

## Compensated compactness for a hyperbolic system of balance laws with three equations

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We consider the problem studied by Lu *et. al.* in [Proc. Am. Math. Soc. **131**, 3511 (2003)] to which we add a source term. This is a system of three balance laws. We use standard techniques, such as viscous approximations and compensated compactness, to prove the existence of solutions.

Keywords: compensated compactness, conservations laws.

Se considera el problema estudiado por Lu *et. al.* en [Proc. Am. Math. Soc. **131**, 3511 (2003)] al cual añadimos un término fuente. Este es un sistema de tres leyes de balance. Se usan técnicas standrad, tal como aproximaciones viscosas y compacidad compensada, para demostrar la existencia de soluciones.

Palabras claves: compacidad compensada, leyes de conservación.

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## 1 The system of balance laws

In this paper we study the following system of balance laws

$$\begin{aligned} v_t - u_x &= \epsilon v_{xx} + g(v, u, s), \\ u_t - \sigma(v, s)_x &= \epsilon u_{xx} + f(v, u), \\ s_t - c_1 s_x + \frac{s - cv}{\tau} &= \epsilon s_{xx}, \end{aligned} \tag{1.1}$$

with initial condition

$$(v(x, 0), u(x, 0), s(x, 0)) = (v_0(x), u_0(x), s_0(x)). \tag{1.2}$$

The Cauchy problem (1.1)–(1.2) was considered by Lu and Klingenberg in [1] without the source terms  $g(v, u, s)$  and  $f(v, u)$ .

The third equation in (1.1) contains a relaxation mechanism with  $cv$  as the equilibrium value for  $s$ . Here,  $\tau$  is the relaxation time and  $\epsilon$  is the viscous parameter. The relaxation and dissipation limit of  $(v, u, s)$  in (1.1) satisfies  $s = cv$ , where the pair  $(v, u)$  is an entropy solution of the equilibrium system

$$\begin{aligned} v_t - u_x &= g(v, u, cv), \\ u_t - \sigma(v, cv)_x &= f(v, u), \end{aligned} \tag{1.3}$$

when  $\epsilon = 0$  and  $\tau \rightarrow 0$ .

## 2 Viscous solution and convergence

In this section we obtain the existence of solutions. We have the following result concerning viscous solutions and their viscous limits.

**Theorem.** *Part I. If*

$C_1$ .

$$\begin{aligned} \|v_0\|_{L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})}, \|u_0\|_{L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})}, \|s_0\|_{L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})} &\leq M_1, \\ \lim_{|x| \rightarrow \infty} \left( \frac{d^i v_0}{dx^i}, \frac{d^i u_0}{dx^i}, \frac{d^i s_0}{dx^i} \right) &= (0, 0, 0). \end{aligned}$$

for  $i = 0, 1$ ;

$C_2$ .

$$|\sigma_s(v, s)| \leq M_2,$$

$$\bar{\sigma}' \geq d > \max \left( 0, c^2 - c + \frac{2c^2 c_1^2}{(M_2 + 1)^2} \right),$$

where  $\bar{\sigma}(v) = \sigma(v, cv)$ .

$C_3$ .

$$\int_{-\infty}^{\infty} \int_0^{v_0(x)} \bar{\sigma}(r) dr dx \leq M_3.$$

$C_4$ .

$$g(v, u, s) = -(\bar{\sigma}'(v))^{-1/2} \frac{(\bar{\sigma}(v) + cv - cs)}{1 + (\bar{\sigma}(v) + cv - cs)^2} \tilde{g}(v, u, s),$$

and  $f(v, u) = -\frac{u}{1+u^2} \tilde{f}(v, u)$ , with  $\tilde{g}(v, u, s) \geq 0$ ,  $\tilde{f}(v, u) \geq 0 \in C^1(\mathbb{R}^3)$ , and these functions are  $o(|(v, u, s)|^2)$  when  $|(v, u, s)| \rightarrow \infty$ ,  $0 \leq \tilde{g}(v, u, s), 0 \leq \tilde{f}(v, u)$  and their partial derivatives go to zero at infinity.

