Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2015,**35B**(2):281–302
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

DECAY RATE FOR DEGENERATE CONVECTION DIFFUSION EQUATIONS IN BOTH ONE AND SEVERAL SPACE DIMENSIONS*

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Abstract We consider degenerate convection-diffusion equations in both one space dimension and several space dimensions. In the first part of this article, we are concerned with the decay rate of solutions of one dimension convection diffusion equation. On the other hand, in the second part of this article, we are concerned with a decay rate of derivatives of solution of convection diffusion equation in several space dimensions.

Key words Degenerate convection-diffusion equations; regularity; decay rate; Lax-Oleinik type inequality

2010 MR Subject Classification

1 Introduction

In this article, we consider degenerate convection diffusion equation in both one and several space dimensions. First part of this article is concerned with the decay rate of solutions to the general degenerate reaction diffusion convection equation

$$\begin{cases} u_t + F(u,x,t)_x + H(u,x,t) = G(u,t)_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The basic assumption on the diffusion function $G(u,t)$ is that it is nonlinear, depends explicitly on t, and non-decreasing in u, that is, $g(u,t) = G_u(u,t) \ge 0$, and thus (1.1) is a strongly

*Received December 24, 2013; revised January 8, 2014. This work was partially supported by the Natural Science Foundation of Zhejiang Province of China (LY12A01030) and the National Natural Science Foundation of China (11271105).

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degenerate parabolic equation. Regarding the flux function, F is a nonlinear flux function which depends on u, x, and t. H is the given source term which can also depends on x and t explicitly. The equation (1.1) appears in several applications in which u stands for a non-negative quantity. The scalar conservation law $u_t + F(u)_x = 0$ is a special example of this type of problems. Other examples occur in several applications, for instance in porous media flow [1] (a special type of diffusion, that is, $G(u) = u^m$ appears to model a non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions) and in sedimentation processes [2].

If $G \equiv 0$, $H \equiv 0$ and the flux function F only depends on u, then the equation (1.1) becomes the classical conservation laws of the form

$$\begin{cases} u_t + F(u)_x = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.2}$$

The asymptotic form of the solution of (1.2) for large time is well known in the literature due to Oleinik [3]. In fact, there are two distinctly different cases: the case where the initial data u_0 is periodic and the case where the initial data u_0 has compact support. In the periodic case u tends to the mean value of u_0 (over one period), at a rate t^{-1}, uniformly in x. On the other hand, if u_0 has a compact support, then u tends uniformly to zero at a rate $t^{-1/2}$, and tends, in L^1 norm, to a particular function called an N-wave, again at a rate $t^{-1/2}$.

On the other hand, if the diffusion function is of porous media type, that is, $G(u) = u^m$ with $F \equiv 0$ and $H \equiv 0$, then equation (1.1) becomes the degenerate diffusion equation of the form

$$\begin{cases} u_t = (u^m)_{xx}, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.3}$$

In [4], the authors shown that the decay rate of the solution of (1.3) is given by

$$\|u(t)\|_{L^\infty} \le C_\infty t^{-1/(m+1)},$$

where C_∞ is a constant depending on m and the L^1 mass of the initial profile u_0.

If the diffusion is linear, that is, $G = u$, $H \equiv 0$ and the flux function F takes a special form u^q, then equation (1.1) becomes the classical convection diffusion equation of the form

$$\begin{cases} u_t + (u^q)_x = u_{xx}, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.4}$$

Large time behavior of solutions of (1.4) is well developed in literature [4–6]. The results obtained in the above mentioned articles may be summarized as follows: There is a critical exponent $q = 2$ such that if $u_0 \in L^1(\mathbb{R})$ is nonnegative with $M = \|u_0\|_{L^1}$, one has the following:

- If $q > 2$, the profile in $L^1(\mathbb{R})$ of the solution of (1.4) with initial data u_0 is the unique solution to the purely diffusive equation

$$u_t = u_{xx}, \quad (x, t) \in \mathbb{R} \times (0, \infty),$$

with initial data $M\delta$, where δ denotes the Dirac mass centered in zero. In addition, there exists a constant K_∞ depending on q and M such that, for $t > 0$,

$$\|u(t)\|_{L^\infty} \le K_\infty t^{-1/2}.$$

• If $q = 2$, then equation (1.4) has a unique nonnegative solution with initial data $M\delta$, which gives the profile in $L^1(\mathbb{R})$ of the solution to (1.4) with initial data u_0. In addition, there exists a constant K_∞ such that, for $t > 0$,

$$\|u(t)\|_{L^\infty} \le K_\infty t^{-1/2}.$$

• If $1 < q < 2$, the profile in $L^1(\mathbb{R})$ of the solution to (1.4) with initial data u_0 is the unique nonnegative entropy solution to the conservation law

$$u_t + (u^q)_x = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty),$$

with initial data $M\delta$ (uniqueness and existence of such solution is proved in [7]). In addition, there exists a constant K_∞ depending on q and M such that, for $t > 0$,

$$\|u(t)\|_{L^\infty} \le K_\infty t^{-1/q}.$$

If the diffusion is non-linear, that is, $G = u^m$, $H \equiv 0$ and the flux function F takes a special form u^q, then the equation (1.1) becomes the classical convection diffusion equation of the form

$$\begin{cases} u_t + (u^q)_x = (u^m)_{xx}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.5}$$

Large time behavior of solutions of (1.5) is well developed in literature [8]. In fact, they proved the following results:

• If $q > m + 1$, then there exists a constant K_∞ depending on q and M such that, for $t > 0$,

$$\|u(t)\|_{L^\infty} \le K_\infty t^{-1/(m+1)}.$$

• If $q \in (1, m+1)$, then there exists a constant K_∞ depending on q and M such that, for $t > 0$,

$$\|u(t)\|_{L^\infty} \le K_\infty t^{-1/q}.$$

It is well known that degenerate parabolic equations do not possess a classical solution. The solution u in general fails to be smooth at the interface between a parabolic region and a region of parabolic degeneracy. Fortunately, if the diffusion function has only one degenerate point, such as the porous media type degeneracy, then the following estimate is known to be optimal

$$\left|(u^{m-1})_x\right| \le M,$$

for the solutions of the Cauchy problem (1.3) with initial data in the one dimentional space [9].

Because it is well known that in general we can not expect the Cauchy problem (1.1) to have a classical solution, we introduce the following standard definition of weak solution for (1.1).

Definition 1.1 A function $u(x,t)$ defined on $R_T = \mathbb{R} \times [0,T]$ will be called a weak solution of the Cauchy problem (1.1) if

(a) u is bounded, continuous in R_T.

(b) $G(u)$ has a bounded generalized derivative with respect to x in R_T.

(c) u satisfies the following identity

$$\iint_{R_T} -u\phi_t + (G(u,x,t)_x - F(u,x,t))\,\phi_x + H(u,x,t)\phi \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \phi(x,0)u_0(x)\mathrm{d}x$$

for all $\phi \in C_0^1(R_T)$ which vanish for large $|x|$ and $t = T$.

Next, we shall introduce the function space C^α in order to state the existing results about the regularity of the solutions of (1.1).

Definition 1.2 For any two points $Q_1 = (x_1, t_1)$ and $Q_2 = (x_2, t_2) \in R_T$, define a distance function as

$$d(Q_1, Q_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2}.$$

Then, we say that $u(x,t) \in C^{(q)}(R_T)$ for $q = 0, \alpha, 1+\alpha, 2+\alpha$ if $|u|_q$ is finite and also $u(x) \in C^{(q)}(\mathbb{R})$ if $|u|_q$ is finite with $t_1 = t_2$, where

$$|u|_0 = \sup_{R_T} |u(x,t)|,$$

$$|u|_\alpha = |u|_0 + \sup_{Q_1, Q_2 \in R_T} \frac{|u(Q_1) - u(Q_2)|}{d(Q_1, Q_2)}, \quad 0 < \alpha \le 1,$$

$$|u|_{1+\alpha} = |u|_0 + |u_x|_\alpha, \quad |u|_{2+\alpha} = |u|_{1+\alpha} + |u_x|_{1+\alpha} + |u_t|_\alpha.$$

The regularity of solutions of (1.1) has been studied by Lu et al [10, 11], in the case where F and H are only functions of u. They proved the following theorem:

Theorem 1.3 Let $u_0(x) \in C^{2+\alpha}$ for $\alpha \in (0,1)$, where $|u_0(x)|_{2+\alpha}$ may depend on ε. Let $H(u,x,t) = 0$, $g(u) \in C^2$. Then, for any fixed $\varepsilon > 0$, there exists a unique smooth solution for the Cauchy problem (2.11) and (2.12) (with F and H only depends on u) in R_T, which satisfies

$$|u^\varepsilon|_{2+\alpha} \le M(\varepsilon).$$

Moreover, if $|u_0^\varepsilon|_1 \le M$, then

$$|G(u^\varepsilon) + \varepsilon u^\varepsilon|_1 \le M.$$

In this article, we present the L^∞ decay rate of solutions of (1.1), using a technique which relies on the splitting of flux function (see Section 2). The analysis depends on the Lax-Oleinik type estimate, which is well known in the context of the conservation laws (1.2) [12]

$$u_x \le \frac{f_2(u)}{t^\alpha}.$$

Once we have the above type estimate, we can use the following Lemma to obtain decay rate of u. For a proof of the Lemma, see [8].

