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# Global existence of entropy solutions for euler equations of compressible fluid flow

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**Abstract**
The main contribution of this paper is to provide a complete proof of the global weak entropy solution existence of the Cauchy problem for the Euler equations of one-dimensional compressible fluid flow and to correct the mistakes in the paper "Global weak solutions of the one-dimensional hydrodynamic model for semiconductors" (Math. Mod. Meth. Appl. Sci., 6(1993), 759–788). Our technique is the method of the artificial viscosity coupled with the theory of compensated compactness, where four families of Lax entropy-entropy flux pair are constructed by means of the classical Fuchsian equation.

**Mathematics Subject Classification** 35L45 · 35L60 · 35L65

## 1 Introduction

In this paper, we consider the existence of global weak entropy solutions for the following nonlinear, nonstrictly hyperbolic systems of two equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + (\frac{1}{2}u^2 + h(\rho))_x = 0, \end{cases} \tag{1.1}$$

with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \;\; \rho_0(x) \geq 0, \tag{1.2}$$

where the nonlinear function $h$ denotes the enthalpy and takes the following form

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$$h(\rho) = \frac{c_1}{\gamma - 1}\rho^{\gamma-1} + \frac{c_2}{\beta - 1}\rho^{\beta-1}, \ 1 < \gamma < 3, \ \beta > 3, \ c_i > 0, \ (i = 1, 2). \tag{1.3}$$

For smooth solutions, system (1.1) is equivalent to the following well-known isentropic equations of gas dynamics (cf. [4, 15, 16])

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \tag{1.4}$$

where the functions $P(\rho)$ and $h(\rho)$ satisfy the relation $h(\rho) = \int^\rho \frac{P'(\tau)}{\tau} d\tau$, but in the case of discontinuous weak solutions, these two systems lead to different shock waves.

System (1.1) was first derived by S. Earnshaw for isentropic flow (cf. [7, 25]) and is also called the Euler equations of one-dimensional compressible fluid flow (cf. [11]). It is a scaling limit system of a Newtonian dynamics with long-range interaction for a continuous distribution of mass in R (cf. [21, 22]) and also a hydrodynamic limit for the Vlasov equation (cf. [2]).

The study of the existence of global weak solutions of the Cauchy problem (1.1)–(1.2) was first started by DiPerna [5] when $c_2 = 0$ in (1.3), i.e. $h(\rho) = \frac{c_1}{\gamma-1}\rho^{\gamma-1}$, $1 < \gamma < 3$ by using the Glimm's scheme method (cf. [9]). For general nonlinear enthalpy function $h(\rho)$ ensuring the existence of a strictly convex entropy of (1.1), the global solution in $L^\infty$ space was obtained in [12] by using the theory of compensated compactness (cf. [1, 8, 20, 24]) coupled with the groundbreaking framework given by DiPerna in [6], in which, four special classes of entropy-entropy flux pair, all the unknown functions being of the single variable $\rho$, of Lax type was constructed (cf. [17] for a simple case).

If system (1.1) has no strictly convex entropy, for instance, the case of $c_1 = 0$, i.e. $h(\rho) = \frac{c_2}{\beta-1}\rho^{\beta-1}$, $\beta > 3$, DiPerna's framework in [6] does not work directly. The bounded global weak solution was proved in [18] by using the theory of compensated compactness coupled with some basic ideas of the kinetic formulation by Lions, Perthame, Souganidis and Tadmor [15, 16].

It is worthwhile to point out that the existence of global weak solutions for the initial boundary value problem of (1.1) with general function $h(\rho)$ in the form (1.3) was studied by Jochmann in [10] by using the compensated compactness method. Unfortunately, it seems that the proof in [10] is incorrect because the condition $\tau(0) > 0$ in (4.19) in [10] is not true. In fact, from the equation

$$\sqrt{\rho h'(\rho)} = \tau\left(\sqrt{\frac{h'(r)}{r}}dr\right)$$

above (4.19) in [10], we have

$$\sqrt{c_1\rho^{\gamma-1} + c_2\rho^{\beta-1}} = \tau\left(\int_0^\rho \sqrt{c_1 r^{\gamma-3} + c_2 r^{\beta-3}}dr\right).$$

Let $t=\int_0^\rho \sqrt{c_1 r^{\gamma-3}+c_2 r^{\beta-3}}dr$. Then $\rho=0$ if $t=0$ and so $\tau(0)=0$.

Moreover, suppose the estimates in Theorem 5 in [10] were correct without the condition $\tau(0)>0$, besides the two classes of weak entropy-entropy flux pairs of Lax type constructed by DiPerna in [4], we might use the same conclusion given in Theorem 5 in [10] to construct another two classes of weak entropy-entropy flux pairs. This should be an amazing discovery because DiPerna's framework (cf. [6]) on four classes of entropy-entropy flux pairs of Lax type would deduce a very simple proof on the global existence of weak solutions for the isentropic gas dynamics system.

The main contribution in this paper is to reconstruct four families of Lax entropy-entropy flux of system (1.1) by means of the classical Fuchsian equation and give a completed proof of Jochmann's global existence of entropy solutions for the Cauchy problem (1.1)–(1.2) with the condition (1.3) by using the DiPerna's framework given in [6].

System (1.1) with the special condition (1.3) on the enthalpy is very interesting because it brings us some challenging new difficulties as follows:

**Difficulty I.** How to construct suitable approximated solutions $(\rho^l, u^l)$ of system (1.1) and to prove the uniformly bounded estimates of $(\rho^l, u^l)$ because system (1.1) with the condition (1.3) has no convex Riemann invariant and the classical invariant region theory (cf. [3]) can not be used directly?

As introduced in [5, 18] or [23], system (1.1) has positive invariant regions ensuring a positive lower bound $\rho\geq c_0>0$ when $c_1\neq 0, c_2=0$ in (1.3), and positive uniformly bounded invariant regions including the vacuum line $\rho=0$ when $c_1=0, c_2\neq 0$ in (1.3). Although system (1.1) has no convex Riemann invariant when $c_1\neq 0, c_2\neq 0$ in (1.3), for large $\rho$, the term $\frac{c_2}{\beta-1}\rho^{\beta-1}$ in (1.3) takes the crucial role and a technique from the maximum principle given in [10] on the initial-boundary value problem can be used here to prove the uniformly bounded estimates of the viscosity solutions of system (1.1).

**Difficulty II.** System (1.1) is hyperbolically degenerate on the vacuum line $\rho=0$ because two eigenvalues coincide. How to construct four families of Lax entropy-entropy flux of system (1.1) on the whole region $\rho\geq 0$?

The main difficulty in studying this system is that the entropy-entropy flux pairs and their partial derivatives of the first and second orders are probably singular on the line $\rho=0$. Fortunately, near the line $\rho=0$, the term $\frac{c_1}{\gamma-1}\rho^{\gamma-1}$ in (1.3) takes the crucial role. Utilizing the advantages of this power function, and through a careful construction of exact solutions of the classical Fuchsian equation, we obtain the entropy-entropy flux pair of Lax type for this system near the line $\rho=0$ ($0<\rho\leq\rho_0, \rho_0$ small) first (cf. [19]), and then extend these entropies to the whole region $\rho\geq 0$ because the original technique about the construction of Lax entropy for a strictly hyperbolic system given in [14] works well when $\rho>\rho_0$.

**Difficulty III.** How to prove the $H^{-1}$ compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t+q(\rho^\varepsilon, u^\varepsilon)_x$ for these entropy-entropy flux pairs $(\eta, q)$ constructed above with respect to the viscosity approximated solutions $(\rho^\varepsilon, u^\varepsilon)$?

The second-order partial derivatives of the two families of Lax entropy-entropy flux of system (1.1) constructed above are still singular on the line $\rho=0$. However, (1.1) has a strictly convex entropy-entropy flux pair (4.11), whose second-order derivatives are also singular on the line $\rho = 0$. Through a careful analysis of solutions of the Fuchsian equation, we obtain the necessary detailed estimates for the major terms of these Lax entropies, and prove the $H^{-1}$ compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x$ by using the strictly convex entropy-entropy flux pair given in (4.11) to control all the bad terms in these entropy-entropy flux pairs of Lax type.

