HYPOCOERCIVITY OF THE LINEARIZED BGK-EQUATION WITH STOCHASTIC COEFFICIENTS

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Abstract. In this paper we study the effect of randomness on a linearized BGK-model in one dimension. We prove exponential decay rate to a global equilibrium. This decay rate can be proven to be independent of the stochastic influence in a physical reasonable norm. We will further discuss the decay rate of the n-th derivative with respect to the stochastic variable of the solutions. Our strategy is based on Lyapunov’s method as it is presented in [AAC16, AAC18]. The matrices we need for a Lyapunov’s estimate now depend on the stochastic variable. This requires a careful analysis of the random effect.

Key words. linear BGK-equation with uncertainties, hypocoercivity, decay estimate, Lyapunov’s direct method

AMS subject classifications. 35B40, 35F25, 35Q62

1. Introduction. This article is concerned with hypocoercivity-estimates of a randomized BGK-model in one dimension. Hypocoercivity was made widely known by Villani [Vil09] for equations of the form \( \frac{d}{dt} f = -Lf \), where the generator \( L \) is not coercive, but where solutions still exhibit exponential decay in time. The long-time behavior has been studied for a large variety of equations. Some considerable examples are the Fokker-Plank equations [AAS15, AE14], kinetic equations [DMS15, FS20, NS15], a multi-species Boltzmann system [DJMZ16] as well as the BGK-equations [AAC16, AAC18, LP19]. Especially in [AAC16, AAC18, AE14] it was an issue to find sharp exponential decay rates.

Uncertainty is natural for many physical equations. This may have various reasons, like modelling errors or blurred measurements. Thus it is not always sufficient to look for the exact solution. Also a careful study of the uncertainty effect and their long-time behavior is required. Such an analysis was made for linear equations in [LJ18, IW17, AJW20], and for the multi-species Boltzmann equation in [DJL19].

Nowadays many numerical methods with the aim to address the issues related to uncertainties have been developed. These can be classified in Traditional methods and Spectral-methods. Well known representatives of the Traditional methods are the Monte-Carlo method, the moment equation approach and the perturbation methods. Probably the best known Spectral-methods are the (Galerkin) generalized polynomial chaos method and the stochastic collocation method.

A review of spectral type methods can be found in [Xiu10]. Let us have a quick look at the latter two methods. The stochastic collocation method is a nonintrusive method, which means that it takes recourse to a deterministic solver. The main steps are

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1 Named after the physicists Bhatnagar-Gross-Krook [BGG54].

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1. choose a set of nodes in the random space
2. solve the deterministic problem at each node
3. construct (polynomial chaos) polynomials that coincide with the solution at each node.

By contrast to this proceeding the generalized polynomial chaos method needs an implementation independent of the deterministic solver, which is called intrusive. Here, very roughly spoken, one first tries to get rid of the stochastic dimension by some orthogonal polynomial expansions and then one needs to solve the resulting system of equations.

One thing spectral methods have in common is that they provide a higher order of accuracy if the solution has a high level of regularity. Thus it is a common procedure to check the derivatives or show boundlessness or even decay in time in some reasonable norm. As we deal with a kinetic equation in this article, we want to point out [LW17], where such a regularity condition has been studied for a large set of kinetic equations. The paper contains the linear BGK-operator with constant velocity and temperature (where only mass is conserved).

This article extends the results in [LW17] to the linearized BGK equation (2.1) which also contains the mean velocity and the temperature of the distribution function and which also satisfies conservation of momentum and energy. We will show exponential decay in time with a rate $-\lambda$ independent of the random variable and $\lambda$ strictly positive in a physical reasonable norm. To do so, we use the technique developed in [AAC16, AAC18]. The advantage of this approach is that we directly inherit the optimization strategies made in these articles. This has to be understood as kind of an a priori estimate, which means that we find sharp decay rates which serve as lower bound for all possible realizations. This means, the slowest possible decay rate which can be realized tends to be sharp in the sense of [AAC16, AAC18].