Lemma 1.4 Consider a non-negative function $u \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$(u^m)_x \le \frac{M}{(1+t)^\alpha},$$

for some positive constants M and α. Then, the decay rate of u is given by

$$0 \le u(x,t) \le \frac{m}{m+1} M^{1/(m+1)} (1+t)^{-\alpha/(m+1)}, \quad \text{for all } x \text{ and } t.$$

The second part of the article deals with degenerate convection diffusion equation in several space dimensions, which is of the form

$$u_t = \Delta u^m + \sum_{i=1}^{N} f_i(u)_{x_i} - S(u), \tag{1.6}$$

with the initial data

$$u(x,0) = u_0(x_1, x_2, \cdots, x_N) \geq 0, \tag{1.7}$$

where N denotes the space dimension. The equation (1.6) arises in several applications in which u stands for a nonnegative quantity. For example, if $f_i \equiv S \equiv 0$, then the equation (1.6) becomes the degenerate diffusion equation of the form

$$\begin{cases} u_t = \Delta u^m, \\ u(x,0) = u_0(x). \end{cases} \tag{1.8}$$

In fact, this equation models the non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions. If the flow is not polytropic, (1.3) is replaced by the more general equation of Newtonian filtration

$$\begin{cases} u_t = \Delta G(u), \\ u(x,0) = u_0(x), \end{cases} \tag{1.9}$$

where $G(u)$ is a nondecreasing smooth function. If the medium has also heat sources, then (1.9) is replaced by an equation of the form

$$\begin{cases} u_t = \Delta G(u) - S(u), \\ u(x,0) = u_0(x). \end{cases} \tag{1.10}$$

The regularity of solutions of convection diffusion equations is well known in one space dimension as we mentioned earlier. However, the regularity property of solutions for the Cauchy problem in the multidimensional space is completely different.

For example, consider the porous media equation (1.8) with bounded, continuous, and nonnegative function $u_0(x_1, x_2, \cdots, x_N)$, $N \geq 2$ on the line $t = 0$. A numerical example constructed by Graveleau shows that if there are holes in supp u_0, then it is possible for ∇u^{m-1} to blow up. The existence and uniqueness of Graveleau's solution was proved later by Aronson and Graveleau by a construction of radially symmetric solutions [1].

On the other hand, Caffareli, Vazquez, and Wolanski [13] showed that ∇u^{m-1} is bounded in $\mathbb{R}^N \times (T, \infty)$ for a suitable large time T. In some sense, this estimate is the best possible as is shown by Graveleau's solution. In [14], authors extended the above mentioned results in the follwoing sense. They obtain a Hölder solution with explicit Hölder exponent for the Cauchy problem (1.6) and (1.7).

In this article, our aim is to provide a decay rate of solutions of the general convection diffusion equation in multi-dimension. To be more precise, in this article we prove the decay rate of derivatives of solutions of (1.6) with the initial condition (1.7).

The rest of the article is organized as follows: In Section 2, we consider the most general degenerate convection diffusion equation in one space dimension. We first first prove the decay rate of derivative of solutions of such equations and then using a standard argument, we obtain the decay rate of solutions of degenerate convection diffusion equations. In Section 3, we move on to multi-dimensional convection diffusion equations and obtain decay rate of derivatives of solutions of those equations. We use maximum principle to obtain such decay rates.

2 Decay Rate in one Space Dimension

In this section, we are interested to find the decay rate of solutions of most general degenerate convection diffusion equation of the form

$$\begin{cases} u_t + F(u,x,t)_x + H(u,x,t) = G(u,t)_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that because the nonlinear diffusion $G(u,t)$ can be degenerate, we cannot expect, in general, smooth solutions of (2.1). Consequently, we are not entitled to calculate derivatives of u and $G(u)$. To overcome this difficulty, we first regularize equation (2.1) by adding small diffusion and then find the estimates independent of ε which in turn help us to estimate the decay rate of solutions of (2.1).

To begin with, we first consider the following equation

$$\begin{cases} u_t + (u^q)_x + cu^n = (u^m)_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $c \geq 0$ is a real number and $u_0 \geq 0$. We have the following theorem.

Theorem 2.1 Let u be a solution of Cauchy problem (2.2) with a non-negative function $u_0 \in L^1(\mathbb{R})$. Then, there exist a constant K depending on m such that $\|u\|_\infty \leq K(1+t)^{-1/(1+m)}$ for any $1 < q \leq 1+m$.

Proof STEP-I. In order to perform our estimates, we need to approximate our solution u by uniformly positive and bounded solutions. Therefore, we ask that $u_0 \in C^\infty(\mathbb{R})$ and $0 < \varepsilon < u_0(x) \leq N$. Once the estimates hold for u^ε, it will hold for u by a simple approximation and limit process. To be more precise, we first consider the following regularized equation,

$$\begin{cases} u^\varepsilon_t + ((u^\varepsilon)^q)_x + c(u^\varepsilon)^n = ((u^\varepsilon)^m)_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u^\varepsilon(x,0) = u^\varepsilon_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where

$$u^\varepsilon_0(x) = u_0 * J^\varepsilon = \int_{\mathbb{R}} u_0(x-y) J^\varepsilon(y) \mathrm{d}y,$$

where J^ε is a standard mollifier and ε is a positive constant. We consider a smooth and bounded initial data u^ε_0 satisfying

$$0 < \varepsilon \leq u^\varepsilon_0(x), \quad x \in \mathbb{R}.$$

Classical results then ensure the existence of a unique classical solution u^ε to (2.3) satisfying

$$0 < c_0(\varepsilon,t) \leq u^\varepsilon(x,t) \leq \|u^\varepsilon_0\|_\infty, \quad (x,t) \in \mathbb{R} \times [0,T),$$

where the lower bound $c_0(\varepsilon,t)$ could tend to zero as ε goes to zero or t goes to infinity.

Next, we introduce the following change of variables

$$v^\varepsilon = (1+t)^\alpha u^\varepsilon.$$

Then, multiplying equation (2.3) by $(1+t)^\alpha$, we have

$$v_t^\varepsilon - \alpha\frac{v^\varepsilon}{(1+t)} + \left(\frac{(v^\varepsilon)^q}{(1+t)^{(q-1)\alpha}}\right)_x + c\frac{(v^\varepsilon)^n}{(1+t)^{\alpha(n-1)}} = \left(\frac{m(v^\varepsilon)^{m-1}}{(1+t)^{(m-1)\alpha}}v_x^\varepsilon\right)_x,$$

which is of the form

$$v_t^\varepsilon + F(v^\varepsilon,t)_x + H(v^\varepsilon,t) = (g(v^\varepsilon,t)v_x^\varepsilon)_x, \tag{2.4}$$

where

$$F(v^\varepsilon,t) = \frac{(v^\varepsilon)^q}{(1+t)^{(q-1)\alpha}}, \quad H(v^\varepsilon,t) = c\frac{(v^\varepsilon)^n}{(1+t)^{\alpha(n-1)}} - \alpha\frac{v^\varepsilon}{(1+t)}$$
$$g(v^\varepsilon,t) = \frac{(v^\varepsilon)^{m-1}}{(1+t)^{(m-1)\alpha}}.$$

STEP-II. Next, we prove the following Lemma:

Lemma 2.2 The solutions of the Cauchy problem (2.4) with

$$F(v,t) = F_1(v,t) + F_2(v,t),$$
$$f_2(v,t) = \frac{F_2(v,t)}{g(v,t)},$$

satisfies

$$v_x \le f_2(v,t),$$

provided the initial data also satisfy the same estimate and

$$(f_2)_v H - f_2 H_v - (f_2)_t - (F_1)_{vv}(f_2)^2 \le 0, \tag{2.5}$$

where for simplicity the superscript ε in v is omitted.

