Bachelorthesis

Compensated Compactness Applied to the Isentropic Euler Equations

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Zusammenfassung

Diese Thesis behandelt den Beweis der Existenz von schwachen Lösungen des Anfangswertproblems der eindimensionalen isentropen Eulergleichungen mit Hilfe der Methode der 'compensated compactness'. Zunächst wird diese Methode für allgemeine Systeme von eindimensionalen Erhaltungsgleichungen eingeführt, hierbei wird der Lösungsbegriff durch die Ausnutzung von Young-Maßen zunächst verallgemeinert. Unter Verwendung von Entropie-Paaren und des div-curl Lemmas wird schließlich eine Vertauschungsrelation im Bezug auf die Young-Maße hergeleitet.
Anschließend wird die Methode auf das Anfangswertproblem der eindimensionalen isentropen Eulergleichungen angewandt. Hierfür werden explizit Entropie-Paare konstruiert und deren Kompaktheits-Eigenschaften nachgewiesen. Letztlich wird mit Hilfe der Kommutationsrelation die Reduktion der maßwertigen Lösungen zu klassischen schwachen Lösungen bewiesen.

Abstract

In this thesis the existence of weak solutions to the initial value problem of the one dimensional isentropic Euler equations is proven using the method of compensated compactness. At first the method of compensated compactness is introduced for general systems of one dimensional conservation laws, here the concept of measure valued solutions is presented by making use of Young measures. Using entropy pairs and the div-curl Lemma a commutation relation regarding the Young measures will be derived.
Subsequently, the method will be applied to the initial value problem of the one dimensional isentropic Euler equations. For this, entropy pairs will be constructed and their compactness properties will be established. Finally, the commutation relation will be used to show the reduction of measure valued solutions to classical weak solutions.

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Contents

1 Introduction

In the late 1950s P. Lax coined the term 'hyperbolic systems of conservation laws' in his equally named paper [15] and lead these problems to become a principal branch in the study of partial differential equations. His student J. Glimm provided a proof of the existence of weak solutions for large times for general systems of one dimensional conservation laws [12], by assuming the total variation of the initial data to be bounded. In the 1970s, F. Murat and L. Tartar began developing the theory of compensated compactness, set to prove the existence of weak solutions to systems of conservation laws for large initial data. The groundwork was the classification of weak continuous non-linear functions, which would allow approximate solutions to converge to weak solutions of the given system of conservation laws. Making use of Young measures, R. DiPerna introduced the idea of measure valued solutions to these systems and subsequently managed to show their reduction to ordinary solutions for the isentropic Euler equations for adiabatic coefficients $\gamma = 1 + \frac{2}{n}$ in [5] . Following this development G.-Q. Cheng [3] filled the gap for all $\gamma \in (1, 5/3]$. The case for all $\gamma \geq 3$ was finally solved by P. Lions, B. Perthame and E. Tadmor in [17]. Lastly P. Lions, B. Perthame and E. Souganidis [16] provided a proof for the remaining cases of $\gamma < 3$.

The aim of this thesis is to prove the existence of weak solutions to the one dimensional isentropic Euler equations for only bounded initial data, using the method of compensated compactness. Moreover, it concentrates on the relevant results needed to present the method of compensated compactness, formulate the existence proof and in doing so presents a summary of the required results.

In Chapter 2 some preliminary results regarding functional analysis and distributions will be summarized.
In Chapter 3 the general problem regarding one dimensional systems of conservation laws will be introduced. Subsequently, the groundwork for the method of compensated compactness in the form of measure valued solutions and Young measures, as well as the div-curl Lemma will be presented.
In Chapter 4 the beforehand presented method of compensated compactness will be applied to the isentropic Euler equations to obtain an existence result for weak solutions to the given initial value problem with only bounded initial data.
Lastly, in Chapter 5 an outlook regarding measure valued solutions in multiple dimensions will be presented.

2 Preliminaries

2.1 Functional Analysis

In this section we will introduce some functional analytical results and definitions, which will be relevant throughout this thesis. The first necessary theorem is a generalization of the Riesz-Representation Theorem, taken from [10].

Theorem 2.1 (Riesz-Representation Theorem). *Let X be a locally compact Hausdorff space. Let $\mu \in M(X)$ be an element of the space of Radon measures on X and f an element of the space of all continuous functions that vanish at infinity $C_0(X)$. Then the mapping $\mu \mapsto I_\mu$, defined as*

$$I_\mu(f) = \int f \, d\mu,$$

is an isometric isomorphism from $M(X)$ to the dual space of $C_0(X)$.

This Riesz-Representation Theorem will allow us to confuse elements from the dual space of $C_0(X)$ with Radon measures in $M(X)$. We will also need the concept of weak and weak* convergence. The following definitions and results are adapted from [2].

Definition 2.2. *Let X be a normed space. A sequence $(x_n)_n \subset X$ converges weakly to $x \in X$ if $x^*(x_n) \to x^*(x)$ for all $x^* \in X^*$, where X^* denotes the dual space of X. In this case we will also write $x_n \rightharpoonup x$.*

Definition 2.3. *Let X be a normed space. A sequence $(x_n^*)_n \subset X^*$ converges weakly* to $x^* \in X^*$ if $x_n^*(x) \to x^*(x)$ for all $x \in X$. In this case we will also write $x_n^* \overset{*}{\rightharpoonup} x^*$.*

If $B_1(0)$ denotes the closed unit ball we have the following two compactness theorems regarding weak and weak* convergence:

Theorem 2.4 (Banach-Alaoglu). *Let X be a separable normed space. Then $B_1(0)(X^*)$ is weak* sequentially compact.*

Theorem 2.5 (Kakutani). *Let X be a reflexiv Banach space. Then $B_1(0)(X)$ is weak sequentially compact.*

Lastly we will introduce Sobolev spaces according to [24]. For this let u be a locally integrable function on $\Omega \subset \mathbb{R}^n$ open. A locally integrable function v on Ω is called the α-weak derivative of u, where $\alpha \in \mathbb{N}^d$ is a multi-index, if

$$\int_\Omega v(x)\phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi(x)\, dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

Here $C_c^\infty(\Omega)$ denotes the space of all compactly supported smooth functions over Ω. In this case we also write $v = D^\alpha u$. A function u is called k-times weakly differentiable, if all its α-weak derivatives for $|\alpha| \leq k$ exist.

Definition 2.6. *Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p \leq \infty$, and $k \in \mathbb{N}_0$. Then the Sobolev space $W^{k,p}(\Omega)$ is defined as*

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{if } |\alpha| \leq k\},$$

where $W^k(\Omega)$ denotes the space of all k-times weakly differentiable functions.

We can further define a norm on $W^{k,p}(\Omega)$ by

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{2.1.1}$$

Together with this norm the Sobolev space $W^{k,p}(\Omega)$ is a Banach space. In the following we will often write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$. The space $H^k(\Omega)$ is a Hilbert space. Furthermore H^{-k} or $W^{-k,p}$ will denote the dual spaces of H_0^k and $W_0^{k,p}$ respectively. Here we have further defined $H_0^k(\Omega)$ and $W_0^{k,p}$, for a bounded set $\Omega \subset \mathbb{R}^n$, to be the set of all $u \in H^k(\Omega)$ or $u \in W^{k,p}$ such that $u = 0$ on $\partial\Omega$.

2.2 Distributions

In this section we will introduce generalized functions or distributions as well as some of their properties which will be needed in Section 4.4.2. In the following, for an open set $\Omega \subset \mathbb{R}^n$, we will denote the set of all smooth functions over Ω with compact support by $D(\Omega)$. An element $\phi \in D(\Omega)$ will be called a test function. Although the spaces $D(\Omega)$ and $C_c^\infty(\Omega)$ contain the same functions, D is equipped with a special inductive limit topology, hence the reason for this distinction. We define distributions according to [24].

Definition 2.7. *A distribution T on Ω is a linear functional on $D(\Omega)$ such that $T(\phi_n) \to 0$ for every sequence $(\phi_n)_n \subset D(\Omega)$ satisfying the following two properties:*

i) $\operatorname{supp}(\phi_n) \subset K \subset \Omega$*, where K is compact and independent of n,*

ii) $\|D^\alpha \phi_n\|_{L^\infty(\Omega)} \to 0$*, as $j \to \infty$ for all $\alpha \in \mathbb{N}^d$.*

In this case we say $\phi_n \to 0$ in distribution.

The space of all distributions on Ω will be denoted by $D'(\Omega)$. We define the α-th derivative of $T \in D'(\Omega)$ as

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \quad \forall \phi \in D(\Omega),$$

for every $\alpha \in \mathbb{N}^d$. Per construction every distribution is infinitely often differentiable.

Example 2.8. Consider $\Omega = \mathbb{R}$ and the Heaviside-function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Since $H \in L^\infty(\Omega) \subset L^1_{loc}(\Omega)$ the Heaviside-function defines a distribution by

$$T_H(\phi) := \int_\Omega H(x)\phi(x)\,dx, \quad \forall \phi \in D(\Omega).$$

According to the above definition we therefore obtain

$$\left(\frac{d}{dx} T_H\right)(\phi) = -T_H(\phi'(x)) = -\int_\Omega H(x)\phi'(x)\,dx = -\int_0^\infty \phi'(x)\,dx = \phi(0),$$

which implies that in the distributional sense $\frac{d}{dx} T_H = \delta_0$, where δ_0 describes the Dirac-Delta distribution. In the following we will often confuse locally integrable functions with the distributions they define, which means we will often wrongfully write $H' = \delta_0$ and interpret H as the above distribution.

3 The Method of Compensated Compactness

3.1 One Dimensional Systems of Conservation Laws

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be open. Given a smooth function $F : \Omega \to \mathbb{R}^n$ we consider the following system of equations

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}F(u(x,t)) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}^+, \tag{3.1.1}$$

where

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}$$

is the unknown vector function and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. The function F is called the flux function and denotes the conservative term. For that reason (3.1.1) is often called a system of one dimensional conservation laws. In the following we will be studying the Cauchy problem corresponding to (3.1.1), which means we would like to find $u : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ satisfying (3.1.1) together with the initial condition

$$u(x,0) = u_0(x), \tag{3.1.2}$$

where $u_0 : \mathbb{R} \to \Omega$ is a given function called the initial data. In the following system (3.1.1) together with the initial data (3.1.2) will be called the initial value problem (3.1.1),(3.1.2).

Definition 3.1. *Let dF denote the Jacobian matrix of the flux function F. We say that system* (3.1.1) *is hyperbolic, if dF has n real eigenvalues λ_n and n corresponding linearly independent eigenvectors. System* (3.1.1) *is called strictly hyperbolic, if all λ_n are distinct. If $\lambda_i = \lambda_j$ coincide at some points or domain, for $i \neq j$, system* (3.1.1) *is called non-strictly hyperbolic or hyperbolically degenerate.*

This definition is according to [13]. A function $u \in C^1(\mathbb{R} \times \mathbb{R}^+; \Omega)$ is called a strong solution of the initial value problem (3.1.1), (3.1.2), if (3.1.1) and (3.1.2) are satisfied pointwise. It can be shown, see for example [24], that in general continuously differentiable solutions of the initial value problem (3.1.1), (3.1.2) do not exist for all times $t \in \mathbb{R}^+$. We must therefore generalize our understanding of solutions. After multiplying (3.1.1) by a function $\phi \in C_c^1(\mathbb{R} \times \mathbb{R}^+; \Omega)$, where C_c^1 denotes the space of all continuously differentiable functions with compact support, and integrating by parts we obtain

$$\int_{-\infty}^{\infty}\int_0^{\infty} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)\, dtdx + \int_{-\infty}^{\infty} u_0(x) \cdot \phi(x,0)\, dx = 0. \tag{3.1.3}$$

Here $\cdot$ denotes the inner product on $\mathbb{R}^n$. Since (3.1.3) is defined even when u is not differentiable we obtain the following definition of a generalized solution, according to [18].

Definition 3.2. *An L^p, $1 < p \leq \infty$, bounded function u is called a weak solution of the initial value problem* (3.1.1), (3.1.2), *with L^p bounded initial data u_0, if* (3.1.3) *holds for all $\phi \in C_c^1(\mathbb{R} \times \mathbb{R}^+; \Omega)$.*

In this thesis we are interested in systems of two conservation laws. In the most general way this corresponds to the following two conservation laws

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t),v(x,t)) = 0, \quad \frac{\partial}{\partial t}v(x,t) + \frac{\partial}{\partial x}g(u(x,t),v(x,t)) = 0, \tag{3.1.4}$$

where $u = u(x,t)$ and $v = v(x,t)$ are in $\mathbb{R}$, with initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \tag{3.1.5}$$

We set $U = (u,v)$ and $F = (f,g)$, such that (3.1.4) can be written as

$$\frac{\partial}{\partial t}U(x,t) + dF(U(x,t))\frac{\partial}{\partial x}U(x,t) = 0. \tag{3.1.6}$$

If we now assume that (3.1.4) or (3.1.6) is strictly hyperbolic then dF has two real and distinct eigenvalues $\lambda_1 < \lambda_2$. Let r_{λ_1}, l_{λ_1} and r_{λ_2}, l_{λ_2} denote the corresponding right and left eigenvectors.

Definition 3.3. *We say that* (3.1.4) *is genuinely non-linear in the* λ_1 *characteristic field, if*

$$\nabla\lambda_1 \cdot r_{\lambda_1} \neq 0$$

and genuinly non-linear in the λ_2 *characteristic field, if*

$$\nabla\lambda_2 \cdot r_{\lambda_2} \neq 0.$$

If $\nabla\lambda_1 \cdot r_{\lambda_1} = 0$ *or* $\nabla\lambda_2 \cdot r_{\lambda_2} = 0$ *in some domain, then system* (3.1.4) *is called linearly degenerate in this domain in the* λ_1 *or* λ_2 *characteristic field.*

The above definition was taken from [18]. We further define Riemann invariants according to [24].

Definition 3.4. *The functions* $w = w(u,v)$ *and* $z = z(u,v)$ *are called Riemann invariants of system* (3.1.4) *corresponding to* λ_1 *and* λ_2 *if they satisfy the equations*

$$\nabla w \cdot r_{\lambda_1} = 0, \quad \nabla z \cdot r_{\lambda_2} = 0. \tag{3.1.7}$$

Following [8], it can be shown that equations (3.1.7) imply that ∇w is a left eigenvector of dF corresponding to λ_2 and ∇z is a left eigenvector of dF corresponding to λ_1. Hence multiplying ∇w and ∇z respectively to the left of (3.1.6) yields

$$\frac{\partial}{\partial t} w(u,v) + \lambda_2(u,v)\frac{\partial}{\partial x} w(u,v) = 0, \quad \frac{\partial}{\partial t} z(u,v) + \lambda_2(u,v)\frac{\partial}{\partial x} z(u,v) = 0.$$

The above transport equations imply the conservation of the quantities w and z along the trajectories given by the solutions of the ordinary differential equations $\frac{dx}{dt} = \lambda_2$ and $\frac{dx}{dt} = \lambda_1$.
The system of conservation laws (3.1.4), (3.1.5) can give rise to physically non-acceptable solutions, see [24]. In order to filter out physically relevant solutions we introduce entropies in accordance to [19].

Definition 3.5. *A function* $\eta \in C^\infty(\mathbb{R} \times \mathbb{R})$ *is called companion, if there exists a smooth function* q *such that*

$$\nabla q(u,v) = \nabla\eta(u,v)\, dF(u,v). \tag{3.1.8}$$

A convex companion is called entropy. In this case the corresponding function q *is called entropy flux and the pair* (η, q) *entropy-entropy flux-pair or simply entropy pair.*

In the following we will often also refer to non-convex companions as entropies, if we want to emphasize the convexity we will sometimes redundantly call entropies convex entropies. For smooth solutions u, v of (3.1.4) condition (3.1.8) for entropy pairs can equivalently be written as an additional conservation law also called the companion law

$$\frac{\partial}{\partial t}\eta(u,v) + \frac{\partial}{\partial x} q(u,v) = 0. \tag{3.1.9}$$

If u, v are only weak solutions the additional conservation law (3.1.9) will instead be imposed as thermodynamic admissibility constraints. This means we will filter out physically relevant solutions, as they must satisfy the additional inequality

$$\frac{\partial}{\partial t}\eta(u,v) + \frac{\partial}{\partial x} q(u,v) \leq 0, \tag{3.1.10}$$

in the sense of distributions, for all non-negative test functions $\phi \in C_c^1(\mathbb{R} \times \mathbb{R}^+)$. The additional conditions (3.1.9) and (3.1.10) are a manifestation of the second law of thermodynamics, which states that the physical entropy of a thermodynamic system must never decrease. We therefore only consider solutions admissible which satisfy this entropy criterion. We end this section with some remarks on entropies and entropy pairs taken from [19]:

i) The entropy flux of each entropy is unique up to an additive constant.

ii) Every strong solution of (3.1.4) satisfies the companion law (3.1.9).

3.2 Young Measures and Measure Valued Solutions

In order to prove the existence of weak solutions to our given system of conservation laws we will construct a convenient sequence of approximate solution and subsequently wish to show that the limit of this approximation satisfies the given system of conservation laws. If the convergence we obtain is strong this limit will in fact be a weak solution, however the approximating sequence will turn out to only have a weak or weak* convergent subsequence. For the limit of this convergent subsequence to be a weak solution, we would need to pass this weak limit through the possibly non-linear flux function. The following example illustrates that we can in general not pass weak or weak* limits through non-linearities.

Example 3.6. Consider the sequence of functions $f_n(x) = \sin(nx)$ for $x \in [0, 2\pi]$. We then know that $f_n \in L^p([0, 2\pi])$ for all $1 \leq p \leq \infty$. Here and in the following we will make use of the fact, that for $1 \leq p < \infty$ and q such that $\frac{1}{p} + \frac{1}{p} = 1$, the dual space of L^p is isometrically isomorphic to L^q, which will allow us to confuse elements of the dual space of L^p with elements of L^q. It can be shown that the weak and weak* limit of f_n is proportional to the signed area under the curve of f_n. We can therefore calculate

$$\frac{1}{2\pi} \int_0^{2\pi} f_n(t)\, dt = 0.$$

Using the above result one can now show that f_n converges weakly to 0 in $L^p([0, 2\pi])$ for $1 < p < \infty$ and weakly* to 0 in $L^\infty([0, 2\pi])$. Now consider the non-linear mapping $x \mapsto x^2$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2(nt)\, dt = \frac{1}{2\pi} \left(\pi - \frac{\sin(4n\pi)}{4n} \right) \to \frac{1}{2}, \quad \text{for } n \to \infty.$$

This implies that the weak and weak* limits of $(f_n)^2$ are different from the square of the weak and weak* limits of f_n.

However in 1937 L.C. Young [28] proposed that for a weakly convergent sequence of measurable functions $u_n : \Omega \to \mathbb{R}^n$ there exists a family of measures ν_x, which at least give the probability distribution of the values of u_n as $n \to \infty$ near x. This means, whereas we cannot make a statement about the limit directly of a weakly convergent sequence under a non-linear function, we can express how the limit behaves on average with respect to these measures ν_x. This is formalized in the following theorem based on [18].

Theorem 3.7 (Young Measure Representation Theorem). *Let $\Omega \subset \mathbb{R}^n$ be measurable. Suppose $u_n : \Omega \to \mathbb{R}^m$ is a sequence of Lebesgue measurable functions. Then there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and a family of positive measures $\nu_x \in M(\mathbb{R}^m)$, depending measurably on x, such that for any continuous $f : \mathbb{R}^m \to \mathbb{R}$ with $\lim_{|\lambda| \to \infty} f(\lambda) = 0$,*

$$f(u_{n_k}) \overset{*}{\rightharpoonup} \langle f(\lambda), \nu_x \rangle = \int_{\mathbb{R}^m} f(\lambda)\, d\nu_x(\lambda) \quad \text{in } L^\infty(\Omega). \tag{3.2.1}$$

If the range of the u_n is contained in some bounded set $G \subset \mathbb{R}^n$, then so is the support of ν_x. If furthermore $\|u_n\|_{L^\infty(\Omega)} \leq M$ for some $M > 0$ and independent of n, then the ν_x are probability measures.

Proof. We set $C_0(\mathbb{R}^m)$ to be the space of all continuous functions on $\mathbb{R}^m$ that vanish at infinity and define a norm on $C_0(\mathbb{R}^m)$ by $\|g\|_{C_0(\mathbb{R}^m)} = \sup_{\lambda \in \mathbb{R}} |g(\lambda)|$. It can be shown that $C_0(\mathbb{R}^m)$ is separable with respect to this norm, hence there exists $\{f_m : m \in \mathbb{N}\} \subset C_0(\mathbb{R}^m)$ with $\{f_m : m \in \mathbb{N}\}$ dense in $C_0(\mathbb{R}^m)$. Furthermore this implies that $(f_1(u_n))_n$ is bounded on Ω. We can therefore extract a subsequence u_{n_k} of u_n such that

$$f_1(u_{n_k}) \overset{*}{\rightharpoonup} \alpha(f_1) \in L^\infty(\Omega).$$

By a diagonal argument we can conclude that there exists some $\alpha(f_m) \in L^\infty(\Omega)$ and a subsequence u_{n_k} of u_n with

$$f_m(u_{n_k}) \overset{*}{\rightharpoonup} \alpha(f_m), \quad \text{for all } m \in \mathbb{N}. \tag{3.2.2}$$

Each f_m defines a linear functional $I(f_m)$ on $L^1(\Omega)$ by

$$I(f_m)(\varphi) = \int_\Omega \varphi(x) \alpha(f_m(x))\, dx = \lim_{k \to \infty} \int_\Omega \varphi(x) f_m(u_{n_k}(x))\, dx.$$

Observe that we can estimate the absolute value of $I(f_m)$ by

$$|I(f_m)(\varphi)| \le \|\alpha(f_m)\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} < \infty, \quad \forall\varphi \in L^1(\Omega),$$

which implies the boundedness of $I(f_m)$. Now choose an arbitrary $f \in C_0(\mathbb{R}^m)$. Because of the density of $\{f_m : m \in \mathbb{N}\}$ we can write $f = \lim_{j\to\infty} f_j$ for some appropriate $(f_j)_j \subset \{f_m : m \in \mathbb{N}\}$. Let $n_{k_1}, n_{k_2} \in \mathbb{N}$. Then using the triangle inequality we obtain

$$\begin{aligned}
&\left|\int_\Omega \left[f(u_{n_{k_1}}(x)) - f(u_{n_{k_2}}(x))\right]\varphi(x)\,dx\right| \\
&\le \left|\int_\Omega \left[f(u_{n_{k_1}}(x)) - f_j(u_{n_{k_1}}(x))\right]\varphi(x)\,dx\right| + \left|\int_\Omega \left[f(u_{n_{k_2}}(x)) - f_j(u_{n_{k_2}}(x))\right]\varphi(x)\,dx\right| \\
&\quad + \left|\int_\Omega \left[f_j(u_{n_{k_1}}(x)) - f_j(u_{n_{k_2}}(x))\right]\varphi(x)\,dx\right| \\
&\le 2\|f - f_j\|_{C_0(\mathbb{R}^m)}\|\varphi\|_{L^1(\Omega)} + \left|\int_\Omega \left[f_j(u_{n_{k_1}}(x)) - f_j(u_{n_{k_2}}(x))\right]\varphi(x)\,dx\right|.
\end{aligned}$$

By the above convergence of $f_j \to f$ we can choose j large enough such that the first term becomes arbitrarily small and in virtue of (3.2.2) we can choose n_{k_1} and n_{k_2} large enough such that the second term is small. Hence the sequence of integrals $(\int_\Omega \varphi f(u_{n_k})\,dx)_k$ is a Cauchy sequence in $\mathbb{R}$ and thus converges for any fixed $\varphi \in L^1(\Omega)$:

$$I(f)(\varphi) = \lim_{k\to\infty}\int_\Omega f(u_{n_k}(x))\varphi(x)\,dx.$$

Since $f \in C_0(\mathbb{R}^m)$ we know that f must be bounded an therefore $I(f)$ is also a bounded linear functional on $L^1(\Omega)$. As the dual of $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$ there exists a unique $\alpha(f) \in L^\infty(\Omega)$ such that

$$I(f)(\varphi) = \int_\Omega \alpha(f)(x)\varphi(x)\,dx, \quad \forall\varphi \in L^1(\Omega). \tag{3.2.3}$$

We now want to show that $\alpha(\cdot)$ is itself a bounded linear functional on $C_0(\mathbb{R}^m)$. Since the above limit of the Cauchy sequence and the representatives in L^∞ are unique we have

$$\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2), \quad \forall f_1, f_2 \in C_0(\mathbb{R}^m)$$

and

$$\alpha(cf) = c\alpha(f), \quad \forall f \in C_0(\mathbb{R}^m), c \in \mathbb{R}.$$

Considering that $\alpha(f)$ is in $L^\infty(\Omega)$, it is in particular locally integrable, and we may hence assume that each point in Ω is a Lebesgue point of $\alpha(f)$. Let $x_0 \in \Omega$ and define

$$\psi(x) = \frac{1}{|B_r(x_0)|}\chi_{B_r(x_0)} \in L^1(\Omega),$$

for some $r > 0$, where $|B_r(x_0)|$ denotes the Lebesgue measure of the open ball with radius r centered at x_0. Using (3.2.3) and the boundedness of $I(f)$ we obtain

$$\fint_{B_r(x_0)} \alpha(f)(x)\,dx \le \|f\|_{C_0(\mathbb{R}^m)}$$

and since x_0 is a Lebesgue point we may let $r \to 0$ in order for the above to imply

$$|\alpha(f)(x_0)| \le \|f\|_{C_0(\mathbb{R}^m)},$$

by the Lebesgue Differentiation Theorem [20] and hence $\alpha(\cdot) \in (C_0(\mathbb{R}^m))^*$. We may now apply the Riesz-Representation Theorem 2.1 which yields some $\nu_{x_0} \in M(\mathbb{R}^m)$, such that

$$\alpha(f)(x_0) = \langle f, \nu_{x_0}\rangle = \int_{\mathbb{R}^m} f(\lambda)d\nu_{x_0}(\lambda).$$

Since x_0 was arbitrary we can now conclude that

$$\lim_{k\to\infty}\int_\Omega f(u_{n_k}(x))\varphi(x)\,dx = \int_\Omega \alpha(f)(x)\varphi(x)\,dx = \int_\Omega \langle f,\nu_x\rangle\varphi(x)\,dx$$

holds for all $\varphi \in L^1(\Omega)$ and thus (3.2.1) is true. Notice that we can find $f \in C_0(\mathbb{R}^m)$ with $f \geq 0$. This implies $\alpha(f)(x) = \langle f,\nu_x\rangle \geq 0$ and hence ν_x must be positive almost everywhere.
The fact that the support of ν_x is contained in the range of u_n is a direct consequence of (3.2.1), since we can simply construct a continuous function f, which vanishes on the range of u. Lastly, if the L^∞-norm of $(u_n)_n$ is uniformly bounded we may choose $f \equiv 1$ in order for (3.2.1) to imply

$$\int_\Omega d\nu_x(\lambda) = 1.$$

Thus the measures ν_x are probability measures. □

We can express the Young measures as

$$\nu_x = \lim_{r\to 0^+}\lim_{k\to\infty} \fint_{B_r(x)} \delta_{u_{n_k}(y)}\,dy,$$

where u_n satisfies the conditions in the above theorem and the limits are to be understood in the weak* sense. Then the integral on the right hand side can be interpreted as the probability distribution of the values $u_{n_k}(y)$ as we select y at random from $B_r(x)$. Therefore the Young measures represent the limiting probability distribution at an infinitely small neighborhood. For a derivation of the above formula see [4]. There are special cases in which we can explicitly construct these Young measures. The following example is taken from [6].