Then, for fixed  $\varepsilon$  and  $\tau$ , the solutions  $(v, u, s)$  of the Cauchy problem (1.1)–(1.2) belong to  $C^2(\mathbb{R} \times [0, T])$ ; that is, they exist in  $(-\infty, \infty) \times [0, T]$  for any given  $T > 0$  and satisfy

$$|v(x, t)|, |u(x, t)|, |s(x, t)| \leq M(\varepsilon, \tau, T), \quad (2.1)$$

$$\|v^2(\cdot, t)\|_{L^1(\mathbb{R})}, \|u^2(\cdot, t)\|_{L^1(\mathbb{R})},$$

$$\|s^2(\cdot, t)\|_{L^1(\mathbb{R})} \leq M, \quad (2.2)$$

$$\left\| \frac{(s - cv)^2}{\tau^{1/2}} \right\|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq M, \quad (2.3)$$

$$\|(\varepsilon)^{1/2} v_x\|_{L^2(\mathbb{R} \times \mathbb{R}^+)}, \|(\varepsilon)^{1/2} u_x\|_{L^2(\mathbb{R} \times \mathbb{R}^+)},$$

$$\|(\varepsilon)^{1/2} s_x\|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq M. \quad (2.4)$$

*Part II. If*

- $C_5$ .
- $\sigma(v, s) = h(v) - cs$ ;
  - $h(v) \in C^3(\mathbb{R})$ ,  $h(0) = 0$ ,  $h' \geq d > 0$ , where  $d$  is a constant;
  - $h'' \neq 0$ ,  $h'' \in L^1 \cap L^\infty(\mathbb{R})$ ;
  - $h''' \in L^\infty(\mathbb{R})$ ,  $\|h'''\|_{L^1} \leq M$ .

Then, there exist a subsequence  $(v^{\varepsilon,\tau}(x, t), u^{\varepsilon,\tau}(x, t), s^{\varepsilon,\tau}(x, t))$  of solutions of the Cauchy problem (1.1)–(1.2) and there exist  $L^2$ -bounded functions  $(\bar{v}, \bar{u}, \bar{s})$  such that

$$(v^{\varepsilon,\tau}(x, t), u^{\varepsilon,\tau}(x, t), s^{\varepsilon,\tau}(x, t)) \rightarrow (\bar{v}, \bar{u}, \bar{s}),$$

a.e.  $(x, t)$ , as  $(\varepsilon, \tau) \rightarrow (0, 0)$ , subject to the condition  $\tau(M_2 + 1)^2 \leq \varepsilon$  and  $(\bar{v}, \bar{u})$  is an entropy solution of the equilibrium system (1.3) with the initial data  $(u_0, v_0)$ .

*Proof.* A unique smooth local solution for the Cauchy problem (1.1)–(1.2), for any fixed  $\varepsilon$  and  $\tau > 0$ , is obtained with the help of an equivalent integral operator and the Banach fixed point theorem; see [7] and [12]. Furthermore, the solution satisfies

$$\left| \frac{\partial^i v}{\partial x^i} \right| + \left| \frac{\partial^i u}{\partial x^i} \right| + \left| \frac{\partial^i s}{\partial x^i} \right| \leq M(t_1, \varepsilon, \tau) < +\infty, \quad (2.5)$$

with  $i = 0, 1, 2$ , where  $M(t_1, \varepsilon, \tau)$  is a positive constant that depends only on  $t_1$ ,  $\varepsilon$  and  $\tau$ , and  $t_1$  depends on  $|v_0|_{L^\infty}$ ,  $|u_0|_{L^\infty}$  and  $|s_0|_{L^\infty}$ . Moreover

$$\lim_{|x| \rightarrow \infty} \left( \frac{\partial^i v}{\partial x^i}, \frac{\partial^i u}{\partial x^i}, \frac{\partial^i s}{\partial x^i} \right) = (0, 0, 0), \quad (2.6)$$

for  $i = 0, 1$ , uniformly in  $t \in [0, t_1]$ .

