Global L^∞ Solutions to System of Isentropic Gas Dynamics in General Nozzle

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Abstract

In this paper, we study the global L^∞ entropy solutions for the Cauchy problem of the isentropic gas dynamics system in a general nozzle with bounded initial date. First we apply for the flux-approximation technique coupled with the classical viscosity method to obtain the L^∞ estimates of the viscosity solutions. Second, we prove the pointwise convergence of the approximation solutions by using the compactness framework and extend the work given in [CHY] for any adiabatic exponent $\gamma > 1$.

Key Words: Global L^∞ solution; isentropic gas flow; general nozzle; flux approximation; compensated compactness

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1 Introduction

We consider the following system of isentropic gas dynamics in a general nozzle

$$\begin{cases} \varrho_t + (\varrho v)_x = -\frac{b'(x)}{b(x)}\varrho v \\ \\ (\varrho v)_t + (\varrho v^2 + P(\varrho))_x = -\frac{b'(x)}{b(x)}\varrho v^2 \end{cases} \tag{1.1}$$

with bounded initial data

$$(\varrho(x,0), v(x,0)) = (\varrho_0(x), v_0(x)), \quad \varrho_0(x) \geq 0, \tag{1.2}$$

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where ϱ is the density of gas, v the velocity, $P = P(\varrho)$ the pressure, $b(x)$ is a slowly variable cross section area at x in the nozzle. For the polytropic gas, P takes the special form $P(\varrho) = \frac{1}{\gamma}\varrho^{\gamma}$, where $\gamma > 1$ is the adiabatic exponent. The nozzle is widely used in some types of steam turbines, rocket engine nozzles, supersonic jet engines and jet streams in astrophysics.

It is well-known that after we have a method to obtain the global existence of solutions for the Cauchy problem of the following homogeneous system

$$\begin{cases} \varrho_t + (\varrho v)_x = 0 \\ (\varrho v)_t + (\varrho v^2 + P(\varrho))_x = 0 \end{cases} \tag{1.3}$$

with the bounded initial data (1.2), the unique difficulty to study the inhomogeneous system (1.1) is to obtain the a-priori L^∞ estimate of the approximation solutions of (1.1), for instance, the a-priori L^∞ estimate of the classical viscosity solutions for the Cauchy problem of the parabolic system

$$\begin{cases} \varrho_t + (\varrho v)_x = -\frac{b'(x)}{b(x)}\varrho v + \varepsilon \varrho_{xx} \\ (\varrho v)_t + (\varrho v^2 + P(\varrho))_x = -\frac{b'(x)}{b(x)}\varrho v^2 + \varepsilon(\varrho v)_{xx} \end{cases} \tag{1.4}$$

with the initial data (1.2).

However, to study the Cauchy problem (1.4) and (1.2), a basic technical difficulty is to obtain the positive, lower estimate of ϱ^ε since system (1.4) is singular when $\varrho = 0$. To overcome this difficulty, we constructed a sequence of the regular hyperbolic systems in [Lu1]

$$\begin{cases} \varrho_t + (-2\kappa v + \varrho v)_x = B(x)(\varrho - 2\kappa)v \\ (\varrho v)_t + (\varrho v^2 - \kappa v^2 + P_1(\varrho, \kappa))_x = B(x)(\varrho - 2\kappa)v^2 \end{cases} \tag{1.5}$$

to approximate system (1.1), where $B(x) = -\frac{b'(x)}{b(x)}$, $\kappa > 0$ denotes a regular perturbation constant and the perturbation pressure

$$P_1(\varrho, \kappa) = \int_{2\kappa}^{\varrho} \frac{t - 2\kappa}{t} P'(t)dt. \tag{1.6}$$

As proved in [Lu2], both systems (1.1) and (1.5) have the same Riemann invariants and the entropy equation. With the help of these special behaviors of system (1.5), for any weak entropy-entropy flux pair $(\eta(\varrho, m), q(\varrho, m))$ of system (1.1) and for a general pressure function $P(\varrho)$, we can easily prove that

$$\eta(\varrho^{\varepsilon,\kappa}, m^{\varepsilon,\kappa})_t + q(\varrho^{\varepsilon,\kappa}, m^{\varepsilon,\kappa})_x \quad \text{are compact in} \quad H^{-1}_{loc}(R \times R^+),$$

with respect to the viscosity solutions $(\varrho^{\varepsilon,\kappa}, m^{\varepsilon,\kappa})$, and do not need to introduce the viscous periodic solutions with respect to the spatial variable x to derive the auxiliary estimate (see $(I.53)$ in [LPS]),

$$\int\int_{K_1} \varepsilon^2 (\varrho_x)^2 dxdt \le C\kappa^2$$

for the special pressure $P(\varrho) = \frac{1}{\gamma}\varrho^\gamma$ and $\gamma > 2$.

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = v - \sqrt{P'(\varrho)}, \quad \lambda_2 = v + \sqrt{P'(\varrho)} \tag{1.7}$$

with corresponding Riemann invariants

$$z(\varrho, v) = \int_l^\varrho \frac{\sqrt{P'(s)}}{s} ds - v, \quad w(\varrho, v) = \int_l^\varrho \frac{\sqrt{P'(s)}}{s} ds + v, \tag{1.8}$$

where l is a constant. Two eigenvalues of system (1.5) are

$$\lambda_1^\kappa = \frac{m}{\varrho} - \frac{\varrho - 2\kappa}{\varrho}\sqrt{P'(\varrho)}, \quad \lambda_2^\kappa = \frac{m}{\varrho} + \frac{\varrho - 2\kappa}{\varrho}\sqrt{P'(\varrho)} \tag{1.9}$$

with corresponding two same Riemann invariants (1.8) (both systems (1.1) and (1.5) have the same Riemann invariants as well as the entropy equations [Lu2]).

