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Existence of global solutions to isentropic gas dynamics equations with a source term

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Abstract In this paper we prove existence of isentropic gas dynamic equations with a source term (1.2). To this end we construct a sequence of regular hyperbolic systems (1.1) to approximate the inhomogeneous system of isentropic gas dynamics (1.2). First, for each fixed approximation parameter δ and very general condition on $P(\rho)$, we establish the existence of entropy solutions for the Cauchy problem (1.1) with bounded initial date (1.4). Second, letting $\epsilon = o(\delta)$, we obtain a complete proof of the H^{-1}_{loc} compactness of weak entropy pairs of system (1.2) in the form $\eta(\rho, u) = \rho H(\rho, u)$ given in Chen-LeFloch (2003). Finally, for the conditions of $P(\rho)$ given in Chen-LeFloch (2003), applied to the results in Theorems 1 and 2, we obtain the global existence of entropy solutions for the Cauchy problem (1.2) with bounded initial date (1.4).

Keywords isentropic gas dynamics system, global entropy solution, compactness in H^{-1}_{loc}, compensated compactness method

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1 Introduction

In this paper, we construct a sequence of regular hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x + a(x)\rho + c\rho u|u| = 0, \end{cases} \tag{1.1}$$

to approximate the inhomogeneous system of isentropic gas dynamics in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x + a(x)\rho + c\rho u|u| = 0, \end{cases} \tag{1.2}$$

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where ρ is the density of gas, u is the velocity, $P = P(\rho)$ means the pressure. The function $a(x)$ corresponds physically to the slope of the topography and $c\rho|u|$ to a friction term and $\delta > 0$ in (1.1) denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho,\delta) = \int_{2\delta}^{\rho} \frac{t-2\delta}{t} P'(t)dt. \tag{1.3}$$

The global existence of L^∞ entropy solutions for inhomogeneous system (1.2) with the polytropic gas, $P(\rho) = d\rho^\gamma$, where $\gamma > 1$ and d is an arbitrary positive constant, and arbitrarily large L^∞ initial data was established in [1].

In this paper, we study more general pressure $P(\rho)$. First, for fixed $\delta > 0$, we establish the existence of entropy solutions for the Cauchy problem (1.1) with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geqslant 2\delta. \tag{1.4}$$

Theorem 1. *Let $|a(x)| \leqslant M$, and let c be a nonnegative constant. Let the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \geqslant 2\delta, P(\rho) \in C^2(0,\infty), P'(\rho) > 0$,*

$$2P'(\rho) + \rho P''(\rho) > 0 \quad as \quad \rho > 0 \tag{1.5}$$

and

$$\int_{c_1}^{\infty} \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \infty, \quad \int_0^{c_1} \frac{\sqrt{P'(\rho)}}{\rho} d\rho < \infty, \quad \forall c_1 > 0. \tag{1.6}$$

Then the Cauchy problem (1.1), (1.4) *has a global bounded entropy solution $(\rho(x,t), u(x,t))$ satisfying*

$$2\delta \leqslant \rho(x,t) \leqslant M(t), \quad |u(x,t)| \leqslant M(t),$$

where $M(t) > 0$ *is bounded for any finite time* t.

Second, with the help of the perturbation parameter $\delta, \epsilon = o(\delta)$, we obtain a simple proof of the H^{-1} compactness in the following theorem.

Theorem 2. *Let all conditions about $P(\rho)$ in Theorem* 1 *be satisfied and the limit*

$$\lim_{\rho\to 0} \frac{(P'(\rho))^{\frac{3}{2}}}{\rho P''(\rho)} = e, \tag{1.7}$$

where $e \geqslant 0$ is a constant. If the weak entropy-entropy flux pair $(\eta(\rho,u), q(\rho,u))$ of system (1.2) *is in the form $\eta(\rho,u) = \rho H(\rho,u)$ and $H_u(\rho,u), H_{uu}(\rho,u), H_{uuu}(\rho,u)$ are continuous on $0 \leqslant \rho \leqslant M(t), |u| \leqslant M(t)$, where $M(t)$ is a positive bounded function given in* (2.11), *then*

$$\eta_t(\rho^{\delta,\epsilon}(x,t), u^{\delta,\epsilon}(x,t)) + q_x(\rho^{\delta,\epsilon}(x,t), u^{\delta,\epsilon}(x,t)) \tag{1.8}$$

is compact in $H^{-1}_{\mathrm{loc}}(R \times R^+)$ as $\epsilon = o(\frac{P'(2\delta)}{2\delta})$ and δ tends to zero, with respect to the viscosity solutions $(\rho^{\delta,\epsilon}(x,t), u^{\delta,\epsilon}(x,t))$ of the Cauchy problem (2.6) *and* (1.4).

