NONCONVEX SCALAR CONSERVATION LAWS IN ONE AND TWO SPACE DIMENSIONS

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1. Introduction

Consider the initial value problem for a scalar conservation law

$$\begin{aligned} &u_t + \nabla . f(u) = 0 \,, \qquad t \geq 0, \qquad \vec{x} \in \mathbb{R}^n \\ &u(x,0) = u_0\,(x)\,. \end{aligned} \tag{1.1}$$

For the nonlinear flux function even smooth initial data in general may not prevent the development of jumps in the solution to (1.1). Thus we consider (1.1) in the sense of distributions.

When trying to understand the qualitative behaviour of solutions to conservation laws in more than one space dimension, as a first step one may consider selfsimilar solutions. This way the problem becomes more tractable, because the number of independent variables is reduced by one. In particular one has considered Riemann problems.

Definition: If (1.1) is invariant under the transformation

$$(c\vec{x}, ct) \rightarrow (\vec{x}, t), \; c > 0,$$

then it is called a Riemann problem.

Thus in two space dimensions initial data in (1.1) which is piecewise constant in sectors meeting at the origin is an example of a two dimensional Riemann problem (2-d R.P.).

These arise naturally in front tracking, a numerical scheme for conservation laws, see e.g. [GK]. The essential feature of this method is that a lower dimensional grid is fitted to and follows the jump surfaces. At the intersection points of these discontinuities 2-d R.P. occur. We mention in passing that one can give a short list of these generic types of such intersection points for two dimensional gas dynamics, which presumably constitutes the pieces that the solution to a 2-d R.P. for the Euler equations is made up of, [GK].

Recently some progress was made in understanding the 2-d R.P. for the scalar conservation law:

$$u_t + f(u)_x + g(u)_y = 0\,. \tag{1.2}$$

One knows existence [CS] and uniqueness [K] of the weak solution satisfying the entropy condition. It was natural to ask next what these solutions for the case of a Riemann problem look like. Wagner [W] constructed the solution for a convex f very close to a convex g. In [HK] this was extended to a generic case with f = g, where f is a quadratic and g a cubic polynomial, see section 2. For general flux functions, for the case f = g, the solution was

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constructed in [CK], see section 3. There something reminiscent of a large time Godunov method for the scalar equation in one space dimension was used. This method inspired two results for the one dimensional scalar equation, which are reported in section 4. We close with some examples in section 5.

2. The Riemann problem for the scalar conservation law in two space dimensions with unequal flux functions

The selfsimilar solution may be described completely by giving the solution in the plane, say t = 1. Far away from the origin, the solution is given by solving a 1-d R.P. across the jumps given in the initial data. [HK] proceeded to describe the interaction of these waves for the two flux functions being a cubic and a quadratic polynomial.

We shall give an illustrative example on what may happen. Suppose we consider the equation

$$u_t + (u^2)_x + (u^3)_y = 0\ .$$

Say across the positive x-axis there was an initial jump that gave rise to a jump followed by a rarefaction wave, see Fig. 2.1.

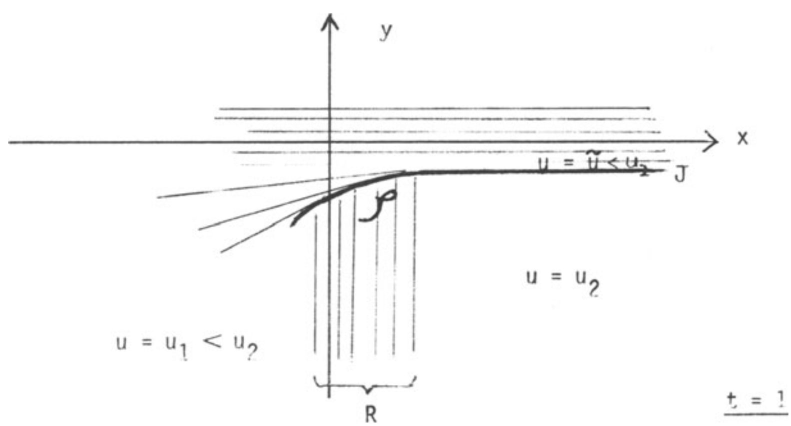

Fig. 2.1 Part of the solution of a particular two dimensional Riemann problem with initial discontinuities across the x-axis and the y-axis.