Example 3.8. Suppose $v : \mathbb{R} \to \mathbb{R}$ is a periodic function with period L. We define $u_n(x) := v(nx)$. Then it is known that the sequences induced by scaling any periodic function converge weak* (in an appropriate L^p space) to a constant given by the signed area under the graph of v, similar to 3.6:

$$u_n \overset{*}{\rightharpoonup} \frac{1}{L}\int_0^L v(s)\,ds.$$

Let $g \in C(\mathbb{R})$ be arbitrary, then

$$g(u_n) \overset{*}{\rightharpoonup} \frac{1}{L}\int_0^L g(v(s))\,ds,$$

since $g(u_n(x)) = g \circ u_n(x)$ and we can interpret this again as a scaling of a periodic function. On the other hand, after applying Theorem 3.7, we obtain

$$\frac{1}{L}\int_0^L g(v(s))\,ds = \langle g(\lambda),\nu\rangle = \int_\mathbb{R} g(\lambda)\,d\nu.$$

Hence $\nu = v\left(\frac{ds}{L}\right)$ is the image of the normalized Lebesgue measure under v.
The Young measures ν_x are defined even when the sequence $(u_n)_n$ is only bounded in some L^p_{loc} with $1 \leq p < \infty$. If the sequence is uniformly bounded in L^∞ this is obviously fulfilled. In case that the domain of this sequence is unbounded the Young measures might not be probability measures. However, Theorem 3.7 still holds for continuous functions f satisfying some growth condition, in which case the convergence will be weak. The following corollary is again based on [18].

Corollary 3.9. *Suppose that $(u_n)_n$ is bounded in $L^p_{loc}(\mathbb{R}^n;\mathbb{R}^m)$, where $1 \leq p < \infty$. Then there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and a family of positive measures $\nu_x \in M(\mathbb{R}^m)$, such that for any bounded set $A \subset \mathbb{R}^n$,*

$$f(u_{n_k}(x)) \rightharpoonup \langle f(\lambda),\nu_x\rangle \quad \text{in } L^1(A), \tag{3.2.4}$$

whenever $f \in C(\mathbb{R}^m)$ satisfies the growth condition

$$\lim_{|\lambda|\to\infty} = \frac{f(\lambda)}{|\lambda|^p} = 0.$$

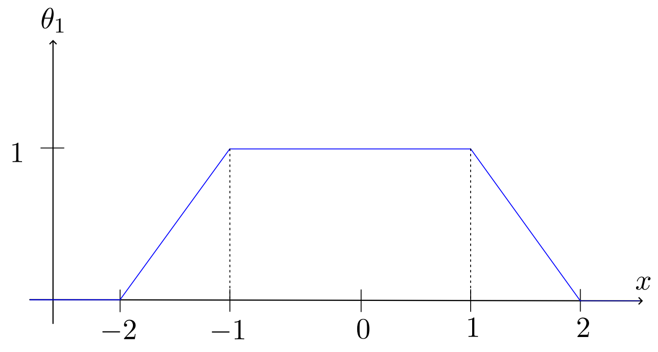

Figure 1: The Function θ_1

Proof. Let $f \in C(\mathbb{R}^m)$. Since the convergence will only be given on a bounded set we may assume that $f \geq 0$, otherwise, since the weak limit is linear, we can shift f upwards by its minimum on the closure of A. We want to apply Theorem 3.7 and for this require a function in $C_0(\mathbb{R}^m)$. In view of this we define $f_m \in C_0(\mathbb{R}^m)$ by $f_m := \theta_m f$, where

$$\theta_m(x) = \begin{cases} 1, & \text{if } |x| \leq m, \\ 1 + m - |x| & \text{if } m \leq |x| \leq m+1, \\ 0 & \text{if } |x| \geq m+1. \end{cases}$$

Then for $A \subset \mathbb{R}^n$ bounded we obtain

$$\begin{aligned} \left| \int_A \left[f_m(u_n(x)) - f(u(x)) \right] \phi(x)\, dx \right| &\leq \|\phi\|_{L^\infty(A)} \int_{\{x \in A : |u_n| \geq m\}} f(u_n(x))\, dx \\ &\leq \|\phi\|_{L^\infty(A)} \|u_n\|_{L^p(A)} \max_{|(x)| \geq m} \left\{ \frac{f(x)}{|x|^p} \right\}, \end{aligned}$$

which by assumption tends to 0 uniformly for $m \to \infty$. Hence

$$\lim_{m \to \infty} \int_A f_m(u_n(x)) \phi(x)\, dx = \int_A f(u_n(x)) \phi(x)\, dx.$$

On the other hand by Theorem 3.7 we obtain a subsequence $(u_{n_k})_k$ and a family of positive measures $\nu_x \in M(\mathbb{R}^m)$ such that

$$\lim_{k \to \infty} \int_A f_m(u_{n_k}(x)) \phi(x)\, dx = \int_A \langle f_m(\lambda), \nu_x \rangle \phi(x)\, dx, \quad \forall \phi \in L^\infty(A),$$

for every $m \in \mathbb{N}$. Now since we assumed f to be positive we know that f_m is monotonically increasing and we can therefore apply the Monotone Convergence Theorem [20] to conclude that

$$\begin{aligned} &\int_A |f(u_{n_k}(x)) - \langle f(\lambda), \nu_x \rangle| |\phi(x)|\, dx \\ &\leq \int_A |f(u_{n_k}(x)) - f_m(u_{n_k}(x))| |\phi(x)|\, dx + \int_A |f_m(u_{n_k}(x)) - \langle f_m(\lambda), \nu_x \rangle |\phi(x)|\, dx \\ &\quad + \int_A |\langle f_m(\lambda), \nu_x \rangle - \langle f(\lambda), \nu_x \rangle |\phi(x)|\, dx \\ &\to 0, \end{aligned}$$

for $m, k \to \infty$, where we also used the above shown convergences. □

We are now able to return to our system of conservation laws (3.1.1). Suppose we have found a sequence of, possibly approximate, solutions $(u_n)_n$ to (3.1.1), namely

$$\frac{\partial}{\partial t} u_n(x,t) + \frac{\partial}{\partial x} F(u_n(x,t)) \to 0, \quad \text{as } n \to \infty, \tag{3.2.5}$$

in the sense of distributions and the sequence $(u_n)_n$ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}^+;\Omega)$. Then Theorem 3.7 tells us that there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that $f(u_{n_k}) \xrightarrow{w^*} \langle f(\lambda),\nu\rangle$ in L^∞ for any continuous function f, since the domain of f is bounded, as a consequence of the uniform boundedness of u_n. Together with (3.2.5) this implies

$$\frac{\partial}{\partial t}\langle\lambda,\nu_{x,t}\rangle + \frac{\partial}{\partial x}\langle F(\lambda),\nu_{x,t}\rangle = 0, \tag{3.2.6}$$

in the sense of distributions and we may therefore interpret $\nu_{x,t}$ as a new type of weak solution, according to [6].

Definition 3.10. *A measurable map $\nu_{x,t} : (x,t) \mapsto \nu_{x,t} \in M(\mathbb{R}^n)$ is called a measure valued solution of system* (3.1.1)*, if*

$$\frac{\partial}{\partial t}\langle\lambda,\nu_{x,t}\rangle + \frac{\partial}{\partial x}\langle F(\lambda),\nu_{x,t}\rangle = 0$$

holds in the sense of distributions, i.e.

$$\int_0^\infty\int_{-\infty}^\infty \left(\langle\lambda,\nu_{x,t}\rangle\cdot\frac{\partial}{\partial t}\phi(x,t) + \langle F(\lambda),\nu_{x,t}\rangle\cdot\frac{\partial}{\partial x}\phi(x,t)\right)dxdt = 0,$$

for all $\phi \in D(\mathbb{R}\times\mathbb{R}^+;\Omega)$.

We can even show, according to [6], that these measure valued solutions also satisfy an admissibility criterion similar to (3.1.10).

Theorem 3.11. *Suppose $\nu_{x,t}$ is the Young measure associated with a sequence $(u_n)_n$ of approximate solutions to system* (3.1.1)*, which are uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}^+;\Omega)$. Then $\nu_{x,t}$ is a measure valued solution of* (3.1.1) *satisfying the admissibility criterion*

$$\frac{\partial}{\partial t}\langle\eta(\lambda),\nu_{x,t}\rangle + \frac{\partial}{\partial x}\langle q(\lambda),\nu_{x,t}\rangle \le 0,$$

in the sense of distributions, for all non-negative test functions in $C_c^1(\mathbb{R}\times\mathbb{R}^+)$ and for all convex entropy pairs (η,q).

Proof. We can calculate

$$\begin{aligned}\frac{\partial}{\partial t}\eta(u_n) + \frac{\partial}{\partial x}q(u_n) = \nabla\eta(u_n)\cdot\left[\frac{\partial}{\partial t}u_n + dF(u_n)\frac{\partial}{\partial x}u_n\right] &= \varepsilon\nabla\eta(u_n)\cdot\frac{\partial^2}{\partial x^2}u_n \\ &= \varepsilon\frac{\partial^2}{\partial x^2}\eta(u_n) - \varepsilon\left(\frac{\partial}{\partial x}u_n\right)^T\nabla^2\eta(u_n)\left(\frac{\partial}{\partial x}u_n\right) \\ &\le \varepsilon\frac{\partial^2}{\partial x^2}\eta(u_n),\end{aligned}$$

since η is convex. Using Theorem 3.7 we obtain

$$\eta(u_n) \overset{*}{\rightharpoonup} \langle\eta(\lambda),\nu_{x,t}\rangle, \quad q(u_n) \overset{*}{\rightharpoonup} \langle q(\lambda),\nu_{x,t}\rangle,$$

which now immediately implies the above assertion, since the right hand side of the above inequality will tend to zero in the sense of distributions. □

In order to obtain a classical weak solution we want to show that the support of the Young measures is contained in a single point. In this case they will reduce to Dirac measures and the convergence in Corollary 3.9 will turn out to be strong. This last theorem is again based on [18].

Theorem 3.12. *Let $1 < p < \infty$ and suppose that the support of ν_x in Corollary 3.9 is a point for almost every $x \in \mathbb{R}^n$. Then there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ which converges strongly to some function u in $L^q_{loc}(\mathbb{R}^n)$ for $1 \le q < p$. Furthermore $\nu_x = \delta_{u(x)}$ almost everywhere.*

Proof. Since $L^p(\mathbb{R}^n)$ is reflexive for $1 < p < \infty$ and u_n is bounded in L^p_{loc} we know by Theorem 2.5 that u_n has a weakly convergent subsequence u_{n_k} converging to u.
From Corollary 3.9 and the given assumptions we know that there exists a function v such that $\nu_x = \delta_{v(x)}$ almost everywhere on $\mathbb{R}^n$. Set $f(\lambda) = \lambda^{(i)}$ for $i = 1, 2, \ldots, m$, where $\lambda^{(i)}$ denotes the i-th component of the vector $\lambda \in \mathbb{R}^m$. Since $p > 1$ the function f satisfies the growth condition in Corollary 3.9 and is obviously continuous. Using (3.2.4) we obtain

$$u^{(i)}_{n_k}(x) = f(u_{n_k}(x)) \rightharpoonup \langle f(\lambda), \nu_x \rangle = v^{(i)}(x)$$

where the superscript (i) again denotes the i-th component of the given vector. The above now implies that $u_{n_k} \rightharpoonup v$ and because of the uniqueness of the weak limit we get $u = v$. We remain to show that this convergence is locally even strong. For this consider $f(\lambda) = |\lambda^{(i)}|^q$ for any $1 \leq q < p$. Then f again satisfies the growth condition and we can apply Corollary 3.9, which yields, for every $\phi \in C_0(\mathbb{R}^n)$ with support in some compact set $K \subset \mathbb{R}^n$,

$$\int_K |u^{(i)}_n(x)|^q \phi(x)\, dx \to \int_K \phi(x) \int_{\mathbb{R}^m} f(\lambda)\, d\nu_x(\lambda) dx = \int_K \phi(x) |u^{(i)}(x)|^q\, dx,$$

as $n \to \infty$, since the support of ν_x was assumed to be a point. This shows the strong convergence of u_n in $L^q_{loc}(\mathbb{R}^n)$. □

Remark 3.13. The strong convergence of a subsequence of $(u_n)_n$ in some L^q_{loc} implies the existence of a subsequence converging almost everywhere by the Theorem of Riesz-Fischer [21].

3.3 The div-curl Lemma

As we have already seen in Example 3.6, we are in general not allowed to pass weak or weak* limits under non-linear functions.

Example 3.14. Consider the two sequences of vector fields u_n and v_n on $[0, 2\pi] \times [0, 2\pi]$ given by

$$u_n = \begin{pmatrix} \cos(nx_1) \\ 0 \end{pmatrix}, \quad v_n = \begin{pmatrix} -\cos(nx_2) \\ 0 \end{pmatrix}.$$

Then these are both in L^2 and converge weakly to 0 in L^2, analogously to Example 3.6. If we now consider the non-linear product of u_n and v_n (the dot-product, meaning non-linear products of each component) we get

$$u_n \cdot v_n = -\cos(nx_1)\cos(nx_2) \to 0,$$

in the sense of distributions, as the oscillations of v_n and u_n are in independent directions.

The example above shows that in some cases the product of two in L^2 weakly convergent sequences can sometimes converge to the product of the weak limits in the sense of distributions. The reason this happens is the compactness of both $\mathrm{div}(u_n)$ and $\mathrm{curl}(v_n)$ in some special space. More precisely both $\mathrm{div}(u_n)$ and $\mathrm{curl}(v_n)$ are bounded in L^2 and hence compact in H^{-1}. This result, called the div-curl lemma, was first found by F. Murat and L. Tartar in 1974 (see for example [27]) and is eponymous for the method presented in this thesis.

Theorem 3.15 (div-curl Lemma). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $(f_n)_n, (g_n)_n \subset L^2(\Omega, \mathbb{R}^m)$ be weakly convergent with limits f, g respectively. Suppose $\{\mathrm{div}(f_n)\}_n$ and $\{\mathrm{curl}(g_n)\}_n$ lie in compact subsets of $H^{-1}_{loc}(\Omega)$ and $H^{-1}_{loc}(\Omega, \mathbb{R}^{m\times m})$. Then*

$$f_n \cdot g_n \to f \cdot g \tag{3.3.1}$$

in the sense of distributions.

Proof. For each $n \in \mathbb{N}$, let $\phi_n \in H^2(\Omega; \mathbb{R}^m) \cap H^1_0(\Omega; \mathbb{R}^m)$ be the solution of the boundary value problem

$$\begin{cases} -\Delta\phi_n = g_n & \text{on } \Omega, \\ \phi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3.2}$$

Since $(g_n)_n$ is weakly convergent it must be bounded in $L^2(\Omega;\mathbb{R}^m)$ and hence $(\phi_n)_n$ is bounded in $H^2(\Omega;\mathbb{R}^m)\cap H_0^1(\Omega;\mathbb{R}^m)$, see [23]. We now define $z_n := -\operatorname{div}\phi_n$ and $y_n := g_n - \nabla z_n$ for all $n\in\mathbb{N}$. Then $(z_n)_n$ is bounded in $H^1(\Omega)$, since $\|z_n\|_{H^1(\Omega)} = \|\operatorname{div}\phi_n\|_{H^1(\Omega)} \le \|\phi_n\|_{H^2(\Omega)} < \infty$. For $1\le i\le m$ we obtain

$$\begin{aligned}
y_k^{(i)} &= g_k^{(i)} - \frac{\partial z_k}{\partial x_i}\\
&= -\sum_{j=1}^{n}\frac{\partial^2\phi_k^{(i)}}{\partial x_j^2} + \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\frac{\partial\phi_k^{(j)}}{\partial x_j}\right)\\
&= \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi_k^{(j)}}{\partial x_i} - \frac{\partial\phi_k^{(i)}}{\partial x_j}\right).
\end{aligned}$$

In view of (3.3.2) it can be shown, since $-\Delta\operatorname{curl}\phi_n = \operatorname{curl} g_n$ and since $\{\operatorname{curl} g_n\}_n$ is contained in a compact set of $H_{loc}^{-1}(\Omega;\mathbb{R}^{m\times m})$ that $\{\operatorname{curl}\phi_n\}_n$ lies in a compact subset of $H_{loc}^{-1}(\Omega;\mathbb{R}^{m\times m})$, by making use of the mapping properties of the inverse Laplacian operator and subsequently the compact embedding of H^1 into H^{-1}. The above calculations then imply that $(y_n)_n$ is contained in a compact set of $L^2(\Omega;\mathbb{R}^m)$. We can now use the reflexivity of the Hilbert space $H^1(\Omega)$ together with Theorem 2.5 in order to obtain (without loss of generality),

$$z_n \rightharpoonup z \quad \text{in } H^1(\Omega), \qquad y_n \to y \quad \text{strongly in } L^2(\Omega;\mathbb{R}^m).$$

Here $z = -\operatorname{div}\phi$ and $y = g - \nabla z$, for $\phi\in H^2(\Omega;\mathbb{R}^m)\cap H_0^1(\Omega;\mathbb{R}^m)$ satisfying

$$\begin{cases} -\Delta\phi = g & \text{on } \Omega,\\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3.3}$$

Let $\psi\in D(\Omega)$, then

$$\int_\Omega f_n(x)\cdot g_n(x)\psi(x)\,dx = \int_\Omega f_n(x)\cdot(y_n(x)+\nabla z_n(x))\psi(x)\,dx.$$

According to the above we have, using the strong convergence of y_n and the weak convergence of f_n in L^2,

$$\int_\Omega f_n(x)\cdot y_n(x)\psi(x)\,dx \to \int_\Omega f(x)\cdot y(x)\psi(x)\,dx,$$

and using the assumption of $\operatorname{div} f_n$ being contained in a compact subset of $H_{loc}^{-1}(\Omega)$, we obtain, using integration by parts and now the weak convergence of z_n and the, without loss of generality, strong convergence of $\operatorname{div} f_n$,

$$\begin{aligned}
\int_\Omega f_n(x)\cdot\nabla z_n(x)\psi(x)\,dx &= -\int_\Omega f_n(x)\cdot\nabla\psi(x)z_n(x)\,dx - \langle\operatorname{div}(f_n), z_n\psi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\
&\to -\int_\Omega f(x)\cdot\nabla\psi(x)z(x) - \langle\operatorname{div}(f), z\psi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\
&= \int_\Omega f(x)\cdot\nabla z(x)\psi(x)\,dx.
\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes a pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Thus we conclude

$$\int_\Omega f_n(x)\cdot g_n(x)\psi(x)\,dx \to \int_\Omega f(x)\cdot(y(x)+\nabla z(x))\psi(x)\,dx = \int_\Omega f(x)\cdot g(x)\psi(x)\,dx.$$

□

The above proof was adopted from [7]. Theorem 3.15 can be used to show a weak continuity of a 2×2 determinant. The following corollary is taken from [18].

Corollary 3.16 (Weak Continuity of a 2×2 Determinant). *Let $\Omega \subset \mathbb{R}\times\mathbb{R}^+$ be a bounded open set and $u_n : \Omega \to \mathbb{R}^4$ a sequence of measurable functions satisfying*

$$u_n \rightharpoonup u, \quad \text{in } (L^2(\Omega))^4$$

and

$$\frac{\partial u_n^{(1)}}{\partial t} + \frac{\partial u_n^{(2)}}{\partial x}, \quad \frac{\partial u_n^{(3)}}{\partial t} + \frac{\partial u_n^{(4)}}{\partial x} \quad \text{are compact in } H^{-1}_{loc}(\Omega).$$

Then there exists a subsequence (still labelled) $(u_n)_n$ such that

$$\begin{vmatrix} u_n^{(1)} & u_n^{(2)} \\ u_n^{(3)} & u_n^{(4)} \end{vmatrix} \to \begin{vmatrix} u^{(1)} & u^{(2)} \\ u^{(3)} & u^{(4)} \end{vmatrix} \quad \text{in the sense of distributions.}$$

Proof. We define two sequences of vector fields $f_n, g_n : \Omega \to \mathbb{R}^2$, by

$$f_n(x,t) = \begin{pmatrix} u_n^{(1)}(x,t) \\ u_n^{(2)}(x,t) \end{pmatrix}, \quad g_n(x,t) = \begin{pmatrix} u_n^{(4)}(x,t) \\ -u_n^{(3)}(x,t) \end{pmatrix}.$$

Then by assumption $(f_n)_n$ and $(g_n)_n$ converge weakly in $L^2(\Omega)$. Moreover we can easily see, since

$$\operatorname{div} f_n = \frac{\partial u_n^{(1)}}{\partial t} + \frac{\partial u_n^{(2)}}{\partial x}$$

and

$$\operatorname{curl} g_n = -\left(\frac{\partial u_n^{(3)}}{\partial t} + \frac{\partial u_n^{(4)}}{\partial x}\right),$$

that $\{\operatorname{div} f_n\}_n$ and $\{\operatorname{curl} g_n\}_n$ are contained in compact subsets of $H^{-1}_{loc}(\Omega)$ and $H^{-1}_{loc}(\Omega;\mathbb{R}^2)$. We may hence apply Theorem 3.15 to obtain the convergence

$$u_n^{(1)}u_n^{(4)} - u_n^{(2)}u_n^{(3)} \to u^{(1)}u^{(4)} - u^{(2)}u^{(3)}$$

in distribution. □

An alternative proof to this weak continuity not relying on Theorem 3.15 can be found in [26]. The conditions needed to apply Theorem 3.15 and Corollary 3.16 respectively can be difficult to verify. We will therefore often make use of the following embedding theorem following [18].