Proof We can rewrite equation (2.4) as

$$v_t + F_1(v,t)_x + H(v,t) = \left(g(v,t)(v_x - f_2(v,t))\right)_x. \tag{2.6}$$

To achieve the desired estimate, we need the following substitution:

$$w = v_x - f_2(v,t).$$

Then, it is easy to see that

$$w_t = v_{xt} - (f_2)_v v_t - (f_2)_t.$$

Note that equation (2.6) can be rewritten as:

$$v_t + F_1(v,t)_x + H(v,t) = \left(g(v,t)w\right)_x. \tag{2.7}$$

Using (2.7), we can rewrite the equation satisfied by w as

$$\begin{aligned} w_t = &- (F_1)_{vv}v_x^2 - (F_1)_v v_{xx} - H_v v_x + (g(v,t)w)_{xx} - (f_2)_t \\ &- (f_2)_v\left(-(F_1)_v v_x - H(v,t) + (g(v,t)w)_x\right) \\ = &- (F_1)_{vv}(w+f_2)^2 - (F_1)_v(w + (f_2)_v v_x) + (g(v,t)w)_{xx} \\ &- (f_2)_t - H_v(w+f_2) - (f_2)_v\left(-(F_1)_v v_x - H(v,t) + (g(v,t)w)_x\right) \end{aligned}$$

$$
\begin{aligned}
&= (g(v,t)w)_{xx} - H_v f_2 + H(v,t)(f_2)_v - (f_2)_t - (F_1)_{vv}(f_2)^2 - a(x,t)w_x - b(x,t)w \\
&\le - a(x,t)w_x - b(x,t)w + g(v,t)w_{xx},
\end{aligned}
$$

where we have used the conditions (2.5) and $a(x,t)$, $b(x,t)$ are given by

$$
\begin{aligned}
a(x,t) &= -(F_1)_{vv}w - 2(F_1)_{vv}f_2 - (F_1)_v - H_v + g(v)_{xx} - (f_2)_v g(v)_x, \\
b(x,t) &= 2g(v)_x - (f_2)_v g(v).
\end{aligned}
$$

Thus, conditions (2.5) gives the following inequality

$$
w_t + a(x,t)w_x + b(x,t)w \le g(v,t)w_{xx}, \tag{2.8}
$$

where a, b are functions of v, v_x and v_{xx}. Therefore, the maximum principle [15] applied to (2.8) give the estimate $w \le 0$ provided $w_0 \le 0$. Hence, we proved that $v_x^\varepsilon \le f_2(v^\varepsilon, t)$. □

STEP-III. Now, we check if conditions (2.5) are true for equation (2.3). Let $F_2 = M, F_1 = F - M$ for a suitable large constan M. It is easy to calculate

$$
\begin{aligned}
-(f_2)_t &= -\alpha(m-1)\frac{M}{u^{m-1}}\frac{1}{(1+t)}, \\
(f_2)_v H &= \alpha(m-1)\frac{M}{u^{m-1}}\frac{1}{(1+t)} - cM(m-1)u^{n-m}, \\
-H_v f_2 &= \frac{\alpha}{(1+t)}\frac{M}{u^{m-1}} - cnMu^{n-m}, \\
-(F_1)_{vv}(f_2)^2 &= -q(q-1)M^2u^{q-2m}\frac{1}{(1+t)^\alpha}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(f_2)_v H - f_2 H_v - (f_2)_t - (F_1)_{vv}(f_2)^2 \\
&= - u^{q-2m}\frac{M^2 q(q-1)}{(1+t)^\alpha} + \frac{\alpha M}{(1+t)u^{m-1}} - c(n+m-1)Mu^{n-m} \le 0
\end{aligned}
$$

for a suitable large M because $\alpha \le 1, 1 < q \le 1+m$, and $c \ge 0$. Therefore, we obtain

$$
((u^\varepsilon)^m)_x \le \frac{M}{(1+t)^\alpha},
$$

which in turn implies

$$
u^\varepsilon(x,t) \le M^{1/(m+1)}(1+t)^{-\alpha/(m+1)}
$$

for any $\alpha \le 1$. Hence, we conclude that

$$
u^\varepsilon(x,t) \le K(1+t)^{-1/(1+m)}
$$

if we choose $\alpha = 1$.

At this point, we use the fact that

$$
u(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t)
$$

exists for any $(x,t) \in \mathbb{R} \times [0,T)$ and conclude

$$
u(x,t) \le K(1+t)^{-1/(1+m)}.
$$

□

We can generalize the above result to more general diffusion functions (that is, it might have finite number of degenerate points). Let us consider the following equation

$$\begin{aligned} &u_t + F(u)_x = G(u)_{xx}, \\ &u(x,0) = u_0(x). \end{aligned} \tag{2.9}$$

Theorem 2.3 Let u be a solution of Cauchy problem (2.9) with a non-negative function $u_0 \in L^1(\mathbb{R})$. Then, there exist a constant K such that $\|u\|_\infty \le K(1+t)^{-1/2}$, provided the flux function F is strictly convex.

Proof As a strategy, we first regularize equation (2.9). In what follows, we consider the equation

$$u_t + F(u)_x = \left((g(u)+\varepsilon)u_x\right)_x, \tag{2.10}$$

where $G'(u) = g(u)$. We prove this theorem as before by using the change of variable $v = (1+t)^\alpha u$. Then, from (2.10) we see that v satisfies the following equation

$$v_t - \alpha \frac{v}{(1+t)} + \left((1+t)^\alpha F(u)\right)_x = \left((g(u)+\varepsilon)u_x\right)_x.$$

We are going to use Lemma 2.2. So, in this case we have

$$\begin{aligned} &F_1 = (1+t)^\alpha F(u), \quad F_2 = M, \quad f_2(v,t) = \frac{M}{g(u)+\varepsilon} \\ &v = (1+t)^\alpha u, \quad H(v,t) = -\alpha \frac{v}{(1+t)}. \end{aligned}$$

Then, we calculate

$$\begin{aligned} &-(f_2)_t = -\frac{\alpha M}{(g(u)+\varepsilon)^2} u g'(u) \frac{1}{(1+t)}, \quad -(F_1)_{vv}(f_2)^2 = -F''(u)\frac{1}{(1+t)^\alpha}\frac{M^2}{(g(u)+\varepsilon)^2} \\ &-H_v f_2 = \frac{\alpha}{(1+t)}\frac{M}{g(u)+\varepsilon}, \quad (f_2)_v H = \alpha M \frac{g'(u)}{(g(u)+\varepsilon)^2}\frac{u}{(1+t)}. \end{aligned}$$

In what follows, after a simple computation, we have to show that

$$-F''(u)\frac{1}{(1+t)^\alpha}\frac{M}{(g(u)+\varepsilon)^2} + \frac{1}{(1+t)}\frac{\alpha}{g(u)+\varepsilon}$$

is negative. In fact, this is true provided that $\alpha \le 1$ and $F''(u) \ge C > 0$ and M is large enough. Then, we have

$$\left((1+t)^\alpha u\right)_x \le \frac{M}{g(u)+\varepsilon},$$

which in turn implies

$$(G(u)+\varepsilon u)_x \le \frac{M}{(1+t)^\alpha},$$

that is, in the limit we have

$$|G(u)| \le \frac{K}{(1+t)^{1/2}}$$

for a suitable constant K. □

Now, we are in a position to state the results corresponding to (2.1). In what follows, we first consider the viscous equation corresponding to (2.1), given by

$$u_t + F(u,x,t)_x + H(u,x,t) = \left((g(u,t)+\varepsilon)u_x\right)_x, \tag{2.11}$$

together with the initial data

$$u(x,0) = u_0^\varepsilon(x) = u_0 * J^\varepsilon = \int_{\mathbb{R}} u_0(x-y)J^\varepsilon(y)\mathrm{d}y, \tag{2.12}$$

where J^ε is a standard mollifier and ε is a positive constant. Throughout this article, we assume that $u_0 \in W^{1,\infty}$. Then, we have

$$u_0^\varepsilon(x) \in C^\infty, \quad |u^\varepsilon(x)| \le M, \quad \left|u_{0,x}^\varepsilon(x)\right| \le M$$
$$\left|\frac{d^i u_0^\varepsilon(x)}{dx^i}\right| \le M(\varepsilon), \quad i = 1, 2, \cdots,$$

where the positive constant M is independent of ε and $M(\varepsilon)$ depends on ε.