The jump J interacts with the rarefaction wave R to form a surface $\mathcal{S}$ which is a smooth continuation of J that bends into R. To the left of $\mathcal{S}$ a new rarefaction appears which is tangential to $\mathcal{S}$, see Fig. 2.1. We may now define an ordinary differential equation for $\mathcal{S}$, show that it is well defined on the interval $[(u_2 + \tilde{u})/2, u_2]$ and satisfies the entropy condition.

Depending on the location of u_1 relative to u_2 and $\tilde{u}$ we have two possibilities:

a) $(u_2 + \tilde{u})/2 < u_l$.

This gives rise to a onesided contact discintinuity (c.d.) with the constant state u_1 on one side and $u_2 + \tilde{u} - u_1$ on the other side.

b) $u_1 < (u_2 + \tilde{u})/2$.

At $Z = \{f'((\tilde{u} + u_2)/2),\ g'((\tilde{u} + u_2)/2)\}$ the shock strength of $\mathcal{S}$ has decayed to zero. Then $\mathcal{S}$ continues on smoothly into a curve Γ given by $(f'(u), g'(u))$, $u_1 < u < (\tilde{u} + u_2)/2$, where the two rarefaction waves meet.

In this way we could construct the solution to our 2-d R.P., see Fig. 2.2 for an example of a solution.

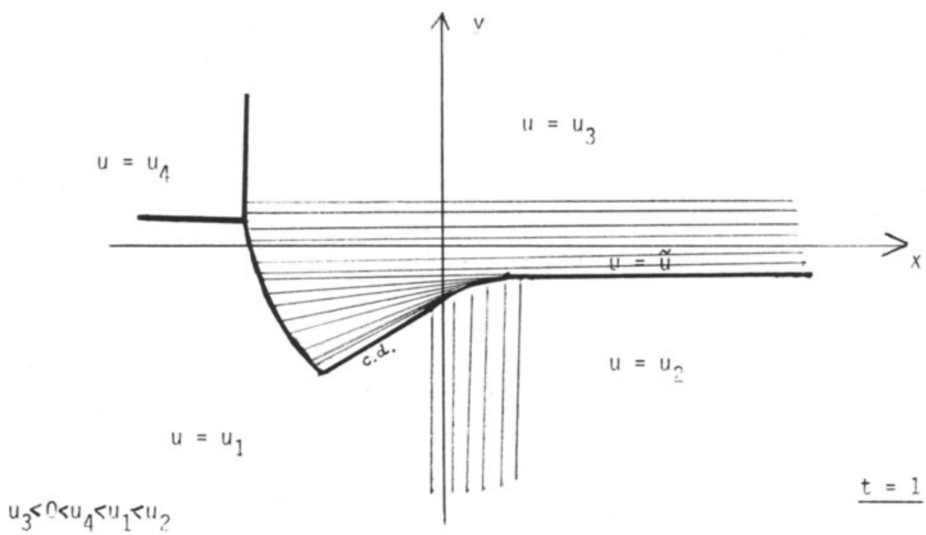

Fig. 2.2 An example of a solution to

$$u_t + (u^2)_x + (u^3)_y = 0$$

with particular initial data constant in each of the four quadrants. Notice how this example contains the piece shown above and mentioned in a) of section 2.

3. The Riemann problem for the scalar conservation law in two space dimensions with equal flux functions

The solution for the 2-d R.P. to (1.2) with f = g is constructed by considering an equivalent problem in one space dimension. Under the coordinate transformation

$\xi = (x + y)/2$ and $\eta = (x - y)/2$ the equation (2.1) for f = g becomes

$$u_t + f(u)_\xi = 0 \tag{3.1}$$

with η a parameter. The initial data for η = const. is piecewise constant with a finite number of jumps. It is easy to see that the 2-d R.P. is now reduced to constructing the solution to (3.1) for $\eta > 0$ and for $\eta < 0$.