Theorem 3. *Let all conditions about $P(\rho)$ in Theorems* 1 *and* 2 *be satisfied. Assume that there exist an exponent $\gamma \in (1,\infty)$, and a smooth function $p(\rho)$, and some real $\theta_1 > 1$ such that*

$$P(\rho) = k\rho^\gamma\big(1 + \rho^{\theta\theta_1} p(\rho)\big), \tag{1.9}$$

where $\theta = \frac{\gamma-1}{2}, k = \frac{(\gamma-1)^2}{4\gamma}$, $p(\rho)$ and $\rho^3 P'''(\rho)$ are bounded as ρ tends to zero. Then there exist global entropy solutions to the Cauchy problem (1.2) *and* (1.4).

Note 1. In Theorem 1, we do not need the condition $a'(x) \geqslant 0$ given in [1].

Note 2. In Theorem 2, the weak entropy in the form $\eta(\rho, u) = \rho H(\rho, u)$ was constructed in [1], however the compactness in H^{-1} space is not proved.

General discussion of this paper: We use the compensated compactness method in this paper. When applying this theory to the equations of isentropic gas dynamics this involves three steps. In the first step, we construct a family of approximate solutions of this system of equations. In this paper this is done by adding viscosity to the gas dynamic equations. When proving existence of these viscosity solutions, the main difficulty is to prove an a-priory lower bound of the solutions $\rho^\epsilon \geqslant c(t,\epsilon) > 0$. In this paper we use a perturbation by δ in the flux of the conservation of mass equation to overcome this difficulty. Secondly, we need to prove the H^{-1}_{loc} compactness for the weak entropy-entropy flux pair. For polytropic gas $= p(\rho) = c\rho^\gamma$, $\gamma > 1$, this compactness is proven by Di Perna, Lions and others by controlling the weak entropy-enropy flux pair by a convex weak entropy (2.18). For general pressure $p(\rho)$ this compactness has been open up to now. In this paper using the perturbation δ and choosing ϵ going to zero faster than δ, we give a simple proof of this compactness. In the third step, we use the weak entropy-entropy flux pairs to prove that the Young measure is a Dirac measure. Here, for pressure equations given in (1.9) we use the result in [1].

2 Proof of Theorem 1

In this section, we prove Theorem 1.

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \tag{2.1}$$

with corresponding right eigenvectors

$$r_1 = (1, u - \sqrt{P'(\rho)})^T, \quad r_2 = (1, u + \sqrt{P'(\rho)})^T, \tag{2.2}$$

and Riemann invariants

$$z(u,v) = -\frac{m}{\rho} + \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds, \quad w(u,v) = \frac{m}{\rho} + \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds, \tag{2.3}$$

where $m = \rho u$. Moreover

$$\nabla\lambda_1 \cdot r_1 = -\frac{4\delta}{\rho^2}\sqrt{P'(\rho)} - \frac{\rho - 2\delta}{2\rho^2\sqrt{P'(\rho)}}(2P'(\rho) + \rho P''(\rho)), \tag{2.4}$$

and

$$\nabla\lambda_2 \cdot r_2 = \frac{4\delta}{\rho^2}\sqrt{P'(\rho)} + \frac{\rho - 2\delta}{2\rho^2\sqrt{P'(\rho)}}(2P'(\rho) + \rho P''(\rho)). \tag{2.5}$$

Considering the Cauchy problem for the related parabolic system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = \epsilon\rho_{xx}, \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x + a(x)\rho + c\rho u|u| = \epsilon(\rho u)_{xx}, \end{cases} \tag{2.6}$$

with the initial data (1.4), we multiply (2.6) by (w_ρ, w_m) and (z_ρ, z_m) respectively to obtain

$$\begin{aligned} w_t+&\lambda_2 w_x + a(x) + \frac{c|u|}{2}(w-z) \\ &= \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x - \frac{\epsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} z_t + &\lambda_1 z_x - a(x) + \frac{c|u|}{2}(z-w) \\ &= \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x - \frac{\epsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2. \end{aligned} \tag{2.8}$$

