Existence of entropy solutions to system of polytropic gas with a class of unbounded sources

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Abstract

In this paper, we first apply the viscosity-flux approximation method coupled with the maximum principle to obtain the a-priori L^∞ estimates for the approximation solutions of the polytropic gas dynamics system with a class of unbounded sources. The key idea is to employ suitable bounded functions $B(x,t), C(x,t)$ to control these unbounded source terms. Second, we prove the pointwise convergence of the approximation solutions by using the compactness framework from the compensated compactness theory and obtain the global existence of entropy solutions for any adiabatic exponent $\gamma > 1$.

Key Words: Global weak solution; Gas dynamics; damping source; friction; flux approximation
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1 Introduction

In this paper we studied the global entropy solutions for the Cauchy problem of the following inhomogeneous system of polytropic gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x + \alpha(x,t,\rho,u) = 0 \end{cases} \tag{1.1}$$

*the corresponding author

with bounded measurable initial data

$$(\rho(x,0),u(x,0))=(\rho_0(x),u_0(x)),\quad \rho_0(x)\ge 0, \tag{1.2}$$

where ρ is the density of gas, u the velocity, $P=\frac{1}{\gamma}\rho^\gamma, \gamma>1$, the pressure and the nonlinear function $\alpha(x,t,\rho,u)$ denotes the source.

System (1.1) has different physical backgrounds [Wh]. When $\alpha(x,t,\rho,u)$ is a linear function of ρu, $\alpha(x,t,\rho,u)=a(x,t)\rho u$, there are many results concerning the influence of damping, corresponding to the case of $a(x,t)\ge 0$, on global existence and singularity formation [Le, SS, KL, KM, Sl]. When $\alpha(x,t,\rho,u)=-\rho E(x,t)+a(x)\rho u$, System (1.1) is corresponding to the one-dimensional hydrodynamic model for semiconductors ((cf. [DM, LY, MN1, HLYY] and the references cited therein). More results on inhomogeneous hyperbolic systems can be found in [CHY, CHHQ, IT, FY, CG, EGM, GL, GMP, MN2, Ga, GK, Jo, JR, LNX, MM, PRV, TW, Zh1, Zh2].

In this paper, we restrict our attention on the following unbounded source functions

$$\alpha(x,t,\rho,u)=a(x,t)|\rho u|,\quad |a(x,t)|\le M+T(t)+X(x), \tag{1.3}$$

where $0\le T(t)\in C(R^+)\cap L^1(R^+), 0\le X(x)\in L^1(R)\cap L^\infty(R)$ and $M\ge 0$ is a constant.

In general, the classical solution of the Cauchy problem for nonlinear hyperbolic system (1.1) exists only locally in time even if the initial data (1.2) are small and smooth. This means that shock waves always appear in the solution for a suitable large time. Since the solution is discontinuous and does not satisfy the given partial differential equations in (1.1) in the classical sense, we have to study the generalized solutions, or functions which satisfy the equations in the sense of distributions.

To study the generalized solutions of the Cauchy problem (1.1) and (1.2), the standard steps of the classical vanishing viscosity method are first to study the approximate solutions $(\rho^\varepsilon(x,t),u^\varepsilon(x,t))$, by adding the small perturbation $\varepsilon>0$ to the right-hand side of (1.1), of the following parabolic system

$$\begin{cases} \rho_t+(\rho u)_x=\varepsilon\rho_{xx}, \\ (\rho u)_t+(\rho u^2+P(\rho))_x+\alpha(x,t,\rho,u)=\varepsilon(\rho u)_{xx}, \end{cases} \tag{1.4}$$

and then to consider the convergence of $(\rho^\varepsilon(x,t),u^\varepsilon(x,t))$ as ε goes to zero.

If we consider the momentum $m=\rho u$ in (1.4) as an independent variable, a basic technical difficulty is to obtain the positive, lower estimate of ρ^ε since $\rho u^2=\frac{m^2}{\rho}$ is singular when $\rho=0$. Moreover, as introduced in [LPS], when we study the convergence of $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$, as ε goes to zero, by applying the theory of the compensated compactness, an essential step is to prove that

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x \quad \text{are compact in} \quad H^{-1}_{loc}(R\times R^+), \tag{1.5}$$

where $(\eta(\rho,m), q(\rho,m))$ is a pair of the weak entropy-entropy flux of (1.1), with respect to the viscosity solutions $(\rho^\varepsilon, m^\varepsilon)$.

For the polytropic gas $P=\frac{1}{\gamma}\rho^\gamma$ and the adiabatic exponent $\gamma\in(1,2]$, the proof of (1.5) is easy because (1.1) has a strictly convex entropy-entropy flux pair

$$(\eta,\ q) = \Big(\frac{m^2}{2\rho} + \frac{1}{\gamma(\gamma-1)}\rho^\gamma,\ \frac{m^3}{2\rho^2} + \frac{1}{\gamma-1}\rho^{\gamma-1}m\Big). \tag{1.6}$$

However, when $\gamma>2$, even if we have a positive lower bound $\rho^\varepsilon(x,t)\ge c(t,c_0,\varepsilon)>0$, as we proved in Theorem 1.0.2 in [Lu3], the proof of (1.5) is still very difficult, where $c(t,c_0,\varepsilon)$ could tend to zero as the time t tends to infinity or ε tends to zero.

