



# The Cauchy problem for multiphase first-contact miscible models with viscous fingering



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## ABSTRACT

In this paper, the Cauchy problem for multiphase first-contact miscible models with viscous fingering, is studied and a global weak solution is obtained by using a new technique from the Div–Curl lemma in the compensated compactness theorem. This work extends the previous works, (Juanes and Blunt, 2006; Barkve, 1989), which provided the analytical solutions and the entropy solutions respectively, of the Riemann problem, and (Lu, 2013), which provided the global solution of the Cauchy problem for the Keyfitz–Kranzer system.

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## 1. Introduction

In this paper, we study the Cauchy problem for multiphase first-contact miscible models with viscous fingering

$$\begin{cases} S_t + f(S, T)_x = 0, \\ (ST - T)_t + (Tf(S, T) - T)_x = 0, \end{cases} \quad (1.1)$$

with bounded measurable initial data

$$(S(x, 0), T(x, 0)) = (S_0(x), T_0(x)), \quad 0 \leq S_0(x) \leq 1, \quad (1.2)$$

where  $S$  is the water saturation,  $C = ST - T$  is the solvent concentration, and  $f$  is the water fractional flow function. System (1.1) is a special case of the following nonstrictly hyperbolic systems of conservation laws

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modelling polymer flooding in enhanced oil recovery

$$\begin{cases} S_t + f(S, T)_x = 0, \\ (ST + \beta(T))_t + (Tf(S, T) + \alpha(T))_x = 0. \end{cases} \quad (1.3)$$

When  $\alpha(T) = aT$ ,  $\beta(T) = bT$ , where  $a, b$  are positive constants, system (1.3) represents a simple model for nonisothermal two-phase flow in a porous medium [1,2], and its Riemann problem was resolved and the entropy conditions were discussed in [1] under suitable conditions on  $f$ . When  $\beta(T) = 0$  and  $\alpha(T) = 0$ , system (1.3) is the famous Keyfitz–Kranzer [3] or Aw–Rascle model [4], and the Riemann problem and the Cauchy problem of system (1.2) were studied in [3–18] and the references cited therein. When  $\alpha(T) = 0$  and  $\beta(T) \neq 0$ , but  $\beta'(T) > 0$ , system (1.3) arises in connection with enhanced oil recovery, and its Riemann problem was resolved in [19]. For general  $\alpha(T) \neq 0$  and  $\beta(T) \neq 0$ , but  $\beta'(T) > 0$ , the Cauchy problem was studied in the recent paper [20].

When  $\beta'(T) < 0$ , system (1.3) is of interest and difficulty in mathematics because the flux functions are singular. For instance, the functions

$$(f(S, T), Tf(S, T) - T) = \left( f\left(S, \frac{C}{S-1}\right), \frac{C}{S-1} \left( f\left(S, \frac{C}{S-1}\right) - 1 \right) \right)$$

in system (1.1) are singular near the line  $S = 1$ .

In [21], the authors studied the analytical solutions of the Riemann problem for system (1.1).

As far as we know, there is no any existence result about the Cauchy problem of system (1.1) or system (1.3) when  $\beta'(T) < 0$ .

In this paper, we obtain the following results.

**Theorem 1.** *Let the initial data  $(S_0(x), T_0(x))$  be bounded,  $0 \leq S_0(x) \leq 1$ , the total variation of the second variant  $T_0(x)$  be bounded, the functions  $f(S, T), \alpha(T), \beta(T)$  be suitable smooth and satisfy  $f(0, T) = f(1, T) = 0$ , meas  $\{S : f_{SS}(S, T) = 0\} = 0$  for any fixed  $T$ ,  $\beta'(T) \leq -1$ .*

- (1) *If meas  $\{T : \beta''(T) = 0\} = 0$  or  $\beta'(T) = bT, b < -1$ , then the Cauchy problem (1.3)–(1.2) has a global bounded entropy solution  $(S(x, t), T(x, t))$  and  $T_x(\cdot, t)$  is bounded in  $L^1(R)$ , namely,  $(S(x, t), T(x, t))$  satisfies system (1.3) and the inequality  $\eta(S, T)_t + q(S, T)_x \leq 0$ , in the sense of distributions, for any smooth, convex, entropy  $\eta(S, C)$  and the corresponding entropy flux  $q(S, C)$ .*
- (2) *If  $\beta'(T) = -T$  and  $\alpha(T) = -T$ , then the Cauchy problem (1.1)–(1.2) has a global bounded **weak** entropy solution  $(S(x, t), T(x, t))$ , namely,  $(S(x, t), T(x, t))$  satisfies system (1.3) and the inequality  $\eta(S, T)_t + q(S, T)_x \leq 0$ , in the sense of distributions, for any smooth, convex, **weak** entropy  $\eta(S, C)$  and the corresponding **weak** entropy flux  $q(S, C)$ .*

**Definition 1.** A **weak** entropy  $\eta(S, C)$  of system (1.3) means  $\eta(1, C) = c_1$ , and a **weak** entropy flux  $q(S, C)$  means  $q(1, C) = c_2$ , where  $c_1, c_2$  are constants.

## 2. Proof of Theorem 1

To prove Theorem 1, we consider the Cauchy problem for the related parabolic system

$$\begin{cases} S_t + f(S, T)_x = \varepsilon S_{xx} \\ (ST + \beta(T) - \delta T)_t + (Tf(S, T) + \alpha(T))_x = \varepsilon(ST + \beta(T) - \delta T)_{xx}, \end{cases} \quad (2.1)$$

with initial data

$$(S^\varepsilon(x, 0), T^\varepsilon(x, 0)) = (S_0^\varepsilon(x), T_0^\varepsilon(x)), \quad (2.2)$$

where  $T_0^\varepsilon(x) = T_0(x) * G^\varepsilon$ ,  $S_0^{\varepsilon,\delta}(x) = (\varepsilon + (1 - 2\varepsilon)S_0(x)) * G^\varepsilon$  are the smooth approximations of  $T_0(x)$ ,  $S_0(x)$ ,  $G^\varepsilon$  is a mollifier and  $\varepsilon, \delta(\varepsilon < \delta, \delta = O(\varepsilon^{\frac{1}{2}}))$  are positive, small perturbation constants. We first have the following lemmas.