Let the matrix $dF(U)$ be

$$dF(U) = \begin{pmatrix} u & \rho \\ h'(\rho) & u \end{pmatrix}. \tag{1.5}$$

By simple calculation, two eigenvalues of (1.1) are

$$\begin{cases} \lambda_1 = u - \sqrt{\rho h'(\rho)} = u - \sqrt{c_1\rho^{\gamma-1} + c_2\rho^{\beta-1}}, \\ \lambda_2 = u + \sqrt{\rho h'(\rho)} = u + \sqrt{c_1\rho^{\gamma-1} + c_2\rho^{\beta-1}} \end{cases} \tag{1.6}$$

with corresponding two Riemann invariants

$$\begin{cases} z = \int_0^\rho \sqrt{\frac{h'(\rho)}{\rho}}d\rho - u = \int_0^\rho \sqrt{c_1\rho^{\gamma-3} + c_2\rho^{\beta-3}}d\rho - u, \\ w = \int_0^\rho \sqrt{\frac{h'(\rho)}{\rho}}d\rho + u = \int_0^\rho \sqrt{c_1\rho^{\gamma-3} + c_2\rho^{\beta-3}}d\rho + u. \end{cases} \tag{1.7}$$

We consider the following corresponding parabolic system of (1.1)

$$\begin{cases} \rho_t + (\rho u)_x = \varepsilon\rho_{xx} \\ u_t + (\frac{1}{2}u^2 + h(\rho))_x = \varepsilon u_{xx} \end{cases} \tag{1.8}$$

with the initial data

$$(\rho^\varepsilon(x,0), u^\varepsilon(x,0)) = (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\overline{\rho_0(x)}, \overline{u_0(x)}) * G^\varepsilon, \tag{1.9}$$

where $\varepsilon > 0$, $G^\varepsilon$ is a suitable mollifier and $(\overline{\rho_0(x)}, \overline{u_0(x)})$ are given as follows

$$\begin{cases} (\overline{\rho_0(x)}, \overline{u_0(x)}) = (\rho_0(-L_\varepsilon) + \varepsilon, u_0(-L_\varepsilon)), \text{ as } x \le -L_\varepsilon, \\ (\overline{\rho_0(x)}, \overline{u_0(x)}) = (\rho_0(x) + \varepsilon, u_0(x)), \text{ as } -Ł_\varepsilon < x < L_\varepsilon, \\ (\overline{\rho_0(x)}, \overline{u_0(x)}) = (\rho_0(L_\varepsilon) + \varepsilon, u_0(L_\varepsilon)), \text{ as } x \ge L_\varepsilon. \end{cases} \tag{1.10}$$

We choose $L_\varepsilon$ to be a positive large constant depending on $\varepsilon$, and $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then $(\rho_0^\varepsilon(x), u_0^\varepsilon(x))$ satisfy

$$\begin{cases} (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho_0(-L_\varepsilon) + \varepsilon, u_0(-L_\varepsilon)), \text{ as } x \le -L_\varepsilon, \\ (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho_0(L_\varepsilon) + \varepsilon, u_0(L_\varepsilon)), \text{ as } x \ge L_\varepsilon, \end{cases} \tag{1.11}$$

$$(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \in C^\infty \times C^\infty, \ \varepsilon \le \rho_0^\varepsilon(x) \le M, \ |u_0^\varepsilon(x)| \le M, \tag{1.12}$$

and

$$(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \to (\rho_0(x), u_0(x)) \text{ a.e., as } \varepsilon \to 0, \tag{1.13}$$

where $M$ is a suitable large constant depending only on the $L^\infty$ bound of $(\rho_0(x), u_0(x))$, but is independent of $\varepsilon$.

We have the following main result:

**Theorem 1** *Let the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \ge 0$, the enthalpy function $h(\rho)$ take the form (1.3). Then for fixed $\varepsilon > 0$, the smooth viscosity solution $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ of the Cauchy problem (1.8), (1.9) exists and satisfies*

$$\begin{cases} \lim_{x\to\infty}(\rho^\varepsilon(x,t), u^\varepsilon(x,t)) = (\rho_0(L_\varepsilon)+\varepsilon, u_0(L_\varepsilon)), \\ \lim_{x\to-\infty}(\rho^\varepsilon(x,t), u^\varepsilon(x,t)) = (\rho_0(-L_\varepsilon)+\varepsilon, u_0(-L_\varepsilon)) \end{cases} \tag{1.14}$$

*and*

$$0 < c(\varepsilon, t) \le \rho^\varepsilon(x,t) \le M, \ |u^\varepsilon(x,t)| \le M, \tag{1.15}$$

*where $M$ is a positive constant, being independent of $\varepsilon$; $c(\varepsilon, t)$ is a positive function, which could tend to zero as $\varepsilon$ tends to zero or $t$ tends to infinity.*

*Moreover, there exists a subsequence (still labelled) $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ which converges almost everywhere on any bounded and open set $\Omega \subset R \times R^+$:*

$$(\rho^\varepsilon(x,t), u^\varepsilon(x,t)) \to (\rho(x,t), u(x,t)), \ \text{as } \varepsilon \downarrow 0^+, \tag{1.16}$$

*where the limit pair of functions $(\rho(x,t), u(x,t))$ is a weak entropy solution of the Cauchy problem (1.1)-(1.2).*

**Definition 1** A pair of bounded functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.1)-(1.2) if

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho\phi_t + \rho u \phi_x \phi dx dt + \int_{-\infty}^\infty \rho_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty u\phi_t + (\frac{1}{2}u^2 + h(\rho))\phi_x dx dt + \int_{-\infty}^\infty \rho_0(x) u_0(x)\phi(x,0)dx = 0, \end{cases} \tag{1.17}$$

holds for all test function $\phi \in C_0^1(R \times R^+)$ and

$$\eta(\rho, u)_t + q(\rho, u)_x \le 0, \tag{1.18}$$

in the sense of distributions for any convex entropy $\eta(\rho, u)$ of system (1.1).

In the next sections, we shall prove Theorem 1. In Sect. 2, we will prove the existence of the viscosity solutions and the uniform $L^\infty$ estimates (1.15). In Sect. 3, we will study the behavior of solutions of Fuchsian equations, which is the basis to construct four families of Lax entropy-entropy flux pair $(\eta, q)$, and to prove the compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x)$ in $H^{-1}_{loc}$ in Sect. 4. Finally, in Sect. 5, we will deduce that the family of positive measures $\nu_{x,t}$, determined by the sequence of viscosity solutions to be Dirac measures and complete the proof of Theorem 1.

## 2 Viscosity solution

In this section, we shall prove the first part about the existence of the viscosity solutions in Theorem 1. The local existence result for the Cauchy problem (1.8) and (1.9) can be easily obtained by applying the contraction mapping principle to an integral representation for a solution, following the standard theory of semilinear parabolic systems [13, 23]. The process to get the local solution clearly shows the behavior (1.14) of the solution.

Whenever we have an a priori upper estimate $\rho^\varepsilon(x,t) \le M$ and $|u^\varepsilon(x,t)| \le M$ given in (1.15) of the local solution, the positive lower bound $\rho^\varepsilon(x,t) \ge c(\varepsilon, t) > 0$ is a direct conclusion of Theorem 1.0.2 in [18], and clearly that the local time $\tau$ can be extended to $T$ step by step since the step time depends only on the $L^\infty$ norm.