For small t, near each jump we may construct the solution to the 1-d R.P. by the method of convex hull (see Fig. 3.1). After some finite time an interaction between two adjacent waves is possible. This interaction may be described qualitatively by using the a timedependent version of the convex hull, see Fig. 3.2. For details see [CK]. One finds that the union of convex hulls given initially gets deformed in a unique way towards the final convex hull, which consists of the solution to 1-d R.P. with the two constant states being the left most state in the initial data for η = const. and the right most state there. This construction is reminiscent of a Godunov scheme which is taken past the time of interaction of the waves.

Using this construction, for the solution of the 2-d R.P. with f = g one may deduce many qualitative features of the solution. One finds that there are no compression waves and thus no shock generation points. Thus jumps may only appear through the bifurcation of jumps and the interaction of jumps. Also for a fixed time t, the number of jumps is bounded uniformly if f has a finite number of inflection points. It seems that for many generic cases, such as polynomial flux functions f, the solution is piecewise smooth.

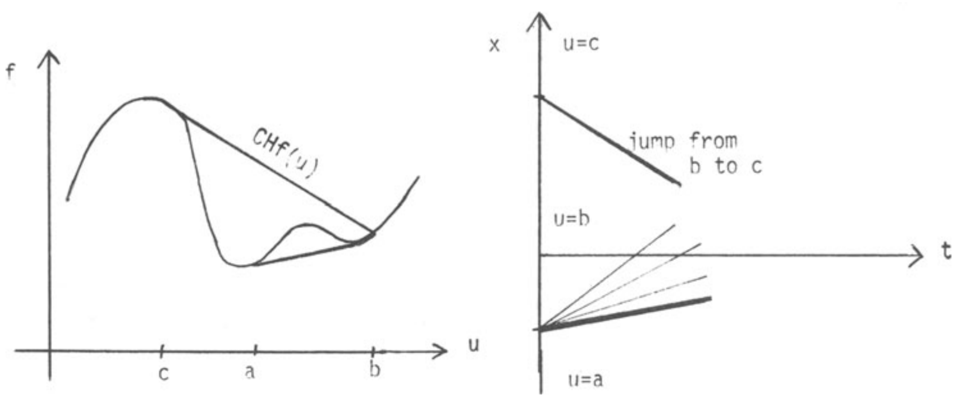

Fig. 3.1 The solution for small time to $u_t + f(u)_x = 0$ and initial data consisting of two jumps is found by solving the individual Riemann problems. To the left is the graph of f together with the convex hull between the jumps. To the right is the solution plane tilted, so that the slopes of the jumps in the convex hull CHf(u) and in the solution plane are parallel.

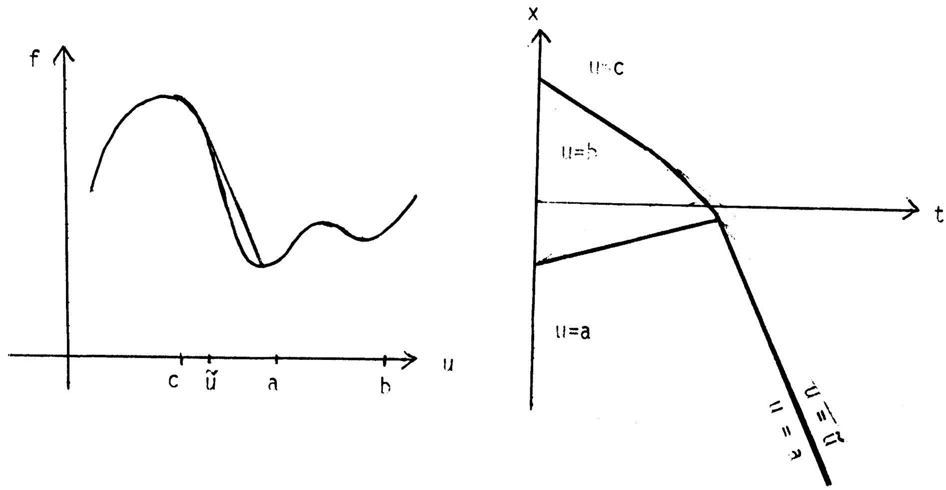

Fig. 3.2 The complete solution to the problem in Fig. 3.1 after the interaction of the waves. This is constructed using a time dependent version of the method of convex hull. The convex hull drawn to the left gives the solution for large time.