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $C \subset W^{-1,q}_{loc}(\Omega)$ a compact set and $B \subset W^{-1,r}_{loc}(\Omega)$ a bounded set for some $1 < q \le p < r < \infty$. Let $D \subset C \cap B$. Then there exists $E \subset W^{-1,p}_{loc}(\Omega)$ compact, such that $D \subset E$.*

Proof. Let $(f_n)_n$ be an arbitrary sequence in D. Then by the assumptions there exists $M \in \mathbb{R}$ such that

$$\|f_n\|_{W^{-1,r}_{loc}(\Omega)} \le M, \quad \|f_n - f_k\|_{W^{-1,q}_{loc}(\Omega)} \to 0 \quad \text{for } n,k \to \infty.$$

The above is only true for a suitable subsequence of $(f_n)_n$, without loss of generality we assumed that $(f_n)_n$ is the said subsequence. For any Ω_1 compactly contained in Ω we choose $\phi \in D(\Omega)$ such that $\phi|_{\Omega_1} \equiv 1$ and set $\hat{f}_n := \phi f_n$. Then clearly $\operatorname{supp} \hat{f}_n \subset \Omega$ and $\hat{f}_n|_{\Omega_1} = f_n|_{\Omega_1}$. We will now show that the above subsequence is even convergent in $W^{-1,p}_{loc}(\Omega)$. For this let u_n, for each $n \in \mathbb{N}$, be a solution of the boundary value problem

$$\begin{cases} -\Delta u_n = \hat{f}_n & \text{on } \Omega, \\ \quad u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3.4}$$

Then it is known, see [23], that

$$\|u_n\|_{W^{1,q}(\Omega)} \le M_1 \|\hat{f}_n\|_{W^{-1,q}(\Omega)}, \quad \|u_n\|_{W^{1,r}(\Omega)} \le M_1 \|\hat{f}_n\|_{W^{-1,r}(\Omega)},$$

for some $M_1 > 0$. We obtain two estimates, since $\hat{f}_n \in W^{-1,q}(\Omega) \cap W^{-1,r}(\Omega)$. Thus

$$\|u_i - u_j\|_{W^{1,q}(\Omega)} \leq M_1 \|\hat{f}_i - \hat{f}_j\|_{W^{-1,q}(\Omega)} \leq M_1 \|f_i - f_j\|_{W^{-1,q}_{loc}(\Omega)} \to 0,$$

for $i, j \to \infty$. Using the interpolation inequality

$$\|v\|_{W^{1,p}(\Omega)} \leq \|v\|^{1-\alpha}_{W^{1,r}(\Omega)} \|v\|_{W^{1,q}(\Omega)},$$

for $\alpha \in (0,1)$, we obtain

$$\|u_i - u_j\|_{W^{1,p}(\Omega)} \leq \|u_i - u_j\|^{1-\alpha}_{W^{1,r}(\Omega)} \|u_i - u_j\|_{W^{1,q}(\Omega)} \leq M_2 \|u_i - u_j\|_{W^{1,q}_{loc}(\Omega)} \to 0,$$

for some constant $M_2 > 0$ as $i, j \to \infty$. From this and the construction of $\hat{f}_n$ it now follows that

$$\begin{aligned}
\|f_i - f_j\|_{W^{-1,p}(\Omega_1)} &= \sup_{\|\phi\|_{W^{1,p'}_0(\Omega)} \leq 1,\ \operatorname{supp}\phi \subset \Omega_1} |(f_i - f_j)(\phi)| \\
&\leq \sup_{\|\phi\|_{W^{1,p'}_0(\Omega)} \leq 1} |(\hat{f}_i - \hat{f}_j)(\phi)| = \|\hat{f}_i - \hat{f}_j\|_{W^{-1,p}(\Omega)} \\
&= \|\Delta u_i - \Delta u_j\|_{W^{-1,p}(\Omega)} \leq \|u_i - u_j\|_{W^{1,p}(\Omega_1)} \to 0,
\end{aligned}$$

as $i, j \to \infty$ for $\frac{1}{p} + \frac{1}{p'} = 1$. Here we used $\|\Delta v\|_{W^{-1,p}(\Omega)} \leq \|v\|_{W^{1,p}(\Omega)}$, since

$$|(\Delta v)(\phi)| = \left| \int_\Omega \nabla v \cdot \nabla \phi \, dx \right| \leq \|v\|_{W^{1,p}(\Omega)} \|\phi\|_{W^{1,p'}_0(\Omega)}.$$

Hence D is contained in a compact set in $W^{-1,p}_{loc}(\Omega)$. □

A similar useful theorem, known as Murat's Lemma is the following taken from [4].

Theorem 3.18 (Murat's Lemma)**.** *Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Suppose $(\phi_n)_n$ is bounded in $W^{-1,p}(\Omega)$, for $p > 2$. Furthermore let $\phi_n = \varphi_n + \psi_n$ for all $n \in \mathbb{N}$, where $(\varphi_n)_n$ is contained in a compact set of $W^{-1,2}(\Omega)$, while $(\psi_n)_n$ lies in a bounded set of $M(\Omega)$. Then $(\phi_n)_n$ lies in a compact set of $W^{-1,2}(\Omega)$.*

Proof. The proof is similar to the proof of Theorem 3.17 and can be found in [4]. □

3.4 Viscosity Solutions

The aim of this section is to find approximate solutions to the hyperbolic system of conservation laws (3.1.1). For this we add a small parabolic perturbation term to (3.1.1), which yields

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} F(u(x,t)) = \varepsilon \frac{\partial^2}{\partial x^2} u(x,t), \quad \text{on } \mathbb{R} \times \mathbb{R}^+. \tag{3.4.1}$$

Physically the parabolic perturbation corresponds to introducing viscosity to the system (3.1.1), hence the reason why solutions of (3.4.1), satisfying some initial data

$$u(x,0) = u_0(x), \quad \text{on } \mathbb{R}, \tag{3.4.2}$$

are called viscosity solutions. We will only state the following existence theorem for this parabolic problem and refer to [24] and [18] for details on the proof.

Theorem 3.19. *For any fixed $\varepsilon > 0$, the initial value problem* (3.4.1), (3.4.2) *with bounded measurable initial data* (3.4.2) *has a local smooth solution $u_\varepsilon(x,t) \in C^\infty(\mathbb{R} \times (0,\tau); \Omega)$ for some small time τ, which only depends on the L^∞-norm of the initial data* (3.4.2)*.*

i) If the solution u_ε satisfies the a priori estimate

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty([0,T])} \leq M(\varepsilon,T), \quad \text{for any } t \in [0,T],$$

then the solution exists on $\mathbb{R} \times [0,T]$.

ii) The solution u_ε satisfies

$$\lim_{|x|\to\infty} u_\varepsilon(x,t) = 0, \quad \textit{if} \lim_{|x|\to\infty} u_0(x) = 0.$$

iii) If one of the equations in (3.4.1) *is of the form*

$$\frac{\partial}{\partial t} w(x,t) + \frac{\partial}{\partial x}(w(x,t)g(u(x,t))) = \varepsilon \frac{\partial^2}{\partial x^2} w(x,t),$$

where $g(u)$ is a continuous function of $u \in \mathbb{R}^n$, then

$$w_\varepsilon(x,t) \geq c(t, c_0, \varepsilon) > 0, \quad \textit{if } w_0(x) \geq c_0 > 0$$

for some positive constant c_0.

If the solution satisfies the a priori estimate $i)$ then it can be extended to T stepwise, as follows:
If we have found a local solution u_ε as above, satisfying $i)$ we consider the initial value problem (3.4.1) but now with initial data $\hat{u}_\varepsilon(x,0) = u(x,\tau)$ which is by $i)$ again bounded in the L^∞-norm. Iterating this process shows that a solution can be extended up to T, since the step time τ is only dependent on the L^∞-norm of each initial data and this norm will stay bounded by assumption.
Having an a priori estimate $i)$ will turn out to be important, as the uniform L^∞-boundedness will yield the existence of a weak* convergent subsequence by Theorem 2.4.

3.5 The Method of Compensated Compactness

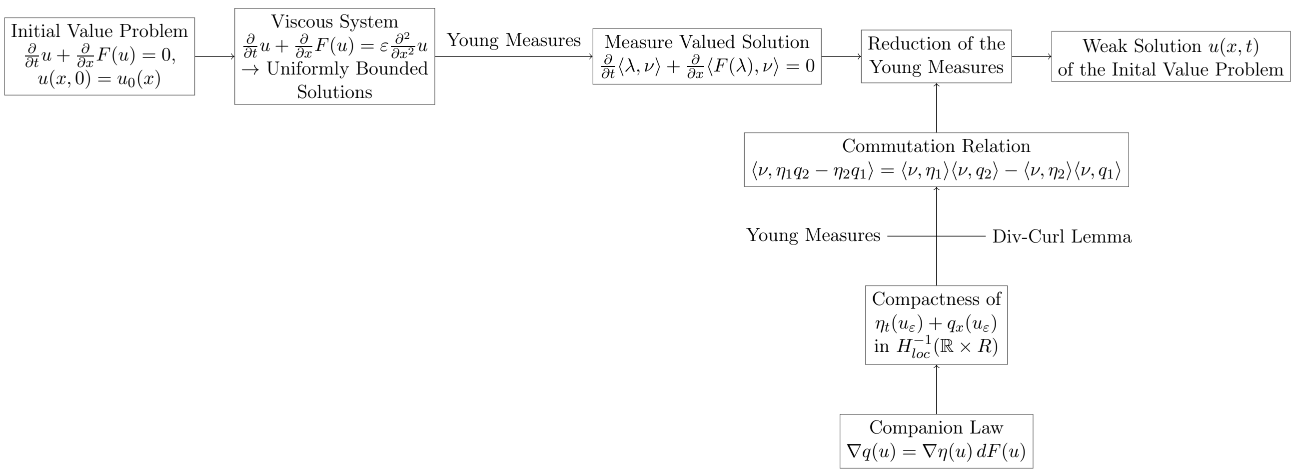

Figure 2: The Method of Compensated Compactness

Using the results obtained in the above sections we are finally able to present the method of compensated compactness. We wish to show the existence of weak solutions to the initial value problem of the system of hyperbolic conservation laws given by (3.1.1), (3.1.2). For this, we construct a sequence of approximate solutions u_ε according to Section 3.4 by adding viscosity to the system. To prove the existence of such an approximate solution, we only need to show that the approximate solution satisfies some a priori L^∞-bound. Theorem 3.19 will then yield a sequence u_ε of smooth solutions. Furthermore, by Theorem 2.4 we can extract a weak* convergent subsequence. We now hope that the limit of this sequence satisfies (3.1.3), however as Example 3.6 showed, it is in general not possible to pass weak or weak* limits under non-linear functions. We are therefore required to use the results found in Section 3.2 in order to see that for now we only obtain a measure valued solution, where the measures are given by Young measures corresponding to the sequence of approximate solutions. In order to find a weak solution of the initial value problem (3.1.1), (3.1.2), we must show that these Young measures reduce to Dirac measures, such that Theorem 3.12 will yield strong convergence in L^p. At this point we need to make use of the additional structure we gave our system of conservation laws (3.1.1) in the form of a companion law (3.1.8). The aim is to use explicit entropy pairs in order to show that the support of the Young measures is confined to a point. This obviously requires us to have a rich family of entropy pairs, which greatly restricts the conservation laws we can apply our method to, to the ones with a suitable flux function. Suppose for now that we have found entropy pairs (η_1, q_1) and (η_2, q_2) of system (3.1.4) (i.e. the system of two conservation laws). The crucial step is to now show that these entropy pairs satisfy the compactness property necessary to apply Corollary 3.16, with respect to the viscosity solutions. Since the viscosity solutions satisfy some uniform bound we can assume that the entropy pairs are compactly supported and hence converge weakly in L^2. We thus only need to verify the compactness relation

$$\frac{\partial \eta_1(u_\varepsilon)}{\partial t} + \frac{\partial q_1(u_\varepsilon)}{\partial x}, \quad \frac{\partial \eta_2(u_\varepsilon)}{\partial t} + \frac{\partial q_2(u_\varepsilon)}{\partial x} \quad \text{are compact in } H^{-1}_{loc}(\Omega).$$

The above compactness would then imply the following commutation relation

$$\langle \eta_1 q_2 - \eta_2 q_1, \nu \rangle = \langle \eta_1, \nu \rangle \langle q_2, \nu \rangle - \langle \eta_2, \nu \rangle \langle q_1, \nu \rangle, \tag{3.5.1}$$

where ν denotes a Young measure, after applying the div-curl Lemma 3.15. It is then a matter of cleverly using our given entropy pairs together with (3.5.1) to show the reduction of the Young measures to Dirac measures. In this case we will obtain strong convergence of a suitable subsequence of our viscosity solutions, which would in return imply that the limit is a weak solution of system (3.1.4).

4 Application to the Isentropic Euler Equations

In this section we will apply the method of compensated compactness, as described in 3.5, to obtain an existence result for weak solutions to the isentropic Euler equations. First of all, we will therefore introduce the Euler equations and derive some basic properties similar to [18].

4.1 The Isentropic Euler Equations

In Eulerian coordinates the flow of a gas is described by

$$\begin{cases} \frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho(x,t)u(x,t)) = 0, \\ \frac{\partial}{\partial t}(\rho(x,t)u(x,t)) + \frac{\partial}{\partial x}\left(\rho(x,t)u^2(x,t) + P(\rho(x,t))\right) = 0, \end{cases} \tag{4.1.1}$$

where in the following we will abbreviate the above by

$$\begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + P(\rho)\right) = 0. \end{cases} \tag{4.1.2}$$

We further introduce some bounded and measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho^{(0)}(x), u^{(0)}(x)), \quad \rho^{(0)}(x) \geq 0. \tag{4.1.3}$$

Physically ρ describes the density of the gas, u the velocity and $P(\rho)$ the pressure of the gas, satisfying $P'(\rho) \geq 0$. The equations (4.1.2) are therefore simply describing the conservation of mass and the conservation of momentum respectively. Substituting $m = \rho u$ yields

$$\begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}m = 0, \\ \frac{\partial}{\partial t}m + \frac{\partial}{\partial x}\left(\frac{m^2}{\rho} + P(\rho)\right) = 0, \end{cases} \tag{4.1.4}$$

where m is called the momentum. Later in this section we will restrict ourselves to the special isentropic pressure law given by

$$P(\rho) = c\rho^\gamma, \quad \gamma > 1,$$

where we call γ the adiabatic coefficient and $c > 0$ denotes a constant. Regarding (3.1.1) the flux function of the above system of conservation laws takes the form

$$F(\rho, m) = \left(m, \frac{m^2}{\rho} + P(\rho)\right),$$

which allows us to calculate

$$dF = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + P'(\rho) & \frac{2m}{\rho} \end{pmatrix}.$$

From this we find that the eigenvalues of dF are given by

$$\lambda_1 = \frac{m}{\rho} - \sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \sqrt{P'(\rho)},$$

and the eigenvectors by

$$r_1 = (1, \lambda_1)^T, \quad r_2 = (1, \lambda_2)^T.$$

Additionally, we want to calculate the Riemann invariants of (4.1.4). Following the same arguments as in 3.1 we obtain that the Riemann invariants $w(\rho, m), z(\rho, m)$ must satisfy

$$\nabla w \, dF = \lambda_2 \nabla w, \quad \nabla z \, dF = \lambda_1 \nabla z. \tag{4.1.5}$$

Writing out equation (4.1.5) explicitly yields

$$\begin{aligned} \nabla w \, dF &= \left(\frac{\partial}{\partial m}w\left(-\frac{m^2}{\rho^2} + P'(\rho)\right), \frac{\partial}{\partial \rho}w + \frac{2m}{\rho}\frac{\partial}{\partial m}w\right) \\ &= \left(\left(\frac{m}{\rho} + \sqrt{P'(\rho)}\right)\frac{\partial}{\partial \rho}w, \left(\frac{m}{\rho} + \sqrt{P'(\rho)}\right)\frac{\partial}{\partial m}w\right). \end{aligned}$$

Form the second equation we obtain the partial differential equation

$$\frac{\partial}{\partial \rho}w + \frac{m}{\rho}\frac{\partial}{\partial m}w = \sqrt{P'(\rho)}\frac{\partial}{\partial m}w,$$

which can be solved to give

$$w(\rho, m) = \frac{m}{\rho} + \int_0^\rho \frac{\sqrt{P'(s)}}{s}\,ds, \tag{4.1.6}$$

and in similar way, upon using (4.1.5), we find

$$z(\rho, m) = \frac{m}{\rho} - \int_0^\rho \frac{\sqrt{P'(s)}}{s}\,ds. \tag{4.1.7}$$

We now assume that the pressure is described by the isentropic pressure law. A straightforward calculation then shows that

$$\nabla\lambda_1 \cdot r_1 = -\frac{(\gamma-1)(\gamma+1)}{4}\rho^{\gamma-3}, \quad \nabla\lambda_2 \cdot r_2 = \frac{(\gamma-1)(\gamma+1)}{4}\rho^{\gamma-3},$$

which implies that (4.1.4) is strictly hyperbolic in the domain where $\rho > 0$ and else non-strict. Furthermore it is genuinely non-linear if $\gamma \in (1, 3]$ and for $\gamma > 3$ no longer genuinely non-linear in the vacuum state $\rho = 0$.

4.2 Existence of Viscosity Solutions

In this section we want to construct a sequence of approximate solutions to the inital value problem (4.1.2), (4.1.3). Regarding Section 3.4 we will consider the viscous approximation to (4.1.4) given by

$$\begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}m = \varepsilon\frac{\partial^2}{\partial x^2}\rho, \\ \frac{\partial}{\partial t}m + \frac{\partial}{\partial x}\left(\frac{m^2}{\rho} + P(\rho)\right) = \varepsilon\frac{\partial^2}{\partial x^2}m, \end{cases} \tag{4.2.1}$$

with initial data

$$(\rho_\varepsilon(x,0), m_\varepsilon(x,0)) = (\rho_\varepsilon^{(0)}(x), m_\varepsilon^{(0)}(x)), \tag{4.2.2}$$

defined by

$$(\rho_\varepsilon^{(0)}(x), m_\varepsilon^{(0)}(x)) = (\rho^{(0)}(x) + \varepsilon, \rho^{(0)}(x)u^{(0)}(x)) * \varphi_\varepsilon,$$

where φ_ε is a mollifier, such that

$$(\rho_\varepsilon^{(0)}(x), m_\varepsilon^{(0)}(x)) \in C^\infty \times C^\infty, \quad (\rho_\varepsilon^{(0)}(x), m_\varepsilon^{(0)}(x)) \to (\rho^{(0)}(x), m^{(0)}(x)),$$

for $\varepsilon \to 0$ almost everywhere. We furthermore assume that the initial data (4.2.2) is bounded by

$$\varepsilon \leq \|\rho_\varepsilon^{(0)}\|_{L^\infty(\mathbb{R})} \leq M_1, \quad \|u_\varepsilon^{(0)}\|_{L^\infty(\mathbb{R})} \leq M_1,$$

where the constant $M_1 \in \mathbb{R}$ only depends on the L^∞-norm of the initial data (4.1.3). Regarding Theorem 3.19 a smooth solution of the initial value problem (4.2.1), (4.2.2) exists, if we can find some a priori bound on the solution. The following theorem is taken from [18], furthermore in order to keep the notation clear we will abbreviate partial derivatives by $f_x := \frac{\partial}{\partial x}f$.

Theorem 4.1 (Existence of Viscosity Solutions)**.** *Let $(\rho^{(0)}(x), u^{(0)}(x))$ be bounded and measurable satisfying $\rho^{(0)}(x) \geq 0$, $P(\rho) \in C^2(0,\infty)$, $P'(\rho) > 0$ and $2P'(\rho) + \rho P''(\rho) \geq 0$ for $\rho > 0$ as well as*

$$\int_c^\infty \frac{\sqrt{P'(\rho)}}{\rho}\,d\rho = \infty, \quad \int_0^c \frac{\sqrt{P'(\rho)}}{\rho}\,d\rho < \infty, \quad \forall c > 0. \tag{4.2.3}$$

Then for any $\varepsilon > 0$, the smooth viscosity solutions $(\rho_\varepsilon(x,t), m_\varepsilon(x,t))$ of the initial value problem (4.2.1), (4.2.2) exist and satisfy

$$0 < c(\varepsilon, t) \leq \rho_\varepsilon(x,t) \leq M_2, \quad \|u_\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} = \left\|\frac{m_\varepsilon}{\rho_\varepsilon}\right\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq M_2, \tag{4.2.4}$$

where $M_2 > 0$ and $c(\varepsilon, t)$ could tend to zero as ε tends to zero or t tends to infinity.

Proof. We will show that $\Sigma := \{(\rho, m) : w(\rho, m) \leq M,\ z(\rho, m) \geq -M\}$, for some $M \in \mathbb{R}$, is an invariant region, meaning if the initial data (4.2.2) is contained in Σ then so will be the solution for all times. This can be done by showing that the Riemann invariants satisfy a parabolic equation and therefore a maximum principle. To obtain this parabolic equation we are going to multiply (4.2.1) by ∇w and ∇z respectively, where w, z denote the Riemann invariants (4.1.6), (4.1.7). Using that ∇w, ∇z are left eigenvectors of dF this yields, after a multiplication from the left with ∇w

$$w_\rho \rho_t + w_m m_t + \lambda_2(w_\rho \rho_x + w_m m_x) = \varepsilon(w_\rho \rho_{xx} + w_m m_{xx}).$$

Furthermore employing

$$w_{xx} = (w_{\rho\rho}\rho_x + w_{m\rho}m_x)\rho_x + w_\rho \rho_{xx} + w_m m_{xx} + (w_{\rho m} + w_{mm}m_x)m_x,$$

yields, after an application of Schwartz Theorem,

$$w_t + \lambda_2 w_x = \varepsilon \left(w_{xx} - w_{\rho\rho}\rho_x^2 - 2w_{m\rho}m_x\rho_x - w_{mm}m_x^2 \right). \tag{4.2.5}$$

A straightforward calculation regarding (4.1.6) implies

$$w_{\rho\rho} = \frac{2m}{\rho^3} + \frac{P''(\rho)}{2\rho\sqrt{P'(\rho)}} - \frac{\sqrt{P'(\rho)}}{\rho^2}, \tag{4.2.6}$$

$$w_{\rho m} = w_{m\rho} = -\frac{1}{\rho^2}, \tag{4.2.7}$$

$$w_{mm} = 0, \tag{4.2.8}$$

and we wish to eliminate m_x. For this recall that

$$\nabla w\, dF = \lambda_2 \nabla w,$$

from which we obtain

$$m_x = \frac{w_x}{w_m} - \left(\lambda_2 - \frac{2m}{\rho}\right)\rho_x = \frac{w_x}{w_m} + \left(\frac{m}{\rho} - \sqrt{P'(\rho)}\right)\rho_x. \tag{4.2.9}$$

Substituting (4.2.6), (4.2.7), (4.2.8), (4.2.9) into (4.2.5) yields

$$\begin{aligned}
&\varepsilon \left(w_{xx} - \left(\frac{2m}{\rho^3} + \frac{P''(\rho)}{2\rho\sqrt{P'(\rho)}} - \frac{\sqrt{P'(\rho)}}{\rho^2} \right)\rho_x^2 - 2\left(-\frac{1}{\rho^2}\left(\frac{w_x}{w_m} + \left(\frac{m}{\rho} - \sqrt{P'(\rho)} \right)\rho_x \right)\rho_x \right) \right) \\
&= \varepsilon \left(w_{xx} - \frac{2m}{\rho^2}\rho_x^2 - \frac{P''(\rho)}{2\rho\sqrt{P'(\rho)}}\rho_x^2 + \frac{\sqrt{P'(\rho)}}{\rho^2}\rho_x^2 + \frac{2}{\rho}w_x\rho_x + \frac{2m}{\rho^3}\rho_x^2 - \frac{2\sqrt{P'(\rho)}}{\rho^2}\rho_x^2 \right) \\
&= \varepsilon \left(w_{xx} - \frac{2P'(\rho) + \rho P''(\rho)}{2\rho^2\sqrt{P'(\rho)}}\rho_x^2 + \frac{2}{\rho}w_x\rho_x \right).
\end{aligned}$$

In a completely similar way we also obtain

$$z_t + \lambda_1 z_x = \varepsilon \left(z_{xx} + \frac{2}{\rho}\rho_x z_x + \frac{2P'(\rho) + \rho P''(\rho)}{2\rho^2\sqrt{P'(\rho)}}\rho_x^2 \right).$$

If we now make use of the assumption $2P'(\rho) + \rho P''(\rho) \geq 0$ this yields

$$w_t + \lambda_2 w_x \leq \varepsilon w_{xx} + \frac{2\varepsilon}{\rho}\rho_x w_x, \quad z_t + \lambda_1 z_x \geq \varepsilon z_{xx} + \frac{2\varepsilon}{\rho}\rho_x z_x. \tag{4.2.10}$$

We can now apply the maximum principle for parabolic equations to (4.2.10) (more specifically to subsolutions) since these are only scalar equations for w and z. This implies that there exists some $M \in \mathbb{R}$ such that

$$w(\rho, m) \leq M, \quad z(\rho, m) \geq -M \tag{4.2.11}$$

and shows that the above defined Σ is in fact an invariant region. From the additional assumption (4.2.3) and the explicit form of the Riemann invariants (4.1.6), (4.1.7), we may now conclude that $0 \leq \rho_\varepsilon \leq M_2$ and $\|u_\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq M_2$ for some constant $M_2 \in \mathbb{R}$. Finally, since u is uniformly bounded Theorem 3.19 *iii*) yields the lower bound on ρ. $\square$

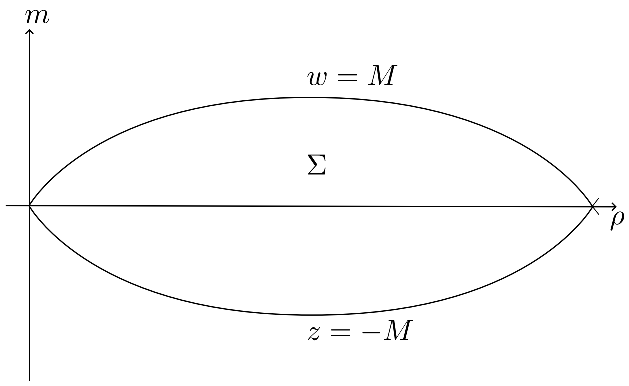

Figure 3: The Invariant Region Σ

Theorem 4.1 obviously now implies the existence of uniformly bounded solutions to the isentropic Euler equations, since

$$P'(\rho) = \gamma c \rho^{\gamma-1} \geq 0,$$

and

$$2P'(\rho) + \rho P''(\rho) = c\rho^{\gamma-1}(\gamma^2 + \gamma) \geq 0$$

and lastly

$$\int \frac{\sqrt{P'(s)}}{s}\, ds = \sqrt{c\gamma}\frac{2s^{\frac{\gamma-1}{2}}}{\gamma - 1},$$

which satisfies (4.2.3).
Occasionally, it is necessary to have a stronger lower bound on the density as in Theorem 4.1, as the lower bound on the density might tend to zero in the limit of vanishing viscosity. It can be shown, see for example [16], that even a uniform lower bound, that might only tend to zero in time, exists for the density.

