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NOTE

The Vacuum Case in Diperna's Paper

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Lemma 4.1 in [*Comm. Math. Phys.* **91** (1983)] by Ron Diperna pertaining to the vacuum case for an existence proof of polytropic gas dynamics using compensated compactness is incomplete as given. Here we give a quick fix of the lemma plus some generalization.

1. INTRODUCTION

In Diperna's famous article [1] he gave an elegant proof for the lower bound of the density for the viscously perturbed isentropic gas dynamic equation $\rho^\epsilon \geq c(t, \epsilon) > 0$, which is based on Lemma 4.1 in [1]. No proof of the lemma is given and as stated it seems not possible to prove it. We believe that actually it is a slip of the pen of Diperna when stating this lemma. We give a proof of his lemma under slightly changed hypotheses which snugly fits into the other results given in [1]. In addition, we will prove the validity of his lemma for more general pressure $p(\rho)$ than in the polytropic gas dynamic case considered by Diperna.

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2. MAIN RESULTS

Consider the following viscous perturbation of the isentropic gas dynamics equations,

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= \epsilon \rho_{xx} \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x &= \epsilon (\rho u)_{xx} \end{aligned}\right\}, \tag{1}$$

with initial data,

$$(\rho, \rho u)|_{t=0} = \left(\rho_0^\epsilon(x), \rho_0^\epsilon(x) u_0^\epsilon(x)\right), \tag{2}$$

where $(\rho_0^\epsilon(x) u_0^\epsilon(x))$ are obtained by smoothing out a pair of bounded functions $(\rho_0(x), u_0(x))$ $(0 \le \rho_0(x) \le M, |u_0(x)| \le M)$ with a mollifier G^ϵ,

$$\left(\rho_0^\epsilon(x), u_0^\epsilon(x)\right) = \left(\bar{\rho}_0(x), \bar{u}_0(x)\right) * G^\epsilon, \tag{3}$$

where

$$\bar{\rho}_0(x) = \begin{cases} \rho_0(x) + \epsilon, & |x| \le L, \\ \bar{\rho}, & |x| > L, \end{cases} \tag{4}$$

$$\bar{u}_0(x) = \begin{cases} u_0(x), & |x| \le L, \\ \bar{u}, & |x| > L. \end{cases} \tag{5}$$

Because the generalized solution of hyperbolic conservation laws is defined in a compact set of the plane $R \times R^+$, we may take L to be large such that the compact set is contained in the region $|x| < L$ and $0 \le t \le T$ for some T. $\bar{\rho} > 0$ and $\bar{u}$ in (4) and (5) are constants as needed in [1]. Therefore,

$$\begin{gathered} \left(\rho_0^\epsilon(x), u_0^\epsilon(x)\right) \in C^\infty \times C^\infty, \\ \epsilon \le \rho_0^\epsilon(x) \le M, \qquad |u_0^\epsilon(x)| \le M, \end{gathered} \tag{6}$$

$$\lim_{|x| \to \infty} \left(\rho_0^\epsilon(x), u_0^\epsilon(x)\right) = (\bar{\rho}, \bar{u}), \tag{7}$$

$$\left(\rho_0^\epsilon(x) - \bar{\rho}, u_0^\epsilon(x) - \bar{u}\right) \in L^2(R) \times L^2(R). \tag{8}$$

By applying the general contraction mapping principle to an integral representation of (1), the following local existence of the Cauchy problems (1) and (2) is obtained:

LEMMA 1. *If $p(\rho) \in C^1$ and the initial data satisfies the conditions* (6) *and* (7), *then for any fixed ϵ, there exists a smooth solution for the Cauchy problem* (1) *and* (2) *in some $R_s = R \times [0, s]$, which satisfies*

$$\frac{\epsilon}{2} \le \rho(x,t) \le M, \qquad |u(x,t)| \le M, \tag{9}$$

and

$$\lim_{|x|\to\infty} (\rho(x,t), u(x,t)) = (\bar{\rho}, \bar{u}), \tag{10}$$

for any fixed $t \in [0, s]$, *where local time* s *depends on the bound of the initial data given in* (6).