**Lemma 2.** *Let  $\alpha(T), \beta(T), f(S, T)$  be suitable smooth functions and  $\beta'(T) \leq -1$ .*

(1) *For any fixed  $\varepsilon > 0$  and  $\delta > 0$ , the Cauchy problem (2.1) with the bounded measurable initial data  $\varepsilon \leq S^\varepsilon(x, 0) \leq 1 - \varepsilon, |T^\varepsilon(x, 0)| \leq M$ , always has a local smooth solution  $(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})(x, t) \in (C^\infty(R \times (0, \tau)), C^\infty(R \times (0, \tau)))$ , for a small time  $\tau$ , which depends only on the  $L^\infty$  norm of the initial data  $(S^\varepsilon(x, 0), T^\varepsilon(x, 0))$ , and the local solution satisfies*

$$\frac{\varepsilon}{2} \leq S^{\varepsilon,\delta} \leq 1 + \varepsilon, \quad |T^{\varepsilon,\delta}(x, t)| \leq M + \varepsilon. \tag{2.3}$$

(2) *If  $f(0, T) = 0, f(1, T) = 1$ , then the local solution  $(S^{\varepsilon,\delta}(x, t), T^{\varepsilon,\delta}(x, t))$  has an a-priori  $L^\infty$  estimate*

$$|T^{\varepsilon,\delta}(x, t)|_{L^\infty} \leq M, \quad 0 < c(t, c_0, \varepsilon, \delta) \leq S^{\varepsilon,\delta}(x, t) \leq 1, \tag{2.4}$$

where  $c(t, c_0, \varepsilon, \delta)$  is a positive function, which could tend to zero as the time  $t$  tends to infinity or  $\varepsilon, \delta$  tends to zero, and the global solution of the Cauchy problem (2.1) and (2.2) exists.

**Proof.** Let  $C^\delta = ST + \beta(T) - \delta T$ . Since  $\beta'(T) \leq -1$ , then for any fixed  $S \in (0, 1 + \varepsilon)$ , there exists a smooth, inverse function  $T = g(S, C^\delta)$  and system (2.1) can be rewritten as

$$\begin{cases} S_t + f(S, g(S, C^\delta))_x = \varepsilon S_{xx}, \\ C_t^\delta + (g(S, C^\delta)f(S, g(S, C^\delta)) + \alpha(g(S, C^\delta)))_x = \varepsilon C_{xx}^\delta, \end{cases} \tag{2.5}$$

which is a standard parabolic system and the local existence result in (1) can be easily obtained by applying the contraction mapping principle to an integral representation for a solution, following the standard theory of semilinear parabolic systems.

To prove the estimates in (2.4), we substitute the first equation in (2.1) into the second, we may rewrite the second equation in (2.1) as

$$T_t + \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x = \varepsilon T_{xx} + \varepsilon \frac{2S_x + \beta''(T)T_x}{S + \beta'(T) - \delta} T_x. \tag{2.6}$$

Then we have the estimate  $|T^\varepsilon(x, t)| \leq M$  by applying the maximum principle to (2.6).

Since  $f(1, T) = 1$  and  $S^\varepsilon(x, 0) \leq 1 - \varepsilon$ , we rewrite the first equation in (2.1) as

$$S_t + f_S(S, T)S_x + f_T(S, T)T_x = \varepsilon S_{xx}, \tag{2.7}$$

where  $f_T(1, T) = 0$ , or

$$S_t + f_S(S, T)S_x + f_{TS}(\theta, T)(S - 1)T_x = \varepsilon S_{xx}, \tag{2.8}$$

where  $\theta \in (S, 1)$ . To prove  $S^\varepsilon(x, 0) \leq 1$ , we make the transformation

$$S = \left( \bar{S} + \frac{M}{L^2}(x^2 + cLe^t) \right) e^{\beta t} + 1, \tag{2.9}$$

where  $c, L, \beta$  are positive constants and  $M$  is the bound of  $S, T$  on  $R \times (0, T_1)$ . By using Eq. (2.8), we have immediately from (2.9)

$$\bar{S}_t + f_S \bar{S}_x - \varepsilon \bar{S}_{xx} + (\beta + f_{TS} T_x) \bar{S} = \frac{2\varepsilon M}{L^2} - cLe^t \frac{M}{L^2} - f_S \frac{2xM}{L^2} - (\beta + f_{TS} T_x) \frac{M}{L^2} (x^2 + cLe^t) \tag{2.10}$$

and

$$\bar{S}(x, 0) = S(x, 0) - 1 - \frac{M}{L^2}(x^2 + cL) < 0, \tag{2.11}$$

$$\bar{S}(\pm L, t) = (S(\pm L, t) - 1)e^{-\beta t} - \frac{M}{L^2}(L^2 + cLe^t) < 0. \tag{2.12}$$

We have from (2.10)–(2.12) that

$$\bar{S}(x, t) < 0 \quad \text{on } (-L, L) \times (0, T_1). \tag{2.13}$$

If (2.13) is violated at a point  $(x, t) \in (-L, L) \times (0, T_1)$ , let  $\bar{t}$  be the least upper bound of the value  $t$  at which  $\bar{S}(x, t) < 0$ ; then by the continuity we see that  $\bar{S}(x, t) = 0$  at some points  $(\bar{x}, \bar{t}) \in (-L, L) \times (0, T_1)$ . So,

$$\bar{S}_t + f_S \bar{S}_x - \varepsilon \bar{S}_{xx} \geq 0 \quad \text{at } (\bar{x}, \bar{t}). \tag{2.14}$$