4.3 Weak Entropies and H^{-1}_{loc} Compactness

In this section we are going to construct entropy pairs corresponding to (4.1.2). Furthermore we are going to verify that these entropy pairs satisfy the compactness relation necessary to apply Corollary 3.16. We do so for the case of the isentropic pressure law $P(\rho) = c\rho^\gamma$, where we set $c = k^2 = \frac{(\gamma-1)^2}{4\gamma}$. For smooth solutions it can be verified, according to [18], that system (4.1.2) can equivalently be written as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + \left(\frac{1}{2}u^2 + \int_0^\rho \frac{P'(s)}{s}\, ds\right)_x = 0, \end{cases} \tag{4.3.1}$$

and hence both systems have the same entropy pairs. This means we obtain the additional system

$$(q_\rho, q_u) = (\eta_\rho, \eta_u)\begin{pmatrix} u & \rho \\ \frac{P'(\rho)}{\rho} & u \end{pmatrix} = \left(\eta_\rho u + \eta_u \frac{P'(\rho)}{\rho}, \eta_\rho \rho + \eta_u u\right),$$

which yields

$$q_{\rho u} = \eta_{\rho u} u + \eta_{uu}\frac{P'(\rho)}{\rho} + \eta_\rho,$$
$$q_{u\rho} = \eta_{\rho\rho}\rho + \eta_\rho + \eta_{u\rho} u.$$

After applying Schwartz Theorem once again the above implies

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{4.3.2}$$

Definition 4.2. *An entropy η of (4.1.2) is called a weak entropy, if η satisfies (4.3.2) together with the boundary condition*

$$\eta(\rho = 0, u) = 0, \quad \eta_\rho(\rho = 0, u) = g(u), \tag{4.3.3}$$

where g is some arbitrary function.

Notice that the coefficients in (4.3.2) are independent of u. This reflects the Galilean invariance of system (4.1.2). The general solution of (4.3.2), (4.3.3) is given by the following theorem taken from [17] with some adaptations from [18].

Theorem 4.3. *Let $\rho \geq 0$, $u, w \in \mathbb{R}$. We define the entropy kernel G as*

$$G(\rho, w) = \left(\rho^{\gamma-1} - w^2\right)_+^\lambda := \sup\left\{0, \left(\rho^{\gamma-1} - w^2\right)^\lambda\right\}, \tag{4.3.4}$$

with $\lambda = \frac{3-\gamma}{2(\gamma-1)}$.

i) The solution of (4.3.2), (4.3.3) is given by

$$\eta(\rho, u) = \int_{\mathbb{R}} g(\xi) G(\rho, \xi - u)\, d\xi = 2^{\frac{2}{\gamma-1}} \rho \int_0^1 \left[\tau(1-\tau)\right]^\lambda g(u + \rho^\theta - 2\rho^\theta \tau)\, d\tau, \tag{4.3.5}$$

where $\theta = \frac{\gamma-1}{2}$.

ii) η is convex in $(\rho, \rho u)$ for all ρ, u, if the function g is convex.

iii) The entropy flux corresponding to (4.3.5) is given by

$$q(\rho, u) = \int_{\mathbb{R}} g(\xi) \left[\theta\xi + (1-\theta)u\right] G(\rho, \xi - u)\, d\xi. \tag{4.3.6}$$

iv) If $g \in C^1(\mathbb{R})$, then

$$|\eta_\rho(\rho, m)| \leq M, \quad |\eta_m(\rho, m)| \leq M,$$

where $M \in \mathbb{R}$ is some constant only depending on the L^∞-bounds of ρ and u.

Proof. We can consider η to be a function of the Riemann invariants w, z of system (4.1.2). Then

$$\nabla q = q_w \nabla w + q_z \nabla z,$$

$$\nabla \eta = \eta_w \nabla w + \eta_z \nabla z.$$

Inserting this into the entropy condition (3.1.8) yields

$$\begin{aligned} q_w \nabla w + q_z \nabla z &= \left(\eta_w \nabla w + \eta_z \nabla z\right) dF \\ &= \eta_w \left(\nabla w\, dF\right) + \eta_z \left(\nabla z\, dF\right) \\ &= \eta_w \left(\lambda_2 \nabla w\right) + \eta_z \left(\lambda_1 \nabla z\right). \end{aligned}$$

Since $\nabla w, \nabla z$ are linearly independent this implies

$$q_w = \lambda_2 \eta_w, \quad q_z = \lambda_1 \eta_z.$$

We now differentiate the first equation with respect to z and the second with respect to w and once again use Schwartz Theorem to obtain

$$\eta_{wz} + \frac{(\lambda_2)_z}{\lambda_2 - \lambda_1} \eta_w - \frac{(\lambda_1)_w}{\lambda_2 - \lambda_1} \eta_z = 0. \tag{4.3.7}$$

Furthermore it can be verified that the following equalities hold

$$\lambda_1 = \frac{3-\gamma}{4} w + \frac{\gamma+1}{4} z, \quad \lambda_2 = \frac{\gamma+1}{4} w + \frac{3-\gamma}{4} z,$$

$$(\lambda_2)_z = (\lambda_1)_w = \frac{3-\gamma}{4}, \quad \lambda_2 - \lambda_1 = \frac{\gamma-1}{2}(w-z).$$

We now make use of the relations $w = u + \rho^\theta$, $z = u - \rho^\theta$ in order to otbtain

$$\begin{aligned}\eta_\rho(\rho,u) &= \eta_w(w,z)w_\eta + \eta_z(w,z)z_\eta \\ &= \eta_w(w,z)\theta\rho^{\theta-1} - \eta_z(w,z)\theta\rho^{\theta-1},\end{aligned}$$

and $\rho = \left(\frac{w-z}{2}\right)^{\frac{1}{\theta}}$. All of the above now show that equation (4.3.7) reduces to the following Euler-Poisson-Darboux equation

$$\eta_{wz} + \frac{\lambda}{w-z}(\eta_w - \eta_z) = 0. \tag{4.3.8}$$

Recall that η is a weak entropy, hence

$$0 = \lim_{\rho\to 0}\eta(\rho,u) = \lim_{(w-z)\to 0}\eta(w,z)$$

and

$$g(w) = \lim_{\eta\to 0}\eta_\rho(\rho,u) = \lim_{(w-z)\to 0}\theta\left(\frac{w-z}{2}\right)^{\frac{\gamma-3}{\gamma-1}}\left(\eta_w(w,z) - \eta_z(w,z)\right),$$

must hold. Thus we have shown (4.3.5) after applying the solution formula to the Euler-Poisson-Darboux equation [29]. Setting $m = \rho u$ allows us to write (4.3.5) equivalently as

$$\eta(\rho,m) = \rho\int_{-1}^{1} g\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz.$$

We are now going to calculate the elements of the Hessian matrix of η, which yields

$$\begin{aligned}\eta_{\rho\rho} &= \int_{-1}^{1}\frac{\gamma^2-1}{4}\rho^{\frac{\gamma-3}{2}}g'\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)z(1-z^2)_+^\lambda\, dz \\ &\quad + \int_{-1}^{1}\left(-\frac{m}{\rho^2} + \frac{\gamma-1}{2}z\rho^{\frac{\gamma-3}{2}}\right)^2 g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz \\ &= \int_{-1}^{1}\frac{\gamma^2-1}{8(\lambda+1)}\rho^{\gamma-2}g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^{\lambda+1}\, dz \\ &\quad + \rho\int_{-1}^{1}\left(-\frac{m}{\rho^2} + \frac{\gamma-1}{2}z\rho^{\frac{\gamma-3}{2}}\right)^2 g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz,\end{aligned}$$

after integrating by parts to make the second derivative of g appear. Analogously we find

$$\eta_{mm} = \frac{1}{\rho}\int_{-1}^{1} g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz,$$

$$\eta_{\rho m} = \eta_{m\rho} = \int_{-1}^{1}\left(-\frac{m}{\rho^2} + \frac{\gamma-1}{2}z\rho^{\frac{\gamma-3}{2}}\right)g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz.$$

Since g is convex if and only if $g'' \geq 0$ we have $\eta_{\rho\rho}, \eta_{mm} \geq 0$ and moreover

$$\begin{aligned}\eta_{\rho\rho}\eta_{mm} - (\eta_{\rho m})^2 &\geq \int_{-1}^{1}\left(-\frac{m}{\rho^2} + \frac{\gamma-1}{2}z\rho^{\frac{\gamma-3}{2}}\right)^2 g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz \\ &\quad \times\int_{-1}^{1} g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz \\ &\quad - \left[\int_{-1}^{1}\left(-\frac{m}{\rho^2} + \frac{\gamma-1}{2}z\rho^{\frac{\gamma-3}{2}}\right)g''\left(\frac{m}{\rho} + z\rho^{\frac{\gamma-1}{2}}\right)(1-z^2)_+^\lambda\, dz\right]^2 \\ &\geq 0,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the last step. Hence η is convex, if g is convex. In order to verify *iii*), a direct calculation shows

$$\begin{aligned}\eta_\rho &= 2^{\frac{2}{\gamma-1}} \int_0^1 [\tau(1-\tau)]^\lambda \, g(u+\rho^\theta - 2\rho^\theta\tau)\, d\tau \\ &\quad + 2^{\frac{2}{\gamma-1}}\rho \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u+\rho^\theta - 2\rho^\theta\tau) \left(\theta\rho^{\theta-1} - 2\theta\rho^{\theta-1}\tau\right) d\tau,\end{aligned}$$

and

$$\eta_u = 2^{\frac{2}{\gamma-1}}\rho \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u+\rho^\theta - 2\rho^\theta\tau)\, d\tau.$$

Using the second equation we obtained after writing system (4.1.2) equivalently as (4.3.1) we get

$$\begin{aligned}q_u = \rho\eta_\rho + u\eta_u = \eta + 2^{\frac{2}{\gamma-1}} & \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u+\rho^\theta - 2\rho^\theta\tau)(1-2\tau)\theta\rho^{\theta+1}\, d\tau \\ + 2^{\frac{2}{\gamma-1}} u\rho & \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u+\rho^\theta - 2\rho^\theta\tau)\, d\tau,\end{aligned}$$

where we also made use of (4.3.5). We now integrate the above with respect to u where we will again use integration by parts and furthermore make use of Fubini's Theorem [20] to obtain

$$\begin{aligned}q &= \int_0^u \eta\, du' + \int_0^u 2^{\frac{2}{\gamma-1}}\theta \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u'+\rho^\theta - 2\rho^\theta\tau)(1-2\tau)\rho^{\theta+1}\, d\tau du' \\ &\quad + 2^{\frac{2}{\gamma-1}} \left[\int_0^1 [\tau(1-\tau)]^\lambda \int_0^u u' g'(u'+\rho^\theta - 2\rho^\theta\tau)\, du' d\tau\right] \\ &= \int_0^u \eta\, du' + \int_0^u 2^{\frac{2}{\gamma-1}}\theta \int_0^1 [\tau(1-\tau)]^\lambda \, g'(u'+\rho^\theta - 2\rho^\theta\tau)(1-2\tau)\rho^{\theta+1}\, d\tau du' \\ &\quad + 2^{\frac{2}{\gamma-1}} \left[\int_0^1 [\tau(1-\tau)]^\lambda u g(u+\rho^\theta - 2\rho^\theta\tau)\, d\tau - \int_0^u \int_0^1 [\tau(1-\tau)]^\lambda g(u'+\rho^\theta - 2\rho^\theta\tau)\, d\tau du'\right] \\ &= \int_0^u \eta\, du' + 2^{\frac{2}{\gamma-1}}\theta \int_0^1 [\tau(1-\tau)]^\lambda g(u+\rho^\theta - 2\rho^\theta\tau)(1-2\tau)\rho^{\theta+1}\, d\tau - \int_0^u \eta\, du' + \eta u.\end{aligned}$$

Plugging in η, substituting and rearranging will then yield (4.3.6). The bounds described in *iv*) are an immediate consequence of the last form of equation (4.3.5) together with the fact that $g \in C^1(\mathbb{R})$. □

The aim is to use these explicit formulas for the entropy pairs to show that they satisfy the necessary compactness relation. The remaining part of this section is based on [18].

Lemma 4.4. *For any weak entropy $\eta(\rho, m)$ of system (4.1.2),*

$$\varepsilon((\rho_\varepsilon)_x, (m_\varepsilon)_x)\nabla^2\eta(\rho_\varepsilon, m_\varepsilon)((\rho_\varepsilon)_x, (m_\varepsilon)_x)^T \tag{4.3.9}$$

is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, where $\nabla\eta$ denotes the Hessian matrix of η and $(\rho_\varepsilon, m_\varepsilon)$ denote the viscosity solutions.

Proof. In the following we will omit the subscript ε from the viscosity solutions. We first show that

$$\eta^\star = \frac{m^2}{2\rho} + \frac{k^2}{\gamma-1}\rho^\gamma$$

is a convex entropy of system (4.1.2). For this we calculate

$$\nabla\eta^\star = \left(-\frac{m^2}{2\rho^2} + \frac{k^2\gamma}{\gamma-1}\rho^{\gamma-1}, \frac{m}{\rho}\right), \quad dF = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + \gamma c\rho^{\gamma-1} & \frac{2m}{\rho}\end{pmatrix},$$

and

$$\nabla\eta^\star\, dF = \left(-\frac{m^3}{\rho^3} + m\gamma c\rho^{\gamma-2}, \frac{3m^2}{2\rho^2} + \frac{k^2\gamma}{\gamma-1}\rho^{\gamma-1}\right)$$
$$= \left(-\frac{m^3}{\rho^3} + m\gamma c\rho^{\gamma-2}, \frac{3m^2}{2\rho^2} + \frac{c\gamma}{\gamma-1}\rho^{\gamma-1}\right).$$

One can now easily verify that $q(\rho,m) = \frac{m^3}{2\rho^2} + \frac{c\gamma m}{\gamma-1}\rho^{\gamma-1}$ is companion to η. Another simple calculation shows

$$\nabla^2\eta^\star(\rho,m) = \begin{pmatrix} \frac{3m^2}{2\rho^3} + k^2\gamma\rho^{\gamma-2} & -\frac{m}{\rho^2} \\ -\frac{m}{\rho^2} & \frac{1}{\rho} \end{pmatrix}.$$

With this we can now conclude

$$\det\left(\nabla^2\eta^\star(\rho,m)\right) = \frac{3m^2}{4\rho^4} + k^2\gamma\rho^{\gamma-3} + \frac{m^2}{\rho^4} \geq 0,$$

since $\rho \geq 0$. Hence $\eta^\star$ is a convex entropy. Another brief calculation shows

$$\varepsilon(\rho_x, m_x)\nabla^2\eta^\star(\rho,m)(\rho_x,m_x)^T = \varepsilon\left[\left(-\frac{3m^2}{2\rho^3} + k^2\gamma\rho^{\gamma-2}\right)\rho_x^2 - \frac{2m}{\rho^2}m_x\rho_x + \frac{1}{\rho}m_x^2\right]$$
$$= \varepsilon k^2\gamma\rho^{\gamma-2}\rho_x^2 + \varepsilon\frac{1}{\rho}\left(\frac{m}{\rho}\rho_x - m_x\right)^2.$$

We wish to verify that for now at least $\varepsilon(\rho_x,m_x)\nabla^2\eta^\star(\rho,m)(\rho_x,m_x)^T$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. For this we multiply $\nabla\eta^\star$ to the left of the viscous system $U_t + F(U)_x = \varepsilon U_{xx}$:

$$\varepsilon\nabla\eta^\star\cdot U_{xx} = \nabla\eta^\star\cdot(U_t + F(U)_x) = \nabla\eta^\star\cdot U_t + \nabla\eta^\star\cdot dF(U)U_x = \eta^\star_t + q^\star_x.$$

If we now write $\nabla\eta^\star\cdot U_{xx} = (\nabla\eta^\star\cdot U_x)_x - U_x\nabla^2\eta^\star U_x^T$, the above becomes

$$\eta^\star_t + q^\star_x - \varepsilon(\nabla\eta^\star\cdot U_x)_x = -\varepsilon(U_x)\nabla^2\eta^\star(U_x)^T.$$

Let $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ be arbitrary with $\phi \geq 0$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}\varepsilon(\rho_x,m_x)\nabla^2\eta^\star(\rho_x,m_x)^T\phi\,dtdx = -\int_{\mathbb{R}}\int_{\mathbb{R}^+}(\eta^\star_t\phi + q^\star_x\phi) - \varepsilon(\nabla\eta^\star\cdot(\rho_x,m_x)^T)_x\phi\,dtdx$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}^+}(\eta^\star\phi_t + q^\star\phi_x) - \varepsilon(\nabla\eta^\star\cdot(\rho_x,m_x)^T)\phi_x\,dtdx$$
$$\leq M(\phi),$$

where we have again used integration by parts, as well as the fact that the viscosity solutions are uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}^+)$, which implies that $\eta^\star$ and hence also $\nabla\eta^\star$ are compactly supported. The above now shows that $\varepsilon(\rho_x,m_x)\nabla^2\eta^\star(\rho,m)(\rho_x,m_x)^T$ is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and hence since

$$k^2\gamma\rho^{\gamma-2}\rho_x^2 + \varepsilon\frac{1}{\rho}\left(\frac{m}{\rho}\rho_x - m_x\right)^2 = k^2\gamma\rho^{\gamma-2}\rho_x^2 + \rho u_x^2,$$

we know that

$$\varepsilon\rho^{\gamma-2}\rho_x^2, \quad \varepsilon\frac{1}{\rho}\left[\frac{m}{\rho}\rho_x - m_x\right]^2, \quad \varepsilon\rho u_x^2, \tag{4.3.10}$$

are locally bounded in $L^1(\mathbb{R}\times\mathbb{R}^+)$. We now calculate all second derivatives of η in order to show that the Hessian matrix of η is locally bounded in the integrable functions. The first derivatives are given by

$$\eta_\rho(\rho,m) = \int_0^1[\tau(1-\tau)]^\lambda g\left(\frac{m}{\rho} + \rho^\theta - 2\rho^\theta\tau\right)d\tau$$
$$+ \int_0^1[\tau(1-\tau)]^\lambda g'\left(\frac{m}{\rho} + \rho^\theta - 2\rho^\theta\tau\right)\left(-\frac{m}{\rho} + (1-2\tau)\theta\rho^\theta\right)d\tau$$

and

$$\eta_m(\rho,m) = \int_0^1 [\tau(1-\tau)]^\lambda g'\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau,$$

in view of (4.3.5). This now yields

$$\begin{aligned}
\eta_{\rho\rho}(\rho,m) &= \int_0^1 [\tau(1-\tau)]^\lambda g'\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\left(-\frac{m}{\rho^2}+(1-2\tau)\theta\rho^{\theta-1}\right)\,d\tau \\
&\quad + \int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\left(-\frac{m}{\rho^2}+(1-2\tau)\theta\rho^{\theta-1}\right)\left(-\frac{m}{\rho}+(1-2\tau)\theta\rho^{\theta}\right)\,d\tau \\
&\quad + \int_0^1 [\tau(1-\tau)]^\lambda g'\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\left(\frac{m}{\rho^2}+(1-2\tau)\theta^2\rho^{\theta-1}\right)\,d\tau \\
&= \frac{m^2}{\rho^3}\int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau \\
&\quad + \theta^2\rho^{2\theta-1}\int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)(1-2\tau)^2\,d\tau \\
&\quad + (\theta+\theta^2)\rho^{\theta-1}\int_0^1 [\tau(1-\tau)]^\lambda g'\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)(1-2\tau)\,d\tau \\
&\quad + 2\theta\frac{m}{\rho}\rho^{\theta-1}\int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)(1-2\tau)\,d\tau \\
&=: I_1+I_2+I_3+I_4
\end{aligned}$$

and analogously

$$\begin{aligned}
\eta_{\rho m}(\rho,m) &= \int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\left(-\frac{m}{\rho^2}+(1-2\tau)\theta\rho^{\theta-1}\right)\,d\tau \\
&=: I_5+I_6
\end{aligned}$$

$$\eta_{mm}(\rho,m) = \int_0^1 [\tau(1-\tau)]^\lambda g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\frac{1}{\rho}\,d\tau =: I_7.$$

In the following we will keep using the uniform boundedness of the viscosity solutions in $L^\infty(\mathbb{R}\times\mathbb{R}^+)$. Since $2\theta-1=\gamma-2$ we obtain the boundedness of $\varepsilon(\rho_x)^2 I_2$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ from (4.3.10). Define

$$\ell(\tau) := \int_0^\tau [s(1-s)]^\lambda(1-2s)\,ds.$$

Since $\lambda>-1$, ℓ is continuous and satisfies $\ell(0)=\ell(1)=0$. Integrating I_3 by parts now yields

$$\begin{aligned}
\frac{1}{(\theta+\theta^2)}I_3 &= \rho^{\theta-1}\int_0^1 [\tau(1-\tau)]^\lambda(1-2\tau)g'\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau \\
&= 2\rho^{2\theta-1}\int_0^1 \ell(\tau)g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau,
\end{aligned}$$

which by the same reasons as above implies the boundedness of $\varepsilon(\rho_x)^2 I_3$. Similarly we obtain the boundedness of $\varepsilon(\rho_x)^2 I_4$ from

$$\begin{aligned}
\frac{1}{2\theta u}I_4 &= \rho^{\theta-1}\int_0^1 [\tau(1-\tau)]\lambda(1-2\tau)g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau \\
&= 2\rho^{2\theta-1}\int_0^1 \ell(\tau)g^{(3)}\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau.
\end{aligned}$$

Using $m_x=\rho u_x+u\rho_x$ we can write

$$\varepsilon\rho_x m_x I_6 = \varepsilon u(\rho_x)^2 I_6 + \varepsilon\rho\rho_x u_x I_6,$$

where the first term is again bounded because of (4.3.10) and we can estimate the second term as

$$\begin{aligned}|\varepsilon\rho\rho_x u_x I_6| &= \left|\varepsilon\theta\rho^\theta\int_0^1[\tau(1-\tau)]^\lambda(1-2\tau)g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau\rho_x u_x\right|\\ &\leq \varepsilon\theta\left(\rho^{2\theta-1}(\rho_x)^2+\rho(u_x)^2\right)\left|\int_0^1[\tau(1-\tau)]^\lambda(1-2\tau)g''\left(\frac{m}{\rho}+\rho^\theta-2\rho^\theta\tau\right)\,d\tau\right|,\end{aligned}$$

which is again finite, as a consequence of (4.3.10). And lastly $\varepsilon\left((\rho_x)^2I_1+2\rho_x m_x I_5+(m_x)^2I_7\right)$ is locally bounded in $L^1(\mathbb{R}\times\mathbb{R}^+)$ because of (4.3.10) and the uniform bounds on the viscosity solutions. □

Lemma 4.5. *Let $K\subset\mathbb{R}\times\mathbb{R}^+$ be bounded. Then*

$$\iint_K\left(\varepsilon(\rho_\varepsilon)_x\right)^2\,dxdt\to 0, \tag{4.3.11}$$

as $\varepsilon\to 0$.

Proof. We again omit the subscript ε and for $\delta\in(0,1)$ we define

$$\Omega_1=\{(x,t)\in\mathbb{R}\times\mathbb{R}^+:\ \rho<\delta\},\quad \Omega_2=\{(x,t)\in\mathbb{R}\times\mathbb{R}^+:\ \rho\geq\delta\}.$$

Then Ω_1 is open. Let $K_1\subset\Omega_1$ be compact and choose $\phi\in C_c^\infty(\Omega_1)$ with $\operatorname{supp}\phi\subset\Omega_1$ compact such that $\phi\equiv 1$ on K_1 and $0\leq\phi\leq 1$. Multiplying (4.2.1) by 2ρ yields

$$(\rho^2)_t+2(\rho^2u)_x-2\rho u\rho_x=\varepsilon(\rho^2)_{xx}-2\varepsilon(\rho_x)^2.$$

Consequently we obtain

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty 2\varepsilon(\rho_x)^2\phi\,dxdt &= \int_0^\infty\int_{-\infty}^\infty(\rho^2\phi_t+2u\rho^2\phi_x+\varepsilon\rho^2\phi_{xx}+2\rho u\rho_x\phi)\,dxdt\\ &= \int_0^\infty\int_{-\infty}^\infty\rho^2(\phi_t+2u\phi_x+\varepsilon\phi_{xx})\,dxdt+2\int_0^\infty\int_{-\infty}^\infty\rho u\rho_x\phi\,dxdt\\ &\leq C\delta^2+2\delta\int_0^\infty\int_{-\infty}^\infty u\rho_x\phi\,dxdt\\ &\leq C\delta^2+2\delta\|u\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}\left(\int_0^\infty\int_{-\infty}^\infty(\rho_x)^2\phi^2\,dxdt\right)^{\frac12}\\ &\leq C\delta^2+C\delta\left(\int_0^\infty\int_{-\infty}^\infty(\rho_x)^2\phi\,dxdt\right)^{\frac12},\end{aligned}$$

for some $C\in\mathbb{R}$, where we have used the Cauchy-Schwarz inequality and $0\leq\phi\leq 1$. This now implies

$$\int_0^\infty\int_{-\infty}^\infty\varepsilon^2(\rho_x)^2\phi\,dxdt=\iint_{\operatorname{supp}\phi}\varepsilon^2(\rho_x)^2\phi\,dxdt\leq C\delta^2,$$

which means

$$\iint_{K_1}\varepsilon^2(\rho_x)^2\,dxdt\leq C\delta^2.$$

We can now use the boundedness of $\varepsilon\rho^{\gamma-2}(\rho_x)^2$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ from before to obtain

$$\iint_{K_2}\varepsilon^2(\rho_x)^2\,dxdt\to 0,\quad \text{as }\varepsilon\to 0,$$

for any bounded set $K_2\subset\Omega_2$ and every fixed $\delta\in(0,1)$ using the boundedness away from zero of ρ. Letting $\delta\to 0$ completes the proof. □

We can now state the main theorem of this section concerning the compactness properties of the weak entropy pairs of system (4.1.2) with respect to the viscosity solutions.

