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THE RELAXATION LIMIT FOR SYSTEMS OF BROADWELL TYPE

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1. Introduction. This paper considers the Cauchy problem for the following systems of Broadwell's type

$$\begin{cases} f_{1t} + f_{1x} = \dfrac{F(f_1, f_2, f_3)}{\tau} \\ f_{2t} - f_{2x} = \dfrac{F(f_1, f_2, f_3)}{\tau} \\ f_{3t} = -\dfrac{F(f_1, f_2, f_3)}{2\tau} \end{cases} \tag{1}$$

When the nonlinear function F takes the special form $f_1 f_2 - f_3^2$, (1) is a simple mathematical model of gas kinetics, the so called Broadwell model [1] (see also [2], [6] and the references therein). It describes an idealization of a discrete velocity gas of particles in one dimension subject to a simple binary collision mechanism.

Let $\rho = f_1 + f_2 + 4f_3$, $m = f_1 - f_2$, $s = f_3$. (1) may be written as follows:

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + (\rho - 4s)_x = 0 \\ s_t + \dfrac{\bar{F}(\rho, m, s)}{\tau} = 0 \end{cases} \tag{2}$$

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The conditions for (ρ, m, s) to be a local Maxwellian are

$$s = \frac{1}{6}\big(2 - \sqrt{1 + 3\frac{m^2}{\rho^2}}\big)\rho \tag{3}$$

for the case $F = f_1 f_2 - f_3^2$. (ρ, m, s) is a local Maxwellian for

$$s = h(\rho) \tag{4}$$

in case $F = h_1(f_3) - h_2(f_1 + f_2 + 4f_3)$ for some nonlinear function h_1 and h_2. The equilibrium system corresponding to (3) are the following Euler equations:

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \big(\rho F(u)\big)_x = 0 \end{cases} \tag{5}$$

where $F(u) = \frac{1}{3}\big(2(1 + 3u^2)^{\frac{1}{2}} - 1\big)$.

The equilibrium system corresponding to (4) is the following p - system:

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \big(\rho - 4h(\rho)\big)_x = 0 \end{cases} \tag{6}$$

The asymptotic relationship between the solutions of the Broadwell model (1) and the solutions of the Euler equations (5) as τ goes to zero has been investigated by many authors (See [2] and the references therein). All authors considered the limit assuming some special structure of the solution, such as continuity [2], Riemann solution, finite number of shock waves.

In this paper, we study the Cauchy problem (2) with bounded L^2 measurable initial data

$$(\rho, m, s)_{|t=0} = \big(\rho_0(x), m_0(x), s_0(x)\big). \tag{7}$$

When the local Maxwellian is given by (4), we show that the solution of the equilibrium system (6) is given by the limit of the solutions of the viscous approximation

$$\begin{cases} \rho_t + m_x = \epsilon \rho_{xx} \\ m_t + (\rho - 4s)_x = \epsilon m_{xx} \\ s_t + \dfrac{\bar{F}(\rho, m, s)}{\tau} = \epsilon s_{xx} \end{cases} \tag{8}$$

as ϵ and τ go to zero. Our method is the compensated compactness. This method has shown itself to be powerful in solving some relaxation limit problems [3], [4], [7], [9], [10], [11]. When dealing with systems of more than two equations it is well known that the one basic difficulty is the a priori estimate independent of the approximate paprameter ϵ in a suitable L^p space ($p > 1$). Since system (2) in general can not be diagonalized by using Riemann invariants, it is not to be expected that viscosity solutions $(\rho^\epsilon, m^\epsilon, s^\epsilon)$ of the Cauchy problem (8) will be bounded in L^∞, uniformly in ϵ, by using the invariant region principle.