To prove the $L^\infty$ estimate (1.15), we multiply (1.8) by $(w_\rho, w_u)$ and $(z_\rho, z_u)$, respectively, and obtain

$$w_t + \lambda_2 w_x = \varepsilon w_{xx} - \varepsilon w_{\rho\rho}\rho_x^2 = \varepsilon w_{xx} - \frac{c_1(\gamma-3)\rho^{\gamma-4} + c_2(\beta-3)\rho^{\beta-4}}{2\sqrt{c_1\rho^{\gamma-3} + c_2\rho^{\beta-3}}}\rho_x^2 \tag{2.1}$$

and

$$z_t + \lambda_1 z_x = \varepsilon z_{xx} - \varepsilon z_{\rho\rho}\rho_x^2 = \varepsilon z_{xx} - \frac{c_1(\gamma-3)\rho^{\gamma-4} + c_2(\beta-3)\rho^{\beta-4}}{2\sqrt{c_1\rho^{\gamma-3} + c_2\rho^{\beta-3}}}\rho_x^2. \tag{2.2}$$

Let

$$\rho^\star = \left(\frac{c_1(3-\gamma)}{c_2(\beta-3)}\right)^{\frac{1}{\beta-\gamma}}, \tag{2.3}$$

then

$$\begin{cases} w_{\rho\rho} = z_{\rho\rho} = \frac{c_1(\gamma-3)\rho^{\gamma-4}+c_2(\beta-3)\rho^{\beta-4}}{2\sqrt{c_1\rho^{\gamma-3}+c_2\rho^{\beta-3}}} < 0, \quad \text{as} \quad 0 < \rho < \rho^\star, \\ w_{\rho\rho} = z_{\rho\rho} = \frac{c_1(\gamma-3)\rho^{\gamma-4}+c_2(\beta-3)\rho^{\beta-4}}{2\sqrt{c_1\rho^{\gamma-3}+c_2\rho^{\beta-3}}} > 0, \quad \text{as} \quad \rho > \rho^\star. \end{cases} \tag{2.4}$$

Similarly to the proof of Lemma 2 given in [10] for the initial boundary value problem, we have with the help of (1.14) that

**Lemma 2** *Let the conditions on the initial data in Theorem 1 be satisfied. Then, for some large constant $C_0 > 0$, the following estimate holds*

$$\left|\int_{\rho^\star}^{\rho^\varepsilon} \sqrt{\frac{h'(\tau)}{\tau}} d\tau\right| + |u^\varepsilon| = \left|\int_{\rho^\star}^{\rho^\varepsilon} \sqrt{c_1\tau^{\gamma-3} + c_2\tau^{\beta-3}} d\tau| + |u^\varepsilon\right| \le C_0. \tag{2.5}$$

***Proof of Lemma 2*** For the simplicity of proof, we omit the superscript $\varepsilon$ and let

$$g(x,t) = \left|\int_{\rho^\star}^{\rho(x,t)} \sqrt{\frac{h'(\tau)}{\tau}} d\tau\right| + |u(x,t)| \tag{2.6}$$

and

$$s_\delta(t) = \max_{x\in(-\infty,\infty)} \left\{\int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau + \int_0^{\rho_0^\varepsilon(x)} \sqrt{\frac{h'(\tau)}{\tau}} d\tau + |u_0^\varepsilon(x)|\right\} + \delta(1+t), \tag{2.7}$$

which exists due to (1.11).

We will prove

$$g(x,t) < s_\delta(t), \quad \text{for all } \ (x,t) \in (-\infty,\infty)\times[0,T), \delta > 0. \tag{2.8}$$

Clearly, (2.8) is true when t=0. We argue by assuming that (2.8) is violated at a point $(x,t)$ in $(-\infty,\infty)\times(0,T)$. Let $\bar{t}$ be the least upper bound of values of $t$ at which $g(x,t) < s_\delta(t)$. Then by the continuity, we see that $g(x,t) = s_\delta(t)$ at some points $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,T)$. Moreover, we can let

$$g(\bar{x},\bar{t}) = \max_{x\in(-\infty,\infty)} g(x,\bar{t}) \tag{2.9}$$

due to the behavior given in (1.14). Then

$$g(\bar{x},\bar{t}) = \left|\int_{\rho^\star}^{\rho(\bar{x},\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau\right| + |u(\bar{x},\bar{t})| = s_\delta(\bar{t}). \tag{2.10}$$

First case: $\rho(\bar{x},\bar{t}) < \rho^\star$. Without loss of generality we assume that $u(\bar{x},\bar{t}) \le 0$ (in a similar way, we may prove the case of $u(\bar{x},\bar{t}) \ge 0$). Then

$$
\begin{aligned}
s_\delta(\bar{t}) &= -\int_{\rho^\star}^{\rho(\bar{x},\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - u(\bar{x},\bar{t}) \\
&= \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - \left( \int_0^{\rho(\bar{x},\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau + u(\bar{x},\bar{t}) \right) \\
&= \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - w(\rho(\bar{x},\bar{t}), u(\bar{x},\bar{t})). \qquad (2.11)
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
\int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - w(\rho(x,\bar{t}), u(x,\bar{t})) &\le |u(x,\bar{t}))| - \int_{\rho^\star}^{\rho(x,\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau \\
\le g(x,\bar{t}) \le g(\bar{x},\bar{t}) &= \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - w(\rho(\bar{x},\bar{t}), u(\bar{x},\bar{t})). \qquad (2.12)
\end{aligned}
$$

then

$$
w(\rho(\bar{x},\bar{t}), u(\bar{x},\bar{t})) \le w(\rho(x,\bar{t}), u(x,\bar{t})), \quad \text{for all } \ x \in (-\infty, \infty) \qquad (2.13)
$$

and thus

$$
\frac{\partial w}{\partial x}(\bar{x},\bar{t}) = 0, \ \frac{\partial^2 w}{\partial x^2}(\bar{x},\bar{t}) \ge 0. \qquad (2.14)
$$

Since $\rho(\bar{x},\bar{t}) < \rho^\star$, we have from (2.1), (2.4) and (2.14) that

$$
\frac{\partial w}{\partial t}(\bar{x},\bar{t}) \ge 0 > -\delta = -s'_\delta(\bar{t}) \qquad (2.15)
$$

or

$$
\frac{\partial (w + s_\delta)}{\partial t}(\bar{x},\bar{t}) > 0. \qquad (2.16)
$$

From (1.11), we have $(w + s_\delta)(\bar{x},\bar{t}) = \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau$. Then there exists some $t_1 \in (0,\bar{t})$ such that

$$
(w + s_\delta)(\bar{x}, t_1) < \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau \qquad (2.17)
$$

or

$$
g(\bar{x}, t_1) \ge \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - w(\bar{x}, t_1) > s_\delta(\bar{x}, t_1) \qquad (2.18)
$$

in contradiction to the definition of $\bar{t}$.

Second case: $\rho(\bar{x}, \bar{t}) \geq \rho^\star$. Without loss of generality we still assume that $u(\bar{x}, \bar{t}) \leq 0$ (in a similar way, we may prove the case of $u(\bar{x}, \bar{t}) \geq 0$). Then

$$
\begin{aligned}
s_\delta(\bar{t}) = g(\bar{x}, \bar{t}) &= \int_{\rho^\star}^{\rho(\bar{x},\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - u(\bar{x}, \bar{t}) \\
&= z(\rho(\bar{x}, \bar{t}), u(\bar{x}, \bar{t})) - \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau \\
\geq g(x, \bar{t}) &= |\int_{\rho^\star}^{\rho(x,\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau| + |u(\bar{x}, \bar{t})| \\
&= \int_0^{\rho(x,\bar{t})} \sqrt{\frac{h'(\tau)}{\tau}} d\tau - \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau + |u(\bar{x}, \bar{t})| \\
&\geq z(\rho(x, \bar{t}), u(x, \bar{t})) - \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau \quad \text{for all } x \in (-\infty, \infty).
\end{aligned}
\tag{2.19}
$$

Thus

$$
\frac{\partial z}{\partial x}(\bar{x}, \bar{t}) = 0, \ \frac{\partial^2 z}{\partial x^2}(\bar{x}, \bar{t}) \leq 0 \tag{2.20}
$$

and

$$
\frac{\partial z}{\partial t}(\bar{x}, \bar{t}) \leq 0 < \delta = s'_\delta(\bar{t}) \tag{2.21}
$$

from (2.2) and $z_{\rho\rho} \geq 0$ when $\rho \geq \rho^\star$.