The local existence of solutions of the Cauchy problem (2.11), (2.12) is standard if $u_0^\varepsilon \in C^{2+\alpha}$ because equation (2.11) is strictly parabolic for any fixed $\varepsilon > 0$. See [15–18] for more details.

We only state and prove the following Lemma, which gives the decay rate of derivative of solutions. To obtain decay rate of solution itself, we can use the argument before. For the simplicity, we omit the superscript ε in u.

Lemma 2.4 Let u be a solution of

$$\begin{cases} u_t + F(u,x,t)_x + H(u,x,t) = ((g(u,t)+\varepsilon)u_x)_x, & (x,t) \in \mathbb{R}\times(0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.13}$$

Moreover, if

$$F(u,x,t) = F_1(u,x,t) + F_2(u,x,t),$$
$$f_2(u,x,t) = \frac{F_2(u,x,t)}{g(u,t)+\varepsilon}$$

where

$$\begin{aligned} &(F_1)_{uu} \ge 0, \quad (F_1)_{xu} f_2 \ge 0, \quad (F_1)_{xx} \ge 0, \quad (f_2)_t \ge 0, \\ &H_x \ge 0, \quad H_u f_2 - (f_2)_u H \ge 0, \quad (f_2)_u (F_1)_x - (F_1)_u (f_2)_x \le 0. \end{aligned} \tag{2.14}$$

then, we obtain

$$u_x \le f_2(u,x,t), \tag{2.15}$$

provided the initial data also satisfy the same estimate.

Proof As we know that

$$F(u,x,t) = F_1(u,x,t) + F_2(u,x,t),$$

and

$$f_2(u,x,t) = \frac{F_2(u,x,t)}{g(u,t)+\varepsilon},$$

then equation (2.13) becomes

$$u_t + F_1(u,x,t)_x + H(u,x,t) = ((g(u,t)+\varepsilon)v)_x, \tag{2.16}$$

where v is given by

$$v(u,x,t) = u_x - f_2(u,x,t).$$

Next, it is easy to see that

$$v_t = u_{xt} - (f_2)_u u_t - (f_2)_t,$$

which implies

$$\begin{aligned}
v_t &= -(F_1)_{uu}u_x^2 - 2(F_1)_{xu}u_x - (F_1)_{xx} - (F_1)_u u_{xx} + ((g(u)+\varepsilon)v)_{xx} - (f_2)_t \\
&\quad -H_u u_x - H_x - (f_2)_u\left(-(F_1)_u u_x - (F_1)_x - H(u,x,t) + ((g(u)+\varepsilon)v)_x\right) \\
&= -(F_1)_{uu}\left((f_2)^2 + v^2 + 2vf_2\right) - 2(F_1)_{xu}(v+f_2) - (F_1)_{xx} \\
&\quad -(F_1)_u\left(v_x + (f_2)_u u_x + (f_2)_x\right) - H_u(v+f_2) - H_x + ((g(u)+\varepsilon)v)_{xx} \\
&\quad +(f_2)_u(F_1)_u u_x + (f_2)_u(F_1)_x + (f_2)_u H - (f_2)_u\left((g(u)+\varepsilon)v\right)_x - (f_2)_t \\
&= ((g(u)+\varepsilon)v)_{xx} - (F_1)_{uu}(f_2)^2 - 2(F_1)_{xu}f_2 - (F_1)_{xx} - (F_1)_u(f_2)_x \\
&\quad -H_u f_2 - H_x + (f_2)_u(F_1)_x + (f_2)_u H - (f_2)_t - a(x,t)v_x - b(x,t)v \\
&\le -a(x,t)v_x - b(x,t)v + (g(u)+\varepsilon)v_{xx}, \qquad (2.17)
\end{aligned}$$

where we have used the set of conditions (2.14) and $a(x,t), b(x,t)$ are given by

$$\begin{aligned}
a(x,t) &= (F_1)_u + (g(u)+\varepsilon)(f_2)_u + 2(g(u)+\varepsilon)_x, \\
b(x,t) &= (F_1)_{uu}v + 2(F_1)_{uu}f_2 + 2(F_1)_{xu} + H_u + (f_2)_u(g(u)+\varepsilon)_x + (g(u)+\varepsilon)_{xx}.
\end{aligned}$$

Thus, the conditions (2.14) and (2.17) give the following inequality

$$v_t + a(x,t)v_x + b(x,t)v \le (g(u,t)+\varepsilon)v_{xx}, \qquad (2.18)$$

where a, b are functions of u, u_x and u_{xx}. Therefore, the maximum principle [15] applied to (2.18) give the estimate $v \le 0$ provided $v_0 \le 0$. Consequently, the estimate (2.15) is proved. □

3 Decay Estimate for Cauchy Problem in Multi-Dimensional Space With Porous Media Type Degenerate Diffusion

In this section, we study the decay rate of solutions to the following degenerate parabolic equation in the N-dimensional space

$$u_t = \Delta u^m + \sum_{i=1}^{N} f_i(u)_{x_i} - S(u), \qquad (3.1)$$

with the initial data

$$u(x,0) = u_0(x_1, x_2, \cdots, x_N) \ge 0. \qquad (3.2)$$

Theorem 3.1 There exists a weak solution u of the Cauchy problem (3.1), (3.2) which satisfies the following decay rates:

CASE-I Let $S=0$ and for each i let $f_i = 0$.

- If $1 < m < 1 + \frac{1}{\sqrt{N}}$, then $(u^p)_{x_i}(x,t) \le \frac{M}{(1+t)^{\alpha}/2}$ for every i, $t>0$, and $\alpha \in (0,1)$ is any constant. Also, p is given by

$$p \ge m - \frac{1}{2} - \frac{\sqrt{1-N(m-1)^2}}{2}.$$

CASE-II Assume that for two positive constants C_1 and C_2, we have

$$f_i = u^l, \quad |S'(u)| \ge C_1 > 0, \quad C_2 \le \left|\frac{S(u)}{u}\right| = |u|^{l-m}.$$

• If $1 < m < 1 + \frac{1}{\sqrt{N}}$, then $(u^p)_{x_i}(x,t) \le \frac{M}{(1+t)^{\alpha/2}}$ for every i, $t > 0$, and $\alpha \ge 1$ is any constant. Also, p is given by

$$p \ge m - \frac{1}{2} - \frac{\sqrt{1 - N(m-1)^2}}{2}.$$

Proof To prove the above theorem, one can add a small positive constant ε to the initial data and consider the problem in uniformly parabolic region $u^\varepsilon \ge \varepsilon$. After obtaining all the necessary bound on u^ε, which will be independent of ε, one can pass to the limit in order to conclude that those results are indeed true for u. This is quite a standard procedure and we will omit these steps. Here, we only give the proof of the uniform estimates.