We have to search for solutions of the system (2) in L^p space. Similar results about zero relaxation systems of three equations are discussed in [11]. In paper [11], we studied the following system:

$$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma(v,s)_x = 0 \\ s_t + \dfrac{s - f(v)}{\tau} = 0 \end{cases} \tag{9}$$

where $\sigma(v,s)$ is a nonlinear function of v and s, but $f(v)$ must be a linear function cv in order to make the technique used in [11] work. System (2) is of a different form. The flux functions are linear, but the zero-th order term is nonlinear. Some authors suggested solving the following system:

$$\begin{cases} u_t^\tau + div(v^\tau) = 0 \\ v_t^\tau + \mu\ div(u^\tau) = \dfrac{1}{\tau}\big(f(u^\tau) - v^\tau\big) \end{cases} \tag{10}$$

as an approximation to the general nonlinear hyperbolic system

$$\big\{\ u_t + div\big(f(u)\big) = 0. \tag{11}$$

So in some sense, the study of the system (2) is more significant than that of (9) in comparing the relationship between (10) and (11).

Another difficulty in applying the compensated compactness to the system (2) is the compactness analysis of the viscosity solutions of the Cauchy problem (8) in L^p. To overcome this difficulty, we adopt the method used in [11] to reduce the equations to two equations and then use the entropy - entropy flux pairs of system (6) as constructed by Jim Shearer [14] and the framework given by Serre and Shearer [13] to realize our aim.

This paper is structured as follows: In Section 2 we consider the existence of viscosity solutions of system (8) with initial data

$$\left(\rho^{\epsilon}, m^{\epsilon}, s^{\epsilon}\right)_{|t=0} = \left(\rho_0^{\epsilon}, m_0^{\epsilon}, s_0^{\epsilon}\right) \tag{12}$$

where $\left(\rho_0^{\epsilon}, m_0^{\epsilon}, s_0^{\epsilon}\right)$ are smooth functions obtained by smoothing the initial data (7) with a mollifier. The existence is based on the standard local existence theory by using the contraction mapping principle to an integral representation of (8) and an a-priori estimation of the local solution depending on ϵ and τ. The a-priori L^{∞} bound depending on ϵ and τ is obtained by the energy method. In section 3, the compensated compactness method is used to study convergence of the viscosity solutions $\left(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}\right)$. First the convergence of $(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})$ is shown, and then using estimate (6), the convergence of $s^{\epsilon,\tau}$. When taking $\delta = O(\epsilon)$, the global weak solution of the equilibrium (6) is obtained as ϵ goes to zero.

In this paper we make the following assumptions on $\bar{F}$ and the initial data:

($\mathbf{A_1}$): $\bar{F}(\rho, m, s) = H(s) - \rho$; $H(s) \in C^3(R)$, $H'(s) \geq 4 + c$ for some constants $c > 0$

Since $H'(s) > 0$, $H(s) = \rho$ has an inverse function $H^{-1}(\rho) = s$. Let $\sigma(\rho) = \rho - 4h(\rho)$, $h(\rho) = H^{-1}(\rho)$ and $\sigma(\rho)$ satisfy all the conditions in [16], namely

($\mathbf{A_2}$): Strict hyperbolicity: $\sigma'(\rho) \geq \sigma_0 > 0$ with $\sigma_0 =$ constant

($\mathbf{A_3}$): Genuine nonlinearity except at a point: $\sigma''(\lambda_0) = 0$ and $\sigma''(\lambda) \neq 0$ for $\lambda \neq \lambda_0$.

($\mathbf{A_4}$): Growth constraints: $\frac{\sigma''}{(\sigma')^{\frac{5}{4}}}, \frac{\sigma'''}{(\sigma')^{\frac{7}{4}}} \in L^2$; $\frac{\sigma''}{(\sigma')^{\frac{3}{2}}}, \frac{\sigma'''}{(\sigma')^2} \in L^{\infty}$ $\frac{\sigma(\rho)}{\Sigma(\rho)} \to 0$ as $|\rho| \to \infty$ and there are constants c_1, c_2 with $c_1 > \frac{1}{2}$ such that $\left(\sigma'(\rho)\right)^{c_1} \leq c_2\left(1 + \Sigma(\rho_0)\right)$, where $\Sigma(\rho_0) = \int_0^{\rho} \sigma(s)ds$.