However, we can choose a large  $\beta$  (depends on the local time) such that the right-hand side of (2.10) is negative, then Eq. (2.10) gives a conclusion contradicting (2.14). So (2.13) is proved. Therefore, for any fixed point  $(x_0, t_0) \in (-L, L) \times (0, T_1)$ ,

$$S(x_0, t_0) = \left( \bar{S}(x_0, t_0) + \frac{M}{L^2}(x_0^2 + cLe^{t_0}) \right) e^{\beta t_0} + 1 < \frac{M}{L^2}(x_0^2 + cLe^{t_0})e^{\beta t_0} + 1, \tag{2.15}$$

which gives the desired estimate  $S(x_0, t_0) \leq 1$  when we let  $L$  go to  $\infty$ .

Since  $f(0, T) = 0$  and  $S^\varepsilon(x, 0) > \varepsilon$ , we rewrite the first equation in (2.1) as

$$S_t + (Sg(S, T))_x = \varepsilon S_{xx}, \tag{2.16}$$

where  $g(S, T) = \frac{f(S, T)}{S}$  is bounded, so the lower bound  $0 < c(t, c_0, \varepsilon, \delta) \leq S^{\varepsilon, \delta}(x, t)$  in (2.4) can be proved by using Theorem 1.0.2 in [22] directly.

Whenever we have an a-priori  $L^\infty$  estimate of the local solution given in (2.4), it is clear that the local time  $\tau$  can be extended to any time  $T_1$  step by step since the step time depends only on the  $L^\infty$  norm. So, Lemma 2 is proved.

**Lemma 3.** (1) *If  $T_0(x)$  is of bounded total variation, then*

$$\int_{-\infty}^{\infty} |T_x^{\varepsilon, \delta}|(x, t) dx \leq \int_{-\infty}^{\infty} |T_x^{\varepsilon, \delta}|(x, 0) dx \leq M. \tag{2.17}$$

(2) *The solutions  $(S^{\varepsilon, \delta}, T^{\varepsilon, \delta})$  satisfy that*

$$\varepsilon(\delta - \beta'(T) - S)(T_x^{\varepsilon, \delta})^2, \varepsilon(S_x^{\varepsilon, \delta})^2 \text{ are bounded in } L^1_{loc}(R \times R^+). \tag{2.18}$$

**Proof.** The conclusion (2.17) in Lemma 3 can be proved by applying for a technique from [23] or [14,16].

To prove (2.18), we multiply  $S + \beta'(T) - \delta$  to (2.6) to obtain

$$(S + \beta'(T) - \delta)T_t + (f + \alpha'(T))T_x = \varepsilon(S + \beta'(T) - \delta)T_{xx} + \varepsilon(2S_x + \beta''(T)T_x)T_x. \tag{2.19}$$

We multiply (2.19) by  $h'(T)$ , the first equation in (2.1) by  $h(T)$  and then add the result to obtain

$$\begin{aligned} & \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T) \right)_t + \left( h(T)f(S, T) + \int_c^T h'(\tau)\alpha'(\tau)d\tau \right)_x \\ & = \varepsilon \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T) \right)_{xx} - \varepsilon h''(T)(S + \beta'(T) - \delta)T_x^2. \end{aligned} \tag{2.20}$$

Let  $K \subset R \times R^+$  be an arbitrary compact set and choose  $\phi \in C_0^\infty(R \times R^+)$  such that  $\phi_K = 1, 0 \leq \phi \leq 1$ .

Multiplying Eq. (2.20) by  $\phi$  and integrating over  $R \times R^+$ , we obtain

$$\begin{aligned} & \int_0^\infty \int_{-\infty}^\infty -\varepsilon h''(T)(S + \beta'(T) - \delta)T_x^2 \phi dx dt \\ &= - \int_0^\infty \int_{-\infty}^\infty Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T)\phi_t + h(T)f(S, T) \\ & \quad + \int_c^T h'(\tau)\alpha'(\tau)d\tau \phi_x + \varepsilon \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T) \right) \phi_{xx} dx dt \leq M(\phi) \end{aligned} \tag{2.21}$$

and hence that

$$\varepsilon(\delta - \beta'(T) - S)(T_x^{\varepsilon, \delta})^2 \text{ is bounded in } L^1_{loc}(R \times R^+) \tag{2.22}$$

if we choose  $h$  as a strictly convex function in (2.22).

We multiply  $g'(S)$  to Eq. (2.7) to obtain (for simplicity, we omit the superscripts  $\varepsilon$  and  $\delta$ )

$$\begin{aligned} & g(S)_t + \left( \int_1^S g'(\tau)f_S(\tau, T)d\tau \right)_x + \left( g'(S)f_T(S, T) - \int_1^S g'(\tau)f_{ST}(\tau, T)d\tau \right) T_x \\ &= \varepsilon g(S)_{xx} - \varepsilon g''(S)S_x^2. \end{aligned} \tag{2.23}$$

Choosing a strictly convex function  $g$  and multiplying a suitable nonnegative test function  $\phi$  to (2.23), we have that  $\varepsilon(S_x^{\varepsilon, \delta})^2$  is bounded in  $L^1_{loc}(R \times R^+)$ . Lemma 3 is proved.

**Lemma 4.**

$$h(T^{\varepsilon, \delta})_x, \quad S_t^{\varepsilon, \delta} + f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta})_x \tag{2.24}$$

are compact in  $H^{-1}_{loc}(R \times R^+)$ , for any smooth function  $h(T)$ .