Furthermore the resulting decay rates are direct computable, which can be very useful from time to time. Moreover we show, that this decay rate $\lambda$ also holds for the decay of the derivatives in the random space. That means, computing such a decay rate $\lambda$ for the underlying BGK equation once, gives us immediately a (lower bound of the) decay rate for the derivatives in the random space without it being necessary to compute new decay rates for its $z$-derivatives.

We will start with the introduction of a linearized BGK-model with uncertainties in one dimension in section 2. Here the linearized BGK equation established in [AAC18] will serve as foundation.

Section 3 is devided in three parts. In the first two subsections we will extend Lyapunov’s direct method in infinite dimensions to stochastic matrices. This is an crucial step on our search for decay rates and directly leads to our first decay estimate presented in this section. Finally, in the third part we deal with decay estimates in $z$-derivatives. The main idea here is to benefit from two Gronwall-like estimate theorems presented first in [LW17]. Our major task in this final subsection is to transform our estimates made so far in the right shape so we can take advantage of [LW17].

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2In these articles the authors present a extension of Lyapunov’s direct method to infinite dimensions. In contrast to the finite dimensions counterpart, which leads to exact decay estimates, the presented approach does not jet fulfill the same goal. Nevertheless this method reveals good and reasonable approximations.
As a consequence, we are not longer able to speak of sharp decay estimates in the derivatives, only of lower bounds.

2. A linearized BGK model with uncertainties. We want to extend the linearized BGK equation established in [AAC18]. Therefore we will add a random component in both, the initial data and the RHS operator. We expect $z$ to be a continuous random variable $(\Omega, \Sigma, P) \mapsto (\mathbb{R}, B)^3$, which maps from a random space to $\mathcal{O} \subseteq \mathbb{R}$, where $\mathcal{O}$ is either $\mathbb{R}$ or an interval. More exact, we consider the equation

\begin{align}
\partial_t h(x, v, t, z) + v \partial_x h(x, v, t, z) = L_z(h(x, v, t, z))
\end{align}

with $L_z$ defined as

\[ L_z(h(x, v, t, z)) := \sigma(z) L(h(x, v, t, z)). \]

Here $\sigma(z)$ is a continuous function form $\mathcal{O} \subseteq \mathbb{R} \to \mathbb{R}$ and $L$ is given as

\[ L := M_1(v) \left[ \left( \frac{3}{2} - \frac{v^2}{2} \right) \omega(x, t, z) + v \mu(x, t, z) + \left( -\frac{1}{2} + \frac{v^2}{2} \right) \tau(x, t, z) \right] - h(x, v, t, z), \]

where we set

\begin{align}
\omega(x, t, z) &:= \int_\mathbb{R} h(x, v, t, z) \, dv \\
\mu(x, t, z) &:= \int_\mathbb{R} vh(x, v, t, z) \, dv \\
\tau(x, t, z) &:= \int_\mathbb{R} v^2 h(x, v, t, z) \, dv \\
M_1(v) &:= (2\pi)^{-\frac{1}{2}} e^{-\frac{v^2}{4}}.
\end{align}

Because of the conservation of mass, momentum and energy (see [AAC18]) we have

\begin{align}
\int_\tilde{T} \omega(x, 0, z) \, dx = 0, & \quad \int_\tilde{T} \mu(x, 0, z) \, dx = 0, & \quad \int_\tilde{T} \tau(x, 0, z) \, dx = 0,
\end{align}

where $\tilde{T} := \frac{L}{2\pi} T$ is the torus of side length $L$. To prepare for the following proofs, we want to rewrite (2.1) into an (infinite dimensional) system of differential equations. In doing so, we will proceed mainly as presented in [AAC18, AAC16].