4. The scalar conservation law in one space dimension without convexity

Consider

$$u_t + f(u)_x = 0\,, \quad f \in C^2 \tag{4.1}$$

with the initial condition

$$u(x,0) = u_0\,(x)\,.$$

A classical method for approximating the solution is due to Godunov [G]. There the initial data is approximated by piecewise constant data on intervals of length Δx. At every jump discontinuity in the new initial data this leads to a Riemann problem. The Riemann problem resolves into a well known fan solution, as mentioned in the previous section. For Godunov's method one solves all the Riemann problems found by the constant states exactly. One takes this exact solution of the piecewise constant initial data up to time Δt, such that two neighbouring fans solutions don't interact. At this time level Δt the solution u is again approximated by a piecewise constant function, obtained by averaging over each cell x:

$$\bar{u} = \frac{1}{\Delta x} \int_{\Delta_x} u(x,\Delta t)\, dx\,.$$

Now we may proceed as before.

The inability to determine the interaction between waves propagating from the points of discontinuity in the Riemann solution leads to small time steps Δt. In [CK], as mentioned in the previous section, a method for determining these interactions at least qualitatively was given. In this section we shall try to combine these ideas with those in [O] to get some estimates, which might play a role in improving large time step Godunov methods.

4.1 A large time flux written as Godunov's flux

By integrating (4.1) over $\Delta x \times [0,t]$ we obtain

$$0 = \int_0^t \int_{\Delta x} u_t + f(u)_x \, dx \, dt$$

$$= \int_{\Delta x} u(x,t) - u(x,0) \, dx + \int_0^t f(u(x_1,s)) - f(u(x_0,s)) \, ds .$$

Take the mean value of u at time t in Δx to be $\bar{u}$ and at time $t = 0$ to u_0. Then we obtain

$$\bar{u} = u_0 - \frac{1}{\Delta x} \int_0^t f(u(x_1,s)) - f(u(x_0,s)) \, ds . \tag{4.2}$$

Now suppose the initial data is piecewise constant with jumps at the points x_i, $i \in \mathbb{Z}$, and $x_i < x_{i+1}$, and values

$$u(x,0) = u_i , \quad x \in [x_i , x_{i+1}] , \quad i = ..., -k, ..., -1,0,1, ..., k, ...$$

For t small enough, the fans emanating from two neighbouring Riemann problems in this initial value problem don't intersect. The solution of a Riemann problem is a function of $(x - x_i)/t$ alone. Thus for t small u is constant along $x = x_i$, say $\tilde{u}_i$, and $\tilde{u}_i$ is only a function of u_{i-1} and u_i, the initial constant states to the left and right of x_i. Thus we may define h^G as (now for convenience we set $i = 0$)

$$h^G(u_{-1},u_0) = f(\tilde{u}_0) = \frac{1}{t} \int_0^t f(u(x_0,t)) \, dt , \quad t \text{ small}$$

and (4.2) becomes

$$\bar{u} = u_0 - \frac{t}{\Delta x} (h^G(u_{-1},u_0) - h^G(u_0, u_1)) , \qquad t \text{ small.} \tag{4.3}$$

Now let t become larger. Then u along $x = x_i$ becomes a function of several initial cells neighbouring x_i and we may define

$$h(u_{-k}, \dots, u_{k-1}) = \frac{1}{t} \int_0^t f(u(x_0,t)) \, dt , \tag{4.4}$$

with the property that if the initial constant values in all the cells are equal we obtain

$$h(u, ..., u) = f(u) .$$

Theorem 1 There exist a u_r in the convex hull of $\{u_{-k}, ..., u_{-1}\}$ and a u_L in the convex hull of $\{u_0, ..., u_{k-1}\}$ such that

$$h(u_{-k}, ..., u_{k-1}) = h^G (u_r,u_L) . \tag{4.5}$$

Proof: Using the notaion in (4.4) we may write (4.2) as

$$\bar{u} = u_0 - \frac{t}{\Delta x} (h(u_{-k+1}, \dots, u_k) - h(u_{-k}, \dots, u_{k-1})) .$$