**Proof.** Since  $T_x^{\varepsilon, \delta}$  are uniformly bounded in  $L^1_{loc}(R \times R^+)$  from the estimate (2.17), then they are compact in  $W^{-1, \alpha}_{loc}(R \times R^+)$ , where  $\alpha \in (1, 2)$ , by Sobolev’s embedding theorem. Moreover,  $T_x^{\varepsilon, \delta}$  are uniformly bounded in  $W^{-1, \infty}_{loc}(R \times R^+)$ , then the Murat compact embedding theorem [24] shows that  $h(T^{\varepsilon, \delta})_x$  are compact in  $H^{-1}_{loc}(R \times R^+)$ , for any smooth function  $h(T)$ .

Since  $\varepsilon(S_x^{\varepsilon, \delta})^2$  are bounded in  $L^1_{loc}(R \times R^+)$ , so the term  $\varepsilon S_{xx}$  in the first equation in (2.1) is compact in  $H^{-1}_{loc}(R \times R^+)$ . Thus  $S_t^{\varepsilon, \delta} + f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta})_x$  are compact in  $H^{-1}_{loc}(R \times R^+)$ . Lemma 4 is proved.

**Lemma 5.**

$$\left( S^{\varepsilon, \delta} h(T^{\varepsilon, \delta}) + \int_c^{T^{\varepsilon, \delta}} h'(\tau)\beta'(\tau)d\tau \right)_t + \left( h(T^{\varepsilon, \delta})f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}) + \int_c^{T^{\varepsilon, \delta}} h'(\tau)\alpha'(\tau)d\tau \right)_x \tag{2.25}$$

are compact in  $H^{-1}_{loc}(R \times R^+)$ , for any smooth function  $h(T)$ .

**Proof.** From (2.20), we have that

$$\begin{aligned} & \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau \right)_t + \left( h(T)f(S, T) + \int_c^T h'(\tau)\alpha'(\tau)d\tau \right)_x \\ &= \varepsilon \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T) \right)_{xx} - \varepsilon h''(T)(S + \beta'(T) - \delta)T_x^2 + \delta(h(T))_t. \end{aligned} \tag{2.26}$$

Since  $\beta'(T) \leq -1$ , we have from the estimate in (2.22) that

$$\varepsilon\delta(T_x^{\varepsilon,\delta})^2 \text{ is bounded in } L^1_{loc}(R \times R^+). \tag{2.27}$$

If we let  $\delta = O(\varepsilon^{\frac{1}{2}})$ , we have from (2.18) that the following terms on the right-hand side of (2.26)

$$\varepsilon \left( Sh(T) + \int_c^T h'(\tau)\beta'(\tau)d\tau - \delta h(T) \right)_{xx} + \delta(h(T))_t \text{ are compact in } H^{-1}_{loc}(R \times R^+) \tag{2.28}$$

and

$$-\varepsilon h''(T)(S + \beta'(T) - \delta)T_x^2 \text{ is bounded in } L^1_{loc}(R \times R^+). \tag{2.29}$$

Thus the right-hand side of (2.26) is compact in  $W^{-1,\alpha}$ ,  $\alpha \in (1, 2)$  by using Sobolev’s embedding theorem. Since the left-hand side of (2.26) is bounded in  $W^{-1,\infty}$ , the proof of Lemma 5 is completed by using Murat’s embedding theorem.

**Lemma 6.** *If meas  $\{T : \beta''(T) = 0\} = 0$  or  $\beta(T) = bT, b < -1$ , then there exists a subsequence (still labelled  $T^{\varepsilon,\delta}(x, t)$ ) such that*

$$T^{\varepsilon,\delta}(x, t) \rightarrow T(x, t) \tag{2.30}$$

*a.e. on any bounded and open set  $\Omega \subset R \times R^+$ .*

*If  $\beta(T) = -T$ , then there exists a subsequence (still labelled  $T^{\varepsilon,\delta}(x, t)$ ) such that  $T^{\varepsilon,\delta}(x, t) \rightarrow T(x, t)$  almost everywhere on the set  $S_+ = \{(x, t) : 0 \leq S(x, t) < 1\}$ , where  $S(x, t)$  is the weak limit of  $S^{\varepsilon,\delta}(x, t)$ .*

**Proof.** First, we may apply for the div-curl lemma in the compensated compactness theory [25] to the following special pairs of functions

$$(c, h(T^{\varepsilon,\delta})), \quad (S^{\varepsilon,\delta}, f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})) \tag{2.31}$$

to obtain

$$\overline{S^{\varepsilon,\delta} \cdot h(T^{\varepsilon,\delta})} = \overline{S^{\varepsilon,\delta} h(T^{\varepsilon,\delta})}, \tag{2.32}$$

where  $\overline{f(\theta^{\varepsilon,\delta})}$  denotes the weak-star limit of  $f(\theta^{\varepsilon,\delta})$ .

Second, letting  $h(T) = T$  and  $h(T) = \beta(T)$  in (2.25) respectively, we can apply for the div-curl lemma to the pairs of functions

$$(c, \beta(T^{\varepsilon,\delta}) - \beta(k)), \quad (S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k) + \beta(T^{\varepsilon,\delta}) - \beta(k), (T^{\varepsilon,\delta} - k)f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \alpha(T^{\varepsilon,\delta})) \tag{2.33}$$

to obtain

$$\begin{aligned} & \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)(\beta(T^{\varepsilon,\delta}) - \beta(k))} + \overline{(\beta(T^{\varepsilon,\delta}) - \beta(k))^2} \\ & = \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k) \cdot \beta(T^{\varepsilon,\delta}) - \beta(k)} + \overline{(\beta(T^{\varepsilon,\delta}) - \beta(k))^2}, \end{aligned} \tag{2.34}$$