Remark 1. From (A_1)

$$\frac{d\left(H^{-1}(\rho)\right)}{d\rho} = \frac{ds}{d\rho} = \frac{1}{H'(s)} \leq \frac{1}{4+c},$$

then $\sigma'(\rho) = 1 - 4h'(\rho) \geq \frac{c}{4+c}$ and $\sigma'(\rho) < 1$ from the positivity of $H'(s)$. So the strict hyperbolicity is derived directly from (A_1).

Remark 2. Consider the special choice of $H(s) \in C^3(R)$ with

$$H(s) = (4+c)s + d|s|^{\alpha-1} \cdot s \tag{13}$$

for $|s|$ large enough, where $d > 0$, $\alpha > 1$ are constants. Then $H^{-1}(\rho) \approx \big(sgn(\rho)\big)(|\rho|^{\frac{1}{\alpha}})$ for $|\rho|$ large enough. Therefore,

$$|\sigma''(\rho)| = \Big|\frac{4H''(s)}{\big(H'(s)\big)^2} \cdot \frac{ds}{d\rho}\Big| = \Big|\frac{4H''(s)}{\big(H'(s)\big)^3}\Big| = O\Big(\frac{1}{|s|^{2\alpha-1}}\Big) = O\Big(\frac{1}{|\rho|^{\frac{2\alpha-1}{\alpha}}}\Big) \quad (14)$$

as $|\rho|$ is large. So $\sigma''(\rho) \in L^1 \cap L^\infty$ if $\alpha > 1$. Similarly, we have $\sigma'''(\rho) \in L^1 \cap L^\infty$ if $\alpha > 1$. So all the conditions in (A_4) are satisfied when $H(s)$ is given by (13). (A_3) is also true if $H(s)$ is given by (13) for all $s \in R$ and $\alpha > 2$.

We have the following assumption about the initial data (7):

$(\mathbf{A_5})$: $\rho_0(x)$, $m_0(x)$, $s_0(x)$ are all bounded in $L^2(R)$ and tend to zero as $|x| \to \infty$ sufficiently fast such that the smooth functions given in (12) satisfy

$$\lim_{|x|\to\infty}\Big(\frac{d^i\rho_0^\epsilon(x)}{dx^i}, \frac{d^i m_0^\epsilon(x)}{dx^i}, \frac{d^i s_0^\epsilon(x)}{dx^i}\Big) = (0,0,0) \quad i = 0,1 \quad (15)$$

$$|\rho_0^\epsilon(x)|_{H^1(R)} \le M(\epsilon), |m_0^\epsilon(x)|_{H^1(R)} \le M(\epsilon), |s_0^\epsilon(x)|_{H^1(R)} \le M(\epsilon). \quad (16)$$

From the basic property of the mollifier, we have that

$$\big(\rho_0^\epsilon(x), m_0^\epsilon(x), s_0^\epsilon(x)\big) \to \big(\rho_0(x), m_0(x), s_0(x)\big) \quad (17)$$

uniformly on any compact set in R as $\epsilon \to 0$, and also the following properties:

$$\begin{cases} |\rho_0^\epsilon(x)|_{L^2} \le |\rho_0(x)|_{L^2} \le M \\ |m_0^\epsilon(x)|_{L^2} \le |m_0(x)|_{L^2} \le M \\ |s_0^\epsilon(x)|_{L^2} \le |s_0(x)|_{L^2} \le M \end{cases} \quad (18)$$

$$\Big|\frac{d^i\rho_0^\epsilon(x)}{dx^i}\Big| \quad , \quad \Big|\frac{d^i m_0^\epsilon(x)}{dx^i}\Big| \quad , \quad \Big|\frac{d^i s_0^\epsilon(x)}{dx^i}\Big| \le M(\epsilon) \quad i = 0,1,2. \quad (19)$$

2. Viscosity solutions. In this section, we consider the existence of the Cauchy problem for the parabolic system (8) with initial data (12). Throughout this section, the solutions depend on the parameters ϵ and τ. For simplicity, we continue to use the notation (ρ, m, s) instead of $\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}$. The local existence of solutions can be obtained by applying the general contraction mapping principle to an integral representation of (8).