To overcome the above difficulty, the authors in [LPS] introduced the viscous periodic solutions with respect to the spatial variable x to derive the auxiliary estimate (see $(I.53)$ in [LPS]),

$$\int\int_{K_1} \varepsilon^2(\rho_x)^2 dxdt \le C\delta^2 \tag{1.7}$$

and to obtain the proof of (1.5).

However, for the parabolic system (1.4) with the source $\alpha(x,t,\rho,u)$, the method in [LPS] is not valid because we meet a new difficulty how to obtain the periodic solutions with respect to the spatial variable x.

In this paper, we will adopt the flux approximation method given in [Lu1, Lu2] and study the approximation solutions $(\rho^{\varepsilon,\delta,\mu}, u^{\varepsilon,\delta,\mu})$ of the following parabolic systems

$$\begin{cases} \rho_t + ((\rho-2\delta)u)_x = \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x + a_\mu(x,t)|\rho u| = \varepsilon(\rho u)_{xx} \end{cases} \tag{1.8}$$

with initial data

$$(\rho^{\varepsilon,\delta,\mu}(x,0), u^{\varepsilon,\delta,\mu}(x,0)) = (\rho_0(x)+2\delta, u_0(x)), \tag{1.9}$$

where $\delta>0$ denotes a regular perturbation constant, $\varepsilon>0$ is the viscosity coefficient, the perturbation pressure

$$P_1(\rho,\delta) = \int_{2\delta}^{\rho} \frac{t-2\delta}{t} P'(t)dt, \tag{1.10}$$

and

$$a_\mu(x,t)=\int_{-\infty}^{\infty}a(y,t)J_\mu(x-y)dy \tag{1.11}$$

for a suitable mollifier J_μ, which satisfies

$$|a_\mu(x,t)|\le M+T(t)+\int_{-\infty}^{\infty}|X(y)|J_\mu(x-y)dy. \tag{1.12}$$

An obvious advantage, of this kind of approximations added on the fluxes, is that we may obtain directly the uniformly, positive bound

$$\rho^{\varepsilon,\delta,\mu}\ge 2\delta>0, \tag{1.13}$$

if we apply the maximum principle to the first equation in (1.8), which gurantees the existence of the approximation solutions $(\rho^{\varepsilon,\delta,\mu},u^{\varepsilon,\delta,\mu})$. Moreover, both systems (1.1) and (1.8) have the same Riemann invariants and the entropy equation. With the help of these special behaviors of system (1.8), we may obtain the uniform L^∞ estimates of $(\rho^{\varepsilon,\delta,\mu},u^{\varepsilon,\delta,\mu})$ as well as the $H^{-1}_{loc}(R\times R^+)$ compactness in (1.5) for any adiabatic exponent $\gamma>1$.

It is worthwhile to point out that, the same problem with a different source $a(x,t)\rho$ was studied in [Ts, HLT].

Precisely, we have the following

Theorem 1 I. *Suppose $\alpha(x,t)$ is measurable and satisfies (1.3), where $0\le T(t)\in C(R^+)\cap L^1(R^+), 0\le X(x)\in L^1(-\infty,+\infty)$, and the initial data satisfy*

$$z(\rho_0(x),u_0(x))\le e^{l_1}-|X(x)|_{L^1(-\infty,+\infty)},\quad w(\rho_0(x),u_0(x))\le e^{l_1}, \tag{1.14}$$

where

$$z(\rho,u)=\int_c^\rho\frac{\sqrt{P'(s)}}{s}ds-u,\quad w(\rho,u)=\int_c^\rho\frac{\sqrt{P'(s)}}{s}ds+u \tag{1.15}$$

are the Riemann invariants of (1.1), $c,l_1>0$ are two constants. Then, for fixed ε,δ,μ, the Cauchy problem (1.8) and (1.9) has a global solution $(\rho^{\varepsilon,\delta,\mu},u^{\varepsilon,\delta,\mu})$ satisfying

$$\begin{cases} z(\rho^{\varepsilon,\delta,\mu},u^{\varepsilon,\delta,\mu})\le e^{l_1+l_2\int_0^t M+T(\tau)d\tau}-l_3\int_{-\infty}^x X_\mu(\tau)d\tau\le e^{l_1+l_2\int_0^t M+T(\tau)d\tau},\\ w(\rho^{\varepsilon,\delta,\mu},u^{\varepsilon,\delta,\mu})\le e^{l_1+l_2\int_0^t M+T(\tau)d\tau}+l_3\int_{-\infty}^x X_\mu(\tau)d\tau\\ \le e^{l_1+l_2\int_0^t M+T(\tau)d\tau}+l_3\int_{-\infty}^{+\infty}X(\tau)d\tau, \end{cases} \tag{1.16}$$

where l_2,l_3 are two suitable positive constants and

$$X_\mu(x)=\int_{-\infty}^{\infty}X(y)J_\mu(x-y)dy. \tag{1.17}$$

II. *There exists a subsequence of $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as ε, δ, μ tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2)*