and to the pairs of functions  $(c, T^{\varepsilon,\delta} - k)$  and

$$\left( S^{\varepsilon,\delta}(\beta(T^{\varepsilon,\delta}) - \beta(k)) + \int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau, h(T^{\varepsilon,\delta})f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \int_0^{T^{\varepsilon,\delta}} h'(\tau)\alpha'(\tau)d\tau \right) \tag{2.35}$$

to obtain

$$\begin{aligned} & \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)(\beta(T^{\varepsilon,\delta}) - \beta(k))} + \overline{(T^{\varepsilon,\delta} - k) \int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau} \\ & = \overline{(T^{\varepsilon,\delta} - k) \cdot S^{\varepsilon,\delta}(\beta(T^{\varepsilon,\delta}) - \beta(k))} + \overline{(T^{\varepsilon,\delta} - k) \cdot \int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau}. \end{aligned} \tag{2.36}$$

Deleting the common term  $\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)(\beta(T^{\varepsilon,\delta}) - \beta(k))}$  in (2.34) and (2.36), we have from (2.32) that

$$\overline{(T^{\varepsilon,\delta} - k) \int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau} - \overline{(\beta(T^{\varepsilon,\delta}) - \beta(k))^2} = \overline{(T^{\varepsilon,\delta} - k)} \cdot \overline{\int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau} - \overline{(\beta(T^{\varepsilon,\delta}) - \beta(k))^2}, \tag{2.37}$$

which is same to the weak limit equation we obtained for the following scalar conservation law with the viscosity (see Theorem 3.1.1 in [22] for the details)

$$T_t + \beta(T)_x = \varepsilon T_{xx}. \tag{2.38}$$

Therefore, If meas  $\{T : \beta''(T) = 0\} = 0$ , then there exists a subsequence (still labelled  $T^{\varepsilon,\delta}(x, t)$ ) such that  $T^{\varepsilon,\delta}(x, t) \rightarrow T(x, t)$ .

If  $\beta(T) = bT$ , we apply for the div-curl lemma to the following pairs of functions

$$(c, T^{\varepsilon,\delta}), \quad ((S^{\varepsilon,\delta}T^{\varepsilon,\delta} + bT^{\varepsilon,\delta}), T^{\varepsilon,\delta}f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \alpha(T^{\varepsilon,\delta})) \tag{2.39}$$

to obtain

$$\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta})^2} + \overline{b(T^{\varepsilon,\delta})^2} = \overline{(S^{\varepsilon,\delta}T^{\varepsilon,\delta} + bT^{\varepsilon,\delta})} \cdot \overline{T^{\varepsilon,\delta}} = \overline{ST^2} + \overline{bT^2}. \tag{2.40}$$

Letting  $h(T) = T^2$  in (2.32), we know that  $\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta})^2} = \overline{S(T^{\varepsilon,\delta})^2}$  and so from (2.40) that

$$(S + b)\overline{(T^{\varepsilon,\delta} - T)^2} = 0. \tag{2.41}$$

Since  $S \leq 1$ , if  $b < -1$ , then there exists a subsequence (still labelled  $T^{\varepsilon,\delta}(x, t)$ ) such that  $T^{\varepsilon,\delta}(x, t) \rightarrow T(x, t)$  a.e. on any bounded and open set  $\Omega \subset R \times R^+$ , and if  $b = -1$ ,  $T^{\varepsilon,\delta}(x, t) \rightarrow T(x, t)$  almost everywhere on the set  $S_+ = \{(x, t) : 0 \leq S(x, t) < 1\}$ . Lemma 6 is proved.

**Lemma 7.** *If meas  $\{S : f_{SS}(S, T) = 0\} = 0$  for any fixed  $T$ , then there exists a subsequence (still labelled  $S^\varepsilon(x, t)$ ) such that*

$$S^{\varepsilon,\delta}(x, t) \rightarrow S(x, t) \tag{2.42}$$

a.e. on any bounded and open set  $\Omega \subset R \times R^+$ .

**Proof.** (I) First, we suppose  $\beta''(T)f_T(S, T) \geq 0$  (or  $\leq 0$ ). Then it is easy to prove that

$$f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})_t + \left( \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T^{\varepsilon,\delta}) d\tau \right)_x \text{ are compact in } H_{loc}^{-1}(R \times R^+). \tag{2.43}$$

In fact, multiplying (2.7) by  $f_S$ , (2.6) by  $f_T$  respectively, then adding the result, we have (for simplicity, we omit the superscript  $\varepsilon, \delta$ )

$$\begin{aligned} f_t + \left( \int_1^S f_S^2(\tau, T) d\tau \right)_x &= \varepsilon f_S S_{xx} + \varepsilon f_T T_{xx} + \frac{2\varepsilon f_T}{S + \beta'(T) - \delta} S_x T_x + \varepsilon \frac{\beta''(T) f_T}{S + \beta'(T) - \delta} T_x^2 \\ &+ \int_1^S 2f_S(\tau, T) f_{ST}(\tau, T) d\tau T_x - f_S f_T T_x - f_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \\ &= \varepsilon (f_S S_x + f_T T_x)_x - \varepsilon (f_{SS} S_x^2 + 2f_{ST} S_x T_x + f_{TT} T_x^2) + \frac{2\varepsilon f_T}{S + \beta'(T) - \delta} S_x T_x \\ &+ \varepsilon \frac{\beta''(T) f_T}{S + \beta'(T) - \delta} T_x^2 + \int_1^S 2f_S(\tau, T) f_{ST}(\tau, T) d\tau T_x - f_S f_T T_x - f_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x. \end{aligned} \tag{2.44}$$