Lemma 1 (Local existence). *If the initial data satisfies the condition* (18), *then for any fixed* $\epsilon > 0$, $\tau > 0$, *the Cauchy problem* (8) *and* (12) *has a unique smooth solution* (ρ, m, s) *satisfying*

$$|\frac{d^i\rho}{dx^i}|, |\frac{d^i m}{dx^i}|, |\frac{d^i s}{dx^i}| \leq M(t_1, \epsilon) \quad on \quad R \times [0, t_1], \tag{1}$$

where $M(t_1, \epsilon)$ *is a positive constant that depends on* t_1 *and* t_1 *on the* L_∞ *norm* $M(\epsilon)$ *of* $\big(\rho_0^\epsilon(x), m_0^\epsilon(x), s_0^\epsilon(x)\big)$ *given in* (18) *for nonnegative integers* $i = 0, 1, 2$. *Moreover, if we assume further that the initial data satisfy* (15), *then*

$$\lim_{|x|\to\infty} \big(\frac{\partial^i\rho}{\partial x^i}, \frac{\partial^i m}{\partial x^i}, \frac{\partial^i s}{\partial x^i}\big) = \big(0, 0, 0\big), \quad i = 0, 1 \tag{2}$$

uniformly in $t \in [0, t_1]$.

To extend the local solution to the global one, the following a priori L^∞ estimate, depending on ϵ, τ, obtained by using the energy method, is essential.

Lemma 2. *If the initial data satisfies* (15), (16), (18), $\bar{F}(\rho, m, s)$ *satisfies the condition* (A_1), *and for any fixed* ϵ, $\tau > 0$ *the solution* $(\rho, m, s) \in C^\infty$ *of the Cauchy problem* (8), (12) *exists in* $R \times (0, T]$. *Then the following estimates hold*

$$|\rho(x,t)| \leq M(\epsilon, \tau, T), \quad |m(x,t)| \leq M(\epsilon, \tau, T), \quad |s(x,t)| \leq M(\epsilon, \tau, T). \tag{3}$$

Proof. Multiply $\rho - 4s$ to the first equation in (8), m to the second, $4H(s) - 4\rho$ to the third and adding the result, we have

$$\begin{aligned} &\big(\frac{\rho^2}{2} + \frac{m^2}{2} + 4\int^s H(s)ds - 4\rho s\big)_t + \big(\rho m - 4sm\big)_x + \frac{4(H(s)-\rho)^2}{\tau} \\ &= \epsilon\big(\frac{\rho^2}{2} + \frac{m^2}{2} + 4\int^s H(s)ds - 4\rho s\big)_{xx} - \epsilon\big(\rho_x^2 + m_x^2 + 4H'(s)s_x^2 - 8\rho_x s_x\big). \end{aligned} \tag{4}$$

The condition (A_1) ensures that the function $\frac{\rho^2}{2} + 4\int^s H(s)ds - 4\rho s$ is strictly convex. Thus integrating (4) on $R \times [0, T]$ and the estimates (1), (2) give us

$$|\rho(.,t)|_{L^2(R)} \leq M, \quad |m(.,t)|_{L^2(R)} \leq M, \quad |s(.,t)|_{L^2(R)} \leq M \tag{5}$$

$$|\frac{(H(s)-\rho)^2}{\tau}|_{L^1(R\times[0,T])} \leq M \tag{6}$$

$$|\epsilon\rho_x^2|_{L^1(R\times[0,T])} \leq M, \quad |\epsilon m_x^2|_{L^1(R\times[0,T])} \leq M, \quad |\epsilon s_x^2|_{L^1(R\times[0,T])} \leq M. \tag{7}$$