LEMMA 2. *Let* (6) *hold and* $p(\rho) \in C^2[0,\infty)$, $p'(\rho) > 0$, $2p'(\rho) + \rho p''(\rho) > 0$ *for* $\rho > 0$ *and* $\int_c^\infty \sqrt{p'(\rho)}/\rho\, d\rho = \infty$ *for any positive constant* c. *Suppose that* $(\rho(x,t), \rho(x,t)u(x,t))$ *is a smooth solution of* (1) *and* (2) *defined in a strip* $T_T = R \times [0,t]$, *which satisfies*

$$0 < \rho(x,t) < M(\epsilon,t), \qquad |u(x,t)| \le M(\epsilon,t).$$

Then

$$0 < \rho(x,t) \le M, \qquad |u(x,t)| \le M, \tag{11}$$

if $\int_0^c \sqrt{p(\rho)}/\rho\, d\rho \le M$ *is finite;*

$$0 < \rho(x,t) \le M, \qquad |\rho u| \le M, \tag{12}$$

if $\int_0^c \sqrt{p(\rho)}/\rho\, d\rho = \infty$ *but* $\rho \int_0^c \sqrt{p(\rho)}/\rho\, d\rho$ *is finite.*

Lemma 2 comes from [3].

Before giving the lower bound of τ, we first prove Diperna's Lemma 4.1 in [1].

LEMMA 3 (Diperna). *If* $\phi(t)$ *is a nonnegative continuous function in* $[0,T]$ *satisfying*

$$\phi(0) > 0, \tag{13}$$

$$\phi(t) - \phi(s) \ge -c_1(t-s)^{1/2}, \quad \text{if } t > s, \tag{14}$$

$$\int_0^T \phi^{-\alpha}(t)\, dt \ge c_2, \quad \text{for } \alpha \ge 2, \tag{15}$$

then $\phi(t) \ge c_3$ *on the interval* $[0,T]$, *where* $c_i (i = 1,2,3)$ *are all positive constants and* c_3 *depends on* c_1, c_2, T, *and* α.

Proof. If $\phi(t) = 0$ at some points $t \in (0,T]$, let $t_1 \le T$ be the least point such that $\phi(t) > 0$ for $t \in [0,t_1]$ and $\phi(t_1) = 0$. Then

$$\phi(t_1) - \phi(s) \ge -c_1(t_1 - s)^{1/2}, \tag{16}$$

and so

$$\phi(s) \le c_1(t_1 - s)^{1/2}, \quad \text{for } 0 \le s < t_1. \tag{17}$$

Thus,

$$\int_0^{t_1} \phi^{-\alpha}(s)\, ds \ge \int_0^{t_1} c_1^{-\alpha}(t_1 - s)^{-\alpha/2}\, ds = \infty, \tag{18}$$

which contradicts (15). The lemma is proved.

We are going to give the lower bound of ρ following Diperna's method except correcting a few of what we consider misprints and we are going to extend the result from γ-law gas to general $p(\rho)$.

Using the normalized convex entropy $(p'(\rho) > 0)$,

$$\eta = \tfrac{1}{2}\rho(u - \bar{u})^2 + Q\sigma(\rho, \bar{\rho}),$$

where $\sigma = \rho \int^{\rho} p(s)/s^2\, ds$, $Q\sigma = \sigma(\rho) - \sigma(\bar{\rho}) - \sigma'(\bar{\sigma})(\sigma - \bar{\sigma})$.

We have from (8) and (10),

$$\int_{-\infty}^{\infty} \tfrac{1}{2}\rho(u - \bar{\rho})^2 + Q\sigma(\rho, \bar{\rho})\, dx \le c. \tag{19}$$

We construct a function $h(\rho)$ in the class of strictly convex, nonnegative C^2 functions with the following properties,

$$h(\bar{\rho}) = h'(\bar{\rho}) = 0,$$

$$h(\rho) = \rho^{-\alpha}, \quad \text{on } \left(0, \frac{\bar{\rho}}{2}\right) \text{ for some } \alpha \ge 2,$$

and

$$h \le c(\rho - \bar{\rho})\rho \text{ near } \bar{\rho},$$

$$\rho^2 h'' \le c\rho, \quad \text{for } \frac{\bar{\rho}}{2} \le \rho \le M^{1/2},$$

$$\rho^2 h'' \le c h(\rho), \quad \text{for } 0 < \rho \le \frac{\bar{\rho}}{2}.$$