Since  $|T_x^{\varepsilon,\delta}|, \varepsilon(S_x^{\varepsilon,\delta})^2$  and  $\varepsilon(\delta - \beta'(T) - S)(T_x^{\varepsilon,\delta})^2$  are bounded in  $L^1_{loc}(R \times R^+)$ , the following terms on the right-hand side of (2.44)

$$\left| -\varepsilon f_{SS} S_x^2 + \int_1^S 2f_S(\tau, T) f_{ST}(\tau, T) d\tau T_x - f_S f_T T_x \right| \leq \varepsilon M S_x^2 + M |T_x| \tag{2.45}$$

are bounded in  $L^1_{loc}(R \times R^+)$ . Since  $f(1, T) = 1$ , then  $f_T(1, T) = f_{TT}(1, T) = 0$  and

$$\begin{aligned} & \left| -\varepsilon f_{TT} T_x^2 - f_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \right| \\ &= \left| \varepsilon (f_{TT}(S, T) - f_{TT}(1, T)) T_x^2 + (f_T(S, T) - f_T(1, T)) \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \right| \\ &= \varepsilon \left| f_{TTS}(\theta_1, T)(S - 1) T_x^2 + f_{TS}(\theta_2, T)(S - 1) \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \right| \\ &\leq \varepsilon M (\delta - \beta'(T) - S)(T_x^{\varepsilon,\delta})^2 + M |T_x| \end{aligned} \tag{2.46}$$

are bounded in  $L^1_{loc}(R \times R^+)$ , where  $\theta_i \in (S, 1), i = 1, 2$ .

If we choose  $g(S) = (1 + \delta - S)^l, l \in (0, 1)$  in (2.23), since the following term in (2.23)

$$|g'(S) f_T(S, T)| = |g'(S)(f_T(S, T) - f_T(1, T))| \leq \alpha(1 + \delta - S)^l |f_{TS}(\theta_3, T)| \tag{2.47}$$

is uniformly bounded, we can prove from (2.23) that

$$\varepsilon(1 + \delta - S)^{l-2} (S_x^{\varepsilon,\delta})^2 \text{ are bounded in } L^1_{loc}(R \times R^+), \tag{2.48}$$

and so the terms on the right-hand side of (2.44)

$$\begin{aligned} & \left| -2\varepsilon f_{ST} S_x T_x + \frac{2\varepsilon f_T}{S + \beta'(T) - \delta} S_x T_x \right| = 2\varepsilon \left| -f_{ST} + \frac{f_T(S, T) - f_T(1, T)}{S + \beta'(T) - \delta} \right| |S_x T_x| \\ &\leq \varepsilon M |S_x T_x| \leq \varepsilon M (1 + \delta - S)^{-1} (S_x^{\varepsilon,\delta})^2 + \varepsilon M (1 + \delta - S) (T_x^{\varepsilon,\delta})^2 \end{aligned} \tag{2.49}$$

are bounded in  $L^1_{loc}(R \times R^+)$ .

Since  $\beta''(T) f_T(S, T) \geq 0$  (or  $\leq 0$ ), we may multiply a suitable nonnegative test function to (2.44), to obtain that

$$\varepsilon \left| \frac{\beta''(T) f_T}{S + \beta'(T) - \delta} \right| T_x^2 \text{ are bounded in } L^1_{loc}(R \times R^+). \tag{2.50}$$

Similarly, we can prove that the left terms on the right-hand side of (2.44)

$$\varepsilon (f_S S_x + f_T T_x)_x \text{ are compact in } H^{-1}_{loc}(R \times R^+), \tag{2.51}$$

so, the right-hand side of (2.44) is compact in  $W^{-1,\alpha}_{loc}(R \times R^+), \alpha \in (1, 2)$ . It is clear that the left-hand side of (2.44) is bounded in  $W^{-1,\infty}_{loc}(R \times R^+)$ , so Murat's theorem gives us again the conclusion in (2.43).

Second, we apply for the div-curl lemma to the pairs of functions

$$(S^{\varepsilon,\delta}, f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})) \quad \text{and} \quad \left( f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}), \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T) d\tau \right) \tag{2.52}$$

to obtain

$$\overline{S^{\varepsilon,\delta} \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T^{\varepsilon,\delta}) d\tau} - \overline{f^2(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})} = \overline{S^{\varepsilon,\delta}} \cdot \overline{\int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T^{\varepsilon,\delta}) d\tau} - \overline{(f(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}))^2}. \tag{2.53}$$



Using the convergence given in Lemma 6, we have

$$\begin{aligned} & \left| S^{\varepsilon,\delta} \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T^{\varepsilon,\delta}) d\tau - S^{\varepsilon,\delta} \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T) d\tau \right| \\ &= \left| S^{\varepsilon,\delta} \int_1^{S^{\varepsilon,\delta}} 2f_S(\tau, \theta) f_{ST}(\tau, \theta) (T^{\varepsilon,\delta} - T) d\tau \right| \leq M |(S^{\varepsilon,\delta} - 1)(T^{\varepsilon,\delta} - T)| \rightarrow 0, \end{aligned} \tag{2.54}$$

where  $M$  is a constant independent of  $\varepsilon, \delta$  and  $\theta \in (T, T^{\varepsilon,\delta})$ ; and

$$\begin{aligned} & |f^2(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) - f^2(S^{\varepsilon,\delta}, T)| = 2|f(S^{\varepsilon,\delta}, \theta) f_T(S^{\varepsilon,\delta}, \theta) (T^{\varepsilon,\delta} - T)| \\ &= |f(S^{\varepsilon,\delta}, \theta) (f_T(S^{\varepsilon,\delta}, \theta) - f_T(1, \theta)) (T^{\varepsilon,\delta} - T)| \leq M |(S^{\varepsilon,\delta} - 1)(T^{\varepsilon,\delta} - T)| \rightarrow 0 \end{aligned} \tag{2.55}$$

since  $f_T(1, \theta) = 0$ , thus we may replace  $T^{\varepsilon,\delta}(x, t)$  in (2.53) by  $T(x, t)$  and obtain

$$\overline{S^{\varepsilon,\delta} \int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T) d\tau - f^2(S^{\varepsilon,\delta}, T)} = \overline{S^{\varepsilon,\delta}} \cdot \overline{\int_1^{S^{\varepsilon,\delta}} f_S^2(\tau, T) d\tau - (f(S^{\varepsilon,\delta}, T))^2}. \tag{2.56}$$

Therefore under the condition  $\beta''(T)f_T(S, T) \geq 0$  (or  $\leq 0$ ), the proof of Lemma 7 can be completed by using the previous results on scalar hyperbolic equation if we consider  $T$  in (2.56) as a parameter [26,22].