Theorem 4.6. *For any weak entropy pair (η, q) of system (4.1.2), the sequence*

$$\eta_t(\rho_\varepsilon, m_\varepsilon) + q_x(\rho_\varepsilon, m_\varepsilon)$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ with respect to the viscosity solutions $(\rho_\varepsilon, m_\varepsilon)$ of (4.1.4).

Proof. We multiply the viscous system (4.1.4) by $\nabla\eta$ to obtain

$$\eta_t(\rho_\varepsilon, m_\varepsilon) + q_x(\rho_\varepsilon, m_\varepsilon) = \varepsilon\eta_{xx}(\rho_\varepsilon, m_\varepsilon) - \varepsilon((\rho_\varepsilon)_x, (m_\varepsilon)_x)\nabla^2\eta(\rho_\varepsilon, m_\varepsilon)((\rho_\varepsilon)_x, (m_\varepsilon)_x)^T. \tag{4.3.12}$$

Using that $W^{-1,\infty}$ is isometrically isomorphic to the dual space of $W^{1,1}_0$ we obtain

$$\begin{aligned}\left|\int_0^\infty\int_{-\infty}^\infty (\eta_t(\rho_\varepsilon, m_\varepsilon)\phi + q_x(\rho_\varepsilon, m_\varepsilon)\phi)\,dxdt\right| &= \left|\int_0^\infty\int_{-\infty}^\infty (\eta(\rho_\varepsilon, m_\varepsilon)\phi_t + q(\rho_\varepsilon, m_\varepsilon)\phi_x)\,dxdt\right| \\ \leq \|\eta\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\phi_t\|_{L^1(\mathbb{R}\times\mathbb{R}^+)} + \|q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\phi_x\|_{L^1(\mathbb{R}\times\mathbb{R}^+)} &\leq C\|\phi\|_{W^{1,1}_0(\mathbb{R}\times\mathbb{R}^+)}\end{aligned}$$

for any $\phi \in W^{1,1}_0(\mathbb{R}\times\mathbb{R}^+)$, hence the left hand side of (4.3.12) is bounded in $W^{-1,\infty}(\mathbb{R}\times\mathbb{R}^+)$. The second term on the right hand side of (4.3.12) was proven to be bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ in Lemma 4.4. Lastly, we can bound $\varepsilon\eta_{xx}(\rho_\varepsilon, m_\varepsilon) \in H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ by

$$\begin{aligned}\|\varepsilon\eta_{xx}(\rho_\varepsilon, m_\varepsilon)\|_{H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)} &= \sup_{\|\phi\|_{H^1_0(\mathbb{R}\times\mathbb{R}^+)}\leq 1}\left|\iint_K \varepsilon\eta_{xx}(\rho_\varepsilon, m_\varepsilon)\phi\,dxdt\right| \\ &\leq \sup_{\|\phi\|_{H^1_0(\mathbb{R}\times\mathbb{R}^+)}\leq 1}\iint_K C\varepsilon(|(\rho_\varepsilon)_x||\phi_x| + (\rho_\varepsilon)^{\frac{1}{2}}|(u_\varepsilon)_x||\phi_x|)\,dxdt \\ &\leq \sup_{\|\phi\|_{H^1_0(\mathbb{R}\times\mathbb{R}^+)}\leq 1} C\left[\|\varepsilon(\rho_\varepsilon)_x\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}\|\phi\|_{H^1_0(\mathbb{R}\times\mathbb{R}^+)} + \|\varepsilon(\rho_\varepsilon)^{\frac{1}{2}}(u_\varepsilon)_x\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}\|\phi\|_{H^1_0(\mathbb{R}\times\mathbb{R}^+)}\right] \\ &= C\left[\left(\iint_K \varepsilon^2((\rho_\varepsilon)_x)^2\,dxdt\right)^{\frac{1}{2}} + \left(\iint_K \varepsilon^2((u_\varepsilon)_x)^2\rho_\varepsilon\,dxdt\right)^{\frac{1}{2}}\right],\end{aligned}$$

for every compact $K \subset \mathbb{R}\times\mathbb{R}^+$. We cannot directly conclude by integration by parts twice, since the test functions are only once weakly differentiable. By Lemma 4.5 we know that the first term tends to zero for $\varepsilon \to 0$ and by (4.3.10) $\varepsilon\rho(u_x)^2$ is locally bounded in L^1, hence the second term tends to zero as well, for $\varepsilon \to 0$. This in particular implies that $\varepsilon\eta_{xx}(\rho_\varepsilon, m_\varepsilon)$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The proof is now completed by applying Murat's Lemma 3.18. ☐

4.4 Existence of Weak Solutions to the Isentropic Euler Equations

In this section we want to prove the existence of a subsequence of viscosity solutions to (4.1.2) converging pointwise, which in return implies the existence of a weak solution to the initial value problem (4.1.2), (4.1.3). Regarding the nature of system (4.1.2) it is eminent that different arguments have to be used for $\gamma > 3, \gamma = 3$ and $\gamma < 3$, since the behavior at the vacuum line changes. Ultimately we will also see that the solution we obtain as a limit of viscosity solutions fulfill the admissibility criterion (3.1.10) and will therefore even be an entropy solution. The main theorem we want to prove is the following:

Theorem 4.7. *Let $\gamma > 1$ and $(\rho_\varepsilon, u_\varepsilon)$ be viscosity solutions given by (4.2.1), (4.2.2). Then there exist subsequences of ρ_ε and u_ε respectively which converge pointwise to ρ and u. In particular there exists a subsequence of $\rho_\varepsilon u_\varepsilon$ converging to ρu.*

In virtue of Section 3.2 this is done by showing the reduction of the Young measures, associated to the sequence of viscosity solutions, to a Dirac measures.

4.4.1 General Results

We will first make us of the results of Sections 3.2, 3.3 and 4.2 to introduce the general starting point for the proofs of Theorem 4.7 similar to our description in Section 3.5. For this let h, g be arbitrary functions. Then, following [17], according to Theorem 4.3

$$\eta_1(\rho,u) = \int_{\mathbb{R}} g(\xi_1)G(\rho,\xi_1-u)\,d\xi_1, \quad q_1(\rho,u) = \int_{\mathbb{R}} g(\xi_1)[\theta\xi_1+(1-\theta)u]G(\rho,\xi_1-u)\,d\xi_1$$

$$\eta_2(\rho,u) = \int_{\mathbb{R}} h(\xi_2)G(\rho,\xi_2-u)\,d\xi_2, \quad q_2(\rho,u) = \int_{\mathbb{R}} h(\xi_2)[\theta\xi_2+(1-\theta)u]G(\rho,\xi_2-u)\,d\xi_2,$$

are weak entropies of system (4.1.2). Theorem 4.6 now implies that $(\eta_i(\rho,u))_t + (q_i(\rho,u))_x$ are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ for all $i=1,2$, where we chose to omit the index ε for the viscosity solutions. Using equation (3.5.1) or respectively Corollary 3.16 we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} g(\xi_1)\langle G(\xi_1)\rangle\,d\xi_1 \int_{\mathbb{R}} h(\xi_2)\langle[\theta\xi_2+(1-\theta)u]G(\xi_2)\rangle\,d\xi_2\\
&\quad-\int_{\mathbb{R}} h(\xi_2)\langle G(\xi_2)\rangle\,d\xi_2 \int_{\mathbb{R}} g(\xi_1)\langle[\theta\xi_1+(1-\theta)u]G(\xi_1)\rangle\,d\xi_1\\
&=\int_{\mathbb{R}} g(\xi_1)h(\xi_2)\langle G(\xi_1)[\theta\xi_2+(1-\theta)u]G(\xi_2)\rangle\,d\xi_1 d\xi_2\\
&\quad-\int_{\mathbb{R}} g(\xi_1)h(\xi_2)\langle G(\xi_2)[\theta\xi_1+(1-\theta)u]G(\xi_1)\rangle\,d\xi_1 d\xi_2.
\end{aligned}$$

Above and in the following we introduce the following notation for the pairing with a Young measure:

$$\langle G(\xi)\rangle = \int G(\rho,\xi-u)\,d\nu_{(x,t)}(\rho,u).$$

Since this holds for arbitrary g, h we can conclude, after simplifying,

$$\frac{\theta}{1-\theta}\left[\frac{\langle G(\xi_1)G(\xi_2)\rangle}{\langle G(\xi_1)\rangle\langle G(\xi_2)\rangle}-1\right] = \frac{1}{\xi_2-\xi_1}\left[\frac{\langle G(\xi_2)u\rangle}{\langle G(\xi_2)\rangle}-\frac{\langle G(\xi_1)u\rangle}{\langle G(\xi_1)\rangle}\right]. \tag{4.4.1}$$

The above equation is only meaningful for all ξ where $G>0$ and $\gamma\neq 3$, the case for $\gamma=3$ will be considered separately. Since $G(\rho,\xi-u)>0$ if and only if $u-\rho^\theta\le\xi\le u+\rho^\theta$ the above is only well defined for $\xi_1,\xi_2\in\mathcal{C}$, where

$$\mathcal{C} = \bigcup_{(\rho,u)\in\operatorname{supp}\nu}\{\xi : z(\rho,u)\le v\le w(\rho,u)\}. \tag{4.4.2}$$

We will now prove two useful Lemmas concerning equation (4.4.1) and the set $\mathcal{C}$. The first Lemma is based on [17] and the second one is taken from [16].

Lemma 4.8. *Suppose equation* (4.4.1) *holds for all* $\xi_1,\xi_2\in\mathcal{C}$. *Then the function*

$$\Pi(\xi) := \frac{\langle uG(\xi)\rangle}{\langle G(\xi)\rangle}$$

is non-increasing for $\gamma>3$ *and non-decreasing for* $\gamma<3$ *on* $\mathcal{C}$.

Proof. For simplicity we set

$$f_0(\xi) = \frac{G(\xi)-\langle G(\xi)\rangle}{\langle G(\xi)\rangle},$$

in order to write (4.4.1) as

$$\frac{\theta}{1-\theta}\langle f_0(\xi_1)f_0(\xi_2)\rangle = \frac{1}{\xi_2-\xi_1}\left[\frac{\langle G(\xi_2)u\rangle}{\langle G(\xi_2)\rangle}-\frac{\langle G(\xi_1)u\rangle}{\langle G(\xi_1)\rangle}\right],$$

for $\xi_1, \xi_2 \in \mathcal{C}$. Since by construction $\langle G(\xi)\rangle$ does not vanish on $\mathcal{C}$ we are allowed to let $\xi_2 \to \xi_1 =: \xi$ on the right hand side of the above equality. This yields the following limit in the sense of a distribution

$$\lim_{\xi_2 \to \xi} \frac{1}{\xi_2 - \xi}\left[\frac{\langle G(\xi_2)u\rangle}{\langle G(\xi_2)\rangle} - \frac{\langle G(\xi)u\rangle}{\langle G(\xi)\rangle}\right] = \frac{\partial}{\partial \xi}\left(\frac{\langle uG(\xi)\rangle}{\langle G(\xi)\rangle}\right).$$

We wish to also send ξ_2 to ξ on the left hand side of (4.4.1) and to do this in the $L^2_{loc}(\mathcal{C})$ sense. For this we need to verify that f_0 and hence G is in $L^2(\mathcal{C})$. A brief calculation using the definition of G shows

$$\begin{aligned}
\|G\|^2_{L^2(\mathcal{C})} &= \int_{\mathbb{R}} \rho^{\frac{3-\gamma}{2}} \left(1 - \frac{u-\xi}{\rho^{2\theta}}\right)_+^{2\lambda} d\xi \\
&= \int_{-1}^{1} \rho^{\frac{5-\gamma}{2}} (1-\tau^2)^{2\lambda}\, d\tau,
\end{aligned}$$

which could diverge if $\gamma > 5$, since the viscosity solutions might not be bounded away from zero for all $\varepsilon > 0$. We therefore now let $\varphi_\alpha(\xi) \geq 0$ be a mollifier and set $f_\alpha := f_0 * \varphi_\alpha$. Then

$$\frac{\theta}{1-\theta}\langle f_\alpha(\xi) f_\alpha(\xi_2)\rangle = \frac{1}{\xi_2 - \xi}\left[\frac{\langle G(\xi_2)u\rangle}{\langle G(\xi_2)\rangle} - \frac{\langle G(\xi)u\rangle}{\langle G(\xi)\rangle}\right] * \varphi_\alpha(\xi) * \varphi_\alpha(\xi_2). \tag{4.4.3}$$

Observe that by definition of $\mathcal{C}$ we have f_0 bounded and consequently $\|f_\alpha\|^2_{L^2(\mathcal{C})} = \|f_0\|^2_{L^2(\mathcal{C})} < \infty$. Now, using the fact that the Young measures are probability measures, since the viscosity solutions are uniformly bounded, we obtain $\|\langle f_\alpha\rangle\|^2_{L^2(\mathcal{C})} < \infty$. Hence the left hand side of Equation (4.4.3) is bounded in the L^2-norm and the right hand side is well defined on $\mathcal{C}$ and therefore smooth after mollifying. We may thus set $\xi_2 = \xi$ and let $\alpha \to 0$ which yields a distribution on the left hand side of (4.4.3). Lastly observe that $\frac{\theta}{1-\theta} = 2\lambda$ and $\lambda > 0$ for $\gamma < 3$ and $\lambda < 0$ for $\gamma > 3$. Hence Π is non-increasing for $\gamma > 3$ and non-decreasing for $\gamma < 3$. $\square$

Recall that the Riemann invariants of system (4.1.2) satisfy parabolic equations and thus obey a maximum principle. We can therefore set $w(\rho, u) \leq w_0$ and $z(\rho, u) \geq z_0$, where $w_0 = \sup \mathcal{C}$ and $z_0 = \inf \mathcal{C}$.

Lemma 4.9. *Assume that $\mathcal{C} \neq \emptyset$. Then*

i) The point $(w_0, z_0) \in \mathbb{R} \times \mathbb{R}$ belongs to the support of the Young measures ν.

ii) $\mathcal{C} = (z_0, w_0)$ and is hence open and connected.

Proof. In view of the definition of $\mathcal{C}$ each state (ρ, u) corresponds to one interval (z, w). Hence, because of the above mentioned maximum principle the interval (z_0, w_0) is the largest in the union over all possible states. Hence *ii*) is a direct consequence of *i*) as soon as we know that $\mathcal{C}$ is connected. Assume that $\mathcal{C}$ is not connected. This implies the existence of $a, b \in \mathbb{R}$ and $\delta > 0$ such that $a \leq b$ and

$$[a, b] \cap \mathcal{C} = \emptyset, \quad \text{but } (a-\delta, a) \cup (b, b+\delta) \subset \mathcal{C}.$$

If such a, b exist then

$$\operatorname{supp} \nu \cap \mathbb{R}^2 \cap (\mathbb{R}^2 - \bar{V}) \cap \{(w, z) : a \leq w, z \leq b\} = \emptyset,$$

where V denotes the vacuum line $V = \{(\rho, u) : \rho = 0\}$. This holds, since the intersection of $[a, b]$ and $\mathcal{C}$ is empty, which implies that for all $\xi \in [a, b]$ there are no $(w, z) \in \operatorname{supp} \nu$ with $z \leq \xi \leq w$. This is in particular true for $\xi = a$ and $\xi = b$. Now choose $\xi_1, \xi_2 \in (a-\delta, a) \subset \mathcal{C}$ and $\xi_3 \in (b, b+\delta) \subset \mathcal{C}$. Then

$$\operatorname{supp}(G(\xi_1)G(\xi_3)) = \operatorname{supp} G(\xi_1) \cap \operatorname{supp} G(\xi_3) = \{(w, z) : z \leq \xi_1 \quad \text{and } \xi_3 \leq w\},$$

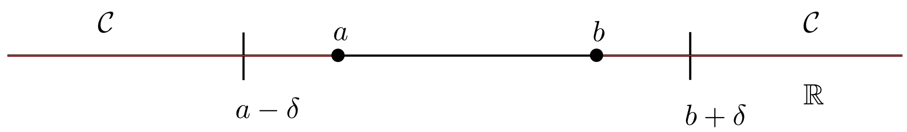

Figure 4: The choice of a and b

since $\xi_1 \leq \xi_3$. Similarly we obtain

$$\operatorname{supp}(G(\xi_2)G(\xi_3)) = \{(w,z) : z \leq b \quad \text{and } \xi_3 \leq w\}.$$

But these are clearly both subsets of $\{(w,z); a \leq w, z \leq b\}$ and therefore cannot contain any points of $\operatorname{supp}\nu$ away from the vacuum. Hence

$$\frac{\langle G(\xi_1)G(\xi_3)\rangle}{\langle G(\xi_1)\rangle\langle G(\xi_3)\rangle} = \frac{\langle G(\xi_3)G(\xi_2)\rangle}{\langle G(\xi_3)\rangle\langle G(\xi_2)\rangle} = 0,$$

which can be shown to imply

$$2\lambda(\Pi(\xi_1) - \Pi(\xi_3)) = -(\xi_1 - \xi_3), \quad 2\lambda(\Pi(\xi_2) - \Pi(\xi_3)) = -(\xi_2 - \xi_3),$$

or equivalently after subtracting the two equations

$$2\lambda(\Pi(\xi_1) - \Pi(\xi_2)) = -(\xi_1 - \xi_2).$$

If $\lambda > 0$ then this implies $\Pi(\xi_1) - \Pi(\xi_2) > 0$ a contradiction to Π being non-decreasing for $\lambda > 0$, if we choose $\xi_2 > \xi_1$. If however $\lambda < 0$ then $\frac{\xi_2 - \xi_1}{2\lambda} < 0$ and hence $\Pi(\xi_1) - \Pi(\xi_2) < 0$ a contradiction to Π being non-increasing for $\lambda < 0$. Suppose now that $(w_0, z_0) \notin \operatorname{supp}\nu$, then $(w_0, z_0) \in \mathbb{R} \setminus \operatorname{supp}\nu$ which is open. We may hence choose ξ_1, ξ_2 close to w_0 and z_0 respectively such that $\langle G(\xi_1)G(\xi_2)\rangle = 0$. Then we can conclude with exactly the same contradiction as before. □

4.4.2 Proof for the Case of $\gamma < 3$

In this section we will prove the reduction of the Young measures for the case of $\gamma < 3$ which corresponds to $\lambda > 0$. The proof is again based on [16]. From equation (4.4.1) we can obtain the following expression

$$\frac{\langle G(\xi_1)G(\xi_2)\rangle}{\langle G(\xi_1)\rangle\langle G(\xi_2)\rangle} = \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \frac{\langle G(\tau)^2\rangle}{\langle G(\tau)\rangle^2}\, d\tau.$$

Using the additivity of the integral this yields the following lemma:

Lemma 4.10. *Let $\mathcal{C} \neq \emptyset$, then for any $\xi_1, \xi_2, \xi_3 \in \mathcal{C}$ we have*

$$(\xi_2 - \xi_3)\langle G(\xi_3)G(\xi_2)\rangle\langle G(\xi_1)\rangle = (\xi_1 - \xi_3)\langle G(\xi_3)G(\xi_1)\rangle\langle G(\xi_2)\rangle + (\xi_2 - \xi_1)\langle G(\xi_1)G(\xi_2)\rangle\langle G(\xi_3)\rangle. \quad (4.4.4)$$

The idea is to differentiate (4.4.4) a fractional amount of times, more precisely $(\lambda + 1)$-times, since the entropy kernel will then be of the form x^{-1} which, interpreted as a distribution, can be written as a linear combination of principle values and Dirac distributions. This implies that after differentiating $(\lambda + 1)$-times, the singularities of G will appear and we will make use of these singularities to show that the support of the Young measures is either contained in the vacuum line or a single point. Since G is smooth on $(u - \rho^\theta, u + \rho^\theta)$ for all (ρ, u) the singularities will appear at $u \pm \rho^\theta$. In the following we will show that the derivatives of G satisfy some regularity, in order to justify our process. Write

$$G(\rho, u, \xi) := \rho^{2\theta\lambda} f\left(\frac{\xi - u}{\rho^\theta}\right),$$

with

$$f(x) = (1 - x^2)_+^\lambda, \quad x \in (-1, 1).$$

Then for any $\mu \in \mathbb{R}$, we obtain

$$D_\xi^\mu G(\rho, u, \xi) = \rho^{(2\lambda - \mu)\theta} D_x^\mu f\left(\frac{\xi - u}{\rho^\theta}\right),$$

where we define the fractional derivative D^μ as $D^\mu g = g * \Phi_{-\mu}$ and the convolution has to be understood in the sense of distributions. Here $\Phi_{-\mu} = \Gamma(-\mu)x_+^{-\mu-1}$, where Γ is the Gamma-function. For more information we refer to [14].

Lemma 4.11. *Let $\lambda > 0$, then:*

i) $D_\xi G \in L^p_{\xi,\rho,u}$, *for all* $p \in \left[1, \frac{1}{1-\lambda}\right)$ *if* $\lambda \le 1$, *and for all* $p \in [1,\infty)$ *if* $\lambda > 1$.

ii) $D^\lambda_\xi G \in L^q_{\xi,\rho,u}$, *for all* $q \in [1,\infty)$.

Proof. We first observe that

$$D_\xi G(\xi,\rho,u) = -2\lambda(\xi-u)(\rho^{2\theta}-(\xi-u)^2)^{\lambda-1}\chi_{(u-\rho^\theta,u+\rho^\theta)}(\xi).$$

For $\lambda \ge 1$ this is simply a rational function on a bounded domain hence $i)$ is surely true, furthermore notice that for $\lambda \in (0,1)$ the prefactor $(\xi-u)$ will only allow integrability up until L^p where $p \in \left[1, \frac{1}{1-\lambda}\right)$, as x^{-s} is only integrable around zero for $s<1$. Similarly we can calculate

$$D^\lambda_\xi G(\xi,\rho,u) = \rho^{\lambda\theta} D^\lambda_x f\left(\frac{\xi-u}{\rho^\theta}\right).$$

Integrating by parts $([\lambda]+1)$-times, where $[\lambda]$ is the greatest integer of λ we obtain, after omitting some constants

$$D^\lambda_x f = f * x_+^{-\lambda-1} = (-1)^{[\lambda]+1} D^{[\lambda]+1} f * x_+^{[\lambda]-\lambda},$$

where we may now notice that $[\lambda]-\lambda > -1$, hence using Young's inequality for convolutions [2], we can conclude that the right hand side is in L^q for all $q \in [1,\infty)$, since $x_+^{[\lambda]-\lambda}$ is in L^p for $p \in \left[1, \frac{1}{\lambda-[\lambda]}\right)$ and $D^{[\lambda]+1} f \in L^q$ for $q \in \left[1, \frac{1}{1+[\lambda]-\lambda}\right)$, where we again used the local integrability condition for x^s. Lastly observe that this implies

$$\begin{aligned}\|D^\lambda_\xi G\|^q_{L^q_{\xi,\rho,u}} &= \iint \rho^{\lambda q\theta}\left|D^\lambda_x f\left(\frac{\xi-u}{\rho^\theta}\right)\right|^q d\xi d\nu(\rho,u)\\ &= \iint \rho^{(\lambda q+1)\theta}\left|D^\lambda_x f(x)\right|^q dx d\nu(\rho,u) < \infty,\end{aligned}$$

for all $q \in [1,\infty)$. □

We now want to analyze the singularities of $D^{\lambda+1}_\xi G(\xi)$ and for this we make use of the fact that the Fourier transform of f is given by

$$\hat f(\xi) = |\xi|^{-\lambda-\frac12} J_{\lambda+\frac12}(|\xi|),$$

where

$$J_{\lambda+\frac12}(s) = \begin{cases} s^{\lambda+\frac12}, & \text{as } s\to 0\\ s^{-\frac12}\left[\cos\left(s-(\lambda+1)\frac{\pi}{2}\right) - \frac{b}{s}\sin\left(s-(\lambda+1)\frac{\pi}{2}\right) + O\left(\frac{1}{s^2}\right)\right], & \text{as } s\to\infty,\end{cases} \tag{4.4.5}$$

is the Bessel function with some normalized constants. We may now apply the Fourier transform to $D^{\lambda+1}_x f(x)$, which yields

$$\begin{aligned} D^{\lambda+1}_x f(x) &= \int_{\mathbb{R}} e^{-i\xi x}(-i\xi)^{\lambda+1}\hat f(\xi)\,d\xi\\ &= \int_{|\xi|\le 1} e^{-i\xi x}(-i\xi)^{\lambda+1}\hat f(\xi)\,d\xi + \int_{|\xi|\ge 1} e^{-i\xi x}(-i\xi)^{\lambda+1}\hat f(\xi)\,d\xi\\ &=: I_1 + I_2.\end{aligned}$$

Using the behavior of J near zero and infinity as described in (4.4.5) we immediately obtain

$$|I_1(x)| \le \int_{|\xi|\le 1} |\xi|^{\lambda+1}\,d\xi \le C,$$

for some $C>0$. We further set

$$I_2(x) = \lim_{A\to\infty} I_2^A(x) = \int_1^A e^{-i\xi x}(-i\xi)^{\lambda+1}\hat f(\xi)\,d\xi,$$

and after applying some trigonometric identities to I_2^A we find

$$\begin{aligned}
I_2^A(x) = &\int_1^A \left[\cos(\xi(x+1)) + e^{(\lambda+1)\pi i}\cos(\xi(1-x)) + i\sin(\lambda\pi)e^{i\xi(1-x)}\right] d\xi \\
&+ b\int_1^A \left[\frac{\sin(\xi(1+x))}{\xi} + e^{(\lambda+1)\pi i}\frac{\sin(\xi(1+x))}{\xi} + \sin(\lambda\pi)\frac{e^{i\xi(1-x)}}{\xi}\right] d\xi \\
&+ \int_1^A O\left(\frac{1}{\xi^2}\right) d\xi.
\end{aligned}$$

In general all the above integrals do not exist in the limit of $A \to \infty$, however we can again make use that the limits exist as distributions, more precisely interpret them as Fourier transforms of tempered distributions. Applying these known results (see for example Appendix (A.1) for some derivations) yields

$$\begin{aligned}
\lim_{A\to\infty} 2I_2^A(x) = &\left[\delta_0(x+1) - \cos(\lambda\pi)\delta_0(1-x) - 2\sin(\lambda\pi)\,\mathrm{PV}(1-x)\right] \\
&+ b\left[H(x+1) - \cos(\lambda\pi)H(1-x) + 2\sin(\lambda\pi)\log|1-x|\right] + S(x),
\end{aligned}$$

where δ_0 describes the Dirac distribution, PV the principle value distribution, H the Heaviside function and S a function in $W^{1,s}$ with $s \in [2,\infty)$. In summary this allows us to write

$$\begin{aligned}
D_\xi^{\lambda+1}G(\xi) = &\frac{1}{2}\rho^{\lambda\theta}\left[\delta_0(\xi - u + \rho^\theta) - \cos(\lambda\pi)\delta_0(\xi - u + \rho^\theta) + 2\sin(\lambda\pi)\,\mathrm{PV}(\xi - u - \rho^\theta)\right] \\
&+ \frac{b}{2}\rho^{(\lambda-1)\theta}\big[H(\xi - u + \rho^\theta) - \cos(\lambda\pi)H(v - u - \rho^\theta) \\
&+ 2\sin(\lambda\pi)\log|u + \rho^\theta - \xi|\big] + \rho^{(\lambda-1)\theta}S(\xi),
\end{aligned}$$

and to now distributionally differentiate (4.4.4) $(\lambda+1)$-times to obtain

$$\begin{aligned}
&(\xi_2 - \xi_3)\langle G(\xi_3)G(\xi_2)\rangle\langle D_{\xi_1}^{\lambda+1}G(\xi_1)\rangle \\
&\quad = (\xi_1 - \xi_3)\langle G(\xi_3)D_{\xi_1}^{\lambda+1}G(\xi_1)\rangle\langle G(\xi_2)\rangle + (\xi_2 - \xi_1)\langle D_{\xi_1}^{\lambda+1}G(\xi_1)G(\xi_2)\rangle\langle G(\xi_3)\rangle \\
&\quad\quad + (\lambda+1)\left[\langle D_{\xi_1}^{\lambda}G(\xi_1)G(\xi_3)\rangle\langle G(\xi_2)\rangle - \langle D_{\xi_1}^{\lambda}G(\xi_1)G(\xi_2)\rangle\langle G(\xi_3)\rangle\right].
\end{aligned} \tag{4.4.6}$$

We will now justify that we are allowed to let $\xi_3 \to \xi_1$ in equation (4.4.6). In the following we will omit the additional variable from the derivative hence setting $D_{\xi_i} = D$ if the variable is clear.