From (2.19) and (2.21), we have respectively

$$
(z - s_\delta)(\bar{x}, \bar{t}) = \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau, \quad \frac{\partial(z - s_\delta)}{\partial t}(\bar{x}, \bar{t}) < 0. \tag{2.22}
$$

Then there exists some $t_1 \in (0, \bar{t})$ such that

$$
(z - s_\delta)(\bar{x}, t_1) > \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau \tag{2.23}
$$

or

$$
g(\bar{x}, t_1) \geq z(\bar{x}, t_1) - \int_0^{\rho^\star} \sqrt{\frac{h'(\tau)}{\tau}} d\tau > s_\delta(\bar{x}, t_1) \tag{2.24}
$$

in contradiction to the definition of $\bar{t}$.

Therefore, Lemma 2, and also the first part about the smooth viscosity solutions in Theorem 1 is proved. □

## 3 Fuchsian equation

To prove the pointwise convergence of $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ in the second part in Theorem 1, we shall first construct enough families of entropy-entropy flux pairs $(\eta, q)$ of system (1.1), then prove the compactness of $\eta_t + q_x$ in $H^{-1}_{loc}(R \times R^+)$, for these entropy-entropy flux pairs, with respect to the sequence of viscosity solutions $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ and last deduce the Young measure, determined by the sequence of viscosity solutions, to be a Dirac measure. In this section, we we shall study the behavior of solutions of Fuchsian equations, which is the basis of four families of Lax entropy-entropy flux pair we will construct in the next section.

A pair $(\eta, q)$ of real-valued maps is an entropy-entropy flux pair of system (1.1) if

$$(q_\rho, q_u) = (u\eta_\rho + h'(\rho)\eta_u, \rho\eta_\rho + u\eta_u). \tag{3.1}$$

Eliminating the $q$ from (3.1), we have

$$\eta_{\rho\rho} = \frac{h'(\rho)}{\rho}\eta_{uu} = (c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})\eta_{uu} \tag{3.2}$$

if $h(\rho) = \frac{c_1}{\gamma-1}\rho^{\gamma-1} + \frac{c_2}{\beta-1}\rho^{\beta-1}$.

If $k$ denotes a positive constant, then the function $\eta = g(\rho)e^{ku}$ solves (3.2) provided that

$$g''(\rho) = k^2(c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})g. \tag{3.3}$$

We introduce new functions $a(\rho), s(\rho)$ and let $g(\rho) = a(\rho)\phi(s(\rho))$. By simple calculations,

$$\begin{cases} g'(\rho) = a'(\rho)\phi(s) + a(\rho)\phi'(s)s'(\rho), \\ g''(\rho) = a''(\rho)\phi(s) + 2a'(\rho)\phi'(s)s'(\rho) + a(\rho)\phi''(s)s'^2(\rho) + a(\rho)\phi'(s)s''(\rho). \end{cases} \tag{3.4}$$

Let the coefficients before $\phi'(s)$ in the second part of (3.4) be zero

$$2a'(\rho)s'(\rho) + a(\rho)s''(\rho) = 0, \tag{3.5}$$

then

$$2\frac{a'(\rho)}{a(\rho)} + \frac{s''(\rho)}{s'(\rho)} = 0, \quad \text{or} \quad s'(\rho)a^2(\rho) = constant. \tag{3.6}$$

Furthermore, let $f(\rho) = (c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})$, and the coefficient $a(\rho)s'^2(\rho)$ before $\phi''(s)$ in the second part of (3.4) be $k^2a(\rho)f(\rho)$. Then

$$\begin{cases} s'(\rho) = kf^{\frac{1}{2}}(\rho),\ s(\rho) = k\int_0^\rho f^{\frac{1}{2}}(\tau)d\tau = k\int_0^\rho \tau^{\frac{\gamma-3}{2}}(c_1 + c_2\rho^{\beta-\gamma})^{\frac{1}{2}}d\tau, \\ a(\rho) = f^{-\frac{1}{4}}(\rho) = \rho^{\frac{3-\gamma}{4}}(c_1 + c_2\rho^{\beta-\gamma})^{-\frac{1}{4}}. \end{cases} \tag{3.7}$$

Thus, for such functions $a(\rho)$ and $s(\rho)$ given in (3.7), we know from (3.3) that $\phi(s)$ satisfies the following equation of Fuchsian type

$$\phi'' - (1 + \frac{A(\rho)}{s^2})\phi = 0, \tag{3.8}$$

where

$$A(\rho) = -\frac{a''(\rho)s^2}{k^2a(\rho)f(\rho)} = -\frac{a''(\rho)(\int_0^\rho f^{\frac{1}{2}}(\tau)d\tau)^2}{f^{\frac{3}{4}}(\rho)}. \tag{3.9}$$

By simple calculations,

$$a'(\rho) = -\frac{1}{4}(c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})^{-\frac{5}{4}}(c_1(\gamma-3)\rho^{\gamma-4} + c_2(\beta-3)\rho^{\beta-4}) \tag{3.10}$$

and

$$\begin{aligned} a''(\rho) &= \frac{5}{16}(c_1\rho^{\gamma-3}+c_2\rho^{\beta-3})^{-\frac{9}{4}}(c_1(\gamma-3)\rho^{\gamma-4}+c_2(\beta-3)\rho^{\beta-4})^2 \\ &\quad -\frac{1}{4}(c_1\rho^{\gamma-3}+c_2\rho^{\beta-3})^{-\frac{5}{4}}(c_1(\gamma-3)(\gamma-4)\rho^{\gamma-5}+c_2(\beta-3)(\beta-4)\rho^{\beta-5}) \\ &= (c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})^{-\frac{9}{4}}I(\rho), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} I(\rho) &= \frac{5}{16}c_1^2(\gamma-3)^2\rho^{2(\gamma-4)} + \frac{10}{16}c_1c_2(\gamma-3)(\beta-3)\rho^{\gamma+\beta-8} \\ &\quad +\frac{5}{16}c_2^2(\beta-3)^2\rho^{2(\beta-4)} - \frac{1}{4}c_1^2(\gamma-3)(\gamma-4)\rho^{2(\gamma-4)} \\ &\quad -\frac{1}{4}c_2^2(\beta-3)(\beta-4)\rho^{2(\beta-4)} - \frac{1}{4}c_1c_2((\gamma-3)(\gamma-4)+(\beta-3)(\beta-4))\rho^{\gamma+\beta-8} \\ &= \frac{1}{16}c_1^2(\gamma+1)(\gamma-3)\rho^{2(\gamma-4)} + \frac{1}{16}c_2^2(\beta+1)(\beta-3)\rho^{2(\beta-4)} \\ &\quad +(\frac{10}{16}(\gamma-3)(\beta-3) - \frac{1}{4}(\gamma-3)(\gamma-4) - \frac{1}{4}(\beta-3)(\beta-4))c_1c_2\rho^{\gamma+\beta-8}. \end{aligned} \tag{3.12}$$

Thus $a''(\rho)$ is negative near the line $\rho = 0$, and

$$a''(\rho) = (c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})^{-\frac{9}{4}} I(x,t) \approx (c_1\rho^{\gamma-3})^{-\frac{9}{4}} \frac{1}{16} c_1^2(\gamma+1)(\gamma-3)\rho^{2(\gamma-4)} \approx O(\rho^{-\frac{9}{4}(\gamma-3)+2(\gamma-4)}) \quad (3.13)$$

near $\rho = 0$. Furthermore,

$$\left(\int_0^\rho f^{\frac{1}{2}}(\tau)d\tau\right)^2 = \left(\int_0^\rho (c_1\tau^{\gamma-3} + c_2\tau^{\beta-3})^{\frac{1}{2}} d\tau\right)^2 \approx O(\rho^{\gamma-1}) \quad (3.14)$$

and

$$f^{\frac{3}{4}}(\rho) = (c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})^{\frac{3}{4}} \approx O(\rho^{\frac{3}{4}(\gamma-3)}). \quad (3.15)$$

Thus $A(\rho)$ is positive near the line $\rho = 0$, and

$$A(\rho) \approx O(\rho^{-\frac{9}{4}(\gamma-3)+2(\gamma-4)+(\gamma-1)-\frac{3}{4}(\gamma-3)}) \approx O(1) \quad (3.16)$$

near the line $\rho = 0$, where $O(1)$ denotes a bounded function.