Then, we added the viscosity terms to the right-hand side of (1.5) and considered the Cauchy problem of the parabolic system

$$\begin{cases} \varrho_t + (-2\kappa v + \varrho v)_x = B(x)(\varrho - 2\kappa)v + \varepsilon \varrho_{xx} \\ (\varrho v)_t + (\varrho v^2 - \kappa v^2 + P_1(\varrho, \kappa))_x = B(x)(\varrho - 2\kappa)v^2 + \varepsilon(\varrho v)_{xx} \end{cases} \tag{1.10}$$

with initial data

$$(\varrho^{\kappa,\varepsilon}(x, 0), v^{\kappa,\varepsilon}(x, 0)) = (\varrho_0(x) + 2\kappa, v_0(x)), \tag{1.11}$$

where $(\varrho_0(x), v_0(x))$ are given in (1.2).

To use the first equation in (1.10), we deduce directly the positive lower bound $\varrho^{\varepsilon,\kappa} \ge 2\kappa > 0$ by the theory of invariant regions [CCS].

Finally, for the nozzle flow with the monotone cross section, which is corresponding to $b'(x) \geq 0$, and for the general pressure function $P(\varrho)$, we made the transformation $z = \varpi + D(x)$, where $D(x)$ is a bounded function to be carefully chosen to control the nonlinear function $B(x)$ so that we might obtain the following inequality on the variable v

$$\varpi_t + b(x,t)\varpi_x + d(x,t)\varpi \leq \varepsilon \varpi_{xx}, \tag{1.12}$$

which gave us the estimate $\varpi \leq 0$ and so the upper estimate $z(\varrho^{\varepsilon,\kappa}, v^{\varepsilon,\kappa}) \leq D(x)$ when the maximum principle was applied to (1.12) (cf. [Lu1] for the details).

Later, instead of the viscosity method, the author introduced a modified Godunov scheme to construct the approximate solutions of (1.1), and obtained the global existence of weak solutions of the Cauchy problem (1.1)-(1.2) for the Laval nozzle, which is corresponding to $b'(x) \cdot x \geq 0$, in [Ts1] and the general nozzle in [Ts2] for the usual gases $1 < \gamma \leq \frac{5}{3}$ under the smallness assumption on $|b(x)|_{L^1(R)}$.

In [CHY], the authors introduced the following approximate system, which is quitely different from the viscosity method introduced in [Lu1],

$$\begin{cases} \varrho_t + (\varrho v)_x = -\frac{b'(x)}{b(x)}\varrho v + \varepsilon \varrho_{xx} \\ \\ (\varrho v)_t + (\varrho v^2 + P(\varrho))_x = -\frac{b'(x)}{b(x)}\varrho v^2 + \varepsilon(\varrho v)_{xx} - 2\varepsilon d(x)\varrho_x, \end{cases} \tag{1.13}$$

to study the general nozzle for more general gases $1 \leq \gamma \leq 3$.

When $\gamma \geq 3$, the technique introduced in [CHY] to obtain the a-priori L^∞ estimates of viscosity solutions does not work because the necessary conditions $b_{12} \leq 0$ and $b_{21} \leq 0$, to guarantee the maximum principle (cf. Lemma 3.1 in [CHY]), are not true.

In [Lu3], the following system of isentropic gas dynamics in the Laval nozzle with the friction (cf. [S])

$$\begin{cases} \varrho_t + (\varrho v)_x = -\frac{b'(x)}{b(x)}\varrho v \\ \\ (\varrho v)_t + (\varrho v^2 + P(\varrho))_x = -\frac{b'(x)}{b(x)}\varrho v^2 - \alpha(x)\varrho v|v| \end{cases}$$

was studied for the polytropic gas $P(\varrho) = \frac{1}{\gamma}\varrho^\gamma$ and γ is limited in $(3,\infty)$ for a technical difficulty; and the initial-boundary value problem of the compressible

Euler equations with friction and heating

$$\begin{cases} (b(x)\varrho)_t + (b(x)\varrho v)_x = 0, \\ (b(x)\varrho v)_t + (b(x)\varrho v^2 + b(x)P)_x = b'(x)P - \alpha\sqrt{b(x)}\varrho v|v|, \\ (b(x)E)_t + (b(x)v(E+P))_x = \beta b(x)q(x) - \alpha\sqrt{b(x)}\varrho v^2|v|, \end{cases}$$

was studied in [CHHQ] , under suitable conditions among the initial data, $b(x)$ and $\alpha(x)$, by using a new version of a generalized Glimm scheme, where ϱ, v, E are, respectively, the density, velocity, total energy and pressure of the gas, α is the coefficient of friction, $q(x)$ is a given function representing the heating effect from the force outside the nozzle.

It is worthwhile to point out that, for a general inhomogeneous system of hyperbolic conservation laws, the Riemann problem was resolved by Isaacson and Temple in [IT]. More results on inhomogeneous hyperbolic systems can be found in [CG, EGM, GL, GMP] and the references cited therein.

In this paper, we apply our method introduced in [Lu1] to give a simple proof of the global existence of the entropy solutions for general nozzle and for any adiabatic exponent $\gamma > 1$.

Mainly, we have the following theorems.