Definition 1 *A pair of bounded functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.1)-(1.2) if*

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho\phi_t + (\rho u)\phi_x \phi dxdt + \int_{-\infty}^\infty \rho_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u\phi_t + (\rho u^2 + P(\rho))\phi_x - \alpha(x,t,\rho,u)\phi dxdt \\ + \int_{-\infty}^\infty \rho_0(x)u_0(x)\phi(x,0)dx = 0 \end{cases} \tag{1.18}$$

holds for all test function $\phi \in C_0^1(R \times R^+)$ and

$$\int_0^\infty \int_{-\infty}^\infty \eta(\rho,m)\phi_t + q(\rho,m)\phi_x - \alpha(x,t,\rho,u)\eta(\rho,m)_m \phi dxdt \geq 0 \tag{1.19}$$

holds for any non-negative test function $\phi \in C_0^\infty(R \times R^+ - \{t=0\})$, where $m = \rho u$ and (η, q) is a pair of convex entropy-entropy flux of system (1.1).

Remark 2. If the nonlinear function $\alpha(x,t,\rho,u)$ is of the C^1 space with respect to the variables, then, without any difficulty, we may prove that Theorem 1 is also true for any $\alpha(x,t,\rho,u)$ satisfying

$$|\alpha(x,t,\rho,u)| \leq |a(x,t)\rho u|, \quad |a(x,t)| \leq M + T(t) + X(x), \tag{1.20}$$

where M is a nonnegative constant, $0 \leq T(t) \in C(R^+) \cap L^1(R^+), 0 \leq X(x) \in C(R) \cap L^1(R)$.

Remark 3. When the conditions (1.3) or (1.20) are changed to

$$\alpha(x,t,\rho,u) = a(x,t)|\rho u|, \quad |a(x,t)| \leq T(t) + X(x), \tag{1.21}$$

although the function $a(x,t)$ could be unbounded, we may deduce a uniformly bounded estimate of solutions with respect to the time. This yields the stability of the solution and is the basis for us to study the asymptotic behavior of solutions when the time goes to infinity.

2 Proof of Theorem 1.

We multiply (1.8) by $(\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial m})$ and $(\frac{\partial z}{\partial \rho}, \frac{\partial z}{\partial m})$, respectively, to obtain

$$\begin{aligned} & z_t + \lambda_1^\delta z_x \\ & = \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 - f_\mu(x,t)u \end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
& w_t + \lambda_2^\delta w_x \\
& = \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 + f_\mu(x,t)u
\end{aligned}
\tag{2.2}
$$

where $f_\mu(x,t) = -a_\mu(x,t)sgn(u)$,

$$
\lambda_1^\delta = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2^\delta = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)} \tag{2.3}
$$

are two eigenvalues of (1.8), $m = \rho u$ denotes the momentum and (w, z) is given by (1.15).

Letting $z = B(x,t) + v$, for a suitable function $B(x,t)$ in (2.1), we have

$$
\begin{aligned}
& v_t + B_t + (u - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})v_x - B_x(B(x,t) + v - \textstyle\int_c^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho) - B_x\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} \\
& = \varepsilon v_{xx} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \tfrac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x B_x + (\tfrac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}B_x)^2] \\
& + \varepsilon B_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x v_x + \tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - f_\mu(x,t)u
\end{aligned}
\tag{2.4}
$$

or

$$
\begin{aligned}
& v_t + B_t + a(x,t)v_x + b(x,t)v + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x,t)B_x] \\
& + (\textstyle\int_c^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})B_x - (1-\varepsilon_1)B(x,t)B_x + f_\mu(x,t)u \le \varepsilon v_{xx},
\end{aligned}
\tag{2.5}
$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a(x,t) = u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b(x,t) = -B_x$.

Similarly, if letting $w = C(x,t) + s$ in (2.2), we have

$$
\begin{aligned}
& s_t + C_t + c(x,t)s_x + d(x,t)s + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C_x^2 - \varepsilon C_{xx} + \varepsilon_1 C(x,t)C_x] \\
& + C_x(\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \textstyle\int_c^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho) + (1-\varepsilon_1)C(x,t)C_x - f_\mu(x,t)u \le \varepsilon s_{xx},
\end{aligned}
\tag{2.6}
$$

where $c(x,t) = u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $d(x,t) = C_x$.