To obtain the estimates in (2.1) we multiply the first equation in (1.1) by  $\bar{\sigma}(v) + cv - cs$ , the second equations by  $u$ , the third equation by  $s - cv$  and add the results; this is the same procedure used by Lu and Klingenberg in [1]. Then we get

$$\begin{aligned}
 & \left( \int_0^v (\bar{\sigma}(r) + cr) dr + \frac{u^2}{2} - csv + \frac{s^2}{2} \right)_t \\
 & + \left( cus - u(\bar{\sigma}(v) + cv) + \frac{c_1 s^2}{2} \right)_x \\
 & - c c_1 v s_x - u(\sigma(v, s) + s - (\sigma(v, cv) + cv))_x + \frac{(s - cv)^2}{\tau} \\
 = & \varepsilon \left( \int_0^v (\bar{\sigma}(r) + cr) dr + \frac{u^2}{2} - csv + \frac{s^2}{2} \right)_{xx} \\
 & - \varepsilon (\bar{\sigma}'(v) + c) v_x^2 - \varepsilon u_x^2 - \varepsilon s_x^2 + 2\varepsilon c s_x v_x \\
 & + \bar{\sigma}(v) g(v, u) + cv g(v, u) - cs g(v, u) + u f(v, u). \tag{2.7}
 \end{aligned}$$

By condition  $C_4$ ,  $\bar{\sigma}(v) g(v, u) + cv g(v, u) - cs g(v, u) + u f(v, u) \leq 0$ , and therefore the inequality (2.7) becomes

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \frac{1}{2} (cv - s)^2 dx + \int_{-\infty}^{\infty} \frac{u^2}{2} dx + \int_0^T \int_{-\infty}^{\infty} \frac{\varepsilon}{2} (u_x)^2 dx dt \\
 & + \int_0^T \int_{-\infty}^{\infty} \varepsilon (cv_x - s_x)^2 dx dt + \int_0^T \int_{-\infty}^{\infty} \frac{(s - cv)^2}{4\tau} dx dt \leq M. \tag{2.8}
 \end{aligned}$$

This proves the estimates (2.2), (2.3) and (2.4). Again, as in [1], differentiating the first equation in (1.1) with respect to  $x$ , multiplying by  $v_x$  and integrating the result on  $\mathbb{R} \times [0, T]$ , we obtain

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left( \frac{v_x(x, T)}{2} \right)^2 dx dt \\
 \leq & \int_{-\infty}^{\infty} \left( \frac{v_x(x, 0)}{2} \right)^2 dx dt + \int_0^T \int_{-\infty}^{\infty} \frac{(u_x)^2}{\varepsilon} dx dt \\
 & + \int_0^T \int_{-\infty}^{\infty} \left( \frac{\partial g(v, u, s)}{\partial v} (v_x)^2 + \frac{\partial g(v, u, s)}{\partial u} u_x v_x \right. \\
 & \quad \left. + \frac{\partial g(v, u, s)}{\partial s} s_x v_x \right) dx dt \\
 \leq & M(\varepsilon),
 \end{aligned}$$

since  $\frac{\partial g(v,u,s)}{\partial v}$ ,  $\frac{\partial g(v,u,s)}{\partial u}$ ,  $\frac{\partial g(v,u,s)}{\partial s} \in C(\mathbb{R}^3)$  and they go to zero at infinity. Then, by (2.4) and the Hölder inequality, we get the above bound. Therefore,

$$\begin{aligned} v^2 &= \left| \int_{-\infty}^x 2v v_x dx \right| \leq \int_{-\infty}^x 2|v| |v_x| dx \\ &\leq \int_{-\infty}^{\infty} v^2 dx + \int_{-\infty}^{\infty} (v_x)^2 dx \\ &\leq M(\epsilon). \end{aligned}$$

Similarly for  $u$  and  $s$ . We get the estimates in (2.1) and the proof of part 1 in Theorem 2 is complete.