Thus for a solution with velocity bounded by M we have

$$h''(\rho)\rho^2(u - \bar{u})^2 \le c\big(\rho(u - \bar{u})^2 + h(\rho)\big). \tag{20}$$

Multiplying the mass equation (the first equation in (1)) by $h'(\rho)$ and integrating over $R \times (0, t)$, we get

$$h_t + (h\rho u)_x - h''\rho u\rho_x = \epsilon h'' - \epsilon h''\rho_x^2,$$

$$\int_{-\infty}^{\infty} h(\rho) - h(\rho_0^\epsilon)\, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h''\rho_x^2\, dx\, dt$$

$$= \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho\rho_x u\, dx\, dt$$

$$= \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho\rho_x(u - \bar{u})\, dx\, dt + \int_0^t \int_{-\infty}^{\infty} \left(\int_{\bar{\rho}}^{\rho} h''(\rho)\rho\, d\rho \right)_x \bar{u}\, dx\, dt$$

$$\leq \int_0^t \int_{-\infty}^{\infty} \frac{\epsilon}{2} h''\rho_x^2 + \frac{2}{\epsilon} h''\rho^2 (u - \bar{u})^2\, dx\, dt.$$

Then from (8), (19), and (20),

$$\int_{-\infty}^{\infty} h(\rho)\, dx + \frac{\epsilon}{2} \int_0^t \int_{-\infty}^{\infty} h''\rho_x^2\, dx\, dt \leq c + \frac{ct}{\epsilon} + \frac{c}{\epsilon} \int_0^t \int_{-\infty}^{\infty} h(\rho)\, dx\, dt, \tag{21}$$

for a suitable positive constant c. Thus,

$$\int_{-\infty}^{\infty} h(\rho)\, dx + \int_0^t \int_{-\infty}^{\infty} h''\rho_x^2\, dx\, dt \leq M(T, \epsilon).$$

The rest of the proof is the same as Diperna's proof. Thus we have

LEMMA 4. *Let* (6), (7), *and* (8) *hold and* $p(\rho) \in C^2(0, \infty)$, $p'(\rho) > 0$, $2p' + \rho p'' > 0$ *for* $\rho > 0$, $\int_c^\infty \sqrt{p'(\rho)}/\rho\, d\rho = \infty$, $\int_0^c \sqrt{p'(\rho)}/\rho\, d\rho$ *is finite for any positive constant* c. *Then*

$$\rho \geq c(\epsilon, t) > 0, \tag{22}$$

for an appropriate function $c(\epsilon, t)$.

Lemmas 1, 2, and 4 give the following global existence of the Cauchy problem (1) and (2).

THEOREM 1. *Let the conditions in Lemma* 4 *be satisfied. Then for any fixed* $\epsilon > 0$, *there exists a smooth solution for the Cauchy problem* (1) *and* (2) *in* $R_T = R \times [0, T]$ (*for arbitrary* T), *which satisfies*

$$0 < c(\epsilon, t) \leq \rho(x, t) \leq M, \qquad |u(x, t) \leq M,$$

$$\lim_{|x| \to \infty} (\rho(x, t), u(x, t)) = (\bar{\rho}, \bar{u}),$$

for any fixed $t \in [0, T]$, *and*

$$(\rho(.,t) - \bar{\rho}, u(.,t) - \bar{u}) \in L^2(R) \times L^2(R).$$

Remark. Conditions (7) and (8) are technical. In [2] they are replaced by a periodicity condition on the initial data for γ-law gas. In [3], conditions (7) and (8) are omitted, but a stronger condition on $p(\rho)$ is introduced: $p'(\rho) - \rho p''(\rho) > 0$, $p'''(\rho) < 0$, for $\rho > 0$.

REFERENCES

1. R. J. Diperna, Convergence of the viscosity method for isentropic gas dynamics, *Comm. Math. Phys.* **91** (1983).
2. P. L. Lions, B. Perthame, and S. E. Sougandis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Commun. Pure Appl. Math.* **49**(6) (1996), 599–638.
3. Y. G. Lu, An explicit lower bound of viscosity solutions to isentropic gas dynamics and to Euler equations, preprint 95-18, Heidelberg University, SFB 359, April 1995.