Note that $\bar{u}$ is a nondecreasing function of all its variables since it is the average of the exact solution. Thus

$$\frac{\partial \bar{u}}{\partial u_k} = -h_k \geq 0 \Rightarrow h_k \leq 0$$

$$\frac{\partial \bar{u}}{\partial u_{k-1}} = -h_{k-1} + h_k \geq 0 \Rightarrow h_{k-1} \leq h_k$$

in general

$$h_1 \leq h_2 \leq \ldots \leq h_k \leq 0 \leq h_0 \leq \ldots \leq h_{-k+2} \leq h_{-k+1} .$$

We prove the theorem by induction. The case k = 1 is immediate. Suppose the claim is true for k. Then add a value of u_{k+1} on the right. Let $u \in [u_k, u_{k+1}]$. Now consider

$$\begin{aligned} g(u) &= h(u_{k+1}, u_k, \ldots, u_{-k+1}) - h(u, u, u_{k-1}, \ldots, u_{-k+1}) \\ &= (u_{k+1} - u)\, h_{k+1} + (u_k - u)\, h_k , \quad \text{by M.V.Th.} . \end{aligned}$$

We have

$$\begin{aligned} g(u_{k+1}) &= (u_k - u_{k+1})\, h_k \\ g(u_k) &= (u_{k+1} - u_k)\, h_{k+1} . \end{aligned}$$

Thus g(u) changes sign in (u_k, u_{k+1}) or vanishes at the endpoints. Thus there exist $\tilde{u}_k$ such that

$$h(u_{k+1}, u_k, \ldots, u_{-k+1}) = h(\tilde{u}_k, u_{k-1}, \ldots, u_{-k+1})$$

and by induction hypothesis the right hand side

$$= h^G(u_r, u_L)$$

for some u_r and u_L .

Next add a value u_{-k} on the left. For $u \in [u_{-k+1}, u_{-k}]$ we consider

$$\tilde{g}(u) = h(u_{k+1}, u_k, \ldots, u_{-k+1}, u_{-k}) - h(u_{k+1}, \ldots, u, u)$$

by the above result

$$= h(u_{k+1}, u_k, \ldots, u_{-k+1}, u_{-k}) - h(\tilde{u}_k, u_{k-1}, \ldots, u, u) .$$

By repeating the above argument we see that $\tilde{g}(u)$ vanishes in $[u_{-k}, u_{-k+1}]$. Thus

$$\begin{aligned} h(u_{k+1}, u_k, \ldots, u_{-k+1}, u_{-k}) &= h(\tilde{u}_k, u_{k-1}, \ldots, \tilde{u}_{-k+1}) \\ &= h^G(u_r, u_L) \end{aligned}$$

some u_r and some u_L, which proves the induction step, and finishes the proof.

To recapitulate, we have shown the following:

Consider

$$u_t + f(u)_x = 0$$

with initial condition

$$t = 0 \qquad \begin{matrix} u_{-2} & & u_{-1} & & u_0 & & u_1 \\ & x_{-1} & & x_0 & & x_1 & \end{matrix} \qquad x . \tag{4.6}$$

Consider the exact solution $u(x_0,t)$. Then

$$\frac{1}{T}\int_0^T f(u(x_0,t))\, dt = h^G(u_r(T), u_L(T))$$

with some u_r in the convex hull of $\{u_0, \ldots, u_{k-1}\}$

and some u_L in the convex hull of $\{u_{-k}, \ldots, u_{-1}\}$,

and where the domain of dependence of $u(x_0,t)$ for $0 \leq t \leq T$ is included in $[u_{-k}, u_{k-1}]$.

4.2. A formula for the solution of the Riemann problem

The following theorem is mentioned in [O], but our proof is different from that given there. We use the notion of convex hull of a function f between u_L and u_r as used in section 3, denoted by $CH_{u_L}^{u_r} f(u)$ and illustrated in Fig. 3.1.