To begin with, let us make the following transformation

$$v = u^n.$$

Then, it follows that

$$\begin{aligned}(u^m)_{xx} &= (u^{m-n}v)_{xx} = u^{m-n}v_{xx} + 2(u^{m-n})_x v_x + v(u^{m-n})_{xx}\\ &= u^{m-n}v_{xx} + 2(m-n)u^{m-n-1}u_x v_x + v((m-n)u^{m-n-1}u_x)_x\\ &= u^{m-n}v_{xx} + \frac{2(m-n)}{n}u^{m-2n}v_x^2 + \frac{m-n}{n}v(u^{m-2n}v_x)_x\\ &= u^{m-n}v_{xx} + \frac{2(m-n)}{n}u^{m-2n}v_x^2 + \frac{m-n}{n}u^{m-2n}vv_{xx}\\ &\quad + \frac{(m-n)(m-2n)}{n}vv_x u^{m-2n-1}u_x\\ &= \frac{m}{n}u^{m-n}v_{xx} + \frac{m(m-n)}{n^2}u^{m-2n}v_x^2, \end{aligned} \tag{3.3}$$

and consequently,

$$\Delta(u^m) = \frac{m}{n}u^{m-n}\Delta v + \frac{m(m-n)}{n^2}u^{m-2n}\sum_i v_{x_i}^2. \tag{3.4}$$

As our aim is to first get an estimate of derivative of u, so we will make the following transformation

$$w = \frac{1}{2}\sum_{i=1}^N v_{x_i}^2.$$

Then, it follows from (3.3) and (3.4) that

$$\begin{aligned}v_t &= nu^{n-1}\Delta(u^m) + \sum_i f_i'(u)v_{x_i} - nu^{n-1}S(u)\\ &= mv^{(m-1)/n}\Delta v + \frac{2m(m-n)}{n}v^{(m-n-1)/n}w + \sum_i f_i'(u)v_{x_i} - nu^{n-1}S(u). \end{aligned} \tag{3.5}$$

In order to calculate the diffusion term more neatly, we let $h(v) = mv^{(m-1)/n}$. Then, it can be shown that (we are not going to specify the range of i and j to avoid clumsy notations)

$$\begin{aligned}(h(v)\Delta v)_{x_i}v_{x_i} &= (h(v)(\Delta v)v_{x_i})_{x_i} - h(v)(\Delta v)v_{x_ix_i}\\ &= \sum_{j\ne i}(h(v)v_{x_jx_j}v_{x_i})_{x_i} + \left(h(v)(\frac{v_{x_i}^2}{2})_{x_i}\right)_{x_i} - h(v)(\Delta v)v_{x_ix_i}\\ &= \sum_{j\ne i}\left(v_{x_jx_j}(h(v)v_{x_i})_{x_i} + h(v)v_{x_i}(v_{x_i})_{x_jx_j}\right) \end{aligned}$$

$$
\begin{aligned}
&+h(v)(\frac{v_{x_i}^2}{2})_{x_ix_i}+(\frac{v_{x_i}^2}{2})_{x_i}h'(v)v_{x_i}-h(v)(\Delta v)v_{x_ix_i}\\
=&\sum_{j\neq i}\left(v_{x_jx_j}h'(v)v_{x_i}^2+h(v)v_{x_jx_j}v_{x_ix_i}+h(v)\left((\frac{v_{x_i}^2}{2})_{x_jx_j}-v_{x_ix_j}^2\right)\right)\\
&+h(v)(\frac{v_{x_i}^2}{2})_{x_ix_i}+(\frac{v_{x_i}^2}{2})_{x_i}h'(v)v_{x_i}-h(v)(\Delta v)v_{x_ix_i}\\
=&\sum_{i,j}\left(h'(v)v_{x_i}^2v_{x_ix_j}+h(v)(\frac{v_{x_i}^2}{2})_{x_jx_j}-h(v)v_{x_ix_j}^2\right). \qquad (3.6)
\end{aligned}
$$

To obtain a equation for w, we need to calculate the following,

$$
\begin{aligned}
(v_{x_i})_tv_{x_i}=&(h(v)\Delta v)_{x_i}v_{x_i}+\frac{2m(m-n)}{n}\left(v^{(m-n-1)/n}w\right)_{x_i}v_{x_i}\\
&+\left(\sum_j f_j'(u)v_{x_j}\right)_{x_i}v_{x_i}-(nu^{n-1}S(u))_{x_i}v_{x_i}\\
=&(h(v)\Delta v)_{x_i}v_{x_i}+\frac{2m(m-n)}{n}v^{(m-n-1)/n}v_{x_i}w_{x_i}\\
&+\frac{2m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}wv_{x_i}^2+\sum_{i,j}f_j'(u)(\frac{v_{x_i}^2}{2})_{x_j}\\
&+\sum_{i,j}\frac{f_j''(u)}{nu^{n-1}}v_{x_i}^2v_{x_j}-\left((n-1)\frac{S(u)}{u}v_{x_i}+S'(u)v_{x_i}\right)v_{x_i}. \qquad (3.7)
\end{aligned}
$$

So, finally combining (3.4)–(3.7), we conclude that w satisfies the following equation

$$
\begin{aligned}
w_t=&2h'(v)(\Delta v)w+h(v)\Delta w-\sum_{i,j}h(v)v_{x_ix_j}^2\\
&+\frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}w_{x_i}+\frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}w^2\\
&+\sum_j f_j'(u)w_{x_j}+\sum_j\frac{f_j''(u)}{nu^{n-1}}v_{x_j}w-2(n-1)\frac{S(u)}{u}w-2S'(u)w. \qquad (3.8)
\end{aligned}
$$

To achieve our goal, we need another transformation, mainly the following,

$$
\theta=\left(w-\frac{1}{(1+t)^\alpha}\right).
$$

We can see that θ satisfies the following equation

$$
\begin{aligned}
&\theta_t+\left(\frac{1}{(1+t)^\alpha}\right)_t\\
=&2h'(v)(\Delta v)\theta+h(v)\Delta\theta-\sum_{i,j=1}^N h(v)v_{x_ix_j}^2+\frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}\theta_{x_i}\\
&+\frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\theta^2+\sum_j f_j'(u)\theta_{x_j}\\
&+\sum_j\frac{2f_j''(u)}{nu^{n-1}}v_{x_j}w-2(n-1)\frac{S(u)}{u}w-2S'(u)w\\
&+2h'(v)(\Delta v)\frac{1}{(1+t)^\alpha}+\frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}. \qquad (3.9)
\end{aligned}
$$

We are going to use the following maximum principle: If a function v satisfies

$$v_t + a(x,t)v_x + b(x,t)v \le \varepsilon v_{xx},$$

where a, b are arbitrary functions of x, t. Then, the maximum principle gives the estimate $v(x,t) \le 0$, provided $v_0(x) \le 0$. Note that, in our setup, to apply maximum principle, one need not to bother about the terms which contains θ, θ_{x_i} etc. Consequently, we are going to look at the terms which are independent of θ.

CASE I At this point, let us assume that $S = 0$ and $f_i = 0$. In what follows, first we claim that

$$\begin{aligned}&2h'(v)(\Delta v)\frac{1}{(1+t)^\alpha} - \sum_{i,j} h(v)v_{x_ix_j}^2 + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}\\ \le& -cv^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}},\end{aligned}$$

where c is a suitable positive constant and $1 < m < 1 + \frac{1}{\sqrt{N}}$. The proof of the claim is a simple computation of the above terms. To begin with, let $P = \Delta v$. Then, it is easy to see that

$$\sum_{i,j=1}^N h(v)v_{x_ix_j}^2 \ge h(v)\sum_{i=1}^N v_{x_ix_j}^2 \ge \frac{h(v)}{N}\left(\sum_{i=1}^N v_{x_ix_j}\right)^2 = \frac{h(v)}{N}P^2. \tag{3.10}$$

Because

$$\begin{aligned}&2h'(v)(\Delta v)\frac{1}{(1+t)^\alpha} - \sum_{i,j} h(v)v_{x_ix_j}^2 + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}\\ \le& 2h'(v)P\frac{1}{(1+t)^\alpha} - \frac{h(v)}{N}P^2 - \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}\\ =& -cmv^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}} - mv^{(m-2n-1)/n}\left(\frac{1}{N}v^2P^2 - \frac{2(m-1)}{n}vP\frac{1}{(1+t)^\alpha}\right.\\ &\left.+\left(\frac{4m(m-n)(m-n-1)}{n^2} - c\right)\frac{1}{(1+t)^{2\alpha}}\right),\end{aligned}$$

for a suitably chosen small positive constant c, we now choose $n \in (m-1, m)$ and assume that

$$\frac{1}{N}\frac{4(m-n)(n-(m-1))}{n^2} > \frac{(m-1)^2}{n^2},$$

that is,

$$\left(m - n - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{(m-1)^2N}{4} \le 0.$$

The above inequality implies

$$m - \frac{1+\sqrt{1-(m-1)^2N}}{2} < n < m - \frac{1-\sqrt{1-(m-1)^2N}}{2}.$$

Using the above informations, equation (3.9) becomes

$$\begin{aligned}&\theta_t + \left(\frac{1}{(1+t)^\alpha}\right)_t\\ \le& 2h'(v)(\Delta v)\theta + h(v)\Delta\theta - cmv^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}\\ &+\frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}\theta_{x_i} + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\theta^2.\end{aligned} \tag{3.11}$$