Then the assumptions on $P(\rho)$ yield

$$w_t + \lambda_2 w_x + a(x) + \frac{c|u|}{2}(w-z) \leqslant \epsilon w_{xx} + \frac{2\epsilon}{\rho}\rho_x w_x, \tag{2.9}$$

$$z_t + \lambda_1 z_x - a(x) + \frac{c|u|}{2}(z-w) \leqslant \epsilon z_{xx} + \frac{2\epsilon}{\rho}\rho_x z_x. \tag{2.10}$$

Making a transformation

$$w = X + Mt, \qquad z = Y + Mt, \tag{2.11}$$

where M is the bound of $a(x)$, from (2.9) and (2.10) we have

$$\begin{cases} X_t + \lambda_2 X_x + \dfrac{c|u|}{2}(X-Y) \leqslant \epsilon X_{xx}, \\ \\ Y_t + \lambda_1 Y_x + \dfrac{c|u|}{2}(Y-X) \leqslant \epsilon Y_{xx}, \end{cases} \tag{2.12}$$

with

$$X|_{t=0} = w|_{t=0} \leqslant M_1, \qquad Y|_{t=0} = \bar{z}|_{t=0} \leqslant M_1. \tag{2.13}$$

Thus the maximum principle (See Lemma 2.4 in [14]) applied to (2.11), (2.12) gives the estimates $w(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \leqslant M_1 + tM, z(\rho^{\epsilon,\delta}, m^{\epsilon,\delta}) \leqslant M_1 + tM$. Moreover, using the first equation in (2.6), we get $\rho^{\epsilon,\delta} \geqslant 2\delta$. Thus we obtain the estimates

$$2\delta \leqslant \rho^{\epsilon,\delta}(x,t) \leqslant M(t), \quad |u^{\epsilon,\delta}(x,t)| \leqslant M(t), \tag{2.14}$$

for a suitable positive function $M(t)$, being independent of ϵ, δ, since $\int_{c_1}^{\infty} \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \infty$ and $\int_0^{c_1} \frac{\sqrt{P'(\rho)}}{\rho} d\rho < \infty$ for any constant $c_1 > 0$.

Thus for fixed $\delta > 0$, it follows from (2.1) that system (1.1) is strictly hyperbolic in the domain $\{(x,t) : \rho > 2\delta\}$, while it is nonstrictly hyperbolic in the domain $\{(x,t) : \rho = 2\delta\}$, since $\lambda_1 = \lambda_2$ when $\rho = 2\delta$. However, from (2.4) and (2.5), both characteristic fields of system (1.1) are genuinely nonlinear in the range $\rho \geqslant 2\delta$.

For smooth solutions, the homogeneous part of system (1.1) is equivalent to the following system

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0, \\ \\ u_t + \left(\dfrac{1}{2}u^2 + \displaystyle\int_{2\delta}^{\rho} \dfrac{(t-2\delta)P'(t)}{t^2} dt\right)_x = 0, \end{cases} \tag{2.15}$$

and particularly, both systems have the same entropy-entropy flux pairs. Thus any entropy-entropy flux pair $(\eta(\rho,m), q(\rho,m))$ of system (1.1) satisfies the additional system

$$q_\rho = u\eta_\rho + \frac{(\rho-2\delta)P'(\rho)}{\rho^2}\eta_u, \quad q_u = (\rho-2\delta)\eta_\rho + u\eta_u. \tag{2.16}$$

Eliminating the q from (2.16), we have

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{2.17}$$

Therefore, systems (1.1) and (1.2) have the same entropies.

It is easy to check that system (1.1) has a convex entropy

$$\eta^\star = \frac{\rho u^2}{2} + \int_{2\delta}^{\rho} \frac{(\rho - t)P'(t)}{t}dt \tag{2.18}$$

with corresponding entropy flux

$$q^\star = \frac{\rho u^3}{2} - \frac{\delta u^3}{3} + u(\rho - 2\delta)\int_{2\delta}^{\rho} \frac{P'(t)}{t}dt. \tag{2.19}$$

We multiply (2.6) by $(\eta^\star_\rho, \eta^\star_m)$ to obtain the boundedness of

$$\epsilon(\rho_x, m_x) \cdot \nabla^2\eta^\star(\rho, m) \cdot (\rho_x, m_x)^T \tag{2.20}$$

in $L^1_{\mathrm{loc}}(R \times R^+)$. Then it follows that

$$\epsilon\frac{P'(\rho)}{\rho}\rho_x^2 + \epsilon\frac{1}{\rho}\left[\frac{m}{\rho}\rho_x - m_x\right]^2 = \epsilon\frac{P'(\rho)}{\rho}\rho_x^2 + \epsilon\rho u_x^2 \tag{2.21}$$

is bounded in $L^1_{\mathrm{loc}}(R \times R^+)$.