Theorem 2 The solution u to the Riemann problem of $u_t + f(u)_x = 0$ with initial constant states $u_L < u_r$ is given by

$$u = u\left(\frac{x}{t}\right) = \left\{ \tilde{u} \in \{[u_L,u_r] \cap \{u : f(u) = CH_{u_L}^{u_r} f(u)\}\} \text{ such that } CH_{u_L}^{u_r} f(\tilde{u}) - \frac{x}{t}\tilde{u} \text{ is the minimum} \right\} . \tag{4.7}$$

Proof: Minimizing $CH_{u_L}^{u_r} f(u) - \frac{x}{t} u$ means that the slope of $CHf(u) - \frac{x}{t} u$ is horizontal at the minimum value of $\tilde{u}$. Hence

$$\frac{d}{du}\left(CHf(u) - \frac{x}{t} u\right)\Big|_{u=\tilde{u}} = 0 ,$$

which implies $f'(\tilde{u}) = \frac{x}{t}$.

This is the definition of a characteristic, i.e. the value of u along a ray through the origin with slope $\frac{x}{t}$ is $\tilde{u}$. Notice that in case that there are two such minima in (4.7), then they are the left and right states bounding a contact discontinuity.

5. Some examples

Consider a special case $\mathcal{B}$ of flux functions as follows: let $f \in C^2$ be such that for all $u_L \in [-M,-1]$ and u_r $[1,M]$, $M > 1$, we have that $\min_{u \in u_L, u_r} \{CH_{u_L}^{u_r}(u) - su , -\varepsilon < s < \varepsilon\}$, remains unchanged. For an example of $f \in \mathcal{B}$ see Fig. 5.1.

As before consider

$$u_t + f(u)_x = 0, \quad \text{fix } f \in \mathcal{B} \tag{5.1}$$

with initial data piecewise constant with jumps as in (4.6), and

$$\begin{aligned} &u_i \in (-M,-1), \quad i \in \{-k, ..., -1\} \\ &u_i \in (1,M), \quad i \in \{0, ..., k\} . \end{aligned} \tag{5.2}$$

Then by theorem 1 one obtains

$$\frac{1}{T}\int_0^T f(u(x_0,t))\, dt = h^G(u_r,u_L)$$

for some $u_L \in [-M,-1]$
$u_r \in [1,M]$.

By theorem 2 we find that $h^G(u_r,u_L) = f(u_{min})$ where $u_{min} \in [-1,1]$ and u_{min} is the absolute minimum of f in $[-M,M]$. Note that by the choice of f, u_{min} is always the same, regardless of u_i in (5.2).

Now we make use of the assumption that not only CHf(u) always attains a minimum in (–1,1), but also CHf(u) – su, $-\varepsilon < s < \varepsilon$. By the change of variable y = x–st equation (5.1) becomes

$$u_t + (f(u) - su)_y = 0 . \tag{5.4}$$

Consider (5.4) with initial condition (5.2) as before. We may again conclude that

$$\frac{1}{T}\int_0^T \{f(u(st,t)) - su\}\, dt = f(u_{min}) - s\, u_{min} ,$$

where u_{min} is the absolute minimum in [–M,M] of f-su.

Thus for (5.1) and initial data (5.2) we conclude that a fan wave dominates the solution for all time, see Fig. 5.1.

We may extend this result to an initial value problem to (5.1) with piecewise continuous initial data:

$$u(x,0) = \begin{cases} u_1(x) \text{ such that } -M < f_1 < 1, & x \in [x_{-k}, x_0] \\ u_2(x) \text{ such that } 1 < f_2 < M, & x \in [x_0, x_k] \\ f_1(x_{-k}) & x < x_{-k} \\ f_s(x_k) & x > x_k \end{cases} \tag{5.5}$$

with u_1 and u_2 continuous functions, for an example see Fig. 5.2.

Now approximate the initial data u_0 in (5.5) by piecewise constant data on intervals of length Δx. For this data the above conclusion holds. Since the flux function is in class $\mathcal{B}$, regardless of the size of Δx in the approximation of u_0 we obtain the same value of u along a fixed ray $y = x_0$-st, $-\varepsilon < s < \varepsilon$. Thus when passing to the limit $\Delta x \to 0$, we find that the solution u to (5.1), (5.5) is also dominated by a fan wave for all time which depends only on the choice of f.