Differentiating (8) with respect to x and then using the estimates (1), (2), (5), (6), (7) and (16), we can immediately get the following energy inequality:

$$\int_{\infty}^{\infty} (\rho_x)^2 + (m_x)^2 + (s_x)^2 dx + \epsilon \int_0^T \int_{-\infty}^{\infty} (\rho_{xx})^2 + (m_{xx})^2 + (s_{xx})^2 dx\, dt$$
$$\leq M\big(\epsilon, \tau, |\rho_0^\epsilon|_{H^1}, |m_0^\epsilon|_{H^1}, |s_0^\epsilon|_{H^1}\big).tag8$$

So the estimates (3) follow from the estimates (5) and (8). The two lemmas in this section give us the following global existence theorem of the Cauchy problem (8), (12):

Theorem 1. *If the initial data* (12) *satisfies* (15), (16), (18), $\bar{F}(\rho, m, s)$ *satisfies the condition* (A_1)*. Then for any fixed* ϵ, $\tau > 0$*, there is a global solution of the Cauchy problem* (8), (12) *such that all the estimates in* (5), (6), (7) *hold.*

3. Strong convergence of viscosity solutions. In this section, we are going to consider the strong convergence of the viscosity solutions $\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}\big)$ for the Cauchy problem (8), (12), when ϵ and τ go to zero related by $\delta = O(\epsilon)$.

We first prove the strong convergence of $\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}\big)$ by using the method of compensated compactness as used in the papers [13] and [14].

To prove the convergence of the viscosity solutions for the following parabolic system

$$\begin{cases} \rho_t + m_x = \epsilon\rho_{xx} \\ m_t + \sigma(\rho)_x = \epsilon m_{xx} \end{cases} \tag{1}$$

J.W. Shearer gives the framework on how to apply the compensated compactness to the corresponding hyperbolic system ($\epsilon = 0$) to obtain a weak solution in L^p space [14] (see also [8]). He first proves global existence of weak solutions in L^p for the system

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \sigma(\rho)_x = 0 \end{cases} \tag{2}$$

in the case of strict hyperbolicity, genuine nonlinearity ($\sigma'' \neq 0$), the same growth constraints on σ as (A_4). This opens the possibility to extend this to study the convergence of physical viscosity solutions, to weaken the genuine nonlinearity assumption ($\sigma'' \neq 0$) such that this may be violated at single points (A_3) [13] and to apply this to our system of three equations.

A pair of functions $\big(\eta, q\big)$ constitute an entropy, entropy flux pair for the hyperbolic system (1) if the following linear hyperbolic differential equations are satisfied:

$$\begin{cases} \eta_\rho = q_m \\ \sigma'(\rho)\eta_m = q_\rho \end{cases} \tag{3}$$

Making a change of variables as in [14]

$$\eta = \tfrac{1}{2}\big(\sigma'\big)^{-\frac{1}{4}}\big(\Phi + \Psi\big), \quad q = \tfrac{1}{2}\big(\sigma'\big)^{\frac{1}{4}}\big(\Phi - \Psi\big) \tag{4}$$

we can obtain from (3)

$$\begin{cases} \Phi_w = a\Psi \\ \Psi_z = -a\Phi \end{cases} \tag{5}$$

where

$$w = m + \int_0^\rho \sqrt{\sigma'(s)}ds, \quad z = m - \int_0^\rho \sqrt{\sigma'(s)}ds \tag{6}$$

are two Riemann invariants and

$$a = a(w - z) = \frac{\sigma''\big(\rho(\frac{w-z}{2})\big)}{8\big(\sigma'\big(\rho(\frac{w-z}{2})\big)\big)^{\frac{3}{2}}}. \tag{7}$$

To prove the convergence of the physical viscosity solutions of (2) with the conditions (A_2), (A_3), (A_4), Serre and Shearer constructed two classes of entropies. One is obtained by choosing a point $(\bar{w}, \bar{z})$ in the (w, z) plane and solving the linear hyperbolic problem (5) with Goursat data given on the lines $w = \bar{w}$ and $z = \bar{z}$. The second class of entropies is obtained by solving the linear hyperbolic problem for Φ, Ψ with continuous, compactly supported initial data on a noncharacteristic line of the form $w - z = \xi_0 =$ *constant* (i.e., $\Phi(w, w - \xi_0) = g(w)$, $\Psi(w, w - \xi_0) = h(w)$ and g, w have compact support in R). From the proof of Lemma 3 in [13], we know that all the estimates about the second class of entropy - entropy flux pairs are the same as for the first class, by compactness of g, h.