If  $(\eta, q)$  is the entropy–entropy flux pair associated with (1.3) constructed in [13], then this pair satisfies the following estimates:

- (I)  $\eta = a^{-1/2}O(1)$ ,  $q = a^{1/2}O(1)$ ;
- (II)  $\eta_u = a^{-1/2}O(1)$ ,  $\eta_v = a^{1/2}O(1)$ ;
- (III)  $\eta_{uu} = a^{-1/2}O(1)$ ,  $\eta_{uv} = a^{1/2}O(1)$ ,  $\eta_{vv} = a^{3/2}O(1)$ ;

where  $a = \bar{\sigma}'(v)$  and  $O(1)$  denotes a  $L^\infty$  function. From (1.1) we get

$$\begin{aligned} \eta_t + q_x &= \varepsilon ((\eta_v v_x + \eta_u u_x)_x + (\eta_u \sigma_s(v, \alpha(v, s)) (s - cv))_x \\ &\quad - \varepsilon (\eta_{vv} (v_x)^2 + 2\eta_{uv} u_x v_x + \eta_{uu} (u_x)^2) \\ &\quad - (\eta_{uv} v_x + \eta_{uu} u_x) (\sigma_s(v, \alpha(v, s)) (s - cv) \\ &\quad + \eta_v g(v, u, s) + \eta_u f(v, u)) \\ &= I_1 + I_2, \end{aligned} \tag{2.9}$$

with  $\alpha(v, s)$  between  $s$  and  $cv$ , and

$$\begin{aligned} I_1 &= (\eta_v v_x + \eta_u u_x)_x + (\eta_u \sigma_s(v, \alpha(v, s)) (s - cv))_x, \\ I_2 &= -(\eta_{uv} v_x + \eta_{uu} u_x) (\sigma_s(v, \alpha(v, s)) (s - cv) \\ &\quad - \varepsilon (\eta_{vv} (v_x)^2 + 2\eta_{uv} u_x v_x + \eta_{uu} (u_x)^2) \\ &\quad + \eta_v g(v, u, s) + \eta_u f(v, u). \end{aligned}$$

From the estimates (2.2), (2.3), (2.4) and the previous equality, we get:

1.  $\eta(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) + q(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})$  is bounded in  $W_{Loc}^{-1,p}(\mathbb{R} \times \mathbb{R}^+)$  for some  $p > 2$  by using estimation (I).
2. Since  $|\sigma_s(v, s)| \leq M$  due to (II) and (2.3), (2.4), then  $(\eta_v v_x + \eta_u u_x)_x + (\eta_u \sigma_s(v, \alpha(v, s))(s - cv))_x$  is compact in  $H_{Loc}^{-1}(\mathbb{R} \times \mathbb{R}^+)$ .
3. From (II), (III), (2.3), (2.4), we have that

$$\begin{aligned} & (\eta_{uv} v_x + \eta_{uu} u_x) \\ & \times (\sigma_s(v, \alpha(v, s))(s - cv) + \varepsilon (\eta_{vv} (v_x)^2 + 2\eta_{uv} u_x v_x + \eta_{uu} (u_x)^2)) \end{aligned}$$

is bounded in  $L_{Loc}^1(\mathbb{R} \times \mathbb{R}^+)$ .

4. From condition  $(C_4)$  of the Theorem 2 we get  $|g(v, u, s)| \leq \frac{M}{(\bar{\sigma}(v))^{1/2}}$  and  $|f(v, u)| \leq M$ . Then we obtain that  $\eta_v g(v, u, s) + \eta_u f(v, u)$  is in  $L_{Loc}^1(\mathbb{R} \times \mathbb{R}^+)$ .

By Murrat’s lemma, see [4],  $\eta(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) + q(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) \in H_{Loc}^{-1}(\mathbb{R} \times \mathbb{R}^+)$ . We can associate a Young measure family to the solutions  $(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})$  found in the *Part I* of this theorem (see [10]). Sharer in [13] proved that the support of this family reduces to one point. Thus, the convergence of  $(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})$  can be obtained. From the estimate in (2.3) we obtain the convergence  $s^{\varepsilon,\tau} \rightarrow cv$ . This finishes the proof.

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