Using the first equation in (1.8), we have the a priori estimate $\rho \ge 2\delta$. Let

$$
B(x,t) = e^{l_1 + l_2\int_0^t M + T(\tau)d\tau} - l_3\int_{-\infty}^x X_\mu(\tau)d\tau, \tag{2.7}
$$

$$
C(x,t) = e^{l_1 + l_2\int_0^t M + T(\tau)d\tau} + l_3\int_{-\infty}^x X_\mu(\tau)d\tau, \tag{2.8}
$$

where $l_i, i = 1, 2, 3$ are suitable positive constants, $X_\mu(x)$ is given by (1.17). Since $|X_\mu(x)|_\infty$ and $\mu|X'_\mu(x)|_\infty$ are uniformly bounded, $|X_\mu(x)|_{L^1(R)} = |X(x)|_{L^1(R)}$ and

$$
\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''} = \frac{2\varepsilon}{\gamma+1}\rho^{-\frac{\gamma-1}{2}} \ge \frac{2\varepsilon}{\gamma+1}(2\delta)^{-\frac{\gamma-1}{2}}, \; B_x = -l_3X_\mu(x), \; B_{xx} = -l_3X'_\mu(x),
$$

we can choose $\varepsilon = o(\delta)$ and suitable relation among $\varepsilon, \varepsilon_1$ and μ such that the following three terms on the left-hand side of (2.5) and (2.6)

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''} B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x,t) B_x > 0 \tag{2.9}$$

and

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''} C_x^2 - \varepsilon C_{xx} + \varepsilon_1 C(x,t) C_x > 0. \tag{2.10}$$

Furthermore, we may obtain the following lemma from (2.5) and (2.6)

<u>**Lemma**</u> **2**

$$\begin{cases} v_t + a(x,t)v_x + b_1(x,t)v + b_2(x,t)s \leq \varepsilon v_{xx}, \\ s_t + c(x,t)s_x + d_1(x,t)s + d_2(x,t)v \leq \varepsilon s_{xx}, \end{cases} \tag{2.11}$$

where

$$\begin{cases} b_1(x,t) = b(x,t) - f_\mu - \frac{1}{2}(M + T(t) + X_\mu(x)), \\ b_2(x,t) = -\frac{1}{2}(M + T(t) + X_\mu(x)) \leq 0, \\ d_1(x,t) = d(x,t) + f_\mu - \frac{1}{2}(M + T(t) + X_\mu(x)), \\ d_2(x,t) = -\frac{1}{2}(M + T(t) + X_\mu(x)) \leq 0 \end{cases} \tag{2.12}$$

when $\gamma > 3$*, and*

$$\begin{cases} b_1(x,t) = b(x,t) - f_\mu - (\frac{1}{2}(M + T(t) + X_\mu(x)) + \frac{3-\gamma}{4} l_3 X_\mu(x)), \\ b_2(x,t) = -(\frac{1}{2}(M + T(t) + X_\mu(x)) + \frac{3-\gamma}{4} l_3 X_\mu(x)) \leq 0, \\ d_1(x,t) = d(x,t) + f_\mu - (\frac{1}{2}(M + T(t) + X_\mu(x)) + \frac{3-\gamma}{4} l_3 X_\mu(x)), \\ d_2(x,t) = -(\frac{1}{2}(M + T(t) + X_\mu(x)) + \frac{3-\gamma}{4} l_3 X_\mu(x)) \leq 0 \end{cases} \tag{2.13}$$

when $1 < \gamma \leq 3$.

Proof of Lemma 2. First, if $\gamma > 3$, we choose $c = 2\delta$ in (1.15),(2.5) and (2.6). Since

$$\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \int_{2\delta}^{\rho} \rho^{\frac{\gamma-3}{2}} d\rho \leq \rho^{\frac{\gamma-3}{2}} \int_{2\delta}^{\rho} 1 d\rho = \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)},$$

the following terms in (2.5)

$$
\begin{aligned}
&L_{1v} = B_t + (\textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})B_x - (1-\varepsilon_1)B(x,t)B_x + f_\mu(x,t)u \\
&\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)(\textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho - B(x,t) - v) \\
&+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - l_3\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau)X_\mu(x) \\
&\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - f_\mu(x,t)v + (1-\varepsilon_1)l_3X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&-(1-\varepsilon_1)l_3^2X_\mu(x)\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau - (M+T(t)+X_\mu(x))(\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho + B(x,t))
\end{aligned}
\tag{2.14}
$$

due to $|f_\mu(x,t)| \leq M + T(t) + X_\mu(x)$.