Since $\rho \geqslant 2\delta$, we get the boundedness of

$$\epsilon\rho_x^2, \quad \epsilon u_x^2 \quad \text{in} \quad L^1_{\mathrm{loc}}(R \times R^+) \tag{2.22}$$

for any fixed $\delta > 0$.

Thus for smooth entropy-entropy flux pairs $(\eta_i(\delta, \rho, u), q_i(\delta, \rho, u))$, $i = 1, 2$, of system (1.1), the following measure equations or the communicate relations are satisfied

$$\begin{aligned}&\langle \nu^\delta_{(x,t)}, \eta_1(\delta)q_2(\delta) - \eta_2(\delta)q_1(\delta)\rangle \\ &= \langle \nu^\delta_{(x,t)}, \eta_1(\delta)\rangle\langle \nu^\delta_{(x,t)}, q_2(\delta)\rangle - \langle \nu^\delta_{(x,t)}, \eta_2(\delta)\rangle\langle \nu^\delta_{(x,t)}, q_1(\delta)\rangle,\end{aligned} \tag{2.23}$$

where $\nu^\delta_{(x,t)}$ is the family of positive probability measures with respect to the viscosity solutions $(\rho^{\epsilon,\delta}, u^{\epsilon,\delta})$ of the Cauchy problem (2.6) and (1.4).

For applying to the framework given by DiPerna in [3] to prove that Young measures are Dirac ones, we construct four families of entropy-entropy flux pairs of Lax's type in the following special form:

$$\eta^1_k = e^{kw}\left(a_1(\rho) + \frac{b_1(\rho, k)}{k}\right), \quad q^1_k = \eta^1_k\left(\lambda_2 + \frac{c_1(\rho, k)}{k} + \frac{d_1(\rho, k)}{k^2}\right); \tag{2.24}$$

$$\eta^2_{-k} = e^{-kw}\left(a_2(\rho) + \frac{b_2(\rho, k)}{k}\right), \quad q^2_{-k} = \eta^2_{-k}\left(\lambda_2 + \frac{c_2(\rho, k)}{k} + \frac{d_2(\rho, k)}{k^2}\right); \tag{2.25}$$

$$\eta^2_k = e^{kz}\left(a_3(\rho) + \frac{b_3(\rho, k)}{k}\right), \quad q^2_k = \eta^2_k\left(\lambda_1 + \frac{c_3(\rho, k)}{k} + \frac{d_3(\rho, k)}{k^2}\right); \tag{2.26}$$

$$\eta^1_{-k} = e^{-kz}\left(a_4(\rho) + \frac{b_4(\rho, k)}{k}\right), \quad q^1_{-k} = \eta^1_{-k}\left(\lambda_1 + \frac{c_4(\rho, k)}{k} + \frac{d_4(\rho, k)}{k^2}\right), \tag{2.27}$$

where w, z are the Riemann invariants of system (1.1) given by (2.3). Notice that all the unknown functions a_i, b_i $(i = 1, 2, 3, 4)$ are only of a single variable ρ. This special simple construction yields an ordinary differential equation of second order with a singular coefficient $1/k$ before the term of the second order derivative. Then the following necessary estimates for functions $a_i(\rho), b_i(\rho,k)$ are obtained by the use of the singular perturbation theory of ordinary differential equations:

$$0 < a_i(\rho) \leqslant M, \quad |b_i(\rho,k)| \leqslant M, \tag{2.28}$$

$$0 < c_i(\rho) \leqslant M, \quad (\text{ or } -M \leqslant c_i(\rho) < 0), \quad |d_i(\rho,k)| \leqslant M, \tag{2.29}$$

uniformly for $2\delta \leqslant \rho \leqslant M_1$, where $i = 1,2,3,4$ and M is a positive constant independent of k.