**Lemma 3** **I.** *For any fixed $k > 0$, let $s_0 = k\int_0^M f^{\frac{1}{2}}(\tau)d\tau > 0$, where $M$ is a upper bound of $\rho$, and the initial data $\phi_1(s_0) > 0$, $\phi_1'(s_0) > 0$ satisfy $s_0\phi_1'(s_0) \le \phi_1(s_0)$. Then there exists a solution $\phi_1(s)$ of Equation (3.8), such that $\phi_1(s) > 0$, $\phi_1'(s) > 0$ for all $s > 0$,*

$$\frac{\phi_1(s_0)}{s_0}s \le \phi_1(s) \le \phi_1(s_0),\ \ \phi_1'(s) \le \phi_1'(s_0) \ \ \text{as} \ \ s \in (0, s_0), \quad (3.17)$$

*and*

$$\frac{\phi_1'(s)}{\phi_1(s)} = 1 + O\left(\frac{1}{s^2}\right), \quad d_1\phi_1(s)e^{-s} = 1 + O\left(\frac{1}{s}\right) \quad (3.18)$$

*as $s$ (more accurately, as $k$) approaches infinity, here and behind $d_i$, i=1,2,...denote suitable positive constants.*

**II.** *Let*

$$\phi_2(s) = \phi_1(s)\int_s^\infty (\phi_1(\tau)^{-2}d\tau > 0. \quad (3.19)$$

*Then $\phi_2(s)$ is a solution of (3.8), and $\phi_2(s) > 0$, $\phi_2'(s) < 0$ for all $s > 0$,*

$$\phi_2(s) \le d_2 + d_3|\ln s|, \quad \text{as}\ s \in (0, s_0), \quad (3.20)$$

*and*

$$\frac{\phi_2'(s)}{\phi_2(s)} = -1 + O\left(\frac{1}{s^2}\right), \quad d_4\phi_2(s)e^s = 1 + O\left(\frac{1}{s}\right) \tag{3.21}$$

*as $s$ (or as $k$) approaches infinity.*

***Proof of Lemma 3*** Since $A(\rho)$ in (3.8) is a continuous function of $\rho$ on $\rho \in (0, \infty)$ and bounded at $\rho = 0$, then there exists a positive constant $M$ such that $|A(\rho)| \leq M$ due to the boundedness of $\rho$ proved in Lemma 2. Moreover, since $A(\rho)$ is positive near the line $\rho = 0$, and $s = k\int_0^\rho \tau^{\frac{\gamma-3}{2}}(c_1 + c_2\rho^{\beta-\gamma})^{\frac{1}{2}}d\tau$, then clearly, we may choose $k$ large enough such that

$$1 + \frac{A(\rho)}{s^2} > \frac{1}{2}, \quad \text{for all} \quad \rho \geq 0. \tag{3.22}$$

Let $z(s) = \frac{\phi_1'(s)}{\phi_1(s)}$. Then from (3.8), $z$ satisfies

$$z'(s) + z^2(s) = 1 + \frac{A}{s^2} > 0, \quad \text{or} \quad \frac{z'(s)}{z^2(s)} + 1 > 0. \tag{3.23}$$

Integrating the second inequality in (3.23) from $s$ to $s_0$, we have

$$\frac{1}{z(s)} - \frac{1}{z(s_0)} + s_0 - s > 0, \quad \text{or} \quad \frac{1}{z(s)} > s > 0 \tag{3.24}$$

due to the condition $s_0 - \frac{1}{z(s_0)} \leq 0$. Thus $\phi_1(s) > 0$, $\phi_1'(s) > 0$ for all $0 < s < s_0$ from the strict positivity of $\phi_1''(s)$. Furthermore, we have $0 < \phi_1'(s) \leq \phi_1'(s_0)$ and $0 < \phi_1(s) \leq \phi_1(s_0)$ as $s \leq s_0$. Using the second inequality in (3.24), we have

$$\frac{\phi_1'(s)}{\phi_1(s)} < \frac{1}{s}. \tag{3.25}$$

Integrating (3.25) from $s$ to $s_0$, we have $\frac{\phi_1(s_0)}{s_0}s \leq \phi_1(s)$ and so the proof of (3.17).

To prove (3.18) (as $k$ becames large), since the strict positivity of $\phi_1''(s)$ on $s \in (0, \infty)$, $\phi_1(s_0) > 0$ and $\phi_1'(s_0) > 0$, clearly the function $\phi_1(s)$ satisfies $\phi_1(s) > 0, \phi_1'(s) > 0$ for all $s > s_0$, and $\phi_1(s), \phi_1'(s)$ both tend to infinite as $s$ approaches infinity. Since $\frac{\phi_1'(s)}{\phi_1(s)}$ has the form $\frac{\infty}{\infty}$ as $s$ goes to infinity, we have by applying the Vol'pert theorem to (3.8),

$$\lim_{s\to\infty} \frac{\phi'(s)}{\phi(s)} = \lim_{s\to\infty} \frac{\phi''(s)}{\phi'(s)} = \lim_{s\to\infty} \frac{(1 + \frac{A}{s^2})\phi}{\phi'(s)}. \tag{3.26}$$

Then

$$\lim_{s\to\infty}\left(\frac{\phi'(s)}{\phi(s)}\right)^2 = 1. \tag{3.27}$$

So we have

$$\lim_{s\to\infty}\frac{\phi_1'(s)}{\phi_1(s)} = 1. \tag{3.28}$$

Using (3.8) again, we have

$$\left(\frac{\phi'(s)}{\phi(s)}\right)' + \left(\frac{\phi'(s)}{\phi(s)}\right)^2 = 1 + \frac{A}{s^2}. \tag{3.29}$$

Let $y = \frac{\phi_1'(s)}{\phi_1(s)} - 1$. Then $y$ satisfies

$$y' + \left(1 + \frac{\phi_1'(s)}{\phi_1(s)}\right) y = \frac{A}{s^2}. \tag{3.30}$$

Integrating Eq. (3.30) from $d$ to $s$ ($d$ is a positive constant), we have

$$e^{\int_d^s(\frac{\phi_1'(t)}{\phi_1(t)}+1)dt}\, y(s) - y(d) = \int_d^s e^{\int_d^\tau\left(\frac{\phi_1'(t)}{\phi_1(t)}+1\right)dt}\frac{A}{\tau^2}d\tau, \tag{3.31}$$

which implies that

$$\frac{\phi_1(s)}{\phi_1(d)}e^{s-d}y(s) - y(d) = \int_d^s \frac{\phi_1(\tau)}{\phi_1(d)}e^{\tau-d}\frac{A}{\tau^2}d\tau, \tag{3.32}$$

and hence

$$y(s) = \frac{\phi_1(d)}{\phi_1(s)}e^{d-s}y(d) + \int_d^s \frac{\phi_1(\tau)}{\phi_1(s)}e^{\tau-s}\frac{A}{\tau^2}d\tau. \tag{3.33}$$

Let $d = \frac{1}{2}s$. Then clearly $\phi_1(d) < \phi_1(s), \phi_1(\tau) < \phi_1(s)$ since $\phi_1' > 0$.