Since

$$
\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \frac{1}{2}(w+z) = \frac{1}{2}(v+s) + e^{l_1+l_2\int_0^t M+T(\tau)d\tau}, \tag{2.15}
$$

we have from (2.14) that

$$
\begin{aligned}
&L_{1v} = -f_\mu(x,t)v - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x)) \\
&+l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + (1-\varepsilon_1)l_3X_\mu(x,t)e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&-(1-\varepsilon_1)l_3^2X_\mu(x)\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau - 2(M+T(t)+X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&+l_3(M+T(t)+X_\mu(x))\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau \\
&\geq -f_\mu(x,t)v - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x)) \\
&+(l_2-2)(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + (1-\varepsilon_1)l_3(\tfrac{1}{2}e^{l_1} - l_3|X_\mu(x)|_{L^1})X_\mu(x) \\
&+(\tfrac{1}{2}(1-\varepsilon_1)l_3 - 2)X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&\geq -f_\mu(x,t)v - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x))
\end{aligned}
\tag{2.16}
$$

if we choose $l_2 \geq 2, l_3 > 4$ and $e^{l_1} \geq 2l_3|X_\mu(x)|_{L^1}$.

Similarly, the following terms in (2.6)

$$L_{1s} = C_t + (\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho)C_x + (1-\varepsilon_1)C(x,t)C_x - f_\mu(x,t)u$$

$$\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)(s + C(x,t) - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho)$$

$$+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + l_3 \textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau)X_\mu(x)$$

$$\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)s + (1-\varepsilon_1)l_3X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$+(1-\varepsilon_1)l_3^2X_\mu(x)\textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau - (M+T(t)+X_\mu(x))(\int_{2\delta}^{\rho} \frac{\sqrt{P'(\rho)}}{\rho} d\rho + C(x,t))$$

$$= f_\mu(x,t)s - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x))$$

$$+l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + (1-\varepsilon_1)l_3X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$+(1-\varepsilon_1)l_3^2X_\mu(x)\textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau - 2(M+T(t)+X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$-l_3(M+T(t)+X_\mu(x))\textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau$$

$$\geq f_\mu(x,t)s - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x))$$

$$+(l_2-2)(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - l_3(M+T(t))\textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau$$

$$+((1-\varepsilon_1)l_3-1)l_3X_\mu(x)\textstyle\int_{-\infty}^{x} X_\mu(\tau)d\tau$$

$$+((1-\varepsilon_1)l_3-2)X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$\geq f_\mu(x,t)s - \tfrac{1}{2}(v+s)(M+T(t)+X_\mu(x)) \tag{2.17}$$

if we choose $l_3 > 2$ and $(l_2-2)e^{l_1} \geq l_3|X_\mu(x)|_{L^1}$.

So, we may choose $l_2 = 3, l_3 = 5, e^{l_1} \geq 10|X_\mu(x)|_{L^1}$ such that both (2.16) and (2.17) are true.

If $1 < \gamma \leq 3$, we let $c = 0$ in (1.15), (2.5) and (2.6). Then

$$z(\rho,u) = \frac{1}{\theta}\rho^\theta - u, \quad w(\rho,u) = \frac{1}{\theta}\rho^\theta + u \tag{2.18}$$

and

$$\rho^\theta = \frac{\theta}{2}(w+z) = \frac{\theta}{2}(v+s) + \theta e^{l_1+l_2\int_0^t M+T(\tau)d\tau}, \tag{2.19}$$

where $\theta = \frac{\gamma-1}{2}$. Moreover,

$$2\delta\rho^{\theta-1} \leq (2\delta)^\theta, \quad \text{when} \quad 1 < \gamma \leq 3. \tag{2.20}$$

Thus the following terms in (2.5)

$$L_{2v} = B_t + (\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)})B_x - (1-\varepsilon_1)B(x,t)B_x + f_\mu(x,t)u$$

$$\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)(\tfrac{1}{\theta}\rho^\theta - B(x,t) - v)$$

$$+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - l_3\int_{-\infty}^x X_\mu(\tau)d\tau)X_\mu(x)$$

$$-l_3X_\mu(x)\tfrac{3-\gamma}{\gamma-1}\rho^\theta - (2\delta)^\theta l_3 X_\mu(x)$$

$$\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - f_\mu(x,t)v - (M+T(t)+X_\mu(x))(\tfrac{1}{\theta}\rho^\theta + B(x,t))$$

$$+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - l_3\int_{-\infty}^x X_\mu(\tau)d\tau)X_\mu(x)$$

$$-l_3X_\mu(x)\tfrac{3-\gamma}{\gamma-1}\rho^\theta - (2\delta)^\theta l_3 X_\mu(x)$$

$$= -f_\mu(x,t)v - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s)$$

$$-(M+T(t)+X_\mu(x)+\tfrac{3-\gamma}{2}l_3X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$-(M+T(t)+X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + (M+T(t)+X_\mu(x))l_3\int_{-\infty}^x X_\mu(\tau)d\tau$$

$$+(1-\varepsilon_1)l_3X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$-(1-\varepsilon_1)l_3^2X_\mu(x)\int_{-\infty}^x X_\mu(\tau)d\tau - (2\delta)^\theta l_3X_\mu(x)$$

$$\geq -f_\mu(x,t)v - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s)$$

$$+(l_2-2)(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + [\tfrac{1}{2}(\tfrac{\gamma-1}{2}-\varepsilon_1)e^{l_1} - (1-\varepsilon_1)l_3|X_\mu(x)|_{L^1}]l_3X_\mu(x)$$

$$+[\tfrac{1}{2}(\tfrac{\gamma-1}{2}-\varepsilon_1)l_3 - (2+(2\delta)^\theta l_3)]X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau}$$

$$\geq -f_\mu(x,t)v - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s) \tag{2.21}$$

if we choose $l_2 \geq 2$, $\frac{\gamma-1}{2}l_3 > 4$ and $\frac{\gamma-1}{2}e^{l_1} \geq 2l_3|X_\mu(x)|_{L^1}$.