Theorem 1 *Let $P(\varrho) = \frac{1}{\gamma}\varrho^\gamma, \gamma \geq 3$. If there exist a positive constant M and a nonnegative function $\beta(x)$ such that*

$$\theta M|B(x)| < \beta(x), \quad \int_{-\infty}^{\infty}\beta(s)ds < \frac{M}{2}, \tag{1.14}$$

then we have

$$z(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) = \frac{(\varrho^{\kappa,\varepsilon}(x,t))^\theta}{\theta} - v^{\kappa,\varepsilon}(x,t) \leq M - \int_{-\infty}^{x}\beta(s)ds \tag{1.15}$$

and

$$w(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) = \frac{(\varrho^{\kappa,\varepsilon}(x,t))^\theta}{\theta} + v^{\kappa,\varepsilon}(x,t) \leq M + \int_{-\infty}^{x}\beta(s)ds \tag{1.16}$$

if the initial data

$$z(\varrho^{\kappa,\varepsilon}(x,0), v^{\kappa,\varepsilon}(x,0)) < M - \int_{-\infty}^{x}\beta(s)ds \tag{1.17}$$

and

$$w(\varrho^{\kappa,\varepsilon}(x,0), v^{\kappa,\varepsilon}(x,0)) < M + \int_{-\infty}^{x} \beta(s)ds, \tag{1.18}$$

where $\theta = \frac{\gamma-1}{2}$ *and* $(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t))$ *are the solutions of the Cauchy problem (1.10) and (1.11).*

<u>Theorem</u> 2 *Let* $P(\varrho) = \frac{1}{\gamma}\varrho^{\gamma}, 1 < \gamma < 3$. *If there exist a positive constant* M *and a nonnegative function* $\beta(x)$ *such that*

$$\frac{(\gamma-1)(\gamma+3)}{4(3-\gamma)} M|B(x)| < \beta(x), \quad \int_{-\infty}^{\infty} \beta(s)ds < \frac{(\gamma-1)M}{4}, \tag{1.19}$$

then we have the same estimates given in (1.15) and (1.16), if the initial data satisfy (1.17) and (1.18).

<u>Theorem</u> 3 *For such functions* $B(x)$ *and the initial data satisfying the conditions in Theorems 1-2, there exists a subsequence of* $(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t))$, *which converges pointwisely to a pair of bounded functions* $(\varrho(x,t), v(x,t))$ *as* κ, ε *tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2).*

Definition 1 *A pair of bounded functions* $(\varrho(x,t), u(x,t))$ *is called a weak entropy solution of the Cauchy problem (1.1)-(1.2) if*

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \varrho\phi_t + (\varrho v)\phi_x - \frac{b'(x)}{b(x)}(\varrho v)\phi dxdt + \int_{-\infty}^\infty \varrho_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \varrho v\phi_t + (\varrho v^2 + P(\varrho))\phi_x - \frac{b'(x)}{b(x)}\varrho v^2\phi dxdt \\ + \int_{-\infty}^\infty \varrho_0(x)v_0(x)\phi(x,0)dx = 0 \end{cases} \tag{1.20}$$

holds for all test function $\phi \in C_0^1(R \times R^+)$ *and*

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \eta(\varrho,m)\phi_t + q(\varrho,m)\phi_x - \frac{b'(x)}{b(x)}\eta(\varrho,m)_\varrho \varrho u \\ &-\frac{b'(x)}{b(x)}\varrho v^2 \eta(\varrho,m)_m \phi dxdt \geq 0 \end{aligned} \tag{1.21}$$

holds for any non-negative test function $\phi \in C_0^\infty(R\times R^+ - \{t=0\})$, *where* $m = \varrho v$ *and* (η, q) *is a pair of convex entropy-entropy flux of system (1.1).*

2 Proof of Theorems 1-3.

In this section, we shall prove Theorems 1-3.

Proof of Theorem 1. We multiply (1.10) by (z_ϱ, z_m) and (w_ϱ, w_m), respectively, where (z, w) are given in (1.8), to obtain

$$\begin{aligned}
&z_t + \lambda_1^\kappa z_x \\
&= \varepsilon z_{xx} + \tfrac{2\varepsilon}{\varrho}\varrho_x z_x - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2 \\
&+B(x)(\varrho - 2\kappa)v(\tfrac{m}{\varrho^2} + \tfrac{\sqrt{P'(\varrho)}}{\varrho}) - \tfrac{1}{\varrho}B(x)(\varrho - 2\kappa)u^2 \\
&= \varepsilon z_{xx} + \tfrac{2\varepsilon}{\varrho}\varrho_x z_x - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2 \\
&+B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&w_t + \lambda_2^\kappa w_x \\
&= \varepsilon w_{xx} + \tfrac{2\varepsilon}{\varrho}\varrho_x w_x - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2 \\
&+B(x)(\varrho - 2\kappa)v(-\tfrac{m}{\varrho^2} + \tfrac{\sqrt{P'(\varrho)}}{\varrho}) + \tfrac{1}{\varrho}B(x)(\varrho - 2\kappa)v^2 \\
&= \varepsilon w_{xx} + \tfrac{2\varepsilon}{\varrho}\varrho_x w_x - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2 \\
&+B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}.
\end{aligned} \tag{2.2}$$