Since $y$ is bounded from (3.28), we have from (3.33) that

$$|y(s)| \le Me^{-\frac{1}{2}s} + \frac{4|A|}{s^2}(1 - e^{-\frac{1}{2}s}), \tag{3.34}$$

where $M$ is the bound of $y$. Therefore

$$y(s) = O\left(\frac{1}{s^2}\right) \tag{3.35}$$

as $s$ approaches infinity. The first part in (3.18) is proved.

Now we prove the remaining estimate in (3.18).

Let $\phi_1(s) = b(s)e^s$. Then

$$\frac{b'(s)}{b(s)} = y(s) = O\left(\frac{1}{s^2}\right) \tag{3.36}$$

for $s$ large. Thus

$$b(s) = b(s_1)e^{\int_{s_1}^{s} y(s)ds}, \tag{3.37}$$

for any fixed $s_1 > 0$.

Since $y(s) = O(\frac{1}{s^2})$ as $s$ approaches infinity, the limit of $\lim_{s\to\infty} b(s)$ exists. Then (3.37) gives

$$\begin{aligned} b(s) &= b(\infty)e^{-\int_s^\infty y(s)ds} \\ &= b(\infty) + b(\infty)(e^{-\int_s^\infty y(s)ds} - 1). \end{aligned} \tag{3.38}$$

Since $\displaystyle\int_s^\infty y(s)ds = O(\frac{1}{s})$ as $s$ approaches infinity, applying the Taylor expansion to $e^{-\int_s^\infty y(s)ds} - 1$ gives

$$b(s) = b(\infty) + O(\frac{1}{s}) \tag{3.39}$$

as $s$ approaches infinity. Thus, we have the proof of Part **I** in Lemma 3.

It is easy to prove that $\phi_2(s)$, given by (3.19), satisfies (3.8). Furthermore, since $\phi_1''(s) > 0$ and the second estimate in (3.18), we have from (3.19) that

$$\begin{aligned} \phi_2'(s) &= \phi_1'(s)\int_s^\infty (\phi_1(\tau)^{-2}d\tau - \frac{1}{\phi_1(s)} \\ &< \int_s^\infty \phi_1'(\tau)(\phi_1(\tau)^{-2}d\tau - \frac{1}{\phi_1(s)} = 0. \end{aligned} \tag{3.40}$$

Thus $\phi_2(s) > 0$, $\phi_2'(s) < 0$ for $s > 0$.

Using the estimates in (3.17) and (3.18), we have

$$\begin{aligned} \phi_2(s) &= \phi_1(s)\left(\int_s^{s_0} (\phi_1(\tau)^{-2}d\tau + \int_{s_0}^\infty (\phi_1(\tau)^{-2}d\tau)\right) \\ &\le d_0 + \phi_1(s)\int_s^{s_0} (\phi_1(\tau)^{-2}d\tau \le d_0 + \int_s^{s_0} (\phi_1(\tau)^{-1}d\tau \\ &\le d_0 + \int_s^{s_0} \frac{s_0}{\phi_1(s_0)\tau}d\tau \le d_2 + d_3|\ln s|. \end{aligned} \tag{3.41}$$

Thus we have the proof of (3.20). In a similar way like the proof of (3.18), we may prove (3.21). **Lemma 3 is proved.** □

## 4 Lax entropy and compactness of $\eta_t + q_x$ in $H^{-1}_{loc}$

Based on the results given in Lemma 3 on the solutions of the Fuchsian equation (3.8), we may construct the entropy-entropy flux pair $(\eta, q)$ of Lax type of system (1.1), and prove the compactness of $\eta_t + q_x$ in $H^{-1}_{loc}(R \times R^+)$, for these entropy-entropy flux pairs.

**Lemma 4** *There exists four families of Lax entropy-entropy flux pair of system (1.1) as follows*

$$\begin{cases} \eta^1_k = a(\rho)\phi_1(s)e^{-s}e^{kw} = e^{kw}(a(\rho) + O(\frac{1}{k})), \\ q^1_k = \eta^1_k\big(\lambda_2 + \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1) + O(\frac{1}{k^2})\big), \end{cases} \tag{4.1}$$

$$\begin{cases} \eta^2_{-k} = a(\rho)\phi_2(s)e^{s}e^{-kw} = e^{-kw}(a(\rho) + O(\frac{1}{k})), \\ q^2_{-k} = \eta^1_{-k}\big(\lambda_2 - \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1) + O(\frac{1}{k^2})\big), \end{cases} \tag{4.2}$$

$$\begin{cases} \eta^2_k = a(\rho)\phi_2(s)e^{s}e^{-kz} = e^{-kz}(a(\rho) + O(\frac{1}{k})), \\ q^2_k = \eta^2_k\big(\lambda_1 + \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1) + O(\frac{1}{k^2})\big), \end{cases} \tag{4.3}$$

*and*

$$\begin{cases} \eta^1_{-k} = a(\rho)\phi_1(s)e^{-s}e^{kz} = e^{kz}(a(\rho) + O(\frac{1}{k})), \\ q^1_{-k} = \eta^1_{-k}\big(\lambda_1 - \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1) + O(\frac{1}{k^2})\big), \end{cases} \tag{4.4}$$

*on* $\rho > 0$ *as* $k$ *approaches infinity.*

***Proof of Lemma 4*** We only prove (4.1) because we can prove (4.2)–(4.4) in a similar way.

From (3.1), we have

$$q_u = \rho\eta_\rho + u\eta_u. \tag{4.5}$$

If

$$\eta_k = g(\rho)e^{ku}, \tag{4.6}$$

then

$$(q_k)_u = \rho g'(\rho)e^{ku} + kug(\rho)e^{ku} \tag{4.7}$$

and hence one entropy flux corresponding to $\eta_k$ is

$$q_k = ug(\rho)e^{ku} + (\rho g' - g)e^{ku}/k. \tag{4.8}$$

Let $\eta_k^1 = a(\rho)\phi_1(s)e^{ku}$. Then using Lemma 3, we have

$$\eta_k^1 = a(\rho)\phi_1(s)e^{-s}e^{kw} = e^{kw}(a(\rho) + O(\frac{1}{k})) \tag{4.9}$$

on $\rho > 0$ as $k$ approaches infinity and the corresponding flux function is of the form

$$\begin{aligned} q_k^1 &= \eta_k^1\Big(u + \frac{\rho g'(\rho) - g(\rho)}{kg(\rho)}\Big) \\ &= \eta_k^1\big(u + f^{\frac{1}{2}}\rho\frac{\phi_1'(s)}{\phi_1(s)} + \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1)\big) \\ &= \eta_k^1\big(u + f^{\frac{1}{2}}\rho + f^{\frac{1}{2}}\rho(\frac{\phi_1'(s)}{\phi_1(s)} - 1) + \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1)\big) \\ &= \eta_k^1\big(\lambda_2 + \frac{1}{k}(\frac{\rho a'(\rho)}{a(\rho)} - 1) + O(\frac{1}{k^2})\big) \end{aligned} \tag{4.10}$$

on $\rho > 0$ as $k$ approaches infinity. **Lemma 4 is proved.** □

**Lemma 5** *For the entropy-entropy flux pairs* $(\eta, q)$ *of Lax type constructed in Sect. 3, the convex entropy-entropy flux*

$$\begin{cases} \eta^\star = \frac{1}{2}u^2 + \int_0^\rho \int_0^y \frac{h'(\tau)}{\tau} d\tau dy, \\ q^\star = \frac{1}{3}u^3 + \rho u \int_0^\rho \frac{h'(\tau)}{\tau} d\tau \end{cases} \tag{4.11}$$

*and two nature entropy-entropy flux pairs*

$$(\eta_1, q_1) = (\rho, \rho u),\ (\eta_2, q_2) = \Big(u, \frac{1}{2}u^2 + h(\rho)\Big), \tag{4.12}$$

$\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x$ *is compact in* $H_{loc}^{-1}(R \times R^+)$ *with respect to the approximated solutions* $(\rho^\varepsilon, u^\varepsilon)$ *constructed by the viscosity method in Sect. 2.*

***Proof of Lemma 5*** It is easy to check that $\eta^\star$ is a convex entropy of system (1.1) with the corresponding entropy flux $q^\star$. Then using this convex entropy, we can easily prove that

$$\varepsilon^{\frac{1}{2}}\partial_x u^\varepsilon,\ \varepsilon^{\frac{1}{2}}\sqrt{\frac{h'(\rho^\varepsilon)}{\rho^\varepsilon}}\partial_x\rho^\varepsilon \text{ are uniformly bounded in } L_{loc}^2(R \times R^+), \tag{4.13}$$

and so the compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x$ in $H^{-1}_{loc}(R \times R^+)$ for the entropy-entropy flux pairs $(\eta^\star, q^\star)$, $(\eta_1, q_1)$ and $(\eta_2, q_2)$ (cf. [19]).