To get rid of the x-derivative, we start with the x-Fourier series of $h(x, v, t, z)$:

\[ h(x, v, t, z) = \sum_{k \in \mathbb{Z}} h_k(v, t, z) e^{ik \frac{2\pi}{L} x}. \]

Inserting this expansion in (2.1), for $t \geq 0$ we get

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\[^3\text{Here } \Sigma \text{ is an } \sigma\text{-algebra on } \Omega \text{ and } B \text{ denotes the borel-}\sigma\text{-algebra.} \]
\[
\frac{\partial}{\partial t} h_k + ik \frac{2\pi}{L} v h_k
= \sigma(z) \left( \mathcal{M}_1(v) \left[ \left( \frac{3}{2} - \frac{v^2}{2} \right) \omega_k + v \mu_k + \left( -\frac{1}{2} + \frac{v^2}{2} \right) \tau_k \right] - h_k \right), \quad k \in \mathbb{Z}
\]
for each spatial mode \( h_k(v, t, z) \) with

\[
\omega_k(t, z) := \int_{\mathbb{R}} h_k(v, t, z) \, dv, \quad \mu_k(t, z) := \int_{\mathbb{R}} v h_k(v, t, z) \, dv,
\]
\[
\tau_k(t, z) := \int_{\mathbb{R}} v^2 h_k(v, t, z) \, dv.
\]

Now we set

\[
g_0(v) := \mathcal{M}_1(v), \quad g_1(v) := v \mathcal{M}_1(v), \quad g_2(v) := \frac{v^2 - 1}{\sqrt{2}} \mathcal{M}_1(v),
\]
so that we can rewrite the equation above as

\[
\frac{\partial}{\partial t} h_k + ik \frac{2\pi}{L} v h_k
= \sigma(z) \left( g_0(v) \omega_k + g_1(v) \mu_k + g_2(v) \frac{1}{\sqrt{2}} (\tau_k - \omega_k) - h_k \right), \quad k \in \mathbb{Z}; \ t \geq 0.
\]

Thanks to

\[
\int_{\mathbb{R}} g_m(v) g_n(v) \mathcal{M}_1^{-1}(v) \, dv = \delta_{mn} \quad \forall \ 0 \leq m, n \leq 2
\]
we can extend \( g_0, g_1, g_2 \) to an orthonormal basis \( \{g_m(v)\}_{m \in \mathbb{N}_0} \) in \( L^2(\mathbb{R}; \mathcal{M}_1^{-1}(v)) \).

**Remark 2.1.** The functions \( g_m(v) \) are the normalized Hermite functions and can be given directly as

\[
g_m(v) := (2\pi m!)^{-\frac{1}{2}} H_m(v) e^{-\frac{v^2}{2}}
\]
with

\[
H_m(v) := (-1)^m e^{\frac{v^2}{2}} \frac{\partial^m}{\partial v^m} e^{-\frac{v^2}{2}}
\]
being the probabilists’ Hermite polynomials. In general, orthogonal polynomials with respect to a positive weight function follow a three-term recursions relation, shown for example in [HB09]. In our case, this relations simplifies to

\[
vg_m(v) = \sqrt{m + 1} g_{m+1}(v) + \sqrt{m} g_{m-1}(v), \quad m \in \mathbb{N}.
\]
Next, we will expand $h_k(\cdot, t, z) \in L^2(\mathbb{R}; M_1^{-1}(v))$ in the orthogonal basis $\{g_m(v)\}_{m \in \mathbb{N}_0}$:

$$h_k(v, t, z) = \sum_{m=0}^{\infty} \hat{h}_{k,m}(t, z) g_m(v) \quad \text{with} \quad \hat{h}_{k,m}(t, z) = \langle h_k(v, \cdot, \cdot), g_m(v) \rangle_{L^2(M_1^{-1})}.$$  

For each $k \in \mathbb{Z}$ the vector $\hat{h}_k(t, z) = (\hat{h}_{k,0}(t, z), \hat{h}_{k,1}(t, z), \ldots)^T \in \ell^2(\mathbb{N}_0)$ contains all Hermite coefficients of $h_k(\cdot, t, z)$. In particular we have

\begin{align}
\hat{h}_{k,0}(t, z) &= \int_{\mathbb{R}} h_k(v, \cdot, \cdot) \underbrace{g_0(v)}_{=M_1^{-1}(v)} \, dv = \omega_k(t, z) \\
\hat{h}_{k,1}(t, z) &= \int_{\mathbb{R}} h_k(v, \cdot, \cdot) \underbrace{g_1(v)}_{=\sigma M_1^{-1}(v)} \, dv = \mu_k(t, z) \\
\hat{h}_{k,2}(t, z) &= \int_{\mathbb{R}} h_k(v, \cdot, \cdot) \underbrace{g_2(v)}_{=\frac{\sqrt{2}}{\sqrt{3}} M_1^{-1}(v)} \, dv = \frac{1}{\sqrt{2}} (\tau_k(t, z) - \omega_k(t, z)).
\end{align}