Keeping in mind that $n \in (m-1, m)$, we see that $v^{(m-2n-1)/n}$ is bounded. If we choose $0 < \alpha \le 1$, then we see that the following estimate is true for large t,

$$-cv^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}} + \frac{\alpha}{(1+t)^{\alpha+1}} \le 0$$

CASE II As before, first we claim that

$$2h'(v)(\Delta v)\frac{1}{(1+t)^{\alpha}} - \sum_{i,j} h(v)v^2_{x_ix_j} + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}} \le 0,$$

where c is a suitable positive constant and $1 < m < 1 + \frac{1}{\sqrt{N}}$. The proof of the claim is a simple computation of the above terms as before. To begin with, let $P = \Delta v$. Then, it is easy to see that

$$\sum_{i,j=1}^{N} h(v)v^2_{x_ix_j} \ge h(v)\sum_{i=1}^{N} v^2_{x_ix_j} \ge \frac{h(v)}{N}\left(\sum_{i=1}^{N} v_{x_ix_j}\right)^2 = \frac{h(v)}{N}P^2. \tag{3.12}$$

Because

$$\begin{aligned} &2h'(v)(\Delta v)\frac{1}{(1+t)^{\alpha}} - \sum_{i,j} h(v)v^2_{x_ix_j} + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}} \\ \le& 2h'(v)P\frac{1}{(1+t)^{\alpha}} - \frac{h(v)}{N}P^2 - \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}}, \end{aligned}$$

we now choose $n \in (m-1, m)$ and assume that

$$\frac{1}{N}\frac{4(m-n)(n-(m-1))}{n^2} > \frac{(m-1)^2}{n^2},$$

that is,

$$\left(m - n - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{(m-1)^2N}{4} \le 0.$$

The above inequality implies

$$m - \frac{1+\sqrt{1-(m-1)^2N}}{2} < n < m - \frac{1-\sqrt{1-(m-1)^2N}}{2}$$

Using the above informations, equation (3.9) becomes

$$\begin{aligned} \theta_t + \left(\frac{1}{(1+t)^{\alpha}}\right)_t \le& 2h'(v)(\Delta v)\theta + h(v)\Delta\theta + \frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}\theta_{x_i} \\ &+\frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\theta^2 + \sum_j f'_j(u)\theta_{x_j} \\ &+\sum_j \frac{2f''_j(u)}{nu^{n-1}}v_{x_j}w - 2(n-1)\frac{S(u)}{u}w - 2S'(u)w. \end{aligned} \tag{3.13}$$

Next, we assume that $f_i(u) = u^l$. Then, it is easy to see that

$$\begin{aligned} \sum_j \frac{2f''_j(u)}{nu^{n-1}}v_{x_j}w \le& \frac{2\sqrt{N}}{n}\max\left|\frac{uf''_j(u)}{u^n}\right|\frac{1}{\sqrt{N}}\sum_j v_{x_j}w \\ \le& \frac{2l(l-1)\sqrt{N}}{m-1}u^{l-m}w^{3/2} \\ =& du^{l-m}(\theta + \frac{1}{t^{\alpha}})^{3/2}. \end{aligned}$$

Now, we have

$$
\begin{aligned}
(\theta+\frac{1}{(1+t)^\alpha})^{3/2} &= \theta(\theta+\frac{1}{(1+t)^\alpha})^{1/2}+\frac{1}{(1+t)^\alpha}(\theta+\frac{1}{(1+t)^\alpha})^{1/2}\\
&= \theta(\theta+\frac{1}{(1+t)^\alpha})^{1/2}+\Big((\theta+\frac{1}{(1+t)^\alpha})^{1/2}-(\frac{1}{(1+t)^\alpha})^{1/2}\Big)\frac{1}{t^\alpha}\\
&\quad +(\frac{1}{(1+t)^\alpha})^{3/2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big((\theta+\frac{1}{(1+t)^\alpha})^{1/2}-(\frac{1}{(1+t)^\alpha})^{1/2}\Big)\frac{1}{(1+t)^\alpha}\\
&=\left(\frac{\theta}{\Big((\theta+\frac{1}{(1+t)^\alpha})^{1/2}+(\frac{1}{(1+t)^\alpha})^{1/2}\Big)}\right)\frac{1}{(1+t)^\alpha}.
\end{aligned}
$$

It is not hard to see that

$$
M(\theta,t)=\left(\frac{\frac{1}{(1+t)^\alpha}}{\Big((\theta+\frac{1}{(1+t)^\alpha})^{1/2}+(\frac{1}{(1+t)^\alpha})^{1/2}\Big)}\right)
$$

is a bounded quantity. We further claim that under certain condition and $\alpha\ge 1$,

$$
\begin{aligned}
-2(n-1)\frac{S(u)}{u}\frac{1}{t^\alpha}+du^{l-m}(\frac{1}{t^\alpha})^{3/2}\le 0,\\
-2S'(u)\frac{1}{t^\alpha}+\frac{\alpha}{t^{\alpha+1}}\le 0.
\end{aligned}
$$

Again, it is not difficult to check that under the following conditions, the above is true.

$$
\begin{aligned}
|S'(u)|\ge C_1>0,\\
0<C_2\le\left|\frac{S(u)}{u}\right|=du^{l-m}.
\end{aligned}
$$

Hence, using the maximum principle, we have, for each i,

$$
((u^p)_{x_i})^2\le\frac{M}{(1+t)^\alpha},\quad\text{that is,}\quad (u^p)_{x_i}\le\frac{M}{(1+t)^{\alpha/2}}.
$$

□

Theorem 3.2 There exists a weak solution u of the Cauchy problem (3.1), (3.2) which satisfies the following decay rates:

CASE-I Let $S=0$ and for each i, let $f_i=0$.

- If $1<m<1+\frac{1}{\sqrt{N-1}}$, then $(u^q)_{x_i}(x,t)\le\frac{M}{(1+t)^{\alpha/2}}$ for every i and any $t>0$, where

$$
q\ge m-\frac{1}{2}-\frac{\sqrt{2-2(m-1)^2(N-1)}}{4}.
$$

CASE-II Assume that for two positive constants C_1 and C_2, we have

$$
f_i=u^l,\quad |S'(u)|\ge C_1>0,\quad C_2\le\left|\frac{S(u)}{u}\right|=|u|^{l-m}.
$$

- If $1<m<1+\frac{1}{\sqrt{N-1}}$, then $(u^q)_{x_i}(x,t)\le\frac{M}{(1+t)^\alpha/2}$ for every i, $t>0$, and $\alpha\ge 1$ is any constant. Also, q is given by

$$
q\ge m-\frac{1}{2}-\frac{\sqrt{2-2(m-1)^2(N-1)}}{4}.
$$

Proof In order to improve the previous result, we need to introduce another variable which we will make precise in the proof below. To start with, we first recall that w satisfies the following equation

$$\begin{aligned} w_t = & 2h'(v)(\Delta v)w + h(v)\Delta w - \sum_{i,j} h(v)v_{x_ix_j}^2 \\ & + \frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}w_{x_i} + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}w^2 \\ & + \sum_j f_j'(u)w_{x_j} + \sum_j \frac{f_j''(u)}{nu^{n-1}}v_{x_j}w - 2(n-1)\frac{S(u)}{u}w - 2S'(u)w. \end{aligned} \tag{3.14}$$

For any constant s, it follows from (3.5) that

$$\begin{aligned} (v^s)_t = & smv^{((m-1)/n)+s-1}\Delta v + \frac{2ms(m-n)}{n}v^{((m-n-1)/n)-1}(v^s w) \\ & + \sum_{i=1}^N f_i'(u)(v^s)_{x_i} - nsu^{ns-1}S(u) \end{aligned} \tag{3.15}$$

Next, we will introduce a new variable z given by

$$z = v^s w.$$

Then, it follows from (3.14) and (3.15) that

$$\begin{aligned} z_t = & \left(s + \frac{2(m-1)}{n}\right)mv^{((m-1)/n)-1}z\Delta v \\ & + \left(\frac{4m(m-n)(m-n-1)}{n^2} + \frac{2ms(m-n)}{n}\right)v^{((m-2n-1)/n)-s}z^2 \\ & - \sum_{i,j=1}^N mv^{((m-1)/n)+s}v_{x_ix_j}^2 + \sum_{i=1}^N f_i'(u)z_{x_i} + \sum_{j=1}^N \frac{2}{n}f_j''(u)u^{1-n}v_{x_j}z \\ & + mv^{((m-1)/n)+s}\Delta(v^{-s}z) + \frac{2ms(m-n)}{n}v^{((m-n-1)/n)+s}\sum_{i=1}^N v_{x_i}(v^{-s}z)_{x_i} \\ & - 2(n-1)\frac{S(u)}{u}wv^s - 2S'(u)wv^s - nsu^{ns-1}S(u)w. \end{aligned} \tag{3.16}$$