Now we prove the compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x$ in $H^{-1}_{loc}(R \times R^+)$ for the Lax entropy-entropy flux pair $(\eta^2_k, q^2_k)$. In a similar way, we may obtain the proof for $(\eta^1_k, q^1_k)$, $(\eta^2_{-k}, q^2_{-k})$ and $(\eta^1_{-k}, q^1_{-k})$.

Let $\eta(\rho, u) = \eta^2_k(\rho, u) = a(\rho)\phi_2(s)e^{ku} = g(\rho)e^{ku}$. Then $g(\rho)$ satisfies (3.3) and

$$g(\rho) = a(\rho)\phi_2(s) \le \rho^{\frac{3-\gamma}{4}}(c_1 + c_2\rho^{\beta-\gamma})^{-\frac{1}{4}}(d_2 + d_3|\ln s|) = O(1), \quad \text{near} \quad \rho = 0, \tag{4.14}$$

and

$$\begin{aligned} g'(\rho) &= g'(M) - \int_\rho^M g''(\rho)d\rho \\ &= g'(M) - \int_\rho^M k^2(c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})g(\rho)d\rho \\ &= g'(M) + O(\rho^{\frac{3-\gamma}{4}+(\gamma-3)+1}|\ln \rho|) \\ &= g'(M) + O(\rho^{\frac{\gamma-3}{2}}\rho^{\frac{\gamma+1}{4}}|\ln \rho|) = O(\rho^{\frac{\gamma-3}{2}}), \quad \text{near} \quad \rho = 0, \end{aligned} \tag{4.15}$$

where $M$ is the upper bound of $\rho$. Thus, by simple calculations,

$$\begin{cases} \eta(\rho, u) = O(1), \ \eta_u(\rho, u) = k\eta(\rho, u) = O(1), \ \eta_\rho(\rho, u) = g'(\rho)e^{ku} = O(\rho^{\frac{\gamma-3}{2}}), \\ \eta_{uu}(\rho, u) = k^2\eta(\rho, u) = O(1), \ \eta_{\rho u}(\rho, u) = kg'(\rho)e^{ku} = O(\rho^{\frac{\gamma-3}{2}}), \\ \eta_{\rho\rho}(\rho, u) = g''(\rho)e^{ku} = k^2(c_1\rho^{\gamma-3} + c_2\rho^{\beta-3})g(\rho)e^{ku} = O(\rho^{\gamma-3}), \\ q^2_k(\rho, u) = ug(\rho)e^{ku} + (\rho g' - g)e^{ku}/k = O(1), \end{cases} \tag{4.16}$$

near $\rho = 0$.

Thus, multiplying system (1.8) by $\nabla\eta^2_k$, we have

$$(\eta^2_k)_t + (q^2_k)_x = \varepsilon(\eta^2_k)_{xx} - \varepsilon\big((\eta^2_k)_{uu}u^2_x + 2(\eta^2_k)_{\rho u}u_x\rho_x + (\eta^2_k)_{\rho\rho}\rho^2_x\big), \tag{4.17}$$

where

$$\varepsilon(\eta^2_k)_{xx} \ \text{ are compact in } \ H^{-1}_{loc}(R \times R^+) \tag{4.18}$$

and

$$\varepsilon\big((\eta^2_k)_{uu}u^2_x + 2(\eta^2_k)_{\rho u}u_x\rho_x + (\eta^2_k)_{\rho\rho}\rho^2_x\big) \tag{4.19}$$

are bounded in $L^1_{loc}(R \times R^+)$ due to the estimates (4.13) and (4.16). Therefore using the Murat's theorem (cf. [20]), we obtain the compactness of $\eta_k^2(\rho^\varepsilon, u^\varepsilon)_t + q_k^2(\rho^\varepsilon, u^\varepsilon)_x$ in $H^{-1}_{loc}(R \times R^+)$. **Lemma 5 is proved.** □

## 5 Reduction of the Young measure $\nu$

In this section, we shall prove that the family of positive measures $\nu_{x,t}$, determined by the sequence of viscosity solutions $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ of the Cauchy problem (1.8), (1.9), must be Dirac measures. Then using the Young measure representation theorem in [20], we get the proof of (1.16), which implies that the function $(\rho(x,t), u(x,t))$ of support set points of these Dirac measures is a weak solution of the Cauchy problem (1.1), (1.2).

Since the viscosity solutions $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ of the Cauchy problem (1.8), (1.9) are uniformly bounded in $L^\infty$ space, we consider the family of compactly supported probability measures $\nu_{x,t}$. Without any loss of generality we may fix $(x,t) \in R \times R^+$ and consider only one measure $\nu$.

For any entropy-entropy flux pairs $(\eta_i, q_i)$, $i = 1, 2$, of system (1.1), satisfying the compactness of $\eta(\rho^\varepsilon, u^\varepsilon)_t + q(\rho^\varepsilon, u^\varepsilon)_x$ in $H^{-1}_{loc}(R \times R^+)$, we have from the basic theorems in the theory of compensated compactness that (cf. [1, 20, 24])

$$\begin{aligned} &\overline{\eta_1(\rho^\varepsilon, u^\varepsilon)} \cdot \overline{q_2(\rho^\varepsilon, u^\varepsilon)} - \overline{\eta_2(\rho^\varepsilon, u^\varepsilon)} \cdot \overline{q_1(\rho^\varepsilon, u^\varepsilon)} \\ &\quad = \overline{\eta_1(\rho^\varepsilon, u^\varepsilon) q_2(\rho^\varepsilon, u^\varepsilon) - \eta_2(\rho^\varepsilon, u^\varepsilon) q_1(\rho^\varepsilon, u^\varepsilon)}. \end{aligned} \tag{5.1}$$

where $\overline{g(\theta^\varepsilon)}$ denotes the weak-star limit of $g(\theta^\varepsilon)$, and the following measure equation:

$$< \nu, \eta_1 >< \nu, q_2 > - < \nu, \eta_2 >< \nu, q_1 >=< \nu, \eta_1 q_2 - \eta_2 q_1 > . \tag{5.2}$$

Let Q denote the smallest characteristic rectangle:

$$Q = \{(\rho, u) : w_- \le w \le w_+,\ z_- \le z \le z_+,\ \rho \ge 0\}. \tag{5.3}$$

We now prove that the support set of $\nu$ is contained in one point $(w^\star, z^\star)$.

We first prove that if supp $\nu$ is contained in the line $\rho = 0$, then the support set of $\nu$ is $(w^\star, z^\star) = (u^\star, -u^\star)$, for a suitable value $u^\star$.