(II) Second, we prove Lemma 7 without the assumption  $\beta''(T)f_T(S, T) \geq 0$  (or  $\leq 0$ ). Let

$$v = g(S), \quad F(v, T) = \int_1^S g'(\tau) f_S(\tau, T) d\tau. \tag{2.57}$$

Then we have from (2.23) that

$$v_t + F(v, T)_x = \varepsilon g(S)_{xx} - \varepsilon g''(S) S_x^2 - \left( g'(S) f_T(S, T) - \int_1^S g'(\tau) f_{ST}(\tau, T) d\tau \right) T_x. \tag{2.58}$$

Using the estimates given in (2.17) and (2.18), for any smooth function  $g(S)$ , we have that

$$v_t^{\varepsilon,\delta} + F(v^{\varepsilon,\delta}, T^{\varepsilon,\delta})_x \text{ are compact in } H_{loc}^{-1}(R \times R^+). \tag{2.59}$$

Multiplying (2.58) by  $F_v$ , (2.6) by  $F_T$  respectively, then adding the result, we have (for simplicity, we omit the superscript  $\varepsilon, \delta$ )

$$\begin{aligned} & F(v, T)_t + \left( \int_1^v F_v^2(\tau, T) d\tau \right)_x = \varepsilon F_v v_{xx} + \varepsilon F_T T_{xx} + \frac{2\varepsilon F_T}{S + \beta'(T) - \delta} S_x T_x + \varepsilon \frac{\beta''(T) F_T}{S + \beta'(T) - \delta} T_x^2 \\ &+ \int_1^v 2F_v(\tau, T) F_{vT}(\tau, T) d\tau T_x - F_v F_T T_x - F_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \\ &- \varepsilon F_v g''(S) S_x^2 - F_v \left( g'(S) f_T(S, T) - \int_0^S g'(\tau) f_{ST}(\tau, T) d\tau \right) T_x \\ &= \varepsilon (F_v v_x + F_T T_x)_x - \varepsilon (F_{vv} v_x^2 + 2F_{vT} v_x T_x + F_{TT} T_x^2) + \frac{2\varepsilon F_T}{S + \beta'(T) - \delta} S_x T_x \\ &+ \varepsilon \frac{\beta''(T) F_T}{S + \beta'(T) - \delta} T_x^2 + \int_1^v 2F_v(\tau, T) F_{vT}(\tau, T) d\tau T_x - F_v F_T T_x - F_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x \\ &- \varepsilon F_v g''(S) S_x^2 - F_v \left( g'(S) f_T(S, T) - \int_0^S g'(\tau) f_{ST}(\tau, T) d\tau \right) T_x. \end{aligned} \tag{2.60}$$

Now we choose  $g(S) = S - 1$ . By simple calculations

$$F_v = F_S \frac{dS}{dv} = f_S(S, T), \quad F_{vv} = \frac{f_{SS}(S, T)}{g'(S)} \tag{2.61}$$

and

$$|F_T| = \left| \int_1^S g'(\tau) f_{ST}(\tau, T) d\tau \right| \leq M(S-1)^2. \quad (2.62)$$

Thus the following terms on the right-hand side of (2.60)

$$\begin{aligned} & \frac{2\varepsilon F_T}{S + \beta'(T) - \delta} S_x T_x + \varepsilon \frac{\beta''(T) F_T}{S + \beta'(T) - \delta} T_x^2 + \int_1^v 2F_v(\tau, T) F_{vT}(\tau, T) d\tau T_x - F_v F_T T_x \\ & - F_T \frac{f + \alpha'(T)}{S + \beta'(T) - \delta} T_x - \varepsilon F_v g''(S) S_x^2 - F_v \left( g'(S) f_T(S, T) - \int_0^S g'(\tau) f_{ST}(\tau, T) d\tau \right) T_x \end{aligned} \quad (2.63)$$

are uniformly bounded in  $L^1_{loc}(R \times R^+)$  and so compact in  $W_{loc}^{-1, \alpha}(R \times R^+)$ , where  $\alpha \in (1, 2)$ , by Sobolev's embedding theorem.

Similarly

$$\varepsilon(F_{vv} v_x^2 + 2F_{vT} v_x T_x + F_{TT} T_x^2) = \varepsilon \left( f_{SS} g'(S) S_x^2 + 2f_{ST} g'(S) S_x T_x + \int_1^S g'(\tau) f_{STT}(\tau, T) d\tau T_x^2 \right) \quad (2.64)$$

are uniformly bounded in  $L^1_{loc}(R \times R^+)$  and so compact in  $W_{loc}^{-1, \alpha}(R \times R^+)$ .

Using the estimates given in (2.17) and (2.18) again, we have that the terms on the right-hand side of (2.60)

$$\varepsilon(F_v v_x + F_T T_x)_x = \varepsilon \left( f_S g'(S) S_x + \int_1^S g'(\tau) f_{ST}(\tau, T) d\tau T_x \right) \quad (2.65)$$

are compact in  $H_{loc}^{-1}(R \times R^+)$ .