In fact, substituting entropies $\eta_k^1 = e^{kw}(a_1(\rho) + b_1(\rho,k)/k)$ into (2.17), we obtain that

$$k[2f(\rho)a_1' + f'(\rho)a_1] + a_1'' + 2f(\rho)b_1' + f'(\rho)b_1 + \frac{b_1''}{k} = 0, \tag{2.30}$$

where $f(\rho) = \frac{\sqrt{P'(\rho)}}{\rho}$. Let

$$2f(\rho)a_1' + f'(\rho)a_1 = 0 \tag{2.31}$$

and

$$a_1'' + 2f(\rho)b_1' + f'(\rho)b_1 + \frac{b_1''}{k} = 0. \tag{2.32}$$

Then

$$a_1 = f(\rho)^{-\frac{1}{2}} > 0 \qquad \rho \geqslant 2\delta. \tag{2.33}$$

The existence of $b_1(\rho,k)$ and its uniform bound $|b_1(\rho,k)| \leqslant M$ on $2\delta \leqslant \rho \leqslant M_1$ with respect to k can be obtained by the following lemma (cf. [7]) (also see Lemma 10.2.1 in [14]):

Lemma 4. *Let $Y(x) \in C^2[0,h]$ be the solution of the equation*

$$F(x,Y,Y') = 0,$$

and functions $f(x,y,z,\lambda), F(x,y,z)$ be continuous on the regions $0 \leqslant x \leqslant h, |y-Y(x)| \leqslant l(x), |z-Y'(x)| \leqslant m(x)$ for some positive functions $l(x), m(x)$ and $\lambda_0 > \lambda > 0$. In addition,

$$|f(x,y,z,\lambda) - F(x,y,z)| \leqslant \epsilon,$$
$$|F(x,y_2,z) - F(x,y_1,z)| \leqslant M|y_2 - y_1|,$$
$$\frac{F(x,y,z_2) - F(x,y,z_1)}{z_2 - z_1} \geqslant L$$

for some positive constants ϵ, M and L.

If $y(x) = y(x,\lambda)$ is a solution of the following ordinary differential equation of second order:

$$\lambda y'' + f(x,y,y',\lambda) = 0,$$

with $y(0) = Y(0)$ and $y'(0)$ being arbitrary, then for sufficiently small $\lambda > 0, \epsilon > 0$ and $P = |y'(0) - Y'(0)|$, $y(x)$ exists for all $0 \leqslant x \leqslant h$ and satisfies

$$|y(x,\lambda) - Y(x)| < \left[\frac{\epsilon}{M} + \lambda\left(\frac{P}{L} + \frac{N}{M}\right)\right] \exp\left(\frac{Mx}{L}\right),$$

where $N = \max_{0 \leqslant x \leqslant h} |Y(x)|$.

Furthermore, we can use Lemma 4 again to obtain the bound of b_1' with respect to k if we differentiate Equation (2.32) with respect to ρ.

By the second equation in (2.16), an entropy flux q_k^1 corresponding to η_k^1 is provided by

$$q_k^1 = \lambda_2 \eta_k^1 + e^{kw}\left(\frac{(\rho - 2\delta)a_1' - a_1}{k} + \frac{(\rho - 2\delta)b_1' - b_1}{k^2}\right), \tag{2.34}$$

where

$$(\rho - 2\delta)a_1' - a_1 = -\frac{(\rho P'' + 2P')(\rho - 2\delta) + 8\delta P'}{4\rho P'' f(\rho)^{\frac{1}{2}}} < 0, \tag{2.35}$$

and $(\rho - 2\delta)b_1' - b_1$ both are bounded uniformly on $\rho \in [2\delta, M_1]$.

In a similar way, we can obtain another entropy-entropy flux pair of Lax type as follows:

$$\begin{cases} \eta_{-k}^2 = e^{-kw}\left(a_2(\rho) + \dfrac{b_2(\rho,k)}{k}\right), \\ q_{-k}^2 = \lambda_2 \eta_{-k}^2 + e^{-kw}\left(\dfrac{a_2 - (\rho - 2\delta)a_2'}{k} + \dfrac{b_2 - (\rho - 2\delta)b_2'}{k^2}\right), \end{cases} \tag{2.36}$$

where $a_2(\rho) = a_1(\rho)$ and $b_2(\rho, k)$ satisfies

$$a_1'' - 2f(\rho)b_2' - f'(\rho)b_2 + \frac{b_2''}{k} = 0; \tag{2.37}$$

$$\begin{cases} \eta_k^2 = e^{kz}\left(a_3(\rho) + \dfrac{b_3(\rho,k)}{k}\right), \\ q_k^2 = \lambda_1 \eta_k^2 + e^{kz}\left(\dfrac{(\rho - 2\delta)a_3' - a_3}{k} + \dfrac{(\rho - 2\delta)b_3' - b_3}{k^2}\right), \end{cases} \tag{2.38}$$