Letting $z = D(x) + \varpi$ in (2.1), where $D(x) = M - \int_{-\infty}^x \beta(s)ds$, we have

$$\begin{aligned}
&\varpi_t + (v - \tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)})(\varpi_x + D'(x)) - B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\
&= \varepsilon\varpi_{xx} + \varepsilon D''(x) + \tfrac{2\varepsilon}{\varrho}\varrho_x\varpi_x + \tfrac{2\varepsilon}{\varrho}\varrho_x D'(x) - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2
\end{aligned} \tag{2.3}$$

or

$$\varpi_t + (v - \tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)})\varpi_x - D'(x)(D(x) + \varpi - \textstyle\int_l^\varrho \tfrac{\sqrt{P'(\varrho)}}{\varrho}d\varrho)$$

$$-D'(x)\tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)} - B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}$$

$$= \varepsilon\varpi_{xx} - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')[\varrho_x^2 - \tfrac{4\varrho\sqrt{P'(\varrho)}}{2P'+\varrho P''}\varrho_x D'(x) + (\tfrac{2\varrho\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x))^2]$$

$$+\varepsilon D''(x) + \tfrac{2\varepsilon}{\varrho}\varrho_x\varpi_x + \tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x)^2 \tag{2.4}$$

or

$$\varpi_t + b(x,t)\varpi_x + d(x,t)\varpi + [-\tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x)^2 - \varepsilon D''(x) - \varepsilon_1 D(x)D'(x)] \le \varepsilon\varpi_{xx}$$

$$-\textstyle\int_l^\varrho \tfrac{\sqrt{P'(\varrho)}}{\varrho}d\varrho D'(x) + (1-\varepsilon_1)D(x)D'(x) + D'(x)(\varrho - 2\kappa)\tfrac{\sqrt{P'(\varrho)}}{\varrho}$$

$$+B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}, \tag{2.5}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $b(x,t) = v - \frac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)} - \frac{2\varepsilon}{\varrho}\varrho_x$ and $d(x,t) = -D'(x)$.

Similarly, letting $w = C(x) + \varpi_1$ in (2.2), where $C(x) = M + \int_{-\infty}^x \beta(s)ds$, we have

$$\begin{aligned}&\varpi_{1t} + (v + \tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)})(\varpi_{1x} + C'(x)) - B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}\\ &= \varepsilon\varpi_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{\varrho}\varrho_x\varpi_{1x} + \tfrac{2\varepsilon}{\varrho}\varrho_x C'(x) - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')\varrho_x^2\end{aligned} \tag{2.6}$$

or

$$\varpi_{1t} + (v + \tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)})\varpi_{1x} + C'(x)(C(x) + \varpi_1 - \textstyle\int_l^\varrho \tfrac{\sqrt{P'(\varrho)}}{\varrho}d\varrho)$$

$$+C'(x)\tfrac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)} - B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}$$

$$= \varepsilon\varpi_{1xx} - \tfrac{\varepsilon}{2\varrho^2\sqrt{P'(\varrho)}}(2P' + \varrho P'')[\varrho_x^2 - \tfrac{4\varrho\sqrt{P'(\varrho)}}{2P'+\varrho P''}\varrho_x C'(x) + (\tfrac{2\varrho\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x))^2]$$

$$+\varepsilon C''(x) + \tfrac{2\varepsilon}{\varrho}\varrho_x\varpi_{1x} + \tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x)^2 \tag{2.7}$$

or

$$\varpi_{1t} + b_1(x,t)\varpi_{1x} + d_1(x,t)\varpi_1 + [-\tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x)] \leq \varepsilon\varpi_{1xx}$$

$$+\textstyle\int_l^\varrho \frac{\sqrt{P'(\varrho)}}{\varrho}d\varrho C'(x) - (1-\varepsilon_1)C(x)C'(x) - C'(x)(\varrho-2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}$$

$$+B(x)(\varrho-2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho}, \tag{2.8}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $b_1(x,t) = v + \frac{\varrho-2\kappa}{\varrho}\sqrt{P'(\varrho)} - \frac{2\varepsilon}{\varrho}\varrho_x$ and $d_1(x,t) = C'(x)$.

Using the first equation in (1.10), we have the a priori estimate $\varrho \geq 2\kappa$. We can choose $\beta(x)$ to be smooth enough, $\varepsilon = o(\kappa)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of (2.5) and (2.8)

$$-\frac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x)^2 - \varepsilon D''(x) - \varepsilon_1 D(x)D'(x) \geq 0 \tag{2.9}$$

and

$$-\frac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) \geq 0. \tag{2.10}$$

When $P(\varrho) = \frac{1}{\gamma}\varrho^\gamma, \gamma \geq 3$, we choose $l = 2\kappa$, then by using the following inequality

$$\frac{1}{\theta}(\varrho-2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} \leq \int_{2\kappa}^{\varrho}\frac{\sqrt{P'(s)}}{s}ds \leq (\varrho-2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} \quad \text{for} \quad \gamma \geq 3, \tag{2.11}$$

we have the following estimate on the terms of (2.5)

$$L = -\textstyle\int_l^\varrho \frac{\sqrt{P'(\varrho)}}{\varrho}d\varrho D'(x) + (1-\varepsilon_1)D(x)D'(x)$$

$$+D'(x)(\varrho-2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + B(x)(\varrho-2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho} \tag{2.12}$$

$$\leq (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho-2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho}.$$

