{F.24}$$

and

[136]:
```
Ri.scalar()
```

[136]:

$$-\frac{12}{L^2}$$

Let us calculate the spatial Ricci scalar:

$$\hat{R} = \gamma^{ij}\hat{R}_{ij} \tag{F.25}$$

For this, we need to redefine the spatial γ matrix and calculate the same objects as above.

[137]:
```
rho, chi, theta, L, l = symbols('rho, chi, theta, L, l')
g = Function('g')(rho)
# create a coordinate four-vector object instantiating
# the Coordinates class
x = Coordinates('x', [rho, chi, theta])
# define a matrix of a metric tensor components
```

```
#Metric = diag((1-2*M/r), -1/(1-2*M/r), -r**2, -r**2*sin(theta)**2)
Metric = diag( 1 / ( (1-rho/l)**4 + (1-rho/l)**2 * rho**2 / L**2), rho**2 / (1-rho/l)**2, rho**2*sin(chi)**2/ (1-rho/l)**2 )
# create a metric tensor object instantiating the MetricTensor class
h = MetricTensor('h', x, Metric)
```

[138]:
```
h(All, All)
```

[138]:

$$\begin{bmatrix} \frac{1}{\left(1-\frac{\rho}{l}\right)^4+\frac{\rho^2\left(1-\frac{\rho}{l}\right)^2}{L^2}} & 0 & 0 \\ 0 & \frac{\rho^2}{\left(1-\frac{\rho}{l}\right)^2} & 0 \\ 0 & 0 & \frac{\rho^2\sin^2(\chi)}{\left(1-\frac{\rho}{l}\right)^2} \end{bmatrix}$$

[139]:
```
Ga_spatial = Christoffel('Ga_spatial',h)
Rm_spatial = Riemann('Rm_spatial',h)
Ri_spatial = Ricci('Ri_spatial', h)
```

The Christoffel symbols can be calcualted as above where

$$Ga(k,l,m) = \hat{\Gamma}_{klm} \tag{F.26}$$

$$Ga(-k,l,m) = \hat{\Gamma}^k_{lm} \tag{F.27}$$

[140]:
```
Ga_spatial(-1,All,All)
```

[140]:

$$\begin{bmatrix} \frac{2L^2(l-\rho)^2+l^2\rho^2-l^2\rho(l-\rho)}{(l-\rho)\left(L^2(l-\rho)^2+l^2\rho^2\right)} & 0 & 0 \\ 0 & -\frac{\rho\left(L^2(l-\rho)^2+l^2\rho^2\right)}{L^2l(l-\rho)} & 0 \\ 0 & 0 & -\frac{\rho\left(L^2(l-\rho)^2+l^2\rho^2\right)\sin^2(\chi)}{L^2l(l-\rho)} \end{bmatrix}$$

In this way, $\hat{\Gamma}^{\rho}_{\rho\rho}, \hat{\Gamma}^{\rho}_{\chi\chi}, \hat{\Gamma}^{\rho}_{\theta\theta}$ can be written in terms of $q = 1-\rho/\ell$ and $a = q^2 + \frac{\rho^2}{L^2}$ by

$$\hat{\Gamma}^{\rho}_{\rho\rho} = \frac{q}{a\ell} + \frac{1}{q\ell} - \frac{\rho}{L^2 a} \tag{F.28}$$

$$\hat{\Gamma}^{\rho}_{\chi\chi} = -\frac{\rho a}{q} \tag{F.29}$$

$$\hat{\Gamma}^{\rho}_{\theta\theta} = -\frac{\rho a \sin^2\chi}{q} \tag{F.30}$$

and thus

$$\hat{\gamma}^{\rho\rho}\hat{\Gamma}^{\rho}_{\rho\rho} = \frac{q^3}{\ell} + \frac{qa}{\ell} - \frac{q^2\rho}{L^2} \tag{F.31}$$

$$\hat{\gamma}^{\chi\chi}\hat{\Gamma}^{\rho}_{\chi\chi} = -\frac{qa}{\rho} \tag{F.32}$$

$$\hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\rho}_{\theta\theta} = -\frac{aq}{\rho} \tag{F.33}$$

[141]:
```
Ga_spatial(-2,All,All)
```

[141]:

$$\begin{bmatrix} 0 & \frac{l}{\rho(l-\rho)} & 0 \\ \frac{l}{\rho(l-\rho)} & 0 & 0 \\ 0 & 0 & -\frac{\sin(2\chi)}{2} \end{bmatrix}$$

Using the latter, we can simplify

$$\hat{\gamma}^{\theta\theta}\hat{\Gamma}^{\chi}_{\theta\theta} = -\frac{q^2}{\rho^2 \sin^2\chi}\frac{\sin 2\chi}{2} = -\frac{q^2 \cot\chi}{\rho^2} \tag{F.34}$$

[142]:
```
Ga_spatial(-3,All,All)
```

[142]:

$$\begin{bmatrix} 0 & 0 & \frac{l}{\rho(l-\rho)} \\ 0 & 0 & \frac{\sin(2\chi)}{2\sin^2(\chi)} \\ \frac{l}{\rho(l-\rho)} & \frac{\sin(2\chi)}{2\sin^2(\chi)} & 0 \end{bmatrix}$$

The spatial Ricci scalar $\hat{R}$ is given by

[143]:
```
Ri_spatial.scalar()
```

[143]:

$$-\frac{6}{L^2}$$

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die diesen Quellen und Hilfsmitteln wörtlich oder sinngemäß entnommenen Ausführungen als solche kenntlich gemacht habe. Die Arbeit wurde keiner anderen Prüfungsbehörde vorgelegt.

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