where $a_3(\rho) = a_1(\rho)$ and $b_3(\rho, k)$ satisfies

$$a_1'' - 2f(\rho)b_3' - f'(\rho)b_3 + \frac{b_3''}{k} = 0; \tag{2.39}$$

$$\begin{cases} \eta_{-k}^1 = e^{-kz}\left(a_4(\rho) + \dfrac{b_4(\rho,k)}{k}\right), \\ q_{-k}^1 = \lambda_1 \eta_{-k}^1 + e^{-kz}\left(\dfrac{a_4 - (\rho - 2\delta)a_4'}{k} + \dfrac{b_4 - (\rho - 2\delta)b_4'}{k^2}\right), \end{cases} \tag{2.40}$$

where $a_4(\rho) = a_1(\rho)$ and $b_4(\rho, k)$ satisfies

$$a_1'' + 2f(\rho)b_4' + f'(\rho)b_4 + \frac{b_4''}{k} = 0. \tag{2.41}$$

Using the argument in Lemma 4 in Equation (2.41), we can get the existence of b_4 and the uniform bounded estimates of b_4, b_4' with respect to k. If making an independent transformation $\rho_1 = \rho - M_1$ to Equations (2.37) and (2.39), where M_1 is the upper bound of ρ, we also obtain the existence of b_2, b_3 and the uniform bounded estimates of b_2, b_3, b_2' and b_3' by Lemma 4 again.

Then the estimates in (2.28)–(2.29) are obtained, and hence Theorem 1 is proved when we use these entropy-entropy flux pairs in (2.24)–(2.27) together with the theory of compensated compactness coupled with DiPerna's framework (cf. [3]).

3 Proofs of Theorem 2 and Theorem 3

In this section, we prove Theorems 2 and 3.

First of all, we recall the proof of Theorem 2 for the case of polytropic gas and the homogeneous system, in which any weak entropy can be represented by the following explicit formula:

$$\eta_0(\rho, u) = \rho \int_0^1 [\tau(1-\tau)]^\lambda g(u + \rho^\theta - 2\rho^\theta \tau)d\tau, \tag{3.1}$$

where $\theta = \frac{\gamma-1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}$ and g is a smooth function. Multiplying $(\eta_{0\rho}, q_{0\rho})$ to the following parabolic system

$$\begin{cases} \rho_t + m_x = \epsilon\rho_{xx}, \\ m_t + \left(\dfrac{m^2}{\rho} + \rho^\gamma\right)_x = \epsilon m_{xx}, \end{cases} \tag{3.2}$$

we have

$$\eta_{0t} + q_{0x} = \epsilon\eta_{0xx} - \epsilon(\rho_x^\epsilon, m_x^\epsilon) \cdot \nabla^2\eta_0(\rho^\epsilon, m^\epsilon) \cdot (\rho_x^\epsilon, m_x^\epsilon)^T, \tag{3.3}$$

where q_0 is the entropy flux corresponding to η_0. Then using the strictly convex entropy

$$\eta_c = \frac{m^2}{2\rho} + \frac{1}{\gamma-1}\rho^\gamma, \tag{3.4}$$

we first obtain the boundedness of

$$\epsilon(\rho_x^\epsilon, m_x^\epsilon) \cdot \nabla^2\eta_0(\rho^\epsilon, m^\epsilon) \cdot (\rho_x^\epsilon, m_x^\epsilon)^T \tag{3.5}$$

in $L^1_{\rm loc}(R\times R^+)$ since the hessian of η_0 is controlled by the hessian of η_c with the help of the explicit formula (3.1), and hence the compactness in $W^{-1,\alpha}_{\rm loc}(R \times R^+)$, for some $\alpha \in (1, 2)$, by the Sobolev embedding theorems. Second, in the case of $1 < \gamma < 2$, since η_c is strictly convex and $\eta_{0\rho}, \eta_{0m}$ are uniformly bounded, we can prove that $\epsilon\eta_{0xx}(\rho^\epsilon, m^\epsilon)$ is compact in $H^{-1}_{\rm loc}(R \times R^+)$. Noticing that the left-hand side in (3.3) is uniformly bounded in $W^{-1,\infty}(R\times R^+)$ with respect to ϵ, we get the proof of Theorem 2 by using Murat's theorem (cf. [18]) for the case of $1 < \gamma < 2$.