In fact, we assume that the support of the Cauchy problem on $w - z = \xi_0$ lies between the points $(\bar{w}, \bar{z})$ and $(\hat{w}, \hat{z})$, where $\bar{w} - \bar{z} = \xi_0 = \hat{w} - \hat{z}$. Clearly, the characteristics for the hyperbolic problem (5) are $w = constant$, $z = constant$. Hence, the solution Ψ, Φ is identically zero in the quadrants $w > \hat{w}$, $z > \bar{z}$ and $w < \bar{w}$, $z < \bar{z}$. We use this solution to get the initial data for a Goursat problem in the quadrant $w < \bar{w}$ with $z > \bar{z}$. Consider continuous, compactly supported Goursat initial data g_1, h_1 satisfying $g_1(z) = \Phi(\hat{w}, z)$ for $z > \bar{z}$, $h_1(w) = \Psi(w, \bar{z})$ for $w < \bar{w}$. Then the solution $\hat{\Phi}$, $\hat{\Psi}$ to the system (5) with Goursat data g_1, h_1 satisfies $\hat{\Phi} = \Phi$, and $\hat{\Psi} = \Phi$ in the quadrant $w < \hat{w}$ with $z > \bar{z}$ by uniqueness of the Goursat problem. Shearer in [14] carefully constructed and estimated the first class of entropy-entropy flux pairs. Roughly speaking, (Φ, Ψ) satisfy the following estimates (in Lemma 2 of Section 5 in [14])

$$|\theta| \le c_1\big(1 + (\sigma'(\rho)\big)^{\frac{1}{4}} \tag{8}$$

for θ having the form $\frac{\partial^i \Phi}{\partial w^j \partial z^{i-j}}$, $\frac{\partial^i \Psi}{\partial w^j \partial z^{i-j}}$, $i, j = 0, 1, 2$ $i \le j$, respectively.

Noticing $\frac{c}{4+c} \le \sigma'(\rho) < 1$ given in Remark 1, we know $|\theta|$ is bounded for all θ given above. Furthermore, the assumptions on the growth constraints of σ, $\big($see $(A_4)\big)$ give us the boundedness of q and of $\frac{\partial^i \eta}{\partial \rho^j \partial m^{i-j}}$, namely

$$|q| \le M, \quad |\frac{\partial^i \eta}{\partial \rho^j \partial m^{i-j}}| \le M \tag{9}$$

for $i, j = 0, 1, 2$ and $j \le i$ and a constant M, where (η, q) is an entropy-entropy flux of (2). For details see [17].

If the compactness of $\eta\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}\big)_t + q\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}\big)_x$ in $H^{-1}_{loc}(R \times R^+)$ is proved with respect to the viscosity solutions for the Cauchy problem (8), (12), the framework of Serre and Shearer in [13] will give us the convergence of $\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}\big)$ as we stated in the beginning of this section.

Lemma 3. *If the conditions in* (A_1)–(A_5) *are satisfied, then* $\eta(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})_t + q(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})_x$ *are compact in* $H^{-1}_{loc}(R \times R^+)$ *with respect to the viscosity solutions* $(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})$ *given by the Cauchy problem* (8), (12), *where* (η, q) *is an entropy-entropy flux pair of* (2) *satisfying the estimates* (9).