Now we may analyze the function L point by point. First, at the points (x,t), where $B(x)\geq 0$,

$$
\begin{aligned}
&L\leq (1-\varepsilon_1)D(x)D'(x)+B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}\\
&=(1-\varepsilon_1)D(x)D'(x)+B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(w-z)\\
&=B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi_1-\varpi)+(1-\varepsilon_1)D(x)D'(x)\\
&+B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(C(x)-D(x))\\
&=-B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi-\varpi_1)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds\\
&\leq -B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi-\varpi_1)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\theta B(x)\textstyle\int_{-\infty}^{x}\beta(s)ds\int_{2\kappa}^{\varrho}\tfrac{\sqrt{P'(s)}}{s}ds\\
&=-B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi-\varpi_1)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{1}{2}\theta B(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(w+z)\\
&=-B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi-\varpi_1)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{1}{2}\theta B(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(\varpi_1+\varpi+C(x)+D(x))\\
&=(\tfrac{1}{2}\theta B(x)\textstyle\int_{-\infty}^{x}\beta(s)ds+B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho})v_1\\
&+(\tfrac{1}{2}\theta B(x)\textstyle\int_{-\infty}^{x}\beta(s)ds-B(x)(\varrho-2\kappa)\tfrac{\sqrt{P'(\varrho)}}{2\varrho})v\\
&-((1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)-\theta MB(x)\int_{-\infty}^{x}\beta(s)ds),
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
&(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)-\theta MB(x)\int_{-\infty}^{x}\beta(s)ds\\
&>\tfrac{M}{2}\beta(x)-\theta MB(x)|\beta(x)|_{L^1(R)}\geq 0
\end{aligned}
\tag{2.14}
$$

due to the conditions $|\beta(x)|_{L^1(R)}<\frac{M}{2}$ and $\theta M|B(x)|<\beta(x)$ in Theorem 1.

Therefore we obtain the following inequality from (2.5),(2.9),(2.13) and (2.14)

$$\varpi_t + b(x,t)\varpi_x + l_1(x,t)\varpi + l_2(x,t)\varpi_1 \leq \varepsilon\varpi_{xx}, \tag{2.15}$$

where $l_1(x,t), l_2(x,t) \leq 0$ are suitable functions.

Second, at the points (x,t), where $B(x) \leq 0$, we have

$$\begin{aligned}
&L \leq (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho - 2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho} \\
&= (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}(\textstyle\int_{2\kappa}^{\varrho} \frac{\sqrt{P'(s)}}{s}ds - z) \\
&\leq (1-\varepsilon_1)D(x)D'(x) - B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}(\varpi + D(x)) \\
&\leq (1-\varepsilon_1)D(x)D'(x) - B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi - \theta B(x)D(x)\textstyle\int_{2\kappa}^{\varrho} \frac{\sqrt{P'(s)}}{s}ds \\
&= (1-\varepsilon_1)D(x)D'(x) - B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi - \frac{1}{2}\theta B(x)D(x)(w+z) \\
&= (1-\varepsilon_1)D(x)D'(x) - B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi - \frac{1}{2}\theta B(x)D(x)(\varpi + \varpi_1 + 2M) \\
&= -(B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + \frac{1}{2}\theta B(x)D(x))\varpi \\
&-\frac{1}{2}\theta B(x)D(x)\varpi_1 - ((1-\varepsilon_1)\beta(x) + \theta M B(x))D(x) \\
&\leq -(B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + \frac{1}{2}\theta B(x)D(x))\varpi \\
&-\frac{1}{2}\theta B(x)D(x)\varpi_1,
\end{aligned} \tag{2.16}$$

where $-\frac{1}{2}\theta B(x)D(x) \geq 0$. Thus we also obtain an inequality

$$\varpi_t + b(x,t)\varpi_x + l_3(x,t)\varpi + l_4(x,t)\varpi_1 \leq \varepsilon\varpi_{xx}, \tag{2.17}$$

where $l_3(x,t), l_4(x,t) \leq 0$ are suitable functions.

Now we choose $l = 2\kappa$ and consider the following terms on the right-hand side of (2.8)

$$\begin{aligned}
L_1 = &\textstyle\int_{2\kappa}^{\varrho} \frac{\sqrt{P'(\varrho)}}{\varrho}d\varrho C'(x) - (1-\varepsilon_1)C(x)C'(x) \\
&-C'(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + B(x)(\varrho - 2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho}.
\end{aligned} \tag{2.18}$$

First, at the points (x,t), where $B(x) \leq 0$,

$$L_1 \leq B(x)(\varrho - 2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho} = B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{2\varrho}(w - z)$$

$$= B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi_1 - \varpi + C(x) - D(x)) \tag{2.19}$$

$$\leq B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{2\varrho}(\varpi_1 - \varpi),$$

where the coefficient before ϖ, $-B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{2\varrho} \geq 0$. So we have an inequality from (2.8),(2.10) and (2.19) that

$$\varpi_{1t} + b_1(x,t)\varpi_{1x} + l_5(x,t)\varpi + l_6(x,t)\varpi_1 \leq \varepsilon\varpi_{1xx}, \tag{2.20}$$

where $l_5(x,t) \leq 0, l_6(x,t)$ are suitable functions. Second, at the points (x,t), where $B(x) \geq 0$,

$$L_1 \leq -(1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho - 2\kappa)v\frac{\sqrt{P'(\varrho)}}{\varrho}$$

$$= -(1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}(w - \textstyle\int_{2\kappa}^{\varrho}\frac{\sqrt{P'(\varrho)}}{\varrho}d\varrho)$$

$$\leq -(1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}(v_1 + C(x))$$

$$\leq -(1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi_1 + \theta B(x)C(x)\textstyle\int_{2\kappa}^{\varrho}\frac{\sqrt{P'(\varrho)}}{\varrho}d\varrho$$

$$= -(1-\varepsilon_1)C(x)\beta(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi_1 + \tfrac{1}{2}\theta B(x)C(x)(w+z)$$

$$= -(1-\varepsilon_1)C(x)\beta(x) + B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho}\varpi_1$$