Since the left-hand side of (2.60)

$$F(v^{\varepsilon, \delta}, T^{\varepsilon, \delta})_t + \left( \int_1^{v^{\varepsilon, \delta}} F_v^2(\tau, T^{\varepsilon, \delta}) d\tau \right)_x \quad (2.66)$$

are uniformly bounded in  $W_{loc}^{-1, \infty}(R \times R^+)$ , then the Murat compact embedding theorem [24] shows that they are also compact in  $H_{loc}^{-1}(R \times R^+)$ .

So, if we replace  $(S^{\varepsilon, \delta}, f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}))$  in (2.56) by  $(v^{\varepsilon, \delta}, F(v^{\varepsilon, \delta}, T^{\varepsilon, \delta}))$ , we can also prove the pointwise convergence of  $v^{\varepsilon, \delta}$  or  $S^{\varepsilon, \delta}$  given in (2.42). Thus Lemma 7 is proved.

Now we are going to prove Theorem 1.

**Proof of Theorem 1.** When  $\text{meas} \{T : \beta''(T) = 0\} = 0$  or  $\beta'(T) = bT, b < -1$ , since the pointwise convergence of  $S^{\varepsilon, \delta}$  and  $T^{\varepsilon, \delta}$  obtained in Lemmas 6 and 7, we may prove that  $(S, T)$  satisfies (1.1) in the sense of distributions by letting  $\varepsilon, \delta$  in (2.1) go to zero directly. Similarly, we can prove the inequality  $\eta(S, T)_t + q(S, T)_x \leq 0$ , in the sense of distributions, for any smooth, convex, entropy  $\eta(S, C)$  and the corresponding entropy flux  $q(S, C)$ . So, the existence (1) in Theorem 1 is proved.

Let  $\alpha(T) = \beta(T) = -1$ . First, we have

$$\begin{aligned} |f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}) - f(S, T)| & \leq |f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}) - f(S, T^{\varepsilon, \delta})| + |f(S, T^{\varepsilon, \delta}) - f(S, T)| \\ & = |f_S(\theta_1, T^{\varepsilon, \delta})(S^{\varepsilon, \delta} - S)| + |f_T(S, \theta_2)(T^{\varepsilon, \delta} - T)| \\ & = |f_S(\theta_1, T^{\varepsilon, \delta})(S^{\varepsilon, \delta} - S)| + |(f_T(S, \theta_2) - f_T(1, \theta_2))(T^{\varepsilon, \delta} - T)| \\ & = |f_S(\theta_1, T^{\varepsilon, \delta})(S^{\varepsilon, \delta} - S)| + |f_{TS}(\theta_3, \theta_2)(S^{\varepsilon, \delta} - 1)(T^{\varepsilon, \delta} - T)| \rightarrow 0, \end{aligned} \quad (2.67)$$

in the sense of distributions, since the convergence obtained in (2.30) and (2.42).

Similarly, letting  $F(S, T) = T(f(S, T) - 1)$ , we have  $F(1, T) = 0$  and

$$|T^{\varepsilon, \delta}(f(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}) - 1) - T(f(S, T) - 1)| = |F(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}) - F(S, T)| \rightarrow 0, \tag{2.68}$$

in the sense of distributions, so the limit  $(S, T)$  satisfies system (1.1) by letting  $\varepsilon, \delta$  go to zero in (2.1) directly.

Second, let  $\eta(S, C) \in C^2$  be a convex, weak entropy of system (1.1) with the corresponding weak entropy flux  $q(S, C)$ , where  $C = ST - T$ . We multiply (2.1) by  $(\eta_S, \eta_C)$  to obtain (for simplicity, we omit the superscripts  $\varepsilon$  and  $\delta$ )

$$\begin{aligned} \eta(S, C)_t + q(S, C)_x &= \varepsilon \eta(S, C)_{xx} - \varepsilon(\eta_{SS}S_x^2 + 2\eta_{SC}S_xC_x + \eta_{CC}C_x^2) + \eta_C(S, C)(\delta T_t - \varepsilon \delta T_{xx}) \\ &\leq \varepsilon \eta(S, C)_{xx} + \delta \eta_C(S, C) \left( 2\varepsilon \frac{S_x}{S - (1 + \delta)} T_x - \frac{f - 1}{S - (1 + \delta)} T_x \right). \end{aligned} \tag{2.69}$$

Since  $\eta(1, C) = c_1, q(1, C) = c_2$ , we can prove that

$$(\eta(S^{\varepsilon, \delta}, T^{\varepsilon, \delta}), q(S^{\varepsilon, \delta}, T^{\varepsilon, \delta})) \rightarrow (\eta(S, C), q(S, C)), \tag{2.70}$$

in the sense of distributions. Moreover,

$$\delta \left| \eta_C(S, C) \frac{f - 1}{S - (1 + \delta)} T_x \right| = \delta \left| \eta_C(S, C) \frac{f(S, T) - f(1, T)}{S - (1 + \delta)} T_x \right| \leq \delta M |T_x| \rightarrow 0, \tag{2.71}$$

and

$$\begin{aligned} 2\varepsilon \delta \left| \eta_C(S, C) \frac{S_x}{S - (1 + \delta)} T_x \right| &= 2\varepsilon \delta \left| (\eta_C(S, C) - \eta_C(1, C)) \frac{S_x}{S - (1 + \delta)} T_x \right| \\ &\leq \varepsilon \delta M |S_x T_x| \rightarrow 0, \end{aligned} \tag{2.72}$$

in the sense of distributions. Therefore, letting  $\varepsilon, \delta$  go to zero in (2.69), we have that  $\eta(S, C)_t + q(S, C)_x \leq 0$  in the sense of distributions, and so complete the proof of Theorem 1.

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