However, for the case of $\gamma > 2$, the entropy η_c is not strictly convex and hence the compactness of $\epsilon\eta_{0xx}(\rho^\epsilon, m^\epsilon)$ in $H^{-1}_{\rm loc}(R \times R^+)$ is not obvious. To overcome this technical difficulty, the authors in [11] used the periodic viscosity solutions in the space variable x to obtain an auxiliary estimate (see (I.53) in [11]):

$$\int_0^T dt \int_0^L dx\epsilon^2(\rho_x^\epsilon)^2 \to 0 \quad \text{as } \epsilon \to 0. \tag{3.6}$$

In this section, with the help of the approximation parameter δ, the compactness of $\epsilon\eta_{0xx}(\rho^\epsilon, m^\epsilon)$ in $H^{-1}_{\rm loc}(R \times R^+)$ can be easily obtained.

Now, we prove Theorem 2 for the inhomogeneous system with more general pressure $P(\rho)$.

We rewrite system (2.6) by the following equivalent system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = \epsilon\rho_{xx}, \\ u_t + \left(\dfrac{1}{2}u^2 + \displaystyle\int_{2\delta}^\rho \frac{(t-2\delta)P'(t)}{t^2}dt\right)_x + a(x) + cu|u| = \epsilon u_{xx} + \frac{2\epsilon}{\rho}\rho_x u_x. \end{cases} \tag{3.7}$$

Let $(\eta(\rho, u), q(\rho, u)), (\eta(\rho, u), q_1(\rho, u, \delta))$ be the entropy-entropy flux pairs of systems (1.1), (1.2) respectively since they have the same entropy equation (2.14), but different entropy fluxes.

Multiplying system (3.7) by (η_ρ, η_u), we have

$$\begin{aligned} \eta(\rho^\epsilon, m^\epsilon)_t &+ q(\rho^\epsilon, m^\epsilon)_x + (a(x) + cu^\epsilon|u^\epsilon|)\eta_u \\ &= \epsilon\eta(\rho^\epsilon, m^\epsilon)_{xx} - (q_1(\rho^\epsilon, m^\epsilon, \delta) - q(\rho^\epsilon, m^\epsilon))_x + \frac{2\epsilon}{\rho^\epsilon}\eta_u\rho_x u_x \\ &\quad - \epsilon(\eta_{\rho\rho}\rho_x^2 + 2\eta_{\rho u}\rho_x u_x + \eta_{uu}u_x^2). \end{aligned} \tag{3.8}$$

By entropy equation (2.14), we have

$$\begin{aligned}\eta_\rho &= \int_0^\rho \frac{P'(\tau)}{\tau^2}\eta_{uu}(\tau,u)d\tau + g(u)\\ &= \int_0^\rho \frac{P'(\tau)}{\tau}H_{uu}(\tau,u)d\tau + g(u),\end{aligned} \tag{3.9}$$

since $\eta(\rho,u)=\rho H(\rho,u)$, where $g(u)$ is an arbitrary smooth function. Furthermore by integrating (3.9), we get

$$\eta = \int_0^\rho \int_0^t \frac{P'(\tau)}{\tau}H_{uu}(\tau,u)d\tau dt + g(u)\rho \tag{3.10}$$

since $\eta(0,u)=0$. Then

$$\eta_u = \int_0^\rho \int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt + g'(u)\rho, \tag{3.11}$$

$$\eta_{\rho u} = \int_0^\rho \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau + g'(u). \tag{3.12}$$

Substituting (3.11), (3.12) into (3.8) and using entropy equation (2.14), we get the following equality

$$\eta(\rho^\epsilon,m^\epsilon)_t + q(\rho^\epsilon,m^\epsilon)_x = I_1+I_2+I_3, \tag{3.13}$$

where

$$I_1 = \epsilon\eta(\rho^\epsilon,m^\epsilon)_{xx} - (q_1(\rho^\epsilon,m^\epsilon,\delta)-q(\rho^\epsilon,m^\epsilon))_x, \tag{3.14}$$

$$I_2 = -\epsilon\left(\frac{P'(\rho^\epsilon)}{\rho^\epsilon}H_{uu}(\rho^\epsilon,u^\epsilon)\rho_x^2 + \rho^\epsilon H_{uu}u_x^2\right) - \rho^\epsilon H_u(a(x)+cu^\epsilon|u^\epsilon|), \tag{3.15}$$

$$I_3 = -2\epsilon\left(\int_0^{\rho^\epsilon}\frac{P'(\tau)}{\tau}H_{uuu}(\tau,u^\epsilon)d\tau - \frac{1}{\rho^\epsilon}\int_0^{\rho^\epsilon}\int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u^\epsilon)d\tau dt\right)\rho_x u_x. \tag{3.16}$$