To proceed further, we need to calculate the following

$$\begin{aligned} \Delta(v^{-s}w) = & \sum_{j=1}^N (v^{-s}z)_{x_jx_j} = \sum_{j=1}^N \left((v^{-s})_{x_jx_j}z + 2(v^{-s})_{x_j}z_{x_j} + v^{-s}z_{x_jx_j}\right) \\ = & \sum_{j=1}^N \left((-sv^{-s-1}v_{x_jx_j} + s(s+1)v^{-s-2}v_{x_j}^2)z + 2(v^{-s})_{x_j}z_{x_j} + v^{-s}z_{x_jx_j}\right) \\ = & \left(-sv^{-s-1}\Delta v + 2s(s+1)v^{-s-2}w\right)z + \sum_{j=1}^N \left(-2sv^{-s-1}v_{x_j}z_{x_j} + v^{-s}z_{x_jx_j}\right), \end{aligned} \tag{3.17}$$

also

$$\sum_{i=1}^N v_{x_i}(v^{-s}z)_{x_i} = \sum_{i=1}^N \left(-sv^{-s-1}v_{x_i}^2 z + v^{-s}v_{x_i}z_{x_i}\right)$$

$$= -2sv^{-s-1}wz + \sum_{i=1}^{N} v^{-s} v_{x_i} z_{x_i}. \tag{3.18}$$

Using (3.17) and (3.18) in equation (3.16), we obtain

$$\begin{aligned} z_t = & \frac{2m(m-1)}{n} v^{(m-n-1)/n} z\Delta v - \sum_{i,j=1}^{N} m v^{((m-1)/n+s)} v_{x_j x_j}^2 \\ & + \left(2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}\right) v^{((m-2n-1)/n)-s} z^2 \\ & + \sum_{i=1}^{N} \left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right) v^{(m-n-1)/n} v_{x_i}\right) z_{x_i} + mv^{(m-1)/n}\Delta z \\ & + \sum_{i=1}^{N} \frac{2}{n} f_i''(u) u^{1-n} v_{x_i} z - 2(n-1)\frac{S(u)}{u} z - 2S'(u) z - ns\frac{S(u)}{u} z \\ \leq & \frac{2m(m-1)}{n} v^{(m-n-1)/n} zP - \frac{m}{N} v^{((m-1)/n+s)} P^2 \\ & + \left(2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}\right) v^{((m-2n-1)/n)-s} z^2 \\ & + \sum_{i=1}^{N} \left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right) v^{(m-n-1)/n} v_{x_i}\right) z_{x_i} + mv^{(m-1)/n}\Delta z \\ & + \sum_{i=1}^{N} \frac{2}{n} f_i''(u) u^{1-n} v_{x_i} z - 2(n-1)\frac{S(u)}{u} z - 2S'(u) z - ns\frac{S(u)}{u} z, \end{aligned} \tag{3.19}$$

where we have also used the estimate (3.10).

CASE I At this point, let us assume that $S = 0$ and $f_i = 0$. In what follows, first we claim that

$$\begin{aligned} & \frac{2m(m-1)}{n} v^{(m-n-1)/n} zP - \frac{m}{N} v^{((m-1)/n+s)} P^2 \\ & + \left(2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}\right) v^{((m-2n-1)/n)-s} z^2 \\ \leq & -cv^{((m-2n-1)/n)-s} z^2, \end{aligned}$$

where c is a suitable positive constant and $1 < m < 1 + \frac{1}{\sqrt{N-1}}$. The proof of the claim is a simple computation of the above terms. We can rewrite

$$\begin{aligned} & \frac{2m(m-1)}{n} v^{(m-n-1)/n} zP - \frac{m}{N} v^{((m-1)/n+s)} P^2 \\ & + \left(2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}\right) v^{((m-2n-1)/n)-s} z^2 \\ = & \frac{2m(m-1)}{n} v^{(m-n-1)/n} zP - \frac{m}{N} v^{((m-1)/n+s)} P^2 - cv^{((m-2n-1)/n)-s} z^2 \\ & + \left((2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}) - c\right) v^{((m-2n-1)/n)-s} z^2. \end{aligned}$$

Next, we assume that

$$\frac{2}{N}\left(\frac{s(m-n)}{n} - s(s+1) - \frac{2(m-n)(m-n-1)}{n^2}\right) > \frac{(m-1)^2}{n^2},$$

that is,

$$2n^2s^2 + 2n(2n-m)s + 4(m-n)(m-n-1) + (m-1)^2N < 0.$$

Now, we see that the left hand side of the above inequality is a parabola in the s variable, so we conclude

$$(2n-m)^2 - 8(m-n)(m-n-1) - 2(m-1)^2N > 0.$$

Next, we set $l = m-n$, then the above inequality after some calculation reads

$$4l^2 - 4(2-m)l + 2(m-1)^2N - m^2 < 0.$$

Then,

$$(2-m)^2 - 2(2-m)N + m^2 > 0,$$

which implies

$$(m-1)^2(N-1) < 1.$$

If we only consider the case $m > 1$, then

$$1 < m < 1 + \frac{1}{\sqrt{N-1}},$$

which essentially implies

$$\frac{3m-2-\sqrt{2-2(m-1)^2(N-1)}}{2} < n < \frac{3m-2+\sqrt{2-2(m-1)^2(N-1)}}{2}.$$

For simplicity, we choose $n = (3m-2)/2$. Then, it follows that

$$(ns)^2 + 2(m-1)(ns) - m(1-\frac{m}{2}) + \frac{N(m-1)^2}{2} < 0.$$

Which implies

$$\frac{-2(m-1)-\sqrt{2-2(m-1)^2(N-1)}}{2} < ns < \frac{-2(m-1)+\sqrt{2-2(m-1)^2(N-1)}}{2}.$$

So, finally if $n = (3m-2)/2$ and ns satisfies the above inequality, we conclude from (3.19)that

$$z_t \le mv^{(m-1)/n}\Delta z + \sum_{i=1}^{N}\left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right)v^{(m-n-1)/n}v_{x_i}\right)z_{x_i} - cv^{((m-2n-1)/n)-s}z^2, \tag{3.20}$$

for a suitable positive constant c depending only on m and s. Let us now make another change of variable

$$\theta = z - \frac{1}{(1+t)^\alpha}.$$

Then, we see from (3.20) that θ satisfies the following equation,

$$\theta_t + (\frac{1}{(1+t)^\alpha})_t \le -cv^{((m-2n-1)/n)-s}\frac{1}{(1+t)^{2\alpha}} + -cv^{((m-2n-1)/n)-s}\theta^2 + \sum_{i=1}^{N}\left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right)v^{(m-n-1)/n}v_{x_i}\right)\theta_{x_i}. \tag{3.21}$$

Keeping in mind that $n \in (m-1, m)$, we see that $v^{(m-2n-1)/n}$ is bounded. If we choose $0 < \alpha \le 1$, then we see that the following estimate is true for large t

$$-cv^{(m-2n-1)/n}\frac{1}{(1+t)^{2\alpha}} + \frac{\alpha}{(1+t)^{\alpha+1}} \le 0.$$

Then, we can apply maximum principle on (3.21) to conclude the proof as before.