$$+\tfrac{1}{2}\theta B(x)C(x)(\varpi + \varpi_1 + C(x) + D(x))$$

$$= (B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + \tfrac{1}{2}\theta B(x)C(x))\varpi_1$$

$$+\tfrac{1}{2}\theta B(x)C(x)\varpi - ((1-\varepsilon_1)\beta(x) - \theta M B(x))C(x)$$

$$\leq (B(x)(\varrho - 2\kappa)\frac{\sqrt{P'(\varrho)}}{\varrho} + \tfrac{1}{2}\theta B(x)C(x))\varpi_1 + \tfrac{1}{2}\theta B(x)C(x)\varpi \tag{2.21}$$

due to $\theta M|B(x)| < \beta(x)$, where the coefficient $\frac{1}{2}\theta B(x)C(x) \geq 0$. So, we also have an inequality

$$\varpi_{1t} + b_1(x,t)\varpi_{1x} + l_7(x,t)\varpi + l_8(x,t)\varpi_1 \leq \varepsilon\varpi_{1xx}, \tag{2.22}$$

where $l_7(x,t) \leq 0, l_8(x,t)$ are suitable functions.

Summing up the analysis above, we have the following two inequalities on ϖ and ϖ_1

$$\begin{cases} \varpi_t + b(x,t)\varpi_x + d(x,t)\varpi + c(x,t)\varpi_1 \leq \varepsilon\varpi_{xx}, \\ \\ \varpi_{1t} + b_1(x,t)\varpi_{1x} + d_1(x,t)\varpi_1 + c_1(x,t)\varpi \leq \varepsilon\varpi_{1xx}, \end{cases} \tag{2.23}$$

where the coefficient functions $c(x,t) \leq 0, c_1(x,t) \leq 0$, so the maximum principle ([Lu5]) on nonlinear coupled parabolic equations gives us the estimates $\varpi(x,t) \leq 0, \varpi_1(x,t) \leq 0$ and the upper bounds of z and w. This completes the Proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2, when $P(\varrho) = \frac{1}{\gamma}\varrho^\gamma, 1 < \gamma < 3$, we let $l = 0$ and rewrite (2.5) and (2.8) as follows:

$$\begin{aligned} &\varpi_t + b(x,t)\varpi_x + d(x,t)\varpi \\ &+[-\tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x)^2 - \varepsilon D''(x) - \varepsilon_1 D(x)D'(x) + 2\kappa\tfrac{\sqrt{P'(\varrho)}}{\varrho}D'(x)] \\ &\leq \varepsilon\varpi_{xx} - \textstyle\int_0^\varrho \tfrac{\sqrt{P'(\varrho)}}{\varrho}d\varrho D'(x) + (1-\varepsilon_1)D(x)D'(x) \\ &+D'(x)\sqrt{P'(\varrho)} + B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\ &= \varepsilon\varpi_{xx} + \tfrac{\gamma-3}{\gamma-1}\varrho^\theta D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\varpi_{1t} + b_1(x,t)\varpi_{1x} + d_1(x,t)\varpi_1 \\ &+[-\tfrac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) - 2\kappa\tfrac{\sqrt{P'(\varrho)}}{\varrho}C'(x)] \\ &\leq \varepsilon\varpi_{1xx} + \textstyle\int_0^\varrho \tfrac{\sqrt{P'(\varrho)}}{\varrho}d\varrho C'(x) - (1-\varepsilon_1)C(x)C'(x) \\ &-C'(x)\sqrt{P'(\varrho)} + B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\ &= \varepsilon\varpi_{1xx} - \tfrac{\gamma-3}{\gamma-1}\varrho^\theta C'(x) - (1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho - 2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}. \end{aligned} \tag{2.25}$$

Since

$$2\kappa\frac{\sqrt{P'(\varrho)}}{\varrho} = 2\kappa\varrho^{\frac{\gamma-3}{2}} \leq (2\kappa)^{\frac{\gamma-1}{2}}, \tag{2.26}$$

we may choose $\beta(x)$ to be sufficiently smooth, $\varepsilon = o(\kappa)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of (2.24) and (2.25)

$$-\frac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}D'(x)^2 - \varepsilon D''(x) - \varepsilon_1 D(x)D'(x) + 2\kappa\frac{\sqrt{P'(\varrho)}}{\varrho}D'(x) \geq 0, \tag{2.27}$$

$$-\frac{2\varepsilon\sqrt{P'(\varrho)}}{2P'+\varrho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) - 2\kappa\frac{\sqrt{P'(\varrho)}}{\varrho}C'(x) \geq 0. \tag{2.28}$$

Furthermore, we consider the terms on the right-hand side of (2.24) and (2.25)

$$K = \tfrac{\gamma-3}{\gamma-1}\varrho^\theta D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \tag{2.29}$$

and

$$K_1 = -\tfrac{\gamma-3}{\gamma-1}\varrho^\theta C'(x) - (1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}. \tag{2.30}$$