Similarly, the following terms in (2.6)

$$
\begin{aligned}
L_{2s} &= C_t + (\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \textstyle\int_{2\delta}^{\rho}\tfrac{\sqrt{P'(\rho)}}{\rho}d\rho)C_x + (1-\varepsilon_1)C(x,t)C_x - f_\mu(x,t)u \\
&\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)(s+C(x,t)-\tfrac{1}{\theta}\rho^\theta) \\
&+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + l_3\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau)X_\mu(x) \\
&-l_3X_\mu(x)\tfrac{3-\gamma}{\gamma-1}\rho^\theta - (2\delta)^\theta l_3X_\mu(x) \\
&\geq l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + f_\mu(x,t)s - (M+T(t)+X_\mu(x))(\tfrac{1}{\theta}\rho^\theta + C(x,t)) \\
&+(1-\varepsilon_1)l_3(e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + l_3\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau)X_\mu(x) \\
&-l_3X_\mu(x)\tfrac{3-\gamma}{\gamma-1}\rho^\theta - (2\delta)^\theta l_3X_\mu(x) \\
&= f_\mu(x,t)s - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s) \\
&-(M+T(t)+X_\mu(x)+\tfrac{3-\gamma}{2}l_3X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} + l_2(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&-(M+T(t)+X_\mu(x))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - (M+T(t)+X_\mu(x))l_3\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau \\
&+(1-\varepsilon_1)l_3X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&+(1-\varepsilon_1)l_3^2X_\mu(x)\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau - (2\delta)^\theta l_3X_\mu(x) \\
&\geq f_\mu(x,t)s - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s) \\
&+(l_2-2)(M+T(t))e^{l_1+l_2\int_0^t M+T(\tau)d\tau} - l_3(M+T(t))\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau \\
&+[(\tfrac{\gamma-1}{2}-\varepsilon_1)l_3 - 2 - (2\delta)^\theta l_3)]X_\mu(x)e^{l_1+l_2\int_0^t M+T(\tau)d\tau} \\
&+((1-\varepsilon_1)l_3-1)l_3X_\mu(x)\textstyle\int_{-\infty}^x X_\mu(\tau)d\tau \\
&\geq f_\mu(x,t)s - (\tfrac{1}{2}(M+T(t)+X_\mu(x)) + \tfrac{3-\gamma}{4}l_3X_\mu(x))(v+s)
\end{aligned} \tag{2.22}
$$

if we choose $\frac{\gamma-1}{2}l_3 > 2$ and $(l_2-2)e^{l_1} \geq l_3|X_\mu(x)|_{L^1}$.

So, we may choose $l_2 = 3, \frac{\gamma-1}{2}l_3 > 4, \frac{\gamma-1}{2}e^{l_1} \geq 2l_3|X_\mu(x)|_{L^1}$ such that both (2.21) and (2.22) are true.

Therefore, the inequalities in (2.11) are proved. Under the conditions given in (1.14), it is clear that $v(x,0) \leq 0, s(x,0) \leq 0$, so, we may apply the maximum principle given in the following Lemma 3 to (2.11) to obtain the estimates $v(x,t) \leq 0, s(x,t) \leq 0$, and so the estimates in (1.16).

Lemma 3 *If $b_2(x,t) \leq 0, d_2(x,t) \leq 0$, and $v(x,0) \leq 0, s(x,0) \leq 0$ at the time $t = 0$, then the maximum principle is true to the functions $v(x,t)$ and $s(x,t)$ given in the inequalities (2.11), namely, $v(x,t) \leq 0, s(x,t) \leq 0$ for all $t > 0$.*