For any $\phi\in C_0^1(R\times R^+)$ with $S=\operatorname{supp}\phi$,

$$\begin{aligned}&\Big|\int_0^\infty\int_{-\infty}^\infty \epsilon\eta(\rho^\epsilon,m^\epsilon)_{xx}\phi dxdt|\\ &\leqslant \epsilon|\left(\eta_\rho\rho_x^\epsilon + H_u\rho^\epsilon u_x^\epsilon\right)\phi_x|dxdt\\ &\leqslant M\left[\left(\int\int_S \epsilon\frac{P'(\rho^\epsilon)}{\rho^\epsilon}(\rho_x^\epsilon)^2\frac{\epsilon\rho^\epsilon}{P'(\rho^\epsilon)}dxdt\right)^{\frac12}\right.\\ &\quad\left.+\left(\int\int_S \epsilon^2(\rho^\epsilon)^2(u_x^\epsilon)^2dxdt\right)^{\frac12}\right]\left(\int\int_S(\phi_x)^2dxdt\right)^{\frac12}\to 0.\end{aligned} \tag{3.17}$$

since $\epsilon = o\big(\frac{P'(2\delta)}{2\delta}\big)$ or $\frac{\epsilon\rho^\epsilon}{P'(\rho^\epsilon)}\to 0$ as $\epsilon,\delta\to 0$. Since $q_1(\rho^\epsilon,m^\epsilon,\delta)-q(\rho^\epsilon,m^\epsilon)$ tends to zero as δ tends to zero, we get the compactness of I_1 in $H_{\rm loc}^{-1}(R\times R^+)$. Using (2.19) and (2.11), we know that I_2 is bounded in $L_{\rm loc}^1(R\times R^+)$, and hence compact in $W_{\rm loc}^{-1,\alpha}(R\times R^+)$, for some $\alpha\in(1,2)$, by the Sobolev embedding theorems. Using the Vol'pert theorem and the limit given in (1.7), we have the following estimates

$$\begin{cases}\left|\int_0^\rho \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau\right| \leqslant M\left|\int_0^\rho\frac{P'(\tau)}{\tau}d\tau\right| \leqslant M_1\sqrt{P'(\rho)},\\ \left|\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt\right| \leqslant M\left|\int_0^\rho\int_0^t\frac{P'(\tau)}{\tau}d\tau dt\right| \leqslant M_1\rho\sqrt{P'(\rho)}.\end{cases} \tag{3.18}$$

Using these estimates together with (2.19), we get the boundedness of I_3 in $L_{\rm loc}^1(R\times R^+)$ and hence the compactness in $W_{\rm loc}^{-1,\alpha}(R\times R^+)$, for some $\alpha\in(1,2)$, by the Sobolev embedding theorems.

Therefore the right-hand side of (3.13) is compact in $W^{-1,\alpha}_{\rm loc}(R \times R^+)$ for some $\alpha \in (1, 2)$, but the left-hand side is bounded in $W^{-1,\infty}(R \times R^+)$. This implies the compactness of $\eta(\rho^\epsilon, m^\epsilon)_t + q(\rho^\epsilon, m^\epsilon)_x$ in $H^{-1}_{\rm loc}(R \times R^+)$ and hence the proof of Theorem 2 by the Murat theorem (cf. [18]).

Proof of Theorem 3. In [1], under the assumptions of Theorem 2 on the pressure function $P(\rho)$ and the compactness of $\eta(\rho^\epsilon, m^\epsilon)_t + q(\rho^\epsilon, m^\epsilon)_x$ in $H^{-1}_{\rm loc}(R \times R^+)$, the authors have established a compactness framework for the viscosity solutions $(\rho^\epsilon, m^\epsilon)$. Based on this framework and the result given in Theorem 2, we can easily prove Theorem 3.

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