First, at the points (x,t), where $B(x) \geq 0$, we have that

$$\begin{aligned}
K &= \tfrac{\gamma-3}{\gamma-1}\varrho^\theta D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\
&= \tfrac{\gamma-3}{4}(w+z)D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)\tfrac{\varrho-2\kappa}{2\varrho}(w-z)\varrho^\theta \\
&= \tfrac{3-\gamma}{4}(\varpi+\varpi_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) \\
&\quad + B(x)\tfrac{\varrho-2\kappa}{2\varrho}(\varpi_1 - \varpi + 2\textstyle\int_{-\infty}^x \beta(s)ds)\varrho^\theta \\
&= (\tfrac{3-\gamma}{4}\beta(x) - B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi + (\tfrac{3-\gamma}{4}\beta(x) + B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi_1 \\
&\quad + \tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x \beta(s)ds) + B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x \beta(s)ds\varrho^\theta,
\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}
&B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x \beta(s)ds\varrho^\theta = B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x \beta(s)ds\tfrac{\theta}{2}(w+z) \\
&= B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x \beta(s)ds\tfrac{\theta}{2}(\varpi_1+\varpi+2M)
\end{aligned} \tag{2.32}$$

and

$$\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds\theta M$$

$$\leq (\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x) + (1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds + \textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x)$$

$$= \beta(x)((\tfrac{1-\gamma}{2}+\varepsilon_1)M + (2-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds) \leq 0 \tag{2.33}$$

because $|\theta MB(x)| < \frac{2(3-\gamma)}{\gamma+3}\beta(x) < \beta(x)$ and $2\int_{-\infty}^{\infty}\beta(s)ds < \frac{\gamma-1}{2}M$ as given in Theorem 2.

Thus we have from (2.31), (2.32) and (2.33) that

$$\begin{aligned} K &\leq (\tfrac{3-\gamma}{4}\beta(x) - B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi + (\tfrac{3-\gamma}{4}\beta(x) + B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi_1 \\ &+B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds\tfrac{\theta}{2}(\varpi_1+\varpi) = l_1(x,t)\varpi + l_2(x,t)\varpi_1, \end{aligned} \tag{2.34}$$

where $l_2(x,t) \geq 0$.

Second, at the points (x,t), where $B(x) \leq 0$, we have that

$$\begin{aligned} K &= \tfrac{\gamma-3}{\gamma-1}\varrho^{\theta}D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\ &= \tfrac{\gamma-3}{4}(w+z)D'(x) + (1-\varepsilon_1)D(x)D'(x) + B(x)\tfrac{\varrho-2\kappa}{\varrho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2} \\ &= \tfrac{3-\gamma}{4}(\varpi+\varpi_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds) \\ &+\tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(\varpi_1-\varpi+2\textstyle\int_{-\infty}^{x}\beta(s)ds)(\varpi+\varpi_1+2M) \\ &= (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - \varpi - 2M))\varpi \\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))\varpi_1 + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}\varpi_1^2 \\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MB(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds, \end{aligned} \tag{2.35}$$

where the coefficient before ϖ_1

$$\begin{aligned} &\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M) \\ &\geq \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma-1}{8}|B(x)|\tfrac{\varrho-2\kappa}{\varrho}(\tfrac{\gamma-1}{2}M + 2M) \\ &= \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma+3}{8}\theta MB(x) \geq 0 \end{aligned} \tag{2.36}$$

because $|\theta MB(x)| < \frac{2(3-\gamma)}{\gamma+3}\beta(x)$;

$$\frac{\gamma-1}{8}B(x)\frac{\varrho-2\kappa}{\varrho}\varpi_1^2 \le 0 \tag{2.37}$$

and

$$\begin{aligned}&\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&+\tfrac{\gamma-1}{2}MB(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds \le 0\end{aligned} \tag{2.38}$$

because the proof of (2.33). Thus we have from (2.35)-(2.38) that

$$K \le l_3(x,t)\varpi + l_4(x,t)\varpi_1, \tag{2.39}$$

where $l_3(x,t), l_4(x,t) \ge 0$ are two suitable functions.

Summing up the analysis above, for any $B(x)$, we have the following inequality

$$\varpi_t + b(x,t)\varpi_x + d(x,t)v + c(x,t)\varpi_1 \le \varepsilon\varpi_{xx}, \tag{2.40}$$

where the coefficient function $c(x,t) \le 0$.

Similarly, we consider K_1 given in (2.30). First, at the points (x,t), where $B(x) \le 0$, we have that

$$\begin{aligned}K_1 &= -\tfrac{\gamma-3}{\gamma-1}\varrho^{\theta}C'(x)-(1-\varepsilon_1)C(x)C'(x)+B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho}\\&= -\tfrac{\gamma-3}{4}(w+z)C'(x)-(1-\varepsilon_1)C(x)C'(x)+B(x)\tfrac{\varrho-2\kappa}{2\varrho}(w-z)\varrho^{\theta}\\&= \tfrac{3-\gamma}{4}(\varpi+\varpi_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&+B(x)\tfrac{\varrho-2\kappa}{2\varrho}(\varpi_1-\varpi+2\textstyle\int_{-\infty}^{x}\beta(s)ds)\varrho^{\theta}\\&= (\tfrac{3-\gamma}{4}\beta(x)-B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi+(\tfrac{3-\gamma}{4}\beta(x)+B(x)\tfrac{\varrho-2\kappa}{2\varrho})\varpi_1\\&+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)+B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds\varrho^{\theta},\end{aligned} \tag{2.41}$$

where

$$B(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^{x}\beta(s)ds\varrho^{\theta} \le 0, \tag{2.42}$$

$$\begin{aligned}&\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&\le (\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x)-(1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds \le 0\end{aligned} \tag{2.43}$$

and

$$\frac{3-\gamma}{4}\beta(x) - B(x)\frac{\varrho-2\kappa}{2\varrho} \geq 0. \tag{2.44}$$

Thus we have from (2.41)-(2.44) that

$$K_1 \leq l_5(x,t)\varpi + l_6(x,t)\varpi_1, \tag{2.45}$$

where $l_5(x,t) \geq 0, l_6(x,t)$ are two suitable functions.