Proof. We rewrite the first and second equation in (8) as follows:

$$\begin{cases} \rho_t + m_x = \epsilon\rho_{xx} \\ m_t + (\rho - 4h(\rho))_x + 4(h(\rho) - s)_x = \epsilon m_{xx}. \end{cases} \tag{10}$$

Multiplying (η_ρ, η_m) to (10), we get

$$\eta_t + q_x + 4\eta_m(h(\rho) - s)_x = \epsilon\eta_{xx} - \epsilon(\eta_{\rho\rho}\rho_x^2 + 2\eta\rho_m\rho_x m_x + \eta_{mm}m_x^2). \tag{11}$$

Since

$$h(\rho) - s = H^{-1}(\rho) - H^{-1}(H(s)) = \big(H^{-1}(\xi)\big)'(\rho - H(s)), \qquad (12)$$

where ξ is between ρ and $H(s)$ and $0 \leq \big(H^{-1}(\xi)\big)' \leq \frac{1}{4+c}$ by Remark 1, we have from (11)

$$\eta_t + q_x = I_1 + I_2, \qquad (13)$$

where

$$\begin{aligned} I_1 &= \epsilon \eta_{xx} - \Big(4\eta_m \big(H^{-1}(\xi)\big)'\big(\rho - H(s)\big)\Big)_x \\ I_2 &= -\epsilon\big(\eta_{\rho\rho}\rho_x^2 + 2\eta\rho_m \rho_x m_x + \eta_{mm} m_x^2\big) \\ &\quad + 4\big(H^{-1}(\xi)\big)'(\eta_{m\rho}\rho_x + \eta_{mm} m_x)\big(\rho - H(s)\big). \end{aligned}$$

From the estimates (6), (7) and (9), we can easily prove that I_1 is compact in $H_{loc}^{-1}(R \times R^+)$. Since

$$\big(\eta_{m\rho}\rho_x + \eta_{mm} m_x\big)\big(\rho - H(s)\big) = \sqrt{\tau}\big(\eta_{m\rho}\rho_x + \eta_{mm} m_x\big)\frac{\rho - H(s)}{\sqrt{\tau}}, \qquad (14)$$

then the estimates (6), (7) and (9) give the boundedness of I_2 in $L_{loc}^1(R \times R^+)$ if $\tau = O(\epsilon)$, so the compactness of I_2 in $W_{loc}^{-1,k}$ for $1 < k < 2$ by the standard embedding theorem. So $\eta_t + q_x$ is compact in $W_{loc}^{-1,k}$ for $1 < k < 2$ from (13). Moreover, $\eta_t + q_x$ is bounded in $W^{-1,\infty}(R \times R^+)$ by the estimate (9), so $\eta_t + q_x$ is compact in $H_{loc}^{-1}(R \times R^+)$. The lemma is proven.

From the lemma and the framework given in [13], we have the strong convergence of $(\rho^{\epsilon,\tau}, m^{\epsilon,\tau})$ in $L_{loc}^p(R^2)$ for any $p < 2$ as ϵ, τ go to zero related by $\tau = O(\epsilon)$ and so the strong convergence of $s^{\epsilon,\tau}$ by the estimate (6).

We end this section by the following theorem:

Theorem 2. *The solutions* $\big(\rho^{\epsilon,\tau}, m^{\epsilon,\tau}, s^{\epsilon,\tau}\big)$ *of the Cauchy problem* (8), (12) *with the assumptions* (A_1)–(A_5) *converge almost everywhere in a compact set* $\Omega \in R \times R^+$ *to an* L^2 *bounded function triple* (τ, m, s) *as* ϵ, τ *go to zero related by* $\tau = O(\epsilon)$. *Moreover,* (ρ, m) *is a weak solution of the Cauchy problem* (6) *with initial data* $\big(\rho_0(x), m_0(x)\big)$ *in* (7).

Remark 3. The basic ideas to use the framework given in [14] to study solutions in L^p space are also used by other authors to some relaxation systems without introducing the viscosity terms (See [15]). In fact, under a linear transformation of variables, the system in [15] is same to what we study in this paper.

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