Proof of Lemma 3: Make a transformation

$$v = (\bar{v} + \frac{N(x^2 + qLe^t)}{L^2})e^{\beta t}, \quad s = (\bar{s} + \frac{N(x^2 + qLe^t)}{L^2})e^{\beta t}, \tag{2.23}$$

where L, q, β are suitable positive constants and N is the upper bound of v, s on $R \times [0,T]$ (N can be obtained by the local existence). The functions $\bar{v}, \bar{s}$, as are easily seen, satisfy the equations

$$\begin{cases} \bar{v}_t + a(x,t)\bar{v}_x - \varepsilon\bar{v}_{xx} + (\beta + b_1(x,t))\bar{v} + b_2(x,t)\bar{s} \\ \qquad \leq -(qLe^t + 2xa(x,t) - 2\varepsilon)\dfrac{N}{L^2} - (\beta + b_1(x,t) + b_2(x,t))\dfrac{N(x^2 + qLe^t)}{L^2}, \\ \\ \bar{s}_t + c(x,t)\bar{s}_x - \varepsilon\bar{s}_{xx} + (\beta + d_1(x,t))\bar{s} + d_2(x,t)\bar{v} \\ \qquad \leq -(qLe^t + 2xa(x,t) - 2\varepsilon)\dfrac{N}{L^2} - (\beta + d_1(x,t) + d_2(x,t))\dfrac{N(x^2 + qLe^t)}{L^2}, \end{cases} \tag{2.24}$$

resulting from (2.11). Moreover

$$\bar{v}(x,0) = v(x,0) - \frac{N(x^2 + qL)}{L^2} < 0, \;\; \bar{s}(x,0) = s(x,0) - \frac{N(x^2 + qL)}{L^2} < 0, \tag{2.25}$$

$$\bar{v}(+L,t) < 0, \; \bar{v}(-L,t) < 0, \; \bar{s}(+L,t) < 0, \; \bar{s}(-L,t) < 0. \tag{2.26}$$

From (2.24),(2.25) and (2.26), we have

$$\bar{v}(x,t) < 0, \quad \bar{s}(x,t) < 0, \quad \text{on} \quad (-L,L) \times (0,T). \tag{2.27}$$

If (2.27) is violated at a point $(x,t) \in (-L,L) \times (0,T)$, let $\bar{t}$ be the least upper bound of values of t at which $\bar{v} < 0$ (or $\bar{s} < 0$); then by the continuity we see that $\bar{v} = 0, \bar{s} \leq 0$ at some points $(\bar{x},\bar{t}) \in (-L,L) \times (0,T)$. So

$$\bar{v}_t \geq 0, \quad \bar{v}_x = 0, \quad -\varepsilon\bar{v}_{xx} \geq 0, \quad \text{at} \quad (\bar{x},\bar{t}). \tag{2.28}$$

If we choose sufficiently large constants q, β (which may depend on the bound of the local existence) such that

$$qL + 2xa(x,t) - 2\varepsilon > 0, \quad \beta + b_1(x,t) + b_2(x,t) > 0 \quad \text{on} \quad (-L,L) \times (0,T). \tag{2.29}$$

(2.28) and (2.29) give a conclusion contradicting the first inequality in (2.24). So (2.27) is proved. Therefore, for any point $(x_0,t_0) \in (-L,L) \times (0,T)$,

$$v(x_0,t_0) < (\frac{N(x_0^2 + qLe_0^t)}{L^2})e^{\beta t_0}, \quad s(x_0,t_0) < (\frac{N(x_0^2 + qLe_0^t)}{L^2})e^{\beta t_0}, \tag{2.30}$$

which gives the desired estimates $v \leq 0, s \leq 0$ if we let L go to infinity. So Lemma 3 is proved.

From the upper estimates in (1.16), we can use the Riemann invariants (1.15) to obtain the uniformly bounded estimates on $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$ directly

$$2\delta \leq \rho^{\varepsilon,\delta,\mu}(x,t) \leq M(t), \quad |u^{\varepsilon,\delta,\mu}(x,t)| \leq M(t), \tag{2.31}$$

for a suitable bounded function $M(t)$, which is independent of ε, δ, μ.

The local existence result of the Cauchy problem (1.8)-(1.9) can be easily obtained by applying the contraction mapping principle to an integral representation of a solution. Following the standard theory of semilinear parabolic systems. Whenever we have an a priori L^∞ estimate (2.31) on the local solution, it is clear that the local time can be extended to an arbitrary time T step by step since the step time depends only on the L^∞ norm. So **Part I of Theorem** 1 is proved.

To complete the proof of Theorem 1, we shall prove that there exists a subsequence of the viscosity-flux approximate solutions $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$ of the Cauchy problem (1.8) and (1.9), which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as ε, δ, μ tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2).

First, by simple calculations, for smooth solutions, the following two systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = 0, \end{cases} \tag{2.32}$$

and

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ u_t + (\frac{1}{2}u^2 + \int_{2\delta}^{\rho} \frac{(t-2\delta)P'(t)}{t^2} dt)_x = 0 \end{cases} \tag{2.33}$$

are equivalent, and particularly, both systems have the same entropy-entropy flux pairs, where $P_1(\rho,\delta)$ is given in (1.10).