Lemma 4.12. *Assume that* (4.4.6) *holds, then*

$$\begin{aligned}
&(\xi_2 - \xi_1)\langle G(\xi_1)G(\xi_2)\rangle\langle D^{\lambda+1}G(\xi_1)\rangle \\
&\quad = (\xi_2 - \xi_1)\langle G(\xi_2)D^{\lambda+1}G(\xi_1)\rangle\langle G(\xi_1)\rangle \\
&\quad\quad + (\lambda+1)\left[\langle D^\lambda G(\xi_1)G(\xi_1)\rangle\langle G(\xi_2)\rangle - \langle D^\lambda G(\xi_1)G(\xi_2)\rangle\langle G(\xi_1)\rangle\right]
\end{aligned} \tag{4.4.7}$$

is true for all $\xi_1, \xi_2 \in \mathcal{C}$.

Proof. Let $\phi, \psi \in D(\mathcal{C})$ and let φ_ε be a standard mollifier. We multiply (4.4.6) by ϕ, ψ and φ_ε and consider each term separately. For the term on the left hand side and the second term on the right hand side of (4.4.6) we notice that G and $\langle G\rangle$ are uniformly, and even Hölder continuous function in ξ and it can be shown that $\langle D^{\lambda+1}G(\xi_1)\rangle$ and $\langle G(\xi_2)D^{\lambda+1}G(\xi_1)\rangle$ are measures on $\mathcal{C}$, see [16]. Hence we obtain the following limit

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iiint \big[(\xi_2 - \xi_3)\langle G(\xi_3)G(\xi_2)\rangle\langle D^{\lambda+1}G(\xi_1)\rangle \\
&\quad - (\xi_2 - \xi_1)\langle D^{\lambda+1}G(\xi_1)G(\xi_2)\rangle\langle G(\xi_3)\rangle\big]\psi(\xi_2)\phi(\xi_1)\varphi_\varepsilon(\xi_1 - \xi_3)\,d\xi_3 d\xi_2 d\xi_1 \\
&= \iint (\xi_2 - \xi_1)\big[\langle G(\xi_1)G(\xi_2)\rangle\langle D^{\lambda+1}G(\xi_1)\rangle - \langle G(\xi_2)D^{\lambda+1}G(\xi_1)\rangle\langle G(\xi_1)\rangle\big]\psi(\xi_2)\phi(\xi_1)\,d\xi_2 d\xi_1,
\end{aligned}$$

as a consequence of the properties of a convolution with a mollifier. For the third term on the right hand side we can use Lemma 4.11 and the fact that G is a continuous function on $\mathcal{C}$ to find

$$\begin{aligned}&\lim_{\varepsilon\to 0}\iiint \left[\langle D^\lambda G(\xi_1)G(\xi_3)\rangle\langle G(\xi_2)\rangle - \langle D^\lambda G(\xi_1)G(\xi_2)\rangle\langle G(\xi_3)\rangle\right]\psi(\xi_2)\phi(\xi_1)\varphi_\varepsilon(\xi_1-\xi_3)\,d\xi_3 d\xi_2 d\xi_1\\ &=\iint \left[\langle D^\lambda G(\xi_1)G(\xi_1)\rangle\langle G(\xi_2)\rangle - \langle D^\lambda G(\xi_1)G(\xi_2)\rangle\langle G(\xi_1)\rangle\right]\,\psi(\xi_2)\phi(\xi_1)\,d\xi_2 d\xi_1.\end{aligned}$$

Lastly, for the first term on the right hand side of (4.4.6) notice that $D_{\xi_1}\varphi_\varepsilon(\xi_1-\xi_3) = -D_{\xi_3}\varphi_\varepsilon(\xi_1-\xi_3)$, which together with an integration by parts yields

$$\begin{aligned}&\iiint (\xi_1-\xi_3)\langle G(\xi_3)D^{\lambda+1}G(\xi_1)\rangle\langle G(\xi_2)\rangle\psi(\xi_2)\phi(\xi_1)\varphi_\varepsilon(\xi_1-\xi_3)\,d\xi_3 d\xi_2 d\xi_1\\ &= -\iiint \langle G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle D_{\xi_1}\left[(\xi_1-\xi_3)\varphi_\varepsilon(\xi_1-\xi_3)\phi(\xi_1)\right]\psi(\xi_2)\,d\xi_3 d\xi_2 d\xi_1\\ &= -\iiint (\xi_1-\xi_3)\langle G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle\varphi_\varepsilon(\xi_1-\xi_3)\psi(\xi_2)\phi'(\xi_1)\,d\xi_3 d\xi_2 d\xi_1\\ &\quad -\iiint \langle G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle\psi(\xi_2)\phi(\xi_1)\varphi_\varepsilon(\xi_1-\xi_3)\,d\xi_3 d\xi_2 d\xi_1\\ &\quad +\iiint (\xi_1-\xi_3)\langle G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle\psi(\xi_2)\phi(\xi_1)D_{\xi_3}\varphi_\varepsilon(\xi_1-\xi_3)\,d\xi_3 d\xi_2 d\xi_1\\ &= -\iiint (\xi_1-\xi_3)\langle G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle\varphi_\varepsilon(\xi_1-\xi_3)\psi(\xi_2)\phi'(\xi_1)\,d\xi_3 d\xi_2 d\xi_1\\ &\quad -\iiint (\xi_1-\xi_3)\langle D_{\xi_3}G(\xi_3)D^\lambda G(\xi_1)\rangle\langle G(\xi_2)\rangle\psi(\xi_2)\phi(\xi_1)\varphi_\varepsilon(\xi_1-\xi_3)\,d\xi_3 d\xi_2 d\xi_1.\end{aligned}$$

At this point we may now conclude similar to the step before by using Lemma 4.11. □

Now we wish to also let $\xi_2\to\xi_1 =: \xi$ in (4.4.7). This is justified by the following:

Lemma 4.13. *Assume that* (4.4.7) *holds, then*

$$\begin{aligned}\langle G(\xi)^2\rangle\langle D^{\lambda+1}G(\xi)\rangle =&\langle D^{\lambda+1}G(\xi)G(\xi)\rangle\langle G(\xi)\rangle\\ &+(\lambda+1)\left[\langle D^\lambda G(\xi)G(\xi)\rangle\langle DG(\xi)\rangle - \langle D^\lambda G(\xi)DG(\xi)\rangle\langle G(\xi)\rangle\right]\end{aligned}\tag{4.4.8}$$

is true for all $\xi\in\mathcal{C}$.

Proof. Let $\phi\in D(\mathcal{C})$ and let φ_ε again be a standard mollifier. We divide (4.4.7) by $(\xi-\xi_2)$ and multiply by ϕ and φ_ε. For the term on the left hand side and the first term on the right hand side we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\iint \left[\langle G(\xi)G(\xi_2)\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle G(\xi_2)D^{\lambda+1}G(\xi)\rangle\langle G(\xi)\rangle\right]\phi(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\\ &=\int \left[\langle G(\xi)^2\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle D^{\lambda+1}G(\xi)G(\xi)\rangle\langle G(\xi)\rangle\right]\phi(\xi)\,d\xi,\end{aligned}$$

where we again used $D_\xi\varphi_\varepsilon(\xi-\xi_2) = -D_{\xi_2}\varphi_\varepsilon(\xi-\xi_2)$ and the fact that

$$\begin{aligned}&\iint \left[\langle G(\xi)G(\xi_2)\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle G(\xi_2)D^{\lambda+1}G(\xi)\rangle\langle G(\xi)\rangle\right]\phi(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\\ &= -\iint \langle DG(\xi)G(\xi_2)\rangle\langle D^\lambda G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi - \iint \langle G(\xi)G(\xi_2)\rangle\langle D^\lambda G(\xi)\rangle\phi'(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\\ &\quad +\iint \langle G(\xi_2)G(\xi)\rangle\langle D^\lambda G(\xi)\rangle\phi(\xi)D_{\xi_2}\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\\ &\quad +\iint \langle G(\xi_2)D^\lambda G(\xi)\rangle\langle DG(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\\ &\quad +\iint \langle G(\xi_2)D^\lambda G(\xi)\rangle\langle G(\xi)\rangle\phi'(\xi)\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi - \iint \langle G(\xi_2)D^\lambda G(\xi)\rangle\langle G(\xi)\rangle\phi(\xi)D_{\xi_2}\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 d\xi\end{aligned}$$

$$= -\iint \langle D^\lambda G(\xi)\rangle \varphi_\varepsilon(\xi-\xi_2)\big[\langle DG(\xi)G(\xi_2)\rangle\phi(\xi) + \langle G(\xi)G(\xi_2)\rangle\phi'(\xi) + \langle G(\xi)D_{\xi_2}G(\xi_2)\rangle\phi(\xi)\big]\,d\xi_2 d\xi$$
$$+ \iint \varphi_\varepsilon(\xi-\xi_2)\big[\langle G(\xi_2)D^\lambda G(\xi)\rangle\langle DG(\xi)\rangle\phi(\xi) + \langle G(\xi_2)D^\lambda G(\xi)\rangle\langle G(\xi)\rangle\phi'(\xi)\big]\,d\xi_2 d\xi$$
$$+ \iint \varphi_\varepsilon(\xi-\xi_2)\langle D_{\xi_2}G(\xi_2)D^\lambda G(\xi)\rangle\langle G(\xi)\rangle\phi(\xi)\,d\xi_2 d\xi,$$

in order to use the integrability of DG and $D^\lambda G$. Letting $\varepsilon \to 0$ and integrating the result by parts again to move the derivatives off the test function ϕ we obtain the claim. We remain to show the convergence of the second term on the right hand side. However, again by using Lemma 4.11, we obtain

$$\int \frac{\langle G(\xi)\rangle - \langle G(\xi_2)\rangle}{\xi-\xi_2}\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 \to \langle DG(\xi)\rangle, \quad \text{in } L^{1+\delta}(\mathcal{C}),$$

and

$$\int \frac{\langle D^\lambda G(\xi)(G(\xi)-G(\xi_2))\rangle}{\xi-\xi_2}\varphi_\varepsilon(\xi-\xi_2)\,d\xi_2 \to \langle D^\lambda G(\xi)DG(\xi)\rangle, \quad \text{in } \ L^{1+\delta}(\mathcal{C}),$$

where $\delta > 0$ was chosen according to Lemma 4.11. □

We are now able to prove one of the main theorems in this section, which shows that even though the $(\lambda+1)$-th derivative of G is not locally integrable, its average with respect to the Young measures is fairly regular.

Theorem 4.14. *Assume $\mathcal{C} \neq \emptyset$. Then for all $\delta \in \big[0, \frac{1}{1-\lambda} - 1\big]$, if $\lambda < 1$ and for all $\delta \in [0,\infty)$ if $\lambda \geq 1$ we have*

$$\langle D^{\lambda+1}G\rangle \in L^{1+\delta}_{loc}(\mathcal{C}).$$

Proof. In view of equation (4.4.8) we only need to consider the integrability of $\langle D^{\lambda+1}G(\xi)G(\xi)\rangle$, $\langle D^\lambda G(\xi)G(\xi)\rangle$ and $\langle D^\lambda G(\xi)DG(\xi)\rangle$. Since G is continuous on $\mathcal{C}$ we immediately obtain $D^\lambda G\,G \in L^q_{loc}$ for all $q \in [1,\infty]$. Furthermore we obtain $D^\lambda G\,DG \in L^{1+\delta}_{loc}$ for δ according to above, since the least behaved parts of $D^\lambda G$ are uniformly bounded in the L^∞-norm or logarithmic terms, which do not affect the $L^{1+\delta}_{loc}$-integrability of DG. In order to verify that $\langle D^{\lambda+1}G\,G\rangle \in L^{1+\delta}_{loc}$ we only need to verify that there is some $M > 0$ such that for any smooth function g:

$$\left|\int \langle \rho^{\lambda\theta}(A\delta_0(\xi-u+\rho^\theta) + B\delta_0(\xi-u-\rho^\theta) + C\,\mathrm{PV}(\xi-u-\rho^\theta))G(\xi)\rangle g(\xi)\,d\xi\right| \leq M\|g\|_{L^p(\mathcal{C})},$$

for $\frac{1}{p} + \frac{1}{1+\delta} = 1$, in view of the Hölder inequality ([2]). Since G vanishes at $\xi = u \pm \rho^\theta$, the first two terms do not contribute. With this we can now bound, after making use of Fubini's Theorem

$$\begin{aligned}\left|\int \langle \rho^{\lambda\theta}\,\mathrm{PV}(\xi-u-\rho^\theta)G(\xi)g(\xi)\rangle\,d\xi\right| &= \left|\left\langle \int \rho^{\lambda\theta}\frac{1}{\xi}\left[G(\xi+u+\rho^\theta,\rho,u)g(\xi+u+\rho^\theta)\right]\,d\xi\right\rangle\right| \\ &= \left|\left\langle \int \rho^{\lambda\theta}\frac{1}{\xi}\left[\Big((-\xi)(\xi+2\rho^\theta)\Big)_+^\lambda g(\xi+u+\rho^\theta)\right]\,d\xi\right\rangle\right| \\ &\leq M\|g\|_{L^p(\mathcal{C})},\end{aligned}$$

for a suitable $M > 0$. □

Lastly, very similar to before, we can differentiate (4.4.7) $(\lambda+1)$-times with respect to ξ_2 and rearrange to obtain the following, where we have also renamed ξ_2 as ξ_1:

Lemma 4.15. *Assume that* (4.4.7) *holds, then*

$$\begin{aligned}(\lambda+1)\langle G(\xi)\rangle\big[\langle D^\lambda G(\xi_1)D^{\lambda+1}G(\xi)\rangle - \langle D^{\lambda+1}G(\xi_1)D^\lambda G(\xi)\rangle\big]& \\ = (\xi-\xi_1)\big[\langle G(\xi)D^{\lambda+1}G(\xi_1)\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle G(\xi)\rangle\langle D^{\lambda+1}G(\xi_1)D^{\lambda+1}G(\xi)\rangle\big]& \\ + (\lambda+1)\big[\langle G(\xi)D^\lambda G(\xi_1)\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle D^{\lambda+1}G(\xi_1)\rangle\langle D^\lambda G(\xi)G(\xi)\rangle\big]&\end{aligned} \tag{4.4.9}$$

is true for all $\xi, \xi_1 \in \mathcal{C}$ in the sense of distribution.

We can heuristically see why (4.4.9) will yield our result. The right hand side of (4.4.9) is again fairly regular, since we have already shown that the first part of the first term is quite regular and also the second term. It is only the second part of the first term where we might multiply two singularities, however this term is compensated by the factor $(\xi - \xi_1)$. The left hand side however is considerably worse, since here we have two multiplications of two singularities with in particular opposite signs, which suggests that the left hand side will yield something non-zero in the limit $\xi_1 \to \xi$. It is exactly this difference in regularity between the left and right hand side which will force the support of the Young measures to be confined to a point.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\operatorname{supp} \varphi \subset (-1, 0)$, $\int \varphi(t)\, dt = 1$ and set

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right), \quad K(t) = t\varphi(t).$$

We will first of all go through all terms on the right hand side of (4.4.9) and show that each of the terms converges to zero in the sense of distributions. For this we first consider the second term on the right hand side.

Lemma 4.16. *Let $\phi \in D(\mathcal{C})$, then*

$$\iint \left[\langle G(\xi)D^\lambda G(\xi_1)\rangle\langle D^{\lambda+1}G(\xi)\rangle - \langle D^{\lambda+1}G(\xi_1)\rangle\langle D^\lambda G(\xi)G(\xi)\rangle\right]\phi(\xi)\varphi_\varepsilon(\xi - \xi_1)\, d\xi_1 d\xi \to 0,$$

as $\varepsilon \to 0$.

Proof. Using the regularity property found in Theorem 4.14 together with Lemma 4.11, the above follows immediately by the properties of a convolution and the construction of φ_ε. □

The next Lemma is going to deal with the first term on the right hand side of (4.4.9).

Lemma 4.17. *Let $\phi \in D(\mathcal{C})$, then*

$$\iint (\xi_1 - \xi)\langle G(\xi)D^{\lambda+1}_{\xi_1}G(\xi_1)\rangle\langle D^{\lambda+1}G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi - \xi_1)\, d\xi_1 d\xi \to 0$$

and

$$\iint (\xi_1 - \xi)\langle G(\xi)\rangle\langle D^{\lambda+1}_{\xi_1}G(\xi_1)D^{\lambda+1}G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi - \xi_1)\, d\xi_1 d\xi \to 0,$$

as $\varepsilon \to 0$.

Proof. Recall that there exist $a, b, c \in \mathbb{R}$ and $S \in W^{1,s}(\mathcal{C})$ for $s \in [2, \infty)$ such that

$$D^{\lambda+1}G(\xi) = \rho^{\lambda\theta}\left[a\delta_z(\xi) + b\delta_w(\xi) + c\,\mathrm{PV}(\xi - w)\right] + S(\xi).$$

The remaining terms will not contribute, as they all have well enough integrability to vanish analogously to as we have seen before. For the first limit notice that the singular terms containing the $(\lambda + 1)$-th derivative are averaged separately with respect to the Young measure. We may hence make use of Theorem 4.14. For this we first consider the terms containing Dirac distributions. Since $\langle D^{\lambda+1}G\rangle \in L^{1+\delta}_{loc}(\mathcal{C})$ for some $\delta \in (0, \delta_\lambda)$, we obtain, using Fubini's Theorem

$$\begin{aligned}
&\iint (\xi_1 - \xi)\langle G(\xi)\rho^{\lambda\theta}\delta_z(\xi_1)\rangle\langle D^{\lambda+1}G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi_1 - \xi)\, d\xi_1 d\xi \\
&= \left\langle \rho^{\lambda\theta}\int (z - \xi)G(\xi)\phi(\xi)\varphi_\varepsilon(\xi - z)G(\xi)\langle D^{\lambda+1}G(\xi)\rangle\, d\xi\right\rangle \\
&\le \left\langle \rho^{\lambda\theta}\int_{\operatorname{supp}\phi} |\xi - z||\varphi_\varepsilon(\xi - z)||\langle D^{\lambda+1}G(\xi)\rangle|\, d\xi\right\rangle \|G\|_{L^\infty(\mathcal{C})}\|\phi\|_{L^\infty(\mathcal{C})} \\
&= \varepsilon\left\langle \rho^{\lambda\theta}\int |K(t)\langle D^{\lambda+1}G(z + \varepsilon t)\rangle|\chi_{\operatorname{supp}\phi}(z + \varepsilon t)\, dt\right\rangle \|G\|_{L^\infty(\mathcal{C})}\|\phi\|_{L^\infty(\mathcal{C})} \\
&\to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Completely analogously we can see that the second term involving a Dirac distribution vanishes as well in the limit of $\varepsilon \to 0$. Lastly we need to consider the term involving the principle value. We again make use of Fubini's Theorem to obtain

$$\begin{aligned}
&\iint \langle \rho^{\lambda\theta}(\xi_1-\xi)G(\xi)\,\mathrm{PV}(\xi_1-w)\langle D^{\lambda+1}G(\xi)\rangle\varphi_\varepsilon(\xi-\xi_1)\phi(\xi)\rangle\,d\xi_1 d\xi \\
&= \left\langle \rho^{\lambda\theta}\iint \mathrm{PV}(\xi_1-w)(\xi_1-\xi)\varphi_\varepsilon(\xi-\xi_1)G(\xi)\phi(\xi)\langle D^{\lambda+1}G(\xi)\rangle\,d\xi_1 d\xi\right\rangle \\
&= \left\langle \rho^{\lambda\theta}\iint \frac{(\xi_1-\xi)\varphi_\varepsilon(\xi-\xi_1)-(w-\xi)\varphi_\varepsilon(\xi-w)}{\xi_1-w}G(\xi)\phi(\xi)\langle D^{\lambda+1}G(\xi)\rangle\,d\xi_1 d\xi\right\rangle,
\end{aligned}$$

where we used

$$(\xi_1-\xi)\varphi_\varepsilon(\xi_1-\xi) = (\xi_1-w)\frac{(\xi_1-\xi)\varphi_\varepsilon(\xi-\xi_1)-(w-\xi)\varphi_\varepsilon(\xi-w)}{\xi_1-w} + (w-\xi)\varphi_\varepsilon(\xi-w)$$

and the fact that the last term in the above vanishes in the principle value $\mathrm{PV}(\xi_1-w)$ since it is not singular at $w=\xi_1$, as it does not depend on ξ_1. Performing a change of variables now yields

$$\begin{aligned}
&\left\langle \rho^{\lambda\theta}\iint \frac{(\xi_1-\xi)\varphi_\varepsilon(\xi-\xi_1)-(w-\xi)\varphi_\varepsilon(\xi-w)}{\xi_1-w}G(\xi)\phi(\xi)\langle D^{\lambda+1}G(\xi)\rangle\,d\xi_1 d\xi\right\rangle \\
&= -\left\langle \rho^{\lambda\theta}\iint \frac{(w-\xi)\varphi_\varepsilon(\xi-w)-(\xi_1+w-\xi)\varphi_\varepsilon(\xi-w-\xi_1)}{\xi_1}G(\xi)\phi(\xi)\langle D^{\lambda+1}G(\xi)\rangle\,d\xi_1 d\xi\right\rangle \\
&= -\varepsilon\left\langle \rho^{\lambda\theta}\iint \frac{t\varphi(-t)-(t'+t)\varphi(-t-t')}{t'}G(w+\varepsilon t)\phi(w+\varepsilon t)\langle D^{\lambda+1}G(w+\varepsilon t)\rangle\,dt' dt\right\rangle.
\end{aligned}$$