Hence, (2.5) is equivalent to

\begin{equation}
\frac{\partial}{\partial t} \hat{h}_k + \frac{i}{L} 2 \pi k \hat{h}_k = \sigma(z) \left( g_0(v) \hat{h}_{k,0} + g_1(v) \hat{h}_{k,1} + g_2(v) \hat{h}_{k,2} - \hat{h}_k \right), \quad k \in \mathbb{Z}; \ t \geq 0.
\end{equation}

Thus, with (2.6) the vector of its Hermite coefficients satisfies

\begin{equation}
\frac{\partial}{\partial t} \hat{h}_k(t, z) + \frac{i}{L} 2 \pi k \hat{h}_k(t, z) = -\sigma(z) L_2 \hat{h}_k(t, z), \quad k \in \mathbb{Z}; \ t \geq 0
\end{equation}

with the operators $L_1$, $L_2$ represented by the (infinite) matrices

\begin{equation}
L_1 := \begin{pmatrix}
0 & \sqrt{1} & 0 & \cdots \\
\sqrt{1} & 0 & \sqrt{2} & 0 \\
0 & \sqrt{2} & 0 & \sqrt{3} \\
\vdots & 0 & \sqrt{3} & \ddots
\end{pmatrix}, \quad L_2 := \text{diag} \ (0, 0, 0, 1, 1, \cdots)
\end{equation}

or

\begin{equation}
\frac{\partial}{\partial t} \hat{h}_k(t, z) = -C_k \hat{h}_k(t, z) \quad k \in \mathbb{Z}; \ t \geq 0 \quad \text{with} \quad C_k := \frac{i}{L} 2 \pi k L_1 + \sigma(z) L_2.
\end{equation}

In the following, we will also need the $n$-th derivative of (2.1) with respect to $z$.

\begin{equation}
(2.12) \quad \partial_{z^{(n)}} \partial_t h(x, v, t, z) + v \partial_{z^{(n)}} \partial_z h(x, v, t, z) = \partial_{z^{(n)}} \left( L_z(h(x, v, t, z)) \right)
\end{equation}
With the same approach as above, this leads to

\[(2.13) \quad \frac{\partial^{(n)}}{\partial z^{(n)}} \frac{\partial}{\partial t} \hat{h}_k(t, z) = \]

\[-ik \frac{2\pi}{L} \frac{\partial^{(n)}}{\partial z^{(n)}} \hat{h}_k(t, z) - \sum_{i=0}^{n} \left( \frac{\partial^{(i)}}{\partial z^{(i)}} \sigma(z) \frac{\partial \hat{h}_k(t, z)}{\partial z^{(n-i)}} \right) \]

for \( k \in \mathbb{Z}; \ t \geq 0 \). Alternatively, directly differentiating (2.11) \( n \) times with respect to \( z \) leads to the same result.

### 3. Decay rate for a linearized BGK model with uncertainties.

The following theorem 3.1 presents a matrix-inequality, which is of major importance in the following decay theorems. The structure of the inequality is motivated by Lyaponov’s method in finite dimensional spaces. Motivated by [AAC18] we define the matrices \( P_k \) as

\[
P_k := \begin{pmatrix}
1 & -\frac{i\alpha}{k} & 0 & 0 \\
\frac{i\alpha}{k} & 1 & -\frac{i\beta}{k} & 0 \\
0 & \frac{i\beta}{k} & 1 & -\frac{i\gamma}{k} \\
0 & 0 & \frac{i\gamma}{k} & 1 \\
0 & 0 & 0 & I
\end{pmatrix} \quad k \in \mathbb{N}
\]

with \( I \