CASE II As before, first we claim that

$$\begin{aligned}&\frac{2m(m-1)}{n}v^{(m-n-1)/n}zP - \frac{m}{N}v^{((m-1)/n+s)}P^2\\&+\left(2ms(s+1) + \frac{4m(m-n)(m-n-1)}{n^2} - \frac{2ms(m-n)}{n}\right)v^{((m-2n-1)/n)-s}z^2\\&\le 0,\end{aligned}$$

where $1 < m < 1 + \frac{1}{\sqrt{N-1}}$. The proof of the claim is a simple computation of the above terms. It is easy to see that if we assume that

$$\frac{2}{N}\left(\frac{s(m-n)}{n} - s(s+1) - \frac{2(m-n)(m-n-1)}{n^2}\right) > \frac{(m-1)^2}{n^2},$$

then the claim is true. Note the above condition implies that ns should satisfy

$$(ns)^2 + 2(m-1)(ns) - m(1-\frac{m}{2}) + \frac{N(m-1)^2}{2} < 0.$$

Which implies

$$\begin{aligned}&\frac{-2(m-1) - \sqrt{2-2(m-1)^2(N-1)}}{2}\\&< ns < \frac{-2(m-1) + \sqrt{2-2(m-1)^2(N-1)}}{2}.\end{aligned}$$

Finally, from (3.19), we have

$$\begin{aligned}z_t \le{}& mv^{(m-1)/n}\Delta z + \sum_{i=1}^{N}\left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right)v^{(m-n-1)/n}v_{x_i}\right)z_{x_i}\\&+\sum_{i=1}^{N}\frac{2}{n}f_i''(u)u^{1-n}v_{x_i}z - 2(n-1)\frac{S(u)}{u}z - 2S'(u)z - ns\frac{S(u)}{u}z.\end{aligned}\tag{3.22}$$

Let us now make another change of variable

$$\theta = z - \frac{1}{(1+t)^\alpha}.$$

Then, we see from (3.22) that θ satisfies the following equation,

$$\begin{aligned}\theta_t + (\frac{1}{(1+t)^\alpha})_t \le{}& mv^{(m-1)/n}\Delta\theta + \sum_{i=1}^{N}\frac{2}{n}f_i''(u)u^{1-n}v_{x_i}(\theta + \frac{1}{(1+t)^\alpha})\\&+\sum_{i=1}^{N}\left(f_i'(u) + 2m\left(\frac{m-n}{n} - s\right)v^{(m-n-1)/n}v_{x_i}\right)\theta_{x_i}\\&-2(n-1)\frac{S(u)}{u}(\theta + \frac{1}{(1+t)^\alpha}) - 2S'(u)(\theta + \frac{1}{(1+t)^\alpha})\\&-ns\frac{S(u)}{u}(\theta + \frac{1}{(1+t)^\alpha}).\end{aligned}\tag{3.23}$$

Next, we assume that $f_i(u) = u^l$. Then, it is easy to see that

$$\begin{aligned}\sum_j \frac{2f_j''(u)}{nu^{n-1}} v_{x_j} z &\leq \frac{2\sqrt{N}}{n} \max\left|\frac{uf_j''(u)}{u^n}\right| \frac{1}{\sqrt{N}} \sum_j v_{x_j} z \\ &\leq \frac{2l(l-1)\sqrt{N}}{m-1} u^{l-m} z^{3/2} \\ &= du^{l-m}(\theta + \frac{1}{t^\alpha})^{3/2}\end{aligned}$$

Now, we have

$$\begin{aligned}(\theta + \frac{1}{(1+t)^\alpha})^{3/2} &= \theta(\theta + \frac{1}{(1+t)^\alpha})^{1/2} + \frac{1}{(1+t)^\alpha}(\theta + \frac{1}{(1+t)^\alpha})^{1/2} \\ &= \theta(\theta + \frac{1}{(1+t)^\alpha})^{1/2} + \left((\theta + \frac{1}{(1+t)^\alpha})^{1/2} - (\frac{1}{(1+t)^\alpha})^{1/2}\right) \frac{1}{t^\alpha} \\ &\quad + (\frac{1}{(1+t)^\alpha})^{3/2},\end{aligned}$$

and

$$\begin{aligned}&\left((\theta + \frac{1}{(1+t)^\alpha})^{1/2} - (\frac{1}{(1+t)^\alpha})^{1/2}\right) \frac{1}{(1+t)^\alpha} \\ = &\left(\frac{\theta}{\left((\theta + \frac{1}{(1+t)^\alpha})^{1/2} + (\frac{1}{(1+t)^\alpha})^{1/2}\right)}\right) \frac{1}{(1+t)^\alpha}.\end{aligned}$$

It is not hard to see that

$$M(\theta, t) = \left(\frac{\frac{1}{(1+t)^\alpha}}{\left((\theta + \frac{1}{(1+t)^\alpha})^{1/2} + (\frac{1}{(1+t)^\alpha})^{1/2}\right)}\right)$$

is a bounded quantity. We further claim that under certain condition and $\alpha \geq 1$,

$$-(2(n-1) + ns)\frac{S(u)}{u}\frac{1}{t^\alpha} + du^{l-m}(\frac{1}{t^\alpha})^{3/2} \leq 0,$$

$$-2S'(u)\frac{1}{t^\alpha} + \frac{\alpha}{t^{\alpha+1}} \leq 0.$$

Again, it is not difficult to check that under the following conditions, the above is true.

$$|S'(u)| \geq C_1 > 0,$$

$$0 < C_2 \leq \left|\frac{S(u)}{u}\right| = du^{l-m}.$$

Hence, using the maximum principle, we have, for each i,

$$((u^q)_{x_i})^2 \leq \frac{M}{(1+t)^\alpha}, \quad \text{that is,} \quad (u^q)_{x_i} \leq \frac{M}{(1+t)^{\alpha/2}},$$

where q is given by

$$q \geq m - \frac{1}{2} - \frac{\sqrt{2 - 2(N-1)(m-1)^2}}{4}.$$

Hence, we conclude the proof. $\square$

References

[1] Aronson D G. The Porous Medium Equation//Lecture notes in Mathematics. Berlin/New York: Springer-Verlag, 1985, **1224**

[2] Bustos M C, Concha F, Bürger R, Tory E M. Sedimentation and thickening, volume 8 of Mathematical Modelling: Theory and Applications. Kluwer Academic Publishers, Dordrecht, 1999. Phenomenological foundation and mathematical theory

[3] Oleinik O A, Kalashnikov A S, Chzou Yui-lin. The Cauchy problem and boundary problems for equations of the type of unsteady filtration. Izv Ahad Nauk SSR Ser Math, 1958, **22**: 667–704

[4] Escobedo M, Zuazua E. Large time behaviour for convection-diffusion equations in $\mathbb{R}^n$. J Funct Anal, 1991, **100**: 119–161

[5] Escobedo M, Vazquez J L, Zuazua E. Asymptotic behaviour and source type solutions for a diffusion-convection equation. Arch Rational Mech Anal, 1993, **124**: 43–65

[6] Laurencot P H. Large time behaviour for diffusion equations with fast convection. Annali di Matematica pura ed applicata, 1998, **175**: 233–251

[7] Liu T P, Pierre M. Source solutions and asymptotic behaviour in conservation laws. J Differ Equas, 1984, **51**: 419–441

[8] Laurencot P H, Simondon F. Large time behaviour for porous medium equations with convection. Proc Royal Soc Edinburgh, 1998, **128A**: 315–336

[9] Aronson D G. Regularity properties of flows through porous media. SIAM J Appl Math, 1969, **17**: 461–467

[10] Jäger W, Lu Y G. Global regularity of solutions for general degenerate parabolic equations in 1-D. J Differ Equas, 1997, **140**: 365–377

[11] Jäger W, Lu Y G. Hölder estimates of solutions for degenerate parabolic equations. Preprint 96-18, SISSA, Trieste and Abstract of the Workshop Hyperbolic Systems of Conservation Laws, Oberwolffach, Germany, April 28- May 4, 1996

[12] Smoller J A. Shock Waves and Reaction-Diffusion Equations. New York/Heidelberg/Berlin: Springer-Verlag, 1982

[13] Caffarelli L A, Vazquez J L, Wolanski N I. Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation. Indiana Univ Math J, 1987, **36**: 373–401

[14] Jäger W, Lu Y G. On solutions to Nonlinear Reaction-Diffusion-Convection Equations with Degenerate Diffusion. J Differ Equas, 2001, **170**: 1–21

[15] Friedman A. Partial Differential Equations of Parabolic type. Prentice-Hall, Englewood Cliffs, NJ, 1964

[16] Itaya N. On the temporally global problem of the generalized Burgers' equation. J Math Kyoto University, 1974, **14**: 129–177

[17] Ladyzhenskaya O A, Solonnikov V A, Ural'ceva N N. Linear and Quasilinear Equations of Parabolic Type//Translation of Mathematical Monographs, Vol 23. Providence, RI: Amer Math Soc, 1968

[18] Nash J. Le probleme de Cauchy pour les equations differentielles d'un iuide general. Bull Soc Math France, 1962, **90**: 487–497