The above can be shown to tend to zero as $\varepsilon \to 0$, since integrating over t' will yield a Hilbert transform [1], which is well defined, acting on a Schwartz function (see Appendix (A.1)) and the remaining terms are tending to a constant in t in the limit of $\varepsilon \to 0$.
To prove convergence of the second expression in our lemma we only need to consider terms that are a product of singular parts of $D^{\lambda+1}G(\xi)$ or $D^{\lambda+1}G(\xi_1)$, since all other terms can be dealt with similar to above. Therefore, we first consider the product of two Dirac distributions:

$$\begin{aligned}
&\iint (\xi_1-\xi)\langle G(\xi)\rangle\langle \rho^{2\lambda\theta}(a\delta_z(\xi_1)+b\delta_w(\xi_1))(a\delta_z(\xi)+b\delta_w(\xi))\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_1)\,d\xi_1 d\xi \\
&= \left\langle \rho^{2\lambda\theta}\int [a(z-\xi)\langle G(\xi)\rangle\varphi_\varepsilon(\xi-z)+b(w-z)\langle G(\xi)\rangle\varphi_\varepsilon(\xi-w)][a\delta_z(\xi)+b\delta_w(\xi)]\phi(\xi)\,d\xi\right\rangle \\
&= \langle \rho^{2\lambda\theta}a[a(z-z)\langle G(w)\rangle\varphi_\varepsilon(z-z)+b(w-z)\langle G(z)\rangle\varphi_\varepsilon(z-w)]\phi(z) \\
&\quad + \rho^{2\lambda\theta}b[a(z-w)\langle G(z)\rangle\varphi_\varepsilon(w-z)+b(w-w)\langle G(w)\rangle\varphi_\varepsilon(w-w)]\phi(w)\rangle \\
&= -ab\left\langle \rho^{2\lambda\theta}\left[\langle G(z)\rangle\phi(z)K\left(\frac{z-w}{\varepsilon}\right)+\langle G(w)\rangle\phi(w)K\left(\frac{w-z}{\varepsilon}\right)\right]\right\rangle \\
&\to 0,
\end{aligned}$$

as $\varepsilon \to 0$ where we have integrated twice over the Dirac distribution and used the fact that K is compactly supported together with the continuity of all remaining terms. In the next step we are going to consider products of a Dirac distribution and a principle value. This yields

$$\begin{aligned}
&\iint (\xi_1-\xi)\langle G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_1)\langle \rho^{2\lambda\theta}(a\delta_z(\xi_1)+b\delta_w(\xi_1))\,\mathrm{PV}(\xi-w) \\
&\qquad + \mathrm{PV}(\xi_1-w)(a\delta_z(\xi)+b\delta_w(\xi))\rangle\,d\xi_1 d\xi \\
&= \Bigg\langle \rho^{2\lambda\theta}\Bigg[a\Bigg(\int (z-\xi)\langle G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-z)\,\mathrm{PV}(w-\xi)\,d\xi \\
&\qquad + \int (\xi_1-z)\langle G(z)\rangle\phi(z)\varphi_\varepsilon(z-\xi_1)\,\mathrm{PV}(w-\xi_1)\,d\xi\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ b\Big(\int (w-\xi)\langle G(\xi)\rangle \phi(\xi)\rho_\varepsilon(\xi-w)\,\mathrm{PV}(\xi-w)\,d\xi \\
&+ \int (\xi_1 - w)\langle G(w)\rangle \phi(w)\varphi_\varepsilon(\xi_1-w)\,\mathrm{PV}(\xi_1-w)\,d\xi_1\Big)\Big]\Big\rangle \\
= \Big\langle \rho^{2\lambda\theta}\Big[\, & a\Big(\int \frac{(z-\xi)\langle G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-z)}{\xi-w}\,d\xi + \int \frac{(\xi_1-z)\langle G(z)\rangle\phi(z)\varphi_\varepsilon(z-\xi_1)}{w-\xi_1}\,d\xi_1\Big) \\
&+ b\Big(\int \langle G(\xi)\phi(\xi)\varphi_\varepsilon(\xi-w)\,d\xi + \int \langle G(w)\rangle\phi(w)\varphi_\varepsilon(\xi_1-w)\,d\xi_1\Big)\Big]\Big\rangle.
\end{aligned}
$$

Since by construction $\int \varphi_\varepsilon(t)\,dt = 1$ the last term in the above is equal to $\langle G(w)\rangle\phi(w)$. Furthermore using $x\,\mathrm{PV}\,x = 1$ and after substituting and rearranging terms the above becomes

$$
\begin{aligned}
& a\Big\langle \rho^{2\lambda\theta}\Big[\int \Big(\frac{(z-w)\phi(w)\langle G(w)\rangle\varphi_\varepsilon(w-z)}{\xi} - \frac{(z-w-\xi)\phi(w+\xi)\langle G(w+\xi)\rangle\varphi_\varepsilon(w-z+\xi)}{\xi}\Big)\,d\xi \\
& + \phi(z)\langle G(z)\rangle \int \frac{(w-z)\varphi_\varepsilon(z-w) - (\xi+w-z)\varphi_\varepsilon(z-w-\xi)}{\xi}\,d\xi\Big]\Big\rangle \\
& - b\big\langle \rho^{2\lambda\theta}\big(\langle G\rangle\phi * \varphi_\varepsilon(w) - (\langle G\rangle\phi)(w)\big)\big\rangle.
\end{aligned}
$$

Again, using the continuity of G, the term belonging to the constant b tends to zero for $\varepsilon \to 0$. Let I denote the first term corresponding to the constant a, then for any $\tau > 0$ we have

$$
\frac{1}{a}(\mathrm{I}) = \big\langle \rho^{2\lambda\theta}\big(\chi_{[0,\tau]}(\rho) + \chi_{[\tau,\infty)}(\rho)\big)A_\varepsilon(w,z)\big\rangle,
$$

in order to separate the cases where the density becomes small, which would mean $w \to z$. Here we have defined

$$
\begin{aligned}
A_\varepsilon(w,z) := \int & \frac{K\left(\frac{\xi+w-z}{\varepsilon}\right)\langle G(w+\xi)\rangle\phi(w+\xi) - K\left(\frac{w-z}{\varepsilon}\right)\langle G(w)\rangle\phi(w)}{\xi}\,d\xi \\
& + \phi(z)\langle G(z)\rangle \int \frac{K\left(\frac{z-w-\xi}{\varepsilon}\right) - K\left(\frac{z-w}{\varepsilon}\right)}{\xi}\,d\xi.
\end{aligned}
$$

Then, by using that K is compactly supported and the fact that because of the difference structure of the above the singularity at $\xi = 0$ is removable, we obtain the uniform bound

$$
|A_\varepsilon(w,z)| \le C, \quad C > 0,
$$

for all possible (w,z). If $\rho \ge \tau > 0$ the compact support of K implies similar to before $A_\varepsilon(w,z) \to 0$, for $\varepsilon \to 0$ and any $\tau > 0$. Let $\rho \le \tau$, then the uniform bound on A_ε implies

$$
\limsup_{\varepsilon\to 0}\left|\frac{1}{a}(\mathrm{I})\right| \le \tau^{2\lambda\theta}C, \quad \text{for all } \tau > 0,
$$

which shows $\mathrm{I} \to 0$, for $\varepsilon \to 0$.
Lastly we need to consider the product of two principle values. For this we make us of

$$
\begin{aligned}
&(\xi_1 - w + w - \xi)\,\mathrm{PV}(w-\xi)\,\mathrm{PV}(w-\xi_1) \\
&= (\xi_1 - w)\,\mathrm{PV}(w-\xi)\,\mathrm{PV}(w-\xi_1) + (w-\xi)\,\mathrm{PV}(w-\xi)\,\mathrm{PV}(w-\xi_1) \\
&= \mathrm{PV}(w-\xi_1) - \mathrm{PV}(w-\xi),
\end{aligned}
$$

in order to obtain

$$
\begin{aligned}
&\big\langle \rho^{2\lambda\theta}(\xi_1-\xi)\langle G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_1)\,\mathrm{PV}(w-\xi)\,\mathrm{PV}(w-\xi_1)\big\rangle \\
&\quad = \big\langle \rho^{2\lambda\theta}(\xi_1 - w + w - \xi)\langle G(\xi)\rangle\varphi_\varepsilon(\xi-\xi_1)\,\mathrm{PV}(w-\xi)\,\mathrm{PV}(w-\xi_1)\big\rangle \\
&\quad = \big\langle \rho^{2\lambda\theta}\langle G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_1)\,[\mathrm{PV}(w-\xi_1) - \mathrm{PV}(w-\xi)]\,\big\rangle,
\end{aligned}
$$

which again tends to zero as ε tends to zero. □

As already discussed before, the last step consists in showing that the left hand side of (4.4.9) converges to something non-zero in the sense of distributions.

Lemma 4.18. *There exist constants $A, B > 0$ such that for all $\phi \in D(\mathcal{C})$,*

$$\begin{aligned} \mathrm{I}_\varepsilon := \iint \langle G(\xi)\rangle \big[\langle D^\lambda_{\xi_1} G(\xi_1) D^{\lambda+1} G(\xi)\rangle - \langle D^{\lambda+1}_{\xi_1} G(\xi_1) D^\lambda G(\xi)\rangle\big]\phi(\xi)\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \\ \to -\big\langle \rho^{2\lambda\theta}\left[A\langle G(z)\rangle\phi(z) - B\langle G(w)\rangle\phi(w)\right]\big\rangle \end{aligned} \tag{4.4.10}$$

as $\varepsilon \to 0$.

Proof. We use $D_\xi\varphi_\varepsilon(\xi-\xi_1) = -D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)$ and integrate (4.4.10) by parts to obtain

$$\begin{aligned} \mathrm{I}_\varepsilon = &- \iint \big[\langle DG(\xi)\rangle\langle D^\lambda_{\xi_1} G(\xi_1) D^\lambda G(\xi)\rangle\phi(\xi)\varphi_\varepsilon(\xi-\xi_1) + \langle G(\xi)\rangle\langle D^\lambda_{\xi_1} G(\xi_1) D^\lambda G(\xi)\rangle\phi(\xi) D_\xi\varphi_\varepsilon(\xi-\xi_1) \\ &\quad + \langle G(\xi)\rangle\langle D^\lambda_{\xi_1} G(\xi_1) D^\lambda G(\xi)\rangle\phi'(\xi)\varphi_\varepsilon(\xi-\xi_1)\big]\, d\xi_2 d\xi \\ &+ \iint \langle G(\xi)\rangle\langle D^\lambda_{\xi_1} G(\xi_1) D^\lambda G(\xi)\rangle\phi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \\ = &- \iint \psi'(\xi)\langle D^\lambda G(\xi) D^\lambda_{\xi_1}(\xi_1)\rangle\varphi_\varepsilon(\xi-\xi_1)\, d\xi d\xi_1 + 2\iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\langle D^\lambda G(\xi) D^\lambda_{\xi_1} G(\xi_1)\rangle\, d\xi_1 d\xi, \end{aligned}$$

where we set $\psi(\xi) = \langle G(\xi)\rangle\phi(\xi)$. Using the relation $2ab = a^2 + b^2 - (a-b)^2$ the above yields

$$\begin{aligned} \mathrm{I}_\varepsilon = &- \iint \psi'(\xi)\langle D^\lambda G(\xi) D^\lambda_{\xi_1} G(\xi_1)\rangle\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \\ &+ \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big[\big\langle|D^\lambda G(\xi)|^2\big\rangle + \big\langle|D^\lambda_{\xi_1} G(\xi_1)|^2\big\rangle\big]\, d\xi_1 d\xi \\ &- \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big\langle\big(D^\lambda G(\xi) - D^\lambda_{\xi_1} G(\xi_1)\big)^2\big\rangle\, d\xi_1 d\xi \\ = &- \iint \psi'(\xi)\langle D^\lambda G(\xi) D^\lambda_{\xi_1} G(\xi_1)\rangle\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \\ &- \iint \psi(\xi) D\varphi_\varepsilon(\xi-\xi_1)\big\langle|D^\lambda_{\xi_1} G(\xi_1)|^2\big\rangle\, d\xi_1 d\xi \\ &- \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big\langle\big(D^\lambda G(\xi) - D^\lambda_{\xi_1} G(\xi_1)\big)^2\big\rangle\, d\xi_1 d\xi, \end{aligned}$$

where we have again used $D_\xi\varphi_\varepsilon(\xi-\xi_1) = -D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)$ in order to see that the first part of the second term in the first equality is equal to zero after integrating by parts. Simplifying then yields

$$\begin{aligned} \mathrm{I}_\varepsilon = &\iint \psi'(\xi)\big[\big\langle|D^\lambda_{\xi_1} G(\xi_1)|^2\big\rangle - \big\langle D^\lambda G(\xi) D^\lambda_{\xi_1} G(\xi_1)\big\rangle\big]\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \\ &- \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big\langle\big(D^\lambda G(\xi) - D^\lambda_{\xi_1} G(\xi_1)\big)^2\big\rangle\, d\xi_1 d\xi, \end{aligned}$$

and applying Lemma 4.11 shows that the first of the two integrals above converges to zero in the limit of ε going to zero. From the expression for $D^{\lambda+1}G$ we obtain

$$D^\lambda G(\xi,\rho,u) = F(\xi,\rho,u) + L(\xi,\rho,u),$$

where L is a continuous function and

$$F(\xi) = \rho^{\lambda\theta}\left[aH(\xi-z) + bH(\xi-w) + c\log|\xi-w|\right] \in L^q(\mathcal{C}),$$

for suitable constants $a, b, c \in \mathbb{R}$. Similar to before, the regularity of L and F reduces the behavior of I_ε to the behavior of

$$\mathrm{II}_\varepsilon = -\iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big\langle(F(\xi) - F(\xi_1))^2\big\rangle\, d\xi_1 d\xi,$$

since the remaining terms will vanish in the limit. We will now consider each term of

$$\begin{aligned}(F(\xi)-F(\xi_1))^2 = \rho^{2\lambda\theta}\big[[a\,(H(\xi-z)-H(\xi_1-z))\\ + b\,(H(\xi-w)-H(\xi_1-w)) + c\,(\log|\xi-w|-\log|\xi_1-w|)\,\big]^2\end{aligned} \tag{4.4.11}$$

separately. For the first term we integrate by parts and use Example 2.8 to obtain

$$\begin{aligned}&\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\,(H(\xi-z)-H(\xi_1-z))\,(H(\xi-w)-H(\xi_1-w))\;d\xi_1 d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\big[H(\xi-z)H(\xi-w)-H(\xi-z)H(\xi_1-w)\right.\\
&\left.-H(\xi_1-z)H(\xi-w)+H(\xi_1-z)H(\xi_1-w)\big]\,d\xi_1 d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)\varphi_\varepsilon(\xi-\xi_1)\big[\delta_z(\xi_1)H(\xi-w)+\delta_w(\xi_1)H(\xi-z)\right.\\
&\left.-\delta_z(\xi_1)H(\xi_1-w)-\delta_w(\xi_1)H(\xi_1-z)\big]\,d\xi_1 d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\int \psi(\xi)\varphi_\varepsilon(\xi-w)[H(\xi-z)-1]\,d\xi\right\rangle+\left\langle \rho^{2\lambda\theta}\int \psi(\xi)[H(\xi-w)-H(z-w)]\varphi_\varepsilon(\xi-z)\,d\xi\right\rangle,\end{aligned}$$

which tends to zero as $\varepsilon\to 0$, since $H(w-z)=1$ as $z\le w$ and the second term is zero because of the choice of the support of φ_ε. Similarly we obtain, again by integrating by parts, for the second term

$$\begin{aligned}&\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\,(H(\xi-z)-H(\xi_1-z))\,(\log|\xi-w|-\log|\xi_1-w|)\;d\xi_1 d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\,(H(\xi-z)\log|\xi-w|+H(\xi_1-z)\log|\xi_1-w|)\;d\xi_1 d\xi\right\rangle\\
&\quad+\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D\varphi_\varepsilon(\xi-\xi_1)H(\xi-z)\log|\xi_1-w|\,d\xi_1 d\xi\right\rangle\\
&\quad+\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)(-D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1))H(\xi_1-z)\log|\xi-w|\,d\xi_1 d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)(-D\varphi_\varepsilon(\xi-\xi_1))H(\xi_1-z)\log|\xi_1-w|\,d\xi_1 d\xi\right\rangle\\
&\quad-\left\langle \rho^{2\lambda\theta}\left[\int \psi(z)\varphi_\varepsilon(z-\xi_1)\log|\xi_1-w|d\xi_1+\iint \psi'(\xi)\varphi_\varepsilon(\xi-\xi_1)H(\xi-z)\log|\xi_1-w|\,d\xi_1 d\xi\right]\right\rangle\\
&\quad+\left\langle \rho^{2\lambda\theta}\int \psi(\xi)\varphi_\varepsilon(\xi-z)\log|\xi-w|\,d\xi\right\rangle\\
&=\left\langle \rho^{2\lambda\theta}\iint \psi'(\xi)\,[H(\xi_1-z)\log|\xi_1-w|-H(\xi-z)\log|\xi_1-w|]\,\varphi_\varepsilon(\xi-\xi_1)\,d\xi_1 d\xi\right\rangle\\
&\quad+\left\langle \rho^{2\lambda\theta}\left[\int \psi(\xi)\varphi_\varepsilon(\xi-z)\log|\xi-w|\,d\xi-\int \psi(z)\varphi_\varepsilon(z-\xi_1)\log|\xi_1-w|\,d\xi_1\right]\right\rangle.\end{aligned}$$

The first term tends to zero as ε tends to zero in view of the integrability properties of ψ' and log. For the second term we only need to notice that the potential singularity of $\log|v-w|$ in the vacuum case is compensated by $\rho^{2\lambda\theta}=(w-z)^{2\lambda}$. Hence the second term tends to zero as well.
Next we need to consider products of Heaviside functions and logarithms. Again, using integration by parts and Example 2.8, we obtain

$$\begin{aligned}&\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\,(H(\xi-w)-H(\xi_1-w))\,(\log|\xi-w|-\log|\xi_1-w|)\;d\xi_1 d\xi\right\rangle\\
&=-\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D\varphi_\varepsilon(\xi-\xi_1)H(\xi_1-w)\log|\xi_1-w|\,d\xi_1 d\xi\right\rangle\end{aligned}$$

$$
\begin{aligned}
&+\left\langle \rho^{2\lambda\theta} \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1) H(\xi-w)\log|\xi-w|\, d\xi_1 d\xi \right\rangle \\
&+\left\langle \rho^{2\lambda\theta} \iint \psi(\xi) D\varphi_\varepsilon(\xi-\xi_1) H(\xi-w)\log|\xi_1-w|\, d\xi_1 d\xi \right\rangle \\
&-\left\langle \rho^{2\lambda\theta} \iint \psi(\xi) D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1) H(\xi_1-w)\log|\xi-w|\, d\xi_1 d\xi \right\rangle \\
=&\left\langle \rho^{2\lambda\theta} \iint \psi'(\xi)\left[H(\xi_1-w)\log|\xi_1-w| - H(\xi-w)\log|\xi_1-w|\right]\varphi_\varepsilon(\xi-\xi_1)\, d\xi_1 d\xi \right\rangle \\
&-\left\langle \rho^{2\lambda\theta} \int \psi(w)\log|\xi_1-w|\varphi_\varepsilon(w-\xi_1)\, d\xi_1 - \int \psi(\xi)\log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi \right\rangle.
\end{aligned}
$$

For the first of the two terms above we may conclude as before to see that it tends to zero in the limit of $\varepsilon \to 0$. For the second term notice that we can write

$$
\begin{aligned}
&\left\langle \rho^{2\lambda\theta}\left[\int \psi(\xi)\log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi - \psi(w)\int \log|\xi_1-w|\varphi_\varepsilon(w-\xi_1)\, d\xi_1\right] \right\rangle \\
&=\left\langle \rho^{2\lambda\theta} \int [\psi(\xi)-\psi(w)]\ \log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi \right\rangle \\
&\quad+\left\langle \rho^{2\lambda\theta}\psi(w)\left[\int \log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi - \int \log|\xi_1-w|\varphi_\varepsilon(w-\xi_1)\, d\xi_1\right] \right\rangle.
\end{aligned}
$$

Using a symmetry argument we obtain that the second term above is identically equal to zero, whereas for the first term we let α be the Hölder exponent of ψ and use the Hölder inequality for Hölder continuous functions to further obtain

$$
\begin{aligned}
&\left|\left\langle \rho^{2\lambda\theta}\int [\psi(\xi)-\psi(w)]\log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi \right\rangle\right| \\
&\le \left\langle \rho^{2\lambda\theta}\int [|\psi(\xi)-\psi(w)|]\log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi \right\rangle \\
&\le \left\langle \rho^{2\lambda\theta}\int |\xi-w|^\alpha\log|\xi-w|\varphi_\varepsilon(\xi-w)\, d\xi \right\rangle \\
&= \left\langle \rho^{2\lambda\theta}\varepsilon^\alpha\int |y|^\alpha|\log|\varepsilon y||\varphi_\varepsilon(y)\, dy \right\rangle \to 0,
\end{aligned}
$$

as $\varepsilon \to 0$, since again the singularity of the logarithm is compensated by the prefactor.
The fourth term we need to consider is given by

$$
\begin{aligned}
-&\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\left[a^2(H(\xi-z)-H(\xi_1-z))^2 + b^2(H(\xi-w)-H(\xi_1-w))^2\right.\right. \\
&\quad\left.\left. + c^2(\log|\xi-w|-\log|\xi_1-w|)^2\right]\right\rangle.
\end{aligned}
$$

Without loss of generality we may now assume that $\xi<\xi_1$, which by the choice of the support of φ_ε yields

$$
\begin{aligned}
(H(\xi_1-z)-H(\xi-z))^2 &= \chi_{[\xi,z]}(\xi_1) \\
(H(\xi_1-w)-H(\xi-w))^2 &= \chi_{[\xi,w]}(\xi_1).
\end{aligned}
$$

Applying integration by parts to the two terms involving Heaviside functions yields

$$
\begin{aligned}
&-\left\langle \rho^{2\lambda\theta}\iint \psi(\xi)D_{\xi_1}\varphi_\varepsilon(\xi-\xi_1)\left[a^2\chi_{[\xi,z]}(\xi_1)+b^2\chi_{[\xi,w]}(\xi_1)\right]\ d\xi_1 d\xi \right\rangle \\
&= \left\langle \rho^{2\lambda\theta}\iint \psi(\xi)\varphi_\varepsilon(\xi-\xi_1)\left[a^2(\delta_\xi(\xi_1)-\delta_z(\xi_1))+b^2(\delta_\xi(\xi_1)-\delta_w(\xi_1)\right]\ d\xi_1 d\xi \right\rangle \\
&= -\left\langle \rho^{2\lambda\theta}\left[a^2\int \psi(\xi)\varphi_\varepsilon(\xi-z)\, d\xi + b^2\int \psi(\xi)\varphi_\varepsilon(\xi-w)\ \ d\xi\right]\right\rangle \\
&\to -\left\langle \rho^{2\lambda\theta}\left[a^2\psi(z)+b^2\psi(w)\right]\ \right\rangle,
\end{aligned}
$$

as $\varepsilon \to 0$.
Lastly we need to deal with the term involving the square of the logarithms. For this choose $R > 0$ large enough such that

$$|\xi + |w_0|| \leq R, \quad \forall \xi \in \operatorname{supp} \psi.$$

Now observe

$$\begin{aligned}
&-\left\langle \rho^{2\lambda\theta} \iint \psi(\xi) D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1) \left(\log|\xi - w| - \log|\xi_1 - w|\right)^2 \, d\xi_1 d\xi \right\rangle \\
&= -\left\langle \rho^{2\lambda\theta} \iint \psi(\xi + w) D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \, d\xi_1 d\xi \right\rangle \\
&= -\left\langle \rho^{2\lambda\theta} \iint [\psi(\xi + w) - \psi(w)\chi_{|\xi| \leq R}] D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \, d\xi_1 d\xi \right\rangle \\
&\quad - \left\langle \rho^{2\lambda\theta} \psi(w) \right\rangle \iint \chi_{|\xi| \leq R} D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \, d\xi_1 d\xi.
\end{aligned}$$

We will now, similar to before, make use of the Hölder continuity of ψ. For this we bound the first of the two terms above by

$$\begin{aligned}
&\left| \left\langle \rho^{2\lambda\theta} \iint \chi_{|\xi| \leq R} (\psi(\xi + w) - \psi(w)) D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \right\rangle \right| \\
&\leq \left\langle \rho^{2\lambda\theta} \frac{1}{\varepsilon^2} \iint \chi_{|\xi - \xi_1| \leq \varepsilon} (\log|\xi| - \log|\xi_1|)^2 |\xi|^\alpha \chi_{|\xi| \leq R} \, d\xi_1 d\xi \right\rangle \\
&\leq \left\langle \rho^{2\lambda\theta} \right\rangle \iint \chi_{|\xi - \xi_1| \ \leq 1} (\log|\varepsilon\xi| - \log|\varepsilon\xi_1|)^2 \varepsilon^\alpha |\xi|^\alpha \chi_{|\xi| \leq \frac{R}{\varepsilon}} \, d\xi_1 d\xi,
\end{aligned}$$

but

$$\iint_{|\xi| \leq 2} \chi_{|\xi - \xi_1| \leq 1} (\log|\xi| - \log|\xi_1|)^2 \varepsilon^\alpha |\xi|^\alpha \chi_{|\xi| \leq \frac{R}{\varepsilon}} \, d\xi_1 d\xi = O(\varepsilon^\alpha) \quad \text{as } \varepsilon \to 0,$$

and by applying the mean value formula to the logarithms we obtain the estimate

$$\begin{aligned}
&\iint_{|\xi| \geq 2} \chi_{|\xi - \xi_1| \leq 1} (\log|\varepsilon\xi| - \log|\varepsilon\xi_1|)^2 \varepsilon^\alpha |\xi|^\alpha \chi_{|\xi| \leq \frac{R}{\varepsilon}} \, d\xi_1 d\xi \\
&= O\left(\int_{-1}^{1} dz \int_{2 \leq |\xi| \leq \frac{R}{\varepsilon}} \varepsilon^\alpha |\xi|^{\alpha - 2} \, d\xi \right) \\
&= O\left(\varepsilon^\alpha \left(\frac{R}{\varepsilon} \right)^{\alpha - 1} \right) = O(\varepsilon).
\end{aligned}$$

For the second term we write

$$\begin{aligned}
&\iint \chi_{|\xi| \leq R} D_{\xi_1} \varphi_\varepsilon(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \, d\xi_1 d\xi = \int_{-R}^{R} \int \frac{1}{\varepsilon^2} \varphi' \left(\frac{\xi - \xi_1}{\varepsilon} \right) (\log|\xi| - \log|\xi_1|)^2 \, d\xi d\xi_1 \\
&= \int_{-\frac{R}{\varepsilon}}^{\frac{R}{\varepsilon}} d\xi \int \varphi'(\xi - \xi_1)(\log|\xi| - \log|\xi_1|)^2 \, d\xi_1 = \int_{-\frac{R}{\varepsilon}}^{\frac{R}{\varepsilon}} d\xi \int \left(\log\left| 1 - \frac{t}{\xi} \right| \right)^2 \varphi'(t) \, dt \\
&= \int |t| \varphi'(t) \left[\int_{-\infty}^{-\frac{|t|\varepsilon}{R}} + \int_{\frac{|t|\varepsilon}{R}}^{\infty} (\log|1 + s|)^2 \frac{1}{s^2} \, ds \right] dt.
\end{aligned}$$

We now integrate the above by parts and send ε to zero to obtain

$$\int |t| \varphi'(t) \left[\int_{-\infty}^{-\frac{|t|\varepsilon}{R}} + \int_{\frac{|t|\varepsilon}{R}}^{\infty} (\log|1 + s|)^2 \frac{1}{s^2} \, ds \right] dt \overset{\varepsilon \to 0}{=} \int \varphi(t) \int_{-\infty}^{\infty} \frac{(\log|1 + s|)^2}{s^2} \, ds dt = C + o(1),$$

where we also used that the support of φ is contained in $(-1, 0)$. $\square$

We are now able to complete the proof of the reduction of the Young measures. Equation (4.4.9) and the just found Lemmas imply

$$\left\langle \rho^{2\lambda\theta} \langle G(w) \rangle \phi(w) \right\rangle = \left\langle \rho^{2\lambda\theta} \langle G(z) \rangle \phi(z) \right\rangle = 0,$$

for all test functions $\phi \in D(\mathcal{C})$. We can now choose a sequence of test functions ϕ_n such that $\phi_n \to \chi_{\mathcal{C}}$, and since $\langle G(\xi) \rangle > 0$ on $\mathcal{C}$ we obtain

$$\left\langle \rho^{2\lambda\theta} \chi_{\mathcal{C}} \right\rangle = 0.$$

It hence follows that the Young measure ν must admit the decomposition

$$\nu_{x,t} = \mu_{x,t} + \alpha \delta_{(w_0, z_0)},$$

where $\mu_{x,t}$ is a measure with $\operatorname{supp} \mu_{x,t} \subset V$, since by Lemma 4.9 we know that $(w_0, z_0) \in \operatorname{supp} \nu$, if $\mathcal{C} \neq \emptyset$. Inserting this expression for ν into (4.4.1) and rearranging yields

$$(\alpha - \alpha^2)(\xi - \xi_1) G(\xi) G(\xi_1) = 0, \quad \text{for all } \xi, \xi_1 \in \mathcal{C}.$$

Hence if $\mathcal{C} \neq \emptyset$ we must have $\alpha = 1$ or $\alpha = 0$, but because $(w_0, z_0) \in \operatorname{supp} \nu$ we must have $\alpha = 1$ and since ν is a probability measure $\mu = 0$. This completes the proof of the reduction of the Young measures. Applying Theorem 3.12 and subsequently Remark 3.13 we obtain a subsequence of u_ε and ρ_ε converging almost everywhere.