In fact, by applying the measure equation (5.2) to $(\eta_1, q_1) = (\eta^\star, q^\star)$ and $(\eta_2, q_2) = (u, \frac{1}{2}u^2 + h(\rho))$, where $(\eta^\star, q^\star)$ is given in (4.11), we have on the line $\rho = 0$,

$$< \nu, \frac{1}{2}u^2 >< \nu, \frac{1}{2}u^2 > - < \nu, u >< \nu, \frac{1}{3}u^3 >= -\frac{1}{12} < \nu, u^4 > \tag{5.4}$$

or

$$< \nu, u^4 > +3(< \nu, u^2 >)^2 - 4 < \nu, u >< \nu, u^3 >= 0, \tag{5.5}$$

which is equivalent to

$$< \nu, (u-\bar{u})^4 >= -3(< \nu, u^2 - \bar{u}^2 >)^2, \tag{5.6}$$

where $\bar{u} =< \nu, u >$ be the weak-star limit of $u^\varepsilon$.

From (5.5), we have

$$< \nu, (u-\bar{u})^4 >= 0, \quad \text{and} \quad < \nu, u^2 - \bar{u}^2 >= 0. \tag{5.7}$$

From $< \nu, u^2 >= \bar{u}^2 =< \nu, u >^2$, we have the pointwise convergence of $u^\varepsilon$ as $\varepsilon$ tends to zero. Thus the supp $\nu$ is contained in one point $(0, u^\star)$ in the region of $(\rho, u)$ or $(u^\star, -u^\star)$ in the $(w, z)$ plane.

Second, we assume that supp $\nu$ is not contained in the line $\rho = 0$. Then $< \nu, \eta_k^1 > > 0$, $< \nu, \eta_{-k}^2 > > 0$, $< \nu, \eta_k^2 > > 0$ and $< \nu, \eta_{-k}^1 > > 0$ where $\eta_k^1, \eta_{-k}^2, \eta_k^2$ and $\eta_{-k}^1$ are given by (4.1)– (4.4).

We introduce two new probability measures $\mu_k^+$ and $\mu_k^-$ on Q, defined by

$$< \mu_k^+, h >=< \nu, h\eta_k^1 > / < \nu, \eta_k^1 > \tag{5.8}$$

and

$$< \mu_k^-, h >=< \nu, h\eta_{-k}^2 > / < \nu, \eta_{-k}^2 >, \tag{5.9}$$

where $h = h(\rho, u)$ denotes an arbitrary continuous function. Clearly $\mu_k^+$ and $\mu_k^-$ both are uniformly bounded with respect to $k$. Then as a consequence of weak-star compactness, there exist probability measures $\mu^\pm$ on Q such that

$$< \mu^\pm, h >= \lim_{k\to\infty} < \mu_k^\pm, h > \tag{5.10}$$

after the selection of an appropriate subsequence.

Moreover, the measures $\mu^+, \mu^-$ are respectively concentrated on the boundary sections of Q associated with $w$, i.e.

$$\text{supp } \mu^+ = Q \bigcap \{(\rho, u) : w = w_+\} \tag{5.11}$$

and

$$\text{supp } \mu^- = Q \bigcap \{(\rho, u) : w = w_-\}. \tag{5.12}$$

In fact, for any function $h(w, z) \in C_0(Q)$, satisfying

$$\text{supp } h(w, z) \subset Q \bigcap \{w \le w_0\}, \tag{5.13}$$

where $w_0 < w_+$ is any number, as $k \to \infty$, we have

$$\begin{aligned}\frac{|<\nu, h\eta_k^1>|}{|<\nu,\eta_k^1>|} &= \frac{|<\nu, he^{kw}(a(\rho)+O(\frac{1}{k}))>|}{|<\nu, e^{kw}(a(\rho)+O(\frac{1}{k}))>|} \\ &\leq \frac{c_1 e^{k(w_o+\delta)}}{c_2 e^{k(w_+-\delta)}} = \frac{c_1}{c_2} e^{k(w_0+2\delta-w_+)} \to 0,\end{aligned} \tag{5.14}$$

where $c_1, c_2$ are two suitable positive constants and $\delta > 0$ satisfies $2\delta < w_+ - w_0$, since $Q$ is the smallest characteristic rectangle of $\nu$. Thus we get the proof of (5.11). Similarly we can prove (5.12).

Let $(\eta_1, q_1) = (\eta_k^1, q_k^1)$ in (5.2). We have

$$< \nu, q_2 > - < \nu, \eta_2 > \frac{< \nu, q_k^1 >}{< \nu, \eta_k^1 >} = \frac{< \nu, \eta_k^1 q_2 - \eta_2 q_k^1 >}{< \nu, \eta_k^1 >}. \tag{5.15}$$

Noticing the estimate (4.10) between $\eta_k^1$ and $q_k^1$, and letting $k \to \infty$ in (5.15), we have

$$< \nu, q_2 > - < \nu, \eta_2 >< \mu^+, \lambda_2 >=< \mu^+, q_2 - \lambda_2\eta_2 > . \tag{5.16}$$

Similarly, let $(\eta_1, q_1) = (\eta_{-k}^2, q_{-k}^2)$ in (5.2) and use the estimate (4.3) between $\eta_{-k}^2$ and $q_{-k}^1$. We have

$$< \nu, q_2 > - < \nu, \eta_2 >< \mu^-, \lambda_2 >=< \mu^-, q_2 - \lambda_2\eta_2 > . \tag{5.17}$$

Let $(\eta_1, q_1) = (\eta_k^1, q_k^1)$ and $(\eta_2, q_2) = (\eta_{-k}^2, q_{-k}^2)$ in (5.2). We have

$$\frac{< \nu, q_{-k}^2 >}{< \nu, \eta_{-k}^2 >} - \frac{< \nu, q_k^1 >}{< \nu, \eta_k^1 >} = \frac{< \nu, \eta_k^1 q_{-k}^2 - \eta_{-k}^2 q_k^1 >}{< \nu, \eta_{-k}^2 >< \nu, \eta_k^1 >}. \tag{5.18}$$

We now prove that $w_- = w_+$. If not, choose $\delta_0 > 0$ such that $2\delta_0 < w_+ - w_-$. Then

$$< \nu, \eta_{-k}^2 > \geq c_1 e^{-k(w_- + \delta_0)}, \quad < \nu, \eta_k^1 > \geq c_2 e^{k(w_+ - \delta_0)} \tag{5.19}$$

for two suitable positive constants $c_1, c_2$ and hence, the right-hand side of (5.18) satisfies

$$\frac{< \nu, \eta_k^1 q_{-k}^2 - \eta_{-k}^2 q_k^1 >}{< \nu, \eta_{-k}^2 >< \nu, \eta_k^1 >} = O(\frac{1}{k}) e^{-k(w_+ - w_- - 2\delta_0)} \to 0, \quad \text{as } k \to \infty, \tag{5.20}$$

resulting from the estimates given by (4.1) and (4.2).

Letting $k \to \infty$ in (5.18), we have $< \nu^+, \lambda_2 >=< \nu^-, \lambda_2 >$. Combining this with (5.16)-(5.17) gives the following relation:

$$< \mu^+, q - \lambda_2\eta >=< \mu^-, q - \lambda_2\eta > \tag{5.21}$$

for any $(\eta, q)$ satisfying that $\eta_t + q_x$ is compact in $H^{-1}_{loc}(R \times R^+)$.

Let $(\eta, q)$ in (5.21) be $(\eta^2_{-k}, q^2_{-k})$. If $w_+ - w_- > 2\delta_0$, we get from the left-hand side of (5.21) that

$$| < \mu^+, q - \lambda_2 \eta > | \leq \frac{c_1}{k} e^{-k(w_+ - \delta_0)} \tag{5.22}$$

and from the right-hand side

$$| < \mu^-, q - \lambda_2 \eta > | \geq \frac{c_2}{k} e^{-k(w_- + \delta_0)} \tag{5.23}$$

for two positive constants $c_1, c_2$. This is impossible and hence, $w_+$ must equal $w_-$.

Similarly to the above proof, we can use $(\eta^2_k, q^2_k)$, $(\eta^1_{-k}, q^1_{-k})$ constructed in Sect. 4 to prove $z_+ = z_-$. Thus the support set of $\nu$ is either $(0, u^\star)$ or another point $(w^\star, z^\star)$.

This completes the proof of Theorem 1.

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## Declarations

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