Second, at the points (x,t), where $B(x) \geq 0$, we have that

$$\begin{aligned}
K_1 &= -\tfrac{\gamma-3}{\gamma-1}\varrho^\theta C'(x) - (1-\varepsilon_1)C(x)C'(x) + B(x)(\varrho-2\kappa)v\tfrac{\sqrt{P'(\varrho)}}{\varrho} \\
&= -\tfrac{\gamma-3}{4}(w+z)\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x \beta(s)ds) + B(x)\tfrac{\varrho-2\kappa}{\varrho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2} \\
&= \tfrac{3-\gamma}{4}(\varpi+\varpi_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x \beta(s)ds) \\
&+\tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(\varpi_1-\varpi+2\textstyle\int_{-\infty}^x\beta(s)ds)(\varpi+\varpi_1+2M) \\
&= (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(\varpi_1+2\textstyle\int_{-\infty}^x\beta(s)ds+2M))\varpi_1 \\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(2\textstyle\int_{-\infty}^x\beta(s)ds-2M))\varpi - \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}\varpi^2 \\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds) + \tfrac{\gamma-1}{2}MB(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x\beta(s)ds,
\end{aligned} \tag{2.46}$$

where

$$-\frac{\gamma-1}{8}B(x)\frac{\varrho-2\kappa}{\varrho}\varpi^2 \leq 0, \tag{2.47}$$

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^x\beta(s)ds) + \tfrac{\gamma-1}{2}MB(x)\tfrac{\varrho-2\kappa}{\varrho}\textstyle\int_{-\infty}^x\beta(s)ds \\
&\leq (\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x) + \textstyle\int_{-\infty}^x\beta(s)ds\beta(x) \leq 0
\end{aligned} \tag{2.48}$$

and the coefficient before ϖ

$$\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}B(x)\tfrac{\varrho-2\kappa}{\varrho}(2\textstyle\int_{-\infty}^x\beta(s)ds-2M) \\
&\geq \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma-1}{8}|B(x)|\tfrac{\varrho-2\kappa}{\varrho}(\tfrac{\gamma-1}{2}M+2M) \\
&= \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma+3}{8}\theta MB(x) \geq 0
\end{aligned} \tag{2.49}$$

because the proof of (2.36). Thus we have from (2.46)-(2.49) that

$$K_1 \leq l_7(x,t)\varpi + l_8(x,t)\varpi_1, \tag{2.50}$$

where $l_7(x,t) \geq 0, l_8(x,t)$ are two suitable functions.

Summing up the analysis above, for any $B(x)$, we have the following inequality

$$\varpi_{1t} + b_1(x,t)\varpi_{1x} + d_1(x,t)\varpi_1 + c_1(x,t)\varpi \leq \varepsilon \varpi_{1xx}, \tag{2.51}$$

where the coefficient function $c_1(x,t) \leq 0$.

Therefore, we may apply the maximum principle to the coupled inequalities (2.40) and (2.51) to obtain the estimates $\varpi(x,t) \leq 0, \varpi_1(x,t) \leq 0$ and so the upper bounds of z and w (see [Lu3] for the details). This completes the Proof of Theorem 2.

Proof of Theorem 3. Since the original system (1.1) and the approximated system (1.5) have the same entropy equation or the same entropies ([Lu2]), also for any weak entropy-entropy flux pair $(\eta(\varrho,v), q(\varrho,v))$ of system (1.1), it was proved in [Lu2] that

$$\eta_t(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) + q_x(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) \tag{2.52}$$

are compact in $H^{-1}_{loc}(R \times R^+)$, then there exists a subsequence of $(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\varrho(x,t), v(x,t))$ as κ, ε tend to zero by using the compactness framework given in [LPS] for $1 < \gamma < 3$ and in [LPT] for $\gamma \geq 3$. It is easy to prove that the limit $(\varrho(x,t), v(x,t))$ satisfies (1.20). Moreover, for any weak convex entropy-entropy flux pair $(\eta(\varrho,v), q(\varrho,v))$ of system (1.1), we multiply (1.10) by (η_ϱ, η_m) to obtain that

$$\begin{aligned}
&\eta_t(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) + q_x(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) + \kappa q_{1x}(\varrho^{\kappa,\varepsilon}(x,t), v^{\kappa,\varepsilon}(x,t)) \\
&= \varepsilon \eta(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon})_{xx} - \varepsilon(\varrho^{\kappa,\varepsilon}_x, m^{\kappa,\varepsilon}_x) \cdot \nabla^2 \eta(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon}) \cdot (\varrho^{\kappa,\varepsilon}_x, m^{\kappa,\varepsilon}_x)^T \\
&+ B(x)(\varrho^{\kappa,\varepsilon} - 2\kappa) m^{\kappa,\varepsilon} \eta_\varrho(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon}) + B(x)(\varrho^{\kappa,\varepsilon} - 2\kappa)(u^{\kappa,\varepsilon})^2 \eta_m(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon}) \\
&\leq \varepsilon \eta(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon})_{xx} + B(x)(\varrho^{\kappa,\varepsilon} - 2\kappa) m^{\kappa,\varepsilon} \eta_\varrho(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon}) \\
&+ B(x)(\varrho^{\kappa,\varepsilon} - 2\kappa)(v^{\kappa,\varepsilon})^2 \eta_m(\varrho^{\kappa,\varepsilon}, m^{\kappa,\varepsilon}),
\end{aligned} \tag{2.53}$$

where $q + \kappa q_1$ is the entropy flux of system (1.5) corresponding to the entropy η. Thus the entropy inequality (1.21) is proved if we multiply a test function to (2.53) and let ε, κ go to zero. **Theorem 3 is proved.**

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