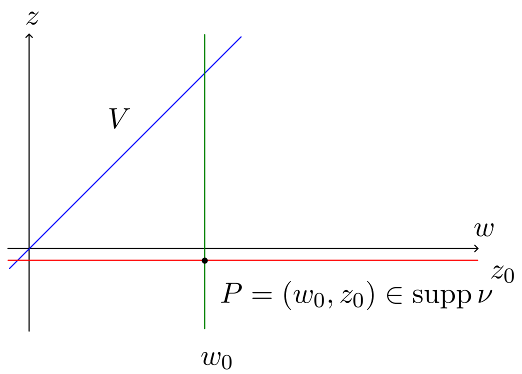

Figure 5: The Support of the Young Measures

4.4.3 Proof for the Case of $\gamma > 3$

The proof of the reduction of the Young measures for the case $\gamma > 3$ is taken from [17] and makes use of the results obtained in Section 4.4.1. The only result we need to verify is the following Lemma:

Lemma 4.19. *Let $\mathcal{B} = (\xi_-, \xi_+)$ be any open connected component of $\mathcal{C}$ and define $u_0 := (\xi_+ + \xi_-)/2$. Then*

$$\lim_{\xi \to \xi_+} \frac{\langle u G(\xi) \rangle}{\langle G(\xi) \rangle} \geq u_0, \quad \lim_{\xi \to \xi_-} \frac{\langle u G(\xi) \rangle}{\langle G(\xi) \rangle} \leq u_0. \tag{4.4.12}$$

Proof. By Lemma 4.8 we already know that in the case of $\gamma > 3$ the function $\langle uG \rangle / \langle G \rangle$ is monotonically decreasing and we will consider the one sided limits. Suppose $G(\xi, \rho, u) > 0$ in an interval $(\xi_+ - \varepsilon, \xi_+)$ for some $\varepsilon > 0$ then

$$u + \rho^\theta \geq \xi_+ - \varepsilon,$$

holds for these values of (ρ, u). Furthermore the values of (ρ, u) satisfy $\xi_- \leq u - \rho^\theta$. This implies

$$\begin{aligned} \lim_{\xi \to \xi_+} \frac{\langle u(\xi) G(\xi) \rangle}{\langle G(\xi) \rangle} &\geq \min\{u : (\rho, u) \in \operatorname{supp} \nu,\ u + \rho^\theta = \xi_+\} \\ &\geq \frac{\xi_+ + \xi_-}{2} = u_0, \end{aligned}$$

by using the above inequalities on ξ_- and ξ_+. The second inequality follows in a similar way. $\square$

The just found Lemma together with Lemma 4.8 implies that $\langle uG\rangle/\langle G\rangle$ is constant on every open connected component of $\mathcal{C}$. Hence $\langle f_\alpha^2(\xi)\rangle$ and thus $f_\alpha(\xi)$ vanish on the support of ν. Letting α tend to zero shows that so does $f_0(\xi)$. From this we obtain

$$f_0(\xi) = \frac{G(\rho, u-\xi)}{\langle G(\xi)\rangle} - 1 = 0, \quad \forall(\rho, u) \in \operatorname{supp}\nu,$$

or equivalently

$$G(\rho, u-\xi) = \int G(\rho, u-\xi)\, d\nu_{x,t}(\rho, u), \quad \forall(\rho, u) \in \operatorname{supp}\nu,$$

which shows $\nu_{x,t} \ = \delta_{x,t}$. The existence of strongly convergent subsequences of u_ε and ρ_ε now follows analogously to Section 4.4.2.

4.4.4 Proof for the Case of $\gamma = 3$

In the case that $\gamma = 3$ we also have $\theta = 1$, hence the Riemann invariants of system (4.1.2) equal the characteristic speeds

$$z = \lambda_1, \quad w = \lambda_2$$

and the families of characteristic speeds totally decouple. Furthermore there now holds

$$q_z = z\eta_z, \quad q_w = w\eta_w$$

and we can therefore find entropy pairs only depending on z, for example $(2z, z^2)$ or $(3z^2, 2z^3)$. Let $(\rho_\varepsilon, u_\varepsilon)$ denote a sequence of viscosity solutions and consider the Riemann invariants $(z_\varepsilon, w_\varepsilon)$ of system (4.1.2) with respect to these viscosity solutions. Following the description in Section 3.5 we obtain

$$4\langle z, \nu\rangle\langle z^3, \nu, \rangle - 3\langle z^2, \nu\rangle\langle z^2, \nu, \rangle = \langle z^4, \nu\rangle. \tag{4.4.13}$$

We will now make use of the identity

$$z^4 - 4z^3\langle z, \nu\rangle + 6z^2\langle z, \nu\rangle^2 - 4z\langle z, \nu\rangle^3 + \langle z, \nu\rangle^4 = (z - \langle z, \nu\rangle)^4 \geq 0.$$

Applying the Young measure ν to the above equation yields

$$\langle z^4, \nu\rangle - 4\langle z^3, \nu\rangle\langle z, \nu\rangle + 6\langle z^2, \nu\rangle\langle z, \nu\rangle^2 - 3\langle z, \nu\rangle^4 \geq 0,$$

which in view of (4.4.13) implies

$$-3\left[\langle z^2, \nu\rangle - \langle z, \nu\rangle^2\right] \geq 0$$

and hence $\langle z^2, \nu\rangle = \langle z, \nu\rangle^2$. We can interpret this as the probability distribution given by the Young measures having zero variance, which implies that the Young measure reduce to Dirac measures. Hence we again obtain the strong convergence of a subsequence of z_ε to $\langle z, \nu\rangle$. Similarly we can show that there exists a subsequence of w_ε converging strongly to $\langle w, \nu\rangle$. In particular we obtain a weak solution of (4.1.2) by

$$\rho = \frac{1}{2}(\langle w, \nu\rangle - \langle z, \nu\rangle) \quad u = \frac{1}{2}(\langle w, \nu\rangle + \langle z, \nu\rangle).$$

4.4.5 Additional Properties of the Solution

Regarding Theorem 3.11 and the above obtained reduction of the Young measures we can conclude that any weak solution of (4.1.2), (4.1.3) obtained via a vanishing viscosity approach satisfies the admissibility criterion (3.1.10). Moreover following [16] we can notice that the reduction of the Young measures and subsequently all Theorems necessary to prove the reduction relied on the uniform L^∞-boundedness of the sequence of approximate solutions. It can therefore be shown, that very similar arguments also apply to a sequence of admissible solutions to (4.1.2), (4.1.3) if they satisfy some uniform L^∞-bound. This additionally proves the stability or compactness of weak admissible solutions to the isentropic Euler equations, at least for the cases $\gamma < 3$. As a conclusion we may therefore state the findings of the before sections:

Theorem 4.20. *The initial value problem* (4.1.2), (4.1.3) *with bounded initial data* (4.1.3) *admits a weak solutions for all adiabatic coefficients* $\gamma > 1$. *Furthermore the solutions obtained via the vanishing viscosity method are admissible solutions in the sense of* (3.1.10).

5 Outlook: Measure Valued Solutions

The discussion in Section 3.5 already showed that the method of compensated compactness is only suitable for systems with a rich family of entropy pairs and furthermore only in one spatial dimension. The compactness arguments and more importantly the application of Young measures, as presented in Chapter 3 work for arbitrary spatial dimension. In the following we will present some results regarding measure valued solutions in multiple dimension for the isentropic Euler equations, to illustrate that although being a very weak form of a solution, they are still a meaningful concept. Measure valued solutions in multiple dimensions can be defined in a variety of ways, in this last section we will follow the construction given in [11].

5.1 Multi-Dimensional Isentropic Euler Equations

Similar to the one dimensional case, the Euler equations describe the conservation of momentum and the conservation of mass of a fluid

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) &= 0, \\ \frac{\partial}{\partial t}\rho + \operatorname{div}(\rho u) &= 0, \end{aligned} \tag{5.1.1}$$

where $u : (0,T) \times \Omega \to \mathbb{R}^n$ and $\rho : (0,T) \times \Omega \to \mathbb{R}$ are the unknown functions and $\Omega \subset \mathbb{R}^n$ is open. In the isentropic case we will again let $P(\rho) = c\rho^\gamma$ where $c > 0$ is some constant and $\gamma > 1$. If the boundary of Ω is non-empty we restrict ourselves to the cases, where no fluid is flowing into our out of the region Ω, which is formalized as the boundary condition $u \cdot n = 0$, where n denotes the outer unit normal vector of Ω. We furthermore introduce some initial data

$$u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x) \quad \text{on } \Omega. \tag{5.1.2}$$

Similar as to explained in Section 3.1, continuously differentiable solutions do not exist for large times. We must therefore consider weak solutions. In case that $\Omega \subset \mathbb{R}^n$ is bounded with the above impermeability boundary condition we can define weak solutions according to [19].

Definition 5.1. *We call* (ρ, u) *a weak solution of the initial value problem* (5.1.1), (5.1.2), *if* $\rho \in L^\infty((0,T) \times \Omega)$, $u \in L^\infty((0,T) \times \Omega; \mathbb{R}^n)$ *and*

$$\int_0^T \int_\Omega \rho \frac{\partial}{\partial t}\psi + \rho u \cdot \nabla \psi \, dxdt = -\int_\Omega \rho_0 \psi(0,\cdot)\, dx,$$

as well as

$$\int_0^T \int_\Omega \rho u \cdot \frac{\partial}{\partial t}\phi + (\rho u \otimes u) : \nabla \phi + \rho^\gamma \operatorname{div} \phi \, dxdt = -\int_\Omega \rho_0 u_0 \cdot \phi(0,\cdot)\, dx,$$

holds for all $\phi \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^n)$ *and* $\psi \in C_c^\infty((0,T) \times \Omega)$.

Here : denotes the Frobenius product of two matrices and $\otimes$ denotes a tensor product. Similarly as described in Section 3.1 we can define entropies in multiple dimensions and again pose a companion law to be satisfied. For weak solutions this companion law is again stated as an admissibility criterion. For more information see [19].

5.2 Measure Valued Solutions to the Isentropic Euler Equations

We will define measure valued solutions to the isentropic Euler equations (5.1.1) similar to Section 3.2, however we will also consider possible concentrations of mass and dissipative effects.

Definition 5.2. *A triple* (ν, m, D) *is called a dissipative measure valued solution of the initial value problem* (5.1.1), (5.1.2) *with the isentropic pressure law with* $1 < \gamma < \infty$ *and the initial data* (5.1.2) *in* $L^1(\Omega)$, *if*

i) ν *is a Young measure,*

ii) the measure $m \in M((0,T) \times \bar{\Omega}; \mathbb{R}^{n\times n})$ *satisfies*

$$m(dx\ dt) = m_t(dx) \otimes dt,$$

for a family of measures $t \mapsto m_t \in L^\infty((0,T); \mathbb{R}^{n\times n})$*.*

iii) $D \in L^\infty(0,T)$ *with* $D \geq 0$ *and*

$$|m_t|(\bar{\Omega}) \leq CD(t), \quad \textit{for almost every } t \in (0,T),$$

and some constant $C > 0$*.*

iv) The momentum equation

$$\int_0^T \int_\Omega \langle \rho u, \nu_{x,t} \rangle \cdot \frac{\partial}{\partial t}\phi + \langle \rho u \otimes u, \nu_{x,t} \rangle : \nabla \phi + \langle \rho^\gamma, \nu_{x,t} \rangle \operatorname{div} \phi \, dxdt$$
$$+ \int_0^T \int_\Omega \nabla \phi : dm_t\ dt = - \int_\Omega \rho_0 u_0 \phi(\cdot, 0)\, dx,$$

holds for all $\phi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^n)$*.*

v) The continuity equation

$$\int_0^T \int_\Omega \langle \rho, \nu_{x,t} \rangle \frac{\partial}{\partial t}\psi + \langle \rho u, \nu_{x,t} \rangle \cdot \nabla \psi \, dxdt = - \int_\Omega \rho_0 \psi(\cdot, 0)\, dx,$$

holds for all $\psi \in C_c^\infty([0,T) \times \Omega)$*.*

vi) The energy inequality

$$\frac{1}{2}\int_\Omega \langle \rho |u|^2, \nu_{x,t} \rangle + \frac{1}{\gamma - 1}\langle \rho^\gamma, \nu_{x,t} \rangle \, dx + D(t) \leq \frac{1}{2}\int_\Omega \rho_0 |u_0|^2 + \frac{1}{\gamma-1}\rho_0^\gamma \, dx,$$

is satisfied for almost every $t \in (0,T)$*.*

Condition *vi*) in the above definition can be interpreted as an admissibility criterion, which physically states that the energy of the system is bounded by the initial energy. It is a special case of the more general entropy admissibility criterion. However it turns out, that a weak solution is entropy admissible, if and only if it satisfies the special energy admissibility (for a proof see [19]). The first terms correspond to the kinetic energy and the second ones to the potential energy. Furthermore the measure m represents the possible concentration of mass at certain points in space and is therefore sometimes referred to as a defect measure. The bounded function D represents the dissipation of energy. Hence in the case $(\nu, 0, 0)$ the measure valued solution is purely oscillatory and will therefore be called an oscillatory measure valued solution.
The advantage of these dissipative measure valued solutions is that a global existence result, at least where approximate solutions are known to exist (this is the case for $\gamma > \frac{3}{2}$), can be established easier.

Theorem 5.3. *Let the initial data* (5.1.2) *be in* $L^2(\Omega)$ *and* $L^\gamma(\Omega)$ *respectively. Then there exists a dissipative measure valued solution of the initial value problem* (5.1.1), (5.1.2) *with the isentropic pressure law.*

Proof. See [11]. □

For a proof we refer to [11], the idea however is to again make use of the Young Measure Representation Theorem, applied to a viscous approximation. Unlike the viscosity solutions found in Section 4.2, the approximate solutions here do not satisfy a uniform L^∞-bound. The non-linear terms might therefore not be suitable test-functions in Theorem 3.7 and must therefore be interpreted as measures themselves. In multiple dimensions even admissibility does not suffice to achieve uniqueness of weak solutions according to [19]. Therefore measure valued solutions will not admit unique solutions either. Suppose however that our initial data (5.1.2) gives rise to a continuously differentiable solution. Then the dissipative measure valued solution corresponding to this initial data will collapse to the strong solution. This result is often referred to as weak-strong uniqueness.

Theorem 5.4. *Let the initial data* (5.1.2) *give rise to a continuously differentiable solution* $(\rho, u) \in (C^1([0,T) \times \Omega), C^1([0,T) \times \Omega; \mathbb{R}^n)$ *of* (5.1.1)*, with* $\rho > 0$ *everywhere and a dissipative measure valued solution* (ν, m, D)*, then* $(\nu, m, D) = (\delta_{(\rho,u)}, 0, 0)$.

Proof. See [11]. □

Notice that weak-strong uniqueness is in this case only possible away from the vacuum.
Although measure valued solutions are a very broad solution concept it can be shown for the incompressible Euler equations, that every measure valued solution stems from a sequence of weak solutions, and hence the two solution concepts carry the same information, see for example [25]. The existence of the former can however be proven much simpler. For the isentropic (compressible) Euler equations (5.1.1) not every measure valued solution comes from a sequence of weak solutions. One may however interpret the requirement of measure valued solutions being generated by weak solutions, as again an admissibility criterion for measure valued solutions, in the sense that measure valued solutions, which were not generated by a sequence of weak solutions, are discarded as unphysical.

6 Summary and Conclusion

Since this thesis focused on the crucial results and concepts needed to apply the method of compensated compactness there are still many additional aspects regarding the theory of compensated compactness which have not been introduced here. In [4] one can find a different approach to constructing viscosity solutions in a more physical way. Besides constructing approximate solutions using the vanishing viscosity approach it is also possible to obtain these approximations in a different way. The paper [3] made use of numerically generated approximate solutions using the Lax-Friedrichs scheme and subsequently proved their convergence to a weak solution. Furthermore, the method of compensated compactness has successfully been applied to a large number of one dimensional systems of conservation laws, see for example [18] or [4] for collections of more applications. It is however equally relevant to mention the restrictions of the method presented. In this thesis the method of compensated compactness was applied to a system of two conservation laws. As already explained in Section 3.5, the crucial requirement is to have a rich family of entropy pairs. This limits the applications of this method to systems of at most two conservation laws, since only then will the governing equations of the entropy pairs not be overdetermined, or larger systems that happen to have a flexible family of entropy pairs. Systems of the latter type are referred to as 'rich systems' and one can find further information in [22]. Regarding these limitations, especially the concept of measure valued solutions, which were shortly presented in Chapter 5, have become a more investigated idea, see for example [11] for a survey. Although existence can be established easier, finding correct selection criteria in order to filter out the relevant solutions remains a vivid field of study. A thorough construction of dissipative measure valued solutions as well as numerical schemes for approximate solutions in multiple dimensions can be found in the monograph [9].

As a summary, in chapter 3 the general problem of solving one dimensional hyperbolic systems of conservation laws was introduced. Consequently, the key results from L.C. Young, L. Tartar, and F. Murat were presented to formulate the method of compensated compactness. In Chapter 4 this method was applied to the isentropic Euler equations to prove the existence of weak entropy solutions to the given initial value problem, where the initial data was only assumed to be bounded.
We may hence conclude that, considering the used references, this thesis introduced the method of compensated compactness and applied this method to the isentropic Euler equations to obtain an existence result and thereby collected the required results for this proof. In one spatial dimension an existence theorem for large initial data has therefore been stated and proven. The existence of entropy admissible solutions for arbitrary bounded initial data to the multi-dimensional Euler equations is still an open problem.

A Appendix

A.1 Further Information on Distributions and Fourier Transforms

In this section we give some additional information regarding distributions in order to justify some of the relations used in Section 4.4.2. The results and definitions are based on [21]. For this we need a different class of test functions:

Definition A.1. *A function $f \in C^\infty(\mathbb{R}^n)$ is called a Schwartz function, if for all $\alpha \in \mathbb{N}^n$ and all $\ell \in \mathbb{N}$ there holds*

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^\ell |D^\alpha f(x)| < \infty. \tag{A.1.1}$$

We will denote the space of all Schwartz functions by $S(\mathbb{R}^n)$.

The condition (A.1.1) implies that f vanishes faster than any polynomial at infinity. Obviously this is a generalization of our before used test functions, as we have $D(\mathbb{R}^n) \subset S(\mathbb{R}^n)$.
Let $\alpha, \beta \in \mathbb{N}^n$, then we can define the (semi)norms

$$q_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} \left| x^\beta D^\alpha f(x) \right|, \quad \forall f \in S(\mathbb{R}^d). \tag{A.1.2}$$

Definition A.2. *We call a linear map $T : S(\mathbb{R}^n) \to \mathbb{C}$ a tempered distribution and write $T \in S'(\mathbb{R}^n)$ if T is continuous with respect to the topology induced by* (A.1.2)*, i.e. there are finitely many α_i, β_i such that*

$$|T(f)| \leq C \sum_{i=1}^N q_{\alpha_i,\beta_i}(f), \quad \forall f \ \in S(\mathbb{R}^n).$$

The sequence of (semi)norms

$$\mu_n(f) = \sup_{x\in\mathbb{R}^n, |\alpha|\leq n} (1+|x|)^n |D^\alpha f(x)|,$$

defines the same topology on S as (A.1.2).
Example A.3. Consider the function $\frac{1}{x} \notin L^1(\mathbb{R})$. We can define a distribution corresponding to $\frac{1}{x}$ by

$$\mathrm{PV}_{\frac{1}{x}}(\phi) := \lim_{\varepsilon\to} \int_{\mathbb{R}-[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\, dx, \quad \forall \phi \in S(\mathbb{R}).$$

Then this is well defined as a tempered distribution and we call it the principle value (distribution).

Definition A.4. *Let $f \in L^1(\mathbb{R}^n)$, we define $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \tag{A.1.3}$$

and call $\hat{f}$ the Fourier transform of f: $\mathcal{F}(f)(\xi) = \hat{f}(\xi)$.

It can be shown that the Fourier transform is a well defined, linear and continuous map between Schwartz spaces. Furthermore the Fourier transform acts on a derivative as

$$\mathcal{F}(D_x^\alpha f)(\xi) = \xi^\alpha \hat{f}(\xi), \tag{A.1.4}$$

where $\xi^\alpha = (\xi_1^{\alpha_1}, \cdots, \xi_n^{\alpha_n})^T$. Furthermore there exists an inverse:

Theorem A.5 (Theorem of Inversion)**.** *Let $f \in S(\mathbb{R}^n)$, then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$$

and the Fourier transform has an inverse $\mathcal{F}^{-1} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ with

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{(2\pi)^n} \mathcal{F}(g(-\cdot))(x).$$

We can even extend the definition of a Fourier transform to tempered distributions.

Definition A.6. *Let $T \in S'(\mathbb{R}^n)$. We then define the Fourier transform of $\mathcal{F}(T) = \hat{T}$ by*

$$\hat{T}(\phi) = T(\hat{\phi}) = \langle T, \hat{\phi} \rangle_{S'(\mathbb{R}^n) \times S(\mathbb{R}^n)}.$$

Example A.7. We can calculate the Fourier transform of the tempered distribution given by the function constantly equal to 1.

$$\langle \hat{1}, \phi \rangle_{S' \times S} = \langle 1, \hat{\phi} \rangle_{S' \times S} = \int_{\mathbb{R}^n} \hat{\phi}(\xi)\, d\xi = (2\pi)^n \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i0 \cdot \xi} \hat{\phi}(\xi)\, d\xi = (2\pi)^n \phi(0),$$

hence $\hat{1} = (2\pi)^n \delta_0$.

Example (A.7) implies $\int_1^A \cos(\xi(x+1))\, d\xi \to \delta_0(x+1)$ for $A \to \infty$. Similarly, upon using 2.8 we can see that

$$\int_1^A \frac{\sin(\xi(1+x))}{\xi}\, d\xi \to H(x+1) \quad \text{for } A \to \infty,$$

since

$$\frac{\partial}{\partial x} \left(\frac{\sin(\xi(x+1))}{\xi} \right) = \cos(\xi(x+1)).$$

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