Finally we give a two dimensional example. Consider

$$u_t + f(u)_x + f(u)_y = 0 \qquad f \in \mathcal{B} \tag{5.6}$$

with initial condition

$$u(x,y,0) = \begin{cases} \varepsilon, & y > -x \\ -\varepsilon, & y < -x \end{cases} \tag{5.7}$$

This is a one dimensional Riemann problem with a fan as a solution. By the above examples we may now give a special perturbation of the constant states (5.7) s.th. the fan wave remains unchanged, but the rest of the solution changes. Let u(x,y,0) be constant on rays through the origin, i.e.

$$u(x,y,0) = u_0(\theta),\ \theta \in [0,2\pi]$$

$$u_0(\theta) = \begin{cases} f_1(\theta) \text{ such that } 1 < f(\theta) < M, & \theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right] \\ f_2(\theta) \text{ such that } -M < f(\theta) < -1, & \theta \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right] \end{cases}$$

with f_1 and f_2 continuous functions.

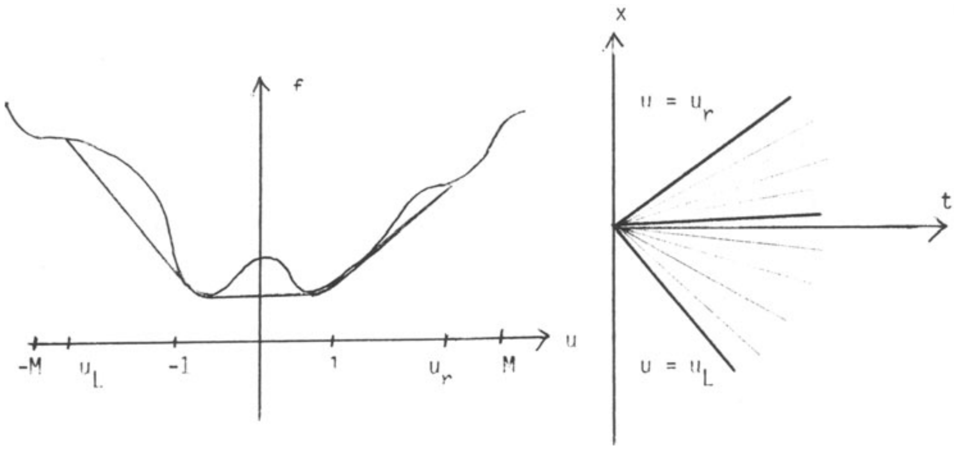

Fig. 5.1 An example of a flux function in class $\mathcal{B}$ together with a convex hull on the left. On the right the fan solution of the corresponding Riemann problem.

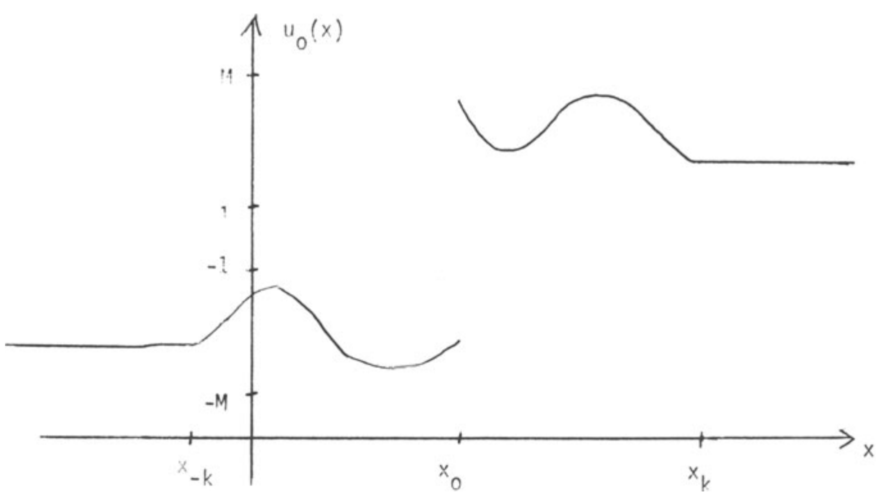

Fig. 5.2 Initial data for which the solution to

$$u_t + f(u)_x = 0\,, \qquad f \in \mathcal{B}$$

will contain the fan drawn in Fig. 5.1.

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