Thus any entropy-entropy flux pair $(\eta(\rho,m), q(\rho,m))$ of system (2.32) satisfies the additional system

$$q_\rho = u\eta_\rho + \frac{(\rho-2\delta)P'(\rho)}{\rho^2}\eta_u, \quad q_u = (\rho-2\delta)\eta_\rho + u\eta_u. \tag{2.34}$$

Eliminating the q from (2.34), we have

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{2.35}$$

Therefore, system (2.32) and system (1.1) have the same entropies.

We recall that for the case of polytropic gas, any weak entropy [Di] can be represented by the following explicit formula:

$$\eta_0(\rho, u) = \rho \int_0^1 [\tau(1-\tau)]^\lambda g(u + \rho^\theta - 2\rho^\theta \tau) d\tau, \tag{2.36}$$

where $\theta = \frac{\gamma-1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}$ and g is a smooth function.

Second, for general pressure $P(\rho)$, we have the following lemma

Lemma 4 *Suppose the viscosity-flux approximate solutions $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$ of the Cauchy problem (1.8) and (1.9) are uniformly bounded in L^∞ space, and the limit*

$$\lim_{\rho \to 0} \frac{(P'(\rho))^{\frac{3}{2}}}{\rho P''(\rho)} = e, \tag{2.37}$$

where $e \geq 0$ is a constant. If the weak entropy-entropy flux pair $(\eta(\rho, u), q(\rho, u))$ of system (1.1) is in the form $\eta(\rho, u) = \rho H(\rho, u)$ and $H_u(\rho, u), H_{uu}(\rho, u), H_{uuu}(\rho, u)$ are continuous on $0 \leq \rho \leq M_1, |u| \leq M_1$, where M_1 is a positive constant, then

$$\eta_t(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t)) + q_x(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t)) \tag{2.38}$$

is compact in $H^{-1}_{loc}(R \times R^+)$ as $\varepsilon = o(\frac{P'(2\delta)}{2\delta})$ and δ, μ tend to zero, with respect to the viscosity solutions $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$ of the Cauchy problem (1.8) and (1.9).

Proof of Lemma 4. For the homogeneous case, namely $\alpha(x,t,\rho,u) = 0$, the proof of Lemma 4 was given in [Lu1]. In a similar way, we may obtain the proof of Lemma when $\alpha(x,t,\rho,u)$ satisfies the condition (1.3).

Clearly, for the polytropic gas, $P(\rho) = \frac{1}{\gamma}\rho^\gamma$ and for any $\gamma > 1$, all the conditions about the pressure function (2.37) and the weak entropies in Lemma 4 are satisfied. Thus we may apply the H^{-1} compactness of (2.38), and the convergence frameworks given in [Chen, DCL, Di, LPS] for $1 < \gamma < 3$ and in [LPT] for $\gamma \geq 3$ to select a subsequence, of $(\rho^{\varepsilon,\delta,\mu}(x,t), u^{\varepsilon,\delta,\mu}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as ε, δ, μ tend to zero.

Finally, it is easy to prove that the limit $(\rho(x,t), u(x,t))$ satisfies (1.18). Moreover, for any weak convex entropy-entropy flux pair $(\eta(\rho, m), q(\rho,)), m = \rho u$, of system (1.1), we multiply (1.8) by (η_ρ, η_m) to obtain that

$$\begin{aligned}
&\eta_t(\rho^{\varepsilon,\delta,\mu}(x,t), m^{\varepsilon,\delta,\mu}(x,t)) + q_x(\rho^{\varepsilon,\delta,\mu}(x,t), m^{\delta,\varepsilon,\mu}(x,t)) + \delta q_{1x}(\rho^{\varepsilon,\delta,\mu}(x,t), m^{\varepsilon,\delta,\mu}(x,t)) \\
&= \varepsilon \eta(\rho^{\varepsilon,\delta,\mu}, m^{\varepsilon,\delta,\mu})_{xx} - \varepsilon(\rho_x^{\varepsilon,\delta,\mu}, m_x^{\varepsilon,\delta,\mu}) \cdot \nabla^2 \eta(\rho^{\varepsilon,\delta,\mu}, m^{\varepsilon,\delta,\mu}) \cdot (\rho_x^{\varepsilon,\delta,\mu}, m_x^{\varepsilon,\delta,\mu})^T \\
&-a_\mu(x,t)|m^{\varepsilon,\delta,\mu}|\eta_m(\rho^{\varepsilon,\delta,\mu}, m^{\varepsilon,\delta,\mu}) \\
&\leq \varepsilon \eta(\rho^{\varepsilon,\delta,\mu}, m^{\varepsilon,\delta,\mu})_{xx} - a_\mu(x,t)|m^{\varepsilon,\delta,\mu}|\eta_m(\rho^{\varepsilon,\delta,\mu}, m^{\varepsilon,\delta,\mu}),
\end{aligned} \tag{2.39}$$

where $q + \delta q_1$ is the entropy flux of system (1.8) corresponding to the entropy η. Thus the entropy inequality (1.19) is proved if we multiply a test function to (2.39) and let ε, δ, μ go to zero. So, **Theorem 1 is proved.**

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