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EXISTENCE OF ENTROPY SOLUTIONS TO SYSTEM OF POLYTROPIC GAS WITH A CLASS OF UNBOUNDED SOURCES

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Abstract. We study the global existence of entropy solutions to a gas dynamics system with a class of unbounded sources. By using the maximum principle, we obtain the uniform L^∞-estimates for the viscosity solutions. The key ingredient here is to introduce two suitable bounded functions to control the unbounded source terms. Then, with the aid of the compensated compactness theory, we prove the convergence of viscosity solutions and existence of global entropy solutions for any adiabatic exponent $\alpha > 1$.
Keywords. Damping source; Friction; Flux approximation; Global weak solution; Gas dynamics.

1. Introduction

The paper is devoted to the study of global weak solutions of the Cauchy problem of a polytropic gas dynamics system with a class of unbounded sources. It is formulated as following

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + \Pi(t,x,\rho,u) = 0, \end{cases} \tag{1.1}$$

with bounded initial data

$$(\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

where $\rho = \rho(t,x)$, $p(\rho) = \frac{1}{\alpha}\rho^\alpha$ ($\alpha > 1$), and $u = u(t,x)$ are the density, pressure and velocity, respectively, of the gas. The nonlinear term $\Pi = \Pi(t,x,\rho,u)$ denotes the source.

System (1.1) is an important model in physics and it can be derived from different physical backgrounds (see, for instance, [1]). During the past decades, there have been many impressive mathematical results on this model. Among them, we only mention those which are related to the main theorem in this paper. When $\Pi(t,x,\rho,u)$ is of the form $\Pi(x,t,\rho,u) = a(t,x)\rho u$ with $a(t,x) \geq 0$, there have been many literatures concerning the damping effects on the global existence and singularity formation of (1.1); see, e.g., [2, 3, 4, 5]. If $\Pi(t,x,\rho,u)$ is of the form $\Pi(t,x,\rho,u) = -\rho E(t,x) + a(x)\rho u$, then (1.1) is corresponding to the hydrodynamic model of

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semiconductors (see [6, 7, 8, 9] and references therein). For more discussions on the inhomogeneous hyperbolic systems, we refer the interested readers to [10, 11, 12, 13, 14] and the references therein.

In the present paper, we restrict our attention on the case that the source Π is unbounded and is of the following form:

$$\Pi(t,x,\rho,u)=a(t,x)|\rho u|, \tag{1.3}$$

where $|a(t,x)|\leq M+P(t)+Q(x)$ with $0\leq P(t)\in C(\mathbf{R}^+)\cap L^1(\mathbf{R}^+)$, $0\leq Q(x)\in L^1(\mathbf{R})\cap L^\infty(\mathbf{R})$, and $M\geq 0$ is a constant.

It is well-known that classical solutions to the Cauchy problem of the nonlinear hyperbolic system (1.1) does not exist globally in time even if the initial data (1.2) are smooth and small. In fact, shock waves always appear in the solutions after a suitable large time. This means that the solutions to (1.1)–(1.2) are discontinuous and do not satisfy (1.1) in the classical sense. Thus we have to study the weak solutions to (1.1)–(1.2), that is, solutions satisfy (1.1)–(1.2) in the sense of distributions.

In order to construct the weak solutions to Cauchy problem (1.1)–(1.2), it is standard that one first construct approximation solutions $(\rho^\varepsilon(t,x),u^\varepsilon(t,x))$ to the following parabolic system:

$$\begin{cases} \rho_t+(\rho u)_x=\varepsilon\rho_{xx}, \\ (\rho u)_t+(\rho u^2+p(\rho))_x+\Pi(t,x,\rho,u)=\varepsilon(\rho u)_{xx}. \end{cases} \tag{1.4}$$

Then, one can obtain the weak solutions from $(\rho^\varepsilon(t,x),u^\varepsilon(t,x))$ by the passage to the limit as ε goes to zero.

To solve (1.4), one may regard the quantity $m=\rho u$ as an independent variable, which leads to a basic technical difficulty that the positive lower bound estimate for ρ^ε is not available since $\rho u^2=\frac{m^2}{\rho}$ is singular when $\rho=0$. Thus, we consider in this paper the flux-viscosity approximation solutions $(\rho^{\varepsilon,\eta,\nu},u^{\varepsilon,\eta,\nu})$ to the following parabolic system (see [15, 16]):

$$\begin{cases} \rho_t+((\rho-2\eta)u)_x=\varepsilon\rho_{xx}, \\ (\rho u)_t+(\rho u^2-\eta u^2+p_1(\rho,\eta))_x+a_\nu(t,x)|\rho u|=\varepsilon(\rho u)_{xx}, \end{cases} \tag{1.5}$$

with initial data

$$(\rho^{\varepsilon,\eta,\nu}(x,0),u^{\varepsilon,\eta,\nu}(x,0))=(\rho_0(x)+2\eta,u_0(x)), \tag{1.6}$$

where $\varepsilon>0$ and $\eta>0$ denote the viscosity coefficient and the regular perturbation constant, respectively, the perturbation function $p_1=p_1(\rho,\eta)$ is of the form:

$$p_1(\rho,\eta)=\int_{2\eta}^{\rho}\frac{t-2\eta}{t}p'(t)dt,$$

and

$$a_\nu(t,x)=\int_{-\infty}^{\infty}a(x',t)j_\nu(x-x')dx',$$

satisfies

$$|a_\nu(t,x)|\leq M+P(t)+\int_{-\infty}^{\infty}|Q(x')|j_\nu(x-x')dx',$$

where j_ν is a suitable mollifier.

It is worth to mentioning that an advantage of this kind of approximations is that one may prove directly the uniformly, positive lower bound:

$$\rho^{\varepsilon,\eta,\nu} \geq 2\eta > 0, \tag{1.7}$$

with the aid of the maximum principle on the density equation in (1.5), which guarantees the existence of the approximation solutions $(\rho^{\varepsilon,\eta,\nu}, u^{\varepsilon,\eta,\nu})$.

More precisely, we have

Theorem 1.1. I. *Assume that $a(t,x)$ is measurable and (1.3) holds with $0 \leq P(t) \in C(\mathbf{R}^+) \cap L^1(\mathbf{R}^+), 0 \leq Q(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, and*

$$z(\rho_0, u_0) \leq e^{k_1} - |Q|_{L^1}, \quad w(\rho_0, u_0) \leq e^{k_1}, \tag{1.8}$$

where

$$z(\rho,u) = -u + \int_c^\rho \frac{\sqrt{p'(s)}}{s} ds, \quad w(\rho,u) = u + \int_c^\rho \frac{\sqrt{p'(s)}}{s} ds \tag{1.9}$$

are the Riemann invariants of (1.1), and c and $k_1 > 0$ are two constants. Then, for any fixed (ε,η,ν), (1.5)-(1.6) admits a global solution $(\rho^{\varepsilon,\eta,\nu}, u^{\varepsilon,\eta,\nu})$ satisfying

$$\begin{cases} z(\rho^{\varepsilon,\eta,\nu}, u^{\varepsilon,\eta,\nu}) \leq e^{k_1 + k_2 \int_0^t M + P(t')dt'}, \\ w(\rho^{\varepsilon,\eta,\nu}, u^{\varepsilon,\eta,\nu}) \leq e^{k_1 + k_2 \int_0^t M + P(t')dt'} + k_3 \int_{-\infty}^{+\infty} Q(t')dt', \end{cases} \tag{1.10}$$

where k_2, k_3 are two suitable positive constants and

$$Q_\nu(x) = \int_{-\infty}^{\infty} Q(x') j_\nu(x - x') dx'. \tag{1.11}$$

II. *There exists a subsequence of $(\rho^{\varepsilon,\eta,\nu}, u^{\varepsilon,\eta,\nu})$, which converges pointwisely to a pair of bounded functions $(\rho(t,x), u(t,x))$ as $(\varepsilon,\eta,\nu) \to (0,0,0)$, and the limit is a weak entropy solution to (1.1)–(1.2)*

Definition 1.1. $(\rho(t,x), u(t,x))$ is called a weak entropy solution to (1.1)–(1.2) if

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho \varphi_t + (\rho u)\varphi_x \varphi dxdt + \int_{-\infty}^\infty \rho_0(x)\varphi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u \varphi_t + (\rho u^2 + P(\rho))\varphi_x - \Pi(x,t,\rho,u)\varphi dxdt \\ + \int_{-\infty}^\infty \rho_0(x) u_0(x) \varphi(x,0)dx = 0 \end{cases} \tag{1.12}$$

holds for all test function $\varphi \in C_0^1(\mathbf{R} \times \mathbf{R}^+)$ and

$$\int_0^\infty \int_{-\infty}^\infty h(\rho,m)\varphi_t + \psi(\rho,m)\varphi_x - \Pi(x,t,\rho,u) h(\rho,m)_m \varphi dxdt \geq 0 \tag{1.13}$$

holds for all non-negative test function $\varphi \in C_0^\infty((\mathbf{R}^+ \setminus \{t=0\}) \times \mathbf{R})$, where $m = \rho u$ and (h,ψ) is a pair of convex entropy-entropy flux of system (1.1).

Example 1.1. When $\Pi(t,x,\rho,u) = a(x)\rho u$, where $a(x) \leq 0$ is corresponding to the sliding friction [17], in general, we could not obtain the uniform L^∞ estimate without the condition $a(x) \in L^1(\mathbf{R})$ (see [18]).

Remark 1.1. If the nonlinear function $\Pi(t,x,\rho,u)$ is of the C^1 class with respect to the variables, then, without any difficulty, we may prove that Theorem 1.1 is also true for any $\Pi(t,x,\rho,u)$ satisfying

$$|\Pi(t,x,\rho,u)| \leq |a(t,x)\rho u|, \quad |a(t,x)| \leq M+P(t)+Q(x), \tag{1.14}$$

where M is a nonnegative constant, $0 \leq P(t) \in C(\mathbf{R}^+) \cap L^1(\mathbf{R}^+), 0 \leq Q(x) \in C(\mathbf{R}) \cap L^1(\mathbf{R})$.

Remark 1.2. When conditions (1.3) or (1.14) are changed to

$$\Pi(t,x,\rho,u) = a(t,x)|\rho u|, \quad |a(t,x)| \leq P(t)+Q(x),$$

although the function $a(t,x)$ could be unbounded, we may deduce a uniformly bounded estimate of solutions with respect to the time.

2. Proof of Theorem 1.1

Recall that $m\rho u$ is the momentum and (w,z) is given by (1.9). Multiplying (1.5) by $(\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial m})$ and $(\frac{\partial z}{\partial \rho}, \frac{\partial z}{\partial m})$, respectively, we obtain that

$$z_t + \lambda_1^\eta z_x = \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x - \tfrac{\varepsilon}{2\rho^2\sqrt{p'(\rho)}}(2p'+\rho p'')\rho_x^2 - f_\nu(t,x)u \tag{2.1}$$

and

$$w_t + \lambda_2^\eta w_x = \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{p'(\rho)}}(2p'+\rho p'')\rho_x^2 + f_\nu(t,x)u, \tag{2.2}$$

where $f_\nu(t,x) = -a_\nu(t,x)sgn(u)$ and

$$\lambda_1^\eta = \frac{m}{\rho} - \frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)}, \quad \lambda_2^\eta = \frac{m}{\rho} + \frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)}$$

are two eigenvalues of (1.5).

We let $z = \Phi(t,x)+v$, for a suitable function $\Phi(t,x)$ in (2.1), and obtain

$$\begin{aligned}
&v_t + \Phi_t + (u - \frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)})v_x - \Phi_x[\Phi(t,x)+v-\int_c^\rho \frac{\sqrt{p'(\rho)}}{\rho}d\rho] - \Phi_x\frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)} \\
&= \varepsilon v_{xx} - \frac{\varepsilon}{2\rho^2\sqrt{p'(\rho)}}(2p'+\rho p'')[\rho_x^2 - \frac{4\rho\sqrt{p'(\rho)}}{2p'+\rho p''}\rho_x\Phi_x + (\frac{2\rho\sqrt{p'(\rho)}}{2p'+\rho p''}\Phi_x)^2] \\
&\quad + \varepsilon\Phi_{xx} + \frac{2\varepsilon}{\rho}\rho_x v_x + \frac{2\varepsilon\sqrt{p'(\rho)}}{2p'+\rho p''}\Phi_x^2 - f_\nu(t,x)u
\end{aligned}$$

or

$$\begin{aligned}
&v_t + \Phi_t + v_x b(t,x) + vc(t,x) + [-\tfrac{2\varepsilon\sqrt{p'(\rho)}}{2p'+\rho p''}\Phi_x^2 - \varepsilon\Phi_{xx} - \varepsilon_1\Phi(t,x)\Phi_x] \\
&+(\textstyle\int_c^\rho \tfrac{\sqrt{p'(\rho)}}{\rho}d\rho - \tfrac{\rho-2\eta}{\rho}\sqrt{p'(\rho)})\Phi_x - (1-\varepsilon_1)\Phi(t,x)\Phi_x + f_\nu(t,x)u \leq \varepsilon v_{xx},
\end{aligned} \tag{2.3}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $b(t,x) = u - \frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$, and $c(t,x) = -\Phi_x$.

Similarly, we let $w = s + \Psi(t,x)$ in (2.2) and have

$$\begin{aligned}
&s_t + \Psi_t + d(t,x)s_x + e(t,x)s + [-\tfrac{2\varepsilon\sqrt{p'(\rho)}}{2p'+\rho p''}\Psi_x^2 - \varepsilon\Psi_{xx} + \varepsilon_1\Psi(t,x)\Psi_x] \\
&+\Psi_x(\tfrac{\rho-2\eta}{\rho}\sqrt{p'(\rho)} - \textstyle\int_c^\rho \tfrac{\sqrt{p'(\rho)}}{\rho}d\rho) + (1-\varepsilon_1)\Psi(t,x)\Psi_x - f_\nu(t,x)u \leq \varepsilon s_{xx},
\end{aligned} \tag{2.4}$$

where $d(t,x)=\frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)}-\frac{2\varepsilon\rho_x}{\rho}+u$ and $e(t,x)=\Psi_x$. In view of the first equation in (1.5), we have the a priori estimate $\rho\geq 2\eta$. Let

$$\Phi(t,x)=e^{k_1+k_2\int_0^t M+P(t')dt'}-k_3\int_{-\infty}^{x}Q_\nu(t')dt',$$

and

$$\Psi(t,x)=e^{k_1+k_2\int_0^t M+P(t')dt'}+k_3\int_{-\infty}^{x}Q_\nu(t')dt',$$

where k_i $(i=1,2,3)$ are suitable positive constants, and $Q_\nu(x)$ is given by (1.11). Since $|Q_\nu|_{L^\infty}$ and $\nu|X'_\nu|_{L^\infty}$ are uniformly bounded, $|Q_\nu|_{L^1}=|Q|_{L^1}$ and

$$\frac{2\varepsilon\sqrt{p'(\rho)}}{2p'+\rho p''}=\frac{2\varepsilon}{\alpha+1}\rho^{-\frac{\alpha-1}{2}}\leq\frac{2\varepsilon}{\alpha+1}(2\eta)^{-\frac{\alpha-1}{2}},\ \Phi_x=-k_3Q_\nu(x),\ \Phi_{xx}=-k_3Q'_\nu(x),$$

we can set $\varepsilon=o(\eta)$ and choose a suitable relation among $\varepsilon,\varepsilon_1$ and ν so that the following two inequalities (i.e., (2.3) and (2.4)) hold:

$$-\frac{2\varepsilon\Phi_x^2\sqrt{p'(\rho)}}{2p'+\rho p''}-\varepsilon\Phi_{xx}-\varepsilon_1\Phi(t,x)\Phi_x>0$$

and

$$-\frac{2\varepsilon\Psi_x^2\sqrt{p'(\rho)}}{2p'+\rho p''}-\varepsilon\Psi_{xx}+\varepsilon_1\Psi(t,x)\Psi_x>0.$$

Furthermore, in view of (2.3) and (2.4), we obtain

Lemma 2.1. *It holds*

$$\begin{cases} v_t+v_xb(t,x)+vb_1(t,x)+sb_2(t,x)\leq\varepsilon v_{xx},\\ s_t+s_xd(t,x)+sd_1(t,x)+vd_2(t,x)\leq\varepsilon s_{xx},\end{cases}\tag{2.5}$$

where

$$\begin{cases} b_1(t,x)=c(t,x)-f_v-\frac{1}{2}(M+P(t)+Q_\nu(x)),\\ b_2(t,x)=-\frac{1}{2}(M+P(t)+Q_\nu(x))\leq 0,\\ d_1(t,x)=e(t,x)+f_v-\frac{1}{2}(M+P(t)+Q_\nu(x)),\\ d_2(t,x)=-\frac{1}{2}(M+P(t)+Q_\nu(x))\leq 0\end{cases}$$

when $\alpha>3$, and

$$\begin{cases} b_1(t,x)=-f_v+c(t,x)-(\frac{1}{2}(M+P(t)+Q_\nu(x))+\frac{3-\alpha}{4}k_3Q_\nu(x)),\\ b_2(t,x)=-(\frac{1}{2}(M+P(t)+Q_\nu(x))+\frac{3-\alpha}{4}k_3Q_\nu(x))\leq 0,\\ d_1(t,x)=e(t,x)+f_v-(\frac{1}{2}(M+P(t)+Q_\nu(x))+\frac{3-\alpha}{4}k_3Q_\nu(x)),\\ d_2(t,x)=-(\frac{1}{2}(M+P(t)+Q_\nu(x))+\frac{3-\alpha}{4}k_3Q_\nu(x))\leq 0\end{cases}$$

when $1<\alpha\leq 3$.

Proof of Lemma 2.1. First, if $\alpha > 3$, we choose $c = 2\eta$ in (1.9), (2.3), and (2.4). Since

$$\int_{2\eta}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho = \int_{2\eta}^{\rho} \rho^{\frac{\alpha-3}{2}} d\rho \leq \rho^{\frac{\alpha-3}{2}} \int_{2\eta}^{\rho} 1 d\rho = \frac{\rho - 2\eta}{\rho} \sqrt{p'(\rho)},$$

the following terms in (2.3)

$$\begin{aligned} L_{1v} &= \Phi_t + (\textstyle\int_{2\eta}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho - \frac{\rho-2\eta}{\rho} \sqrt{p'(\rho)})\Phi_x - (1-\varepsilon_1)\Phi(t,x)\Phi_x + f_v(t,x)u \\ &\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_v(t,x)(\textstyle\int_{2\eta}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho - \Phi(t,x) - v) \\ &\quad + (1-\varepsilon_1)(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} - k_3^2 \textstyle\int_{-\infty}^{x} Q_v(t')dt')Q_v(x) \\ &\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} - f_v(t,x)v + (1-\varepsilon_1)k_3 Q_v(x) e^{k_1+k_2\int_0^t M+P(t')dt'} \\ &\quad - (1-\varepsilon_1)k_3^2 Q_v(x)\textstyle\int_{-\infty}^{x} Q_v(t')dt' - (M+P(t)+Q_v(x))(\textstyle\int_{2\eta}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho + \Phi(t,x)) \end{aligned} \tag{2.6}$$

due to $|f_v(t,x)| \leq M + P(t) + Q_v(x)$. Since

$$\int_{2\eta}^{\rho} \frac{\sqrt{p'(\rho)}}{\rho} d\rho = \frac{1}{2}(w+z) = \frac{1}{2}(v+s) + e^{k_1+k_2\int_0^t M+P(t')dt'},$$

we have from (2.6) that

$$\begin{aligned} L_{1v} &\geq -f_v(t,x)v - \tfrac{1}{2}(v+s)(M+P(t)+Q_v(x)) \\ &\quad + k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + (1-\varepsilon_1)k_3 Q_v(t,x) e^{k_1+k_2\int_0^t M+P(t')dt'} \\ &\quad - (1-\varepsilon_1)k_3^2 Q_v(x)\textstyle\int_{-\infty}^{x} Q_v(t')dt' - 2(M+P(t)+Q_v(x))e^{k_1+k_2\int_0^t M+P(t')dt'} \\ &\quad + k_3(M+P(t)+Q_v(x))\textstyle\int_{-\infty}^{x} Q_v(t')dt' \\ &\geq -\tfrac{1}{2}(M+P(t)+Q_v(x))(v+s) - f_v(t,x)v \\ &\quad + (k_2-2)(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + (1-\varepsilon_1)k_3(\tfrac{1}{2}e^{k_1} - k_3|Q_v(x)|_{L^1})Q_v(x) \\ &\quad + (\tfrac{1}{2}(1-\varepsilon_1)k_3 - 2)Q_v(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\ &\geq -f_v(t,x)v - \tfrac{1}{2}(v+s)(M+P(t)+Q_v(x)) \end{aligned} \tag{2.7}$$

if we choose $k_2 \geq 2, k_3 > 4$ and $e^{k_1} \geq 2k_3|Q_\nu(x)|_{L^1}$. Similarly, the following terms in (2.4)

$$\begin{aligned}
L_{1s} &= \Psi_t + \Psi_x\left(\tfrac{\rho-2\eta}{\rho}\sqrt{p'(\rho)} - \textstyle\int_{2\eta}^{\rho} \tfrac{\sqrt{p'(\rho)}}{\rho}d\rho\right) + (1-\varepsilon_1)\Psi(t,x)\Psi_x - f_\nu(t,x)u \\
&\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_\nu(t,x)\left(s+\Psi(t,x) - \textstyle\int_{2\eta}^{\rho}\tfrac{\sqrt{p'(\rho)}}{\rho}d\rho\right) \\
&+(1-\varepsilon_1)\left(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} + k_3^2\textstyle\int_{-\infty}^{x} Q_\nu(t')dt'\right)Q_\nu(x) \\
&\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_\nu(t,x)s + (1-\varepsilon_1)k_3Q_\nu(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&+(1-\varepsilon_1)k_3^2Q_\nu(x)\textstyle\int_{-\infty}^{x} Q_\nu(t')dt' - (M+P(t)+Q_\nu(x))\left(\textstyle\int_{2\eta}^{\rho}\tfrac{\sqrt{p'(\rho)}}{\rho}d\rho + \Psi(t,x)\right) \\
&= -\tfrac{1}{2}(M+P(t)+Q_\nu(x))(v+s) + f_\nu(t,x)s \\
&+k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + (1-\varepsilon_1)k_3Q_\nu(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&+(1-\varepsilon_1)k_3^2Q_\nu(x)\textstyle\int_{-\infty}^{x} Q_\nu(t')dt' - 2(M+P(t)+Q_\nu(x))e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&-k_3(M+P(t)+Q_\nu(x))\textstyle\int_{-\infty}^{x} Q_\nu(t')dt' \\
&\geq f_\nu(t,x)s - \tfrac{1}{2}(v+s)(M+P(t)+Q_\nu(x)) \\
&+(k_2-2)(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} - k_3(M+P(t))\textstyle\int_{-\infty}^{x} Q_\nu(t')dt' \\
&+((1-\varepsilon_1)k_3-1)k_3Q_\nu(x)\textstyle\int_{-\infty}^{x} Q_\nu(t')dt' \\
&+((1-\varepsilon_1)k_3-2)Q_\nu(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&\geq -\tfrac{1}{2}(M+P(t)+Q_\nu(x))(v+s) + f_\nu(t,x)s
\end{aligned} \tag{2.8}$$

if we choose $k_3 > 2$ and $(k_2-2)e^{k_1} \geq k_3|Q_\nu(x)|_{L^1}$. So, we may choose $k_2 = 3, k_3 = 5, e^{k_1} \geq 10|Q_\nu(x)|_{L^1}$ such that both (2.7) and (2.8) are true. When $1 < \alpha \leq 3$, set $c = 0$ in (1.9), (2.3), and (2.4). Then

$$z(\rho,u) = \frac{\rho^\theta}{\theta} - u, \quad w(\rho,u) = \frac{\rho^\theta}{\theta} + u$$

and

$$\rho^\theta = \frac{\theta(w+z)}{2} = \frac{\theta(v+s)}{2} + \theta e^{k_1+k_2\int_0^t M+P(t')dt'},$$

where $\theta = \frac{\alpha-1}{2}$. Moreover, $(2\eta)^\theta \geq 2\eta\rho^{\theta-1}$ if $1 < \alpha \leq 3$. Thus the following terms in (2.3)

$$
\begin{aligned}
&L_{2v} = \Phi_t + (\textstyle\int_0^\rho \frac{\sqrt{p'(\rho)}}{\rho}d\rho - \frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)})\Phi_x - (1-\varepsilon_1)\Phi(t,x)\Phi_x + f_v(t,x)u \\
&\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_v(t,x)(\tfrac{1}{\theta}\rho^\theta - \Phi(t,x) - v) \\
&+(1-\varepsilon_1)(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} - k_3^2\textstyle\int_{-\infty}^x Q_v(t')dt')Q_v(x) \\
&-k_3 Q_v(x)\tfrac{3-\alpha}{\alpha-1}\rho^\theta - (2\eta)^\theta k_3 Q_v(x) \\
&\geq -f_v(t,x)v - (M+P(t)+Q_v(x))(\tfrac{1}{\theta}\rho^\theta + \Phi(t,x)) + k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&+(1-\varepsilon_1)(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} - k_3^2\textstyle\int_{-\infty}^x Q_v(t')dt')Q_v(x) \\
&-k_3 Q_v(x)\tfrac{3-\alpha}{\alpha-1}\rho^\theta - (2\eta)^\theta k_3 Q_v(x) \\
&= -f_v(t,x)v - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s) \\
&-(M+P(t)+Q_v(x) + \tfrac{3-\alpha}{2}k_3 Q_v(x))e^{k_1+k_2\int_0^t M+P(t')dt'} + k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&-(M+P(t)+Q_v(x))e^{k_1+k_2\int_0^t M+P(t')dt'} + (M+P(t)+Q_v(x))k_3\textstyle\int_{-\infty}^x Q_v(t')dt' \\
&+(1-\varepsilon_1)k_3 Q_v(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&-(1-\varepsilon_1)k_3^2 Q_v(x)\textstyle\int_{-\infty}^x Q_v(t')dt' - (2\eta)^\theta k_3 Q_v(x) \\
&\geq -f_v(t,x)v - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s) \\
&+(k_2-2)(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + [\tfrac{1}{2}(\tfrac{\alpha-1}{2}-\varepsilon_1)e^{k_1} - (1-\varepsilon_1)k_3|Q_v(x)|_{L^1}]k_3 Q_v(x) \\
&+[\tfrac{1}{2}(\tfrac{\alpha-1}{2}-\varepsilon_1)k_3 - (2+(2\eta)^\theta k_3)]Q_v(x)e^{k_1+k_2\int_0^t M+P(t')dt'} \\
&\geq -f_v(t,x)v - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s)
\end{aligned}
\tag{2.9}
$$

if we choose $k_2 \geq 2$, $\frac{\alpha-1}{2}k_3 > 4$ and $\frac{\alpha-1}{2}e^{k_1} > 2k_3|Q_v(x)|_{L^1}$. Similarly, the following terms in (2.4)

$$
\begin{aligned}
L_{2s} &= \Psi_t + \Psi_x(\frac{\rho-2\eta}{\rho}\sqrt{p'(\rho)} - \int_{2\eta}^\rho \frac{\sqrt{p'(\rho)}}{\rho}d\rho) + (1-\varepsilon_1)\Psi(t,x)\Psi_x - f_v(t,x)u \\
&\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_v(t,x)(s+\Psi(t,x) - \frac{1}{\theta}\rho^\theta) \\
&+(1-\varepsilon_1)(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} + k_3^2\int_{-\infty}^x Q_v(t')dt')Q_v(x) - k_3 Q_v(x)\frac{3-\alpha}{\alpha-1}\rho^\theta - (2\eta)^\theta k_3 Q_v(x)
\end{aligned}
$$

$$\geq k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} + f_v(t,x)s - (M+P(t)+Q_v(x))(\tfrac{1}{\theta}\rho^\theta + \Psi(t,x))$$

$$+(1-\varepsilon_1)(k_3 e^{k_1+k_2\int_0^t M+P(t')dt'} + k_3^2\int_{-\infty}^x Q_v(t')dt')Q_v(x)$$

$$-k_3 Q_v(x)\tfrac{3-\alpha}{\alpha-1}\rho^\theta - (2\eta)^\theta k_3 Q_v(x)$$

$$= f_v(t,x)s - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s)$$

$$-(M+P(t)+Q_v(x)+\tfrac{3-\alpha}{2}k_3 Q_v(x))e^{k_1+k_2\int_0^t M+P(t')dt'} + k_2(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'}$$

$$-(M+P(t)+Q_v(x))e^{k_1+k_2\int_0^t M+P(t')dt'} - (M+P(t)+Q_v(x))k_3\int_{-\infty}^x Q_v(t')dt'$$

$$+(1-\varepsilon_1)k_3 Q_v(x)e^{k_1+k_2\int_0^t M+P(t')dt'}$$

$$+(1-\varepsilon_1)k_3^2 Q_v(x)\int_{-\infty}^x Q_v(t')dt' - (2\eta)^\theta k_3 Q_v(x)$$

$$\geq f_v(t,x)s - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s)$$

$$+(k_2-2)(M+P(t))e^{k_1+k_2\int_0^t M+P(t')dt'} - k_3(M+P(t))\int_{-\infty}^x Q_v(t')dt'$$

$$+[(\tfrac{\alpha-1}{2}-\varepsilon_1)k_3 - 2 - (2\eta)^\theta k_3)]Q_v(x)e^{k_1+k_2\int_0^t M+P(t')dt'}$$

$$+((1-\varepsilon_1)k_3-1)k_3 Q_v(x)\int_{-\infty}^x Q_v(t')dt'$$

$$\geq f_v(t,x)s - (\tfrac{1}{2}(M+P(t)+Q_v(x)) + \tfrac{3-\alpha}{4}k_3 Q_v(x))(v+s) \tag{2.10}$$

if we choose

$$\frac{\alpha-1}{2}k_3 > 2$$

and

$$(k_2-2)e^{k_1} \geq k_3|Q_v(x)|_{L^1}.$$

Thus, we may choose $k_2 = 3$, $\frac{\alpha-1}{2}k_3 > 4$, and

$$\frac{\alpha-1}{2}e^{k_1} \geq 2k_3|Q_v(x)|_{L^1}$$

such that both (2.9) and (2.10) are true. Therefore, inequalities in (2.5) are proved. From the conditions in (1.8), we can conclude that $v(0,x) \leq 0$ and $s(0,x) \leq 0$. Thus, by applying the maximum principle given in the following Lemma to (2.5), we have the estimates $v(t,x) \leq 0, s(t,x) \leq 0$, and the estimates in (1.10).

Lemma 2.2. *If $b_2(t,x) \leq 0, d_2(t,x) \leq 0$, and $v(0,x) \leq 0, s(0,x) \leq 0$ at the time $t = 0$, then the maximum principle is true to the functions $v(t,x)$ and $s(t,x)$ given in inequalities (2.5), namely, $v(t,x) \leq 0, s(t,x) \leq 0$ for all $t > 0$.*

Proof of Lemma 2.2. We make a transformation

$$v=(\tilde v+\frac{N(x^2+qLe^t)}{L^2})e^{\beta t},\quad s=(\tilde s+\frac{N(x^2+qLe^t)}{L^2})e^{\beta t},$$

where L,β, and q are some positive constants, and N is the upper bound of v,s on $\mathbf{R}\times[0,T]$. Clearly, from (2.5), functions $\tilde v,\tilde s$ satisfy the equations

$$\begin{cases} \tilde v_t+\tilde v_x b(t,x)-\varepsilon\tilde v_{xx}+(b_1(t,x)+\beta)\tilde v+b_2(t,x)\tilde s \\ \qquad \leq -(qLe^t+2xb(t,x)-2\varepsilon)\dfrac{N}{L^2}-(b_1(t,x)+b_2(t,x)+\beta)\dfrac{N(x^2+qLe^t)}{L^2}, \\ \tilde s_t+d(t,x)\tilde s_x-\varepsilon\tilde s_{xx}+(d_1(t,x)+\beta)\tilde s+d_2(t,x)\tilde v \\ \qquad \leq -(qLe^t+2xd(t,x)-2\varepsilon)\dfrac{N}{L^2}-(\beta+d_1(t,x)+d_2(t,x))\dfrac{N(x^2+qLe^t)}{L^2}. \end{cases} \tag{2.11}$$

Moreover

$$\tilde v(0,x)=v(0,x)-\frac{N(x^2+qL)}{L^2}<0,\ \ \tilde s(0,x)=s(0,x)-\frac{N(x^2+qL)}{L^2}<0, \tag{2.12}$$

$$\tilde v(t,+L)<0,\ \tilde v(t,-L)<0,\ \tilde s(t,+L)<0,\ \tilde s(t,-L)<0. \tag{2.13}$$

From (2.11), (2.12), and (2.13), we have

$$\tilde v(t,x)<0,\quad \tilde s(t,x)<0,\quad \text{on}\quad (0,T)\times(-L,L). \tag{2.14}$$

If (2.14) is violated at a point $(t,x)\in(0,T)\times(-L,L)$, let $\tilde t$ be the least upper bound of values of t at which $\tilde v<0$ (or $\tilde s<0$); then we have by the continuity that $\tilde v=0,\tilde s\leq 0$ at some points $(\tilde t,\tilde x)\in(0,T)\times(-L,L)$. Thus,

$$\tilde v_t\geq 0,\quad \tilde v_x=0,\quad -\varepsilon\tilde v_{xx}\geq 0,\quad \text{at}\quad (\tilde t,\tilde x). \tag{2.15}$$

Now, choosing sufficiently large constants q,β (which may depend on the bound of the local existence), we have

$$qL+2xb(t,x)-2\varepsilon>0,\quad \beta+b_1(t,x)+b_2(t,x)>0\quad \text{on}\quad (0,T)\times(-L,L). \tag{2.16}$$

(2.15) and (2.16) give a conclusion contradicting the first inequality in (2.11). So (2.14) is proved. Therefore, for any point $(t_0,x_0)\in(0,T)\times(-L,L)$,

$$(\frac{N(x_0^2+qLe_0^t)}{L^2})e^{\beta t_0},\quad >v(t_0,x_0),(\frac{N(x_0^2+qLe_0^t)}{L^2})e^{\beta t_0}>s(t_0,x_0),$$

we take L go to infinity and obtain the desired estimates $v\leq 0,s\leq 0$. Thus, Lemma 2.2 is proved.

After we have the estimates in (1.10), by using the Riemann invariants (1.9), we can obtain the uniformly bounded estimates on $(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))$ directly

$$2\eta\leq\rho^{\varepsilon,\eta,\nu}(t,x)\leq M(t),\quad |u^{\varepsilon,\eta,\nu}(t,x)|\leq M(t), \tag{2.17}$$

for a suitable bounded function $M(t)$, which is independent of ε,η, and ν.

By applying the contraction mapping principle to an integral representation of a solution, we may first obtain the local existence result of the Cauchy problem (1.5)-(1.6). After we have the a priori L^∞-estimate (2.17) on the local solution, we can extend the local time to an arbitrary time T step by step and obtain a global solution. Thus, we have the proof of Part I in Theorem 1.1 .

To complete the proof of Theorem 1.1, we prove the pointwise convergence of a subsequence of $(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))$, as $(\varepsilon,\eta,\nu)\to(0,0,0)$, and the limit $(\rho(t,x),u(t,x))$ is a weak entropy solution to Cauchy problem (1.1)-(1.2).

For general pressure function $p(\rho)$, we have the following.

Lemma 2.3. *Suppose that the viscosity-flux approximate solutions $(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))$ to (1.5)-(1.6) are uniformly bounded in L^∞ space, and the limit*

$$\lim_{\rho\to 0}\frac{(p'(\rho))^{\frac{3}{2}}}{\rho p''(\rho)}=e, \tag{2.18}$$

where $e\geq 0$ is a constant. If the weak entropy-entropy flux pair $(h(\rho,u),\psi(\rho,u))$ of system (1.1) is in the form $h(\rho,u)=\rho H(\rho,u)$ and $H_u(\rho,u),H_{uu}(\rho,u),H_{uuu}(\rho,u)$ are continuous on $0\leq\rho\leq M_1,|u|\leq M_1$, where M_1 is a positive constant, then

$$h_t(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))+\psi_x(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x)) \tag{2.19}$$

is compact in $H^{-1}_{loc}(\mathbf{R}^+\times\mathbf{R})$ as $\varepsilon=o\left(\frac{p'(2\eta)}{2\eta}\right)$ and η,ν tend to zero, with respect to the viscosity solutions $(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))$ to (1.5)–(1.6).

Proof of Lemma 2.3. For the homogeneous case, namely $\Pi(x,t,\rho,u)=0$, the proof of Lemma 2.3 was given in [15]. Analogously, we may prove Lemma 2.3 for the case that $\Pi(x,t,\rho,u)$ satisfies condition (1.3).

Clearly, for the polytropic gas, $p(\rho)=\frac{1}{\alpha}\rho^\alpha$ and for any $\alpha>1$, all the conditions about pressure function (2.18) and the weak entropies in Lemma 2.3 are satisfied. Thus, we may apply the H^{-1} compactness of (2.19), and the convergence frameworks given in [19, 20, 21, 22] for $1<\alpha<3$ and in [23] for $\alpha\geq 3$ to prove that $(\rho^{\varepsilon,\eta,\nu}(t,x),u^{\varepsilon,\eta,\nu}(t,x))$ has a subsequence which converges pointwisely to bounded functions $(\rho(t,x),u(t,x))$ as $(\varepsilon,\eta,\nu)\to(0,0,0)$.

Finally, it is not difficult to demonstrate that limit $(\rho(t,x),u(t,x))$ satisfies (1.12). Moreover, for any weak convex entropy-entropy flux pair $(h(\rho,m),\psi(\rho,m)),m=\rho u$, of system (1.1), we multiply (1.5) by (η_ρ,η_m) to obtain that

$$\begin{aligned}
&h_t(\rho^{\varepsilon,\eta,\nu}(t,x),m^{\varepsilon,\eta,\nu}(t,x))+\psi_x(\rho^{\varepsilon,\eta,\nu}(t,x),m^{\eta,\varepsilon,\nu}(t,x))+\eta\psi_{1x}(\rho^{\varepsilon,\eta,\nu}(t,x),m^{\varepsilon,\eta,\nu}(t,x))\\
&=\varepsilon h(\rho^{\varepsilon,\eta,\nu},m^{\varepsilon,\eta,\nu})_{xx}-\varepsilon(\rho_x^{\varepsilon,\eta,\nu},m_x^{\varepsilon,\eta,\nu})\cdot\nabla^2 h(\rho^{\varepsilon,\eta,\nu},m^{\varepsilon,\eta,\nu})\cdot(\rho_x^{\varepsilon,\eta,\nu},m_x^{\varepsilon,\eta,\nu})^T\\
&-a_\nu(t,x)|m^{\varepsilon,\eta,\nu}|h_m(\rho^{\varepsilon,\eta,\nu},m^{\varepsilon,\eta,\nu})\\
&\leq\varepsilon h(\rho^{\varepsilon,\eta,\nu},m^{\varepsilon,\eta,\nu})_{xx}-a_\nu(t,x)|m^{\varepsilon,\eta,\nu}|h_m(\rho^{\varepsilon,\eta,\nu},m^{\varepsilon,\eta,\nu}),
\end{aligned} \tag{2.20}$$

where $\psi+\eta\psi_1$ is the entropy flux of system (1.5) corresponding to entropy h. Thus we can prove the entropy inequality (1.13) if we multiply a test function to (2.20) and let $(\varepsilon,\eta,\nu)\to(0,0,0)$. This completes the proof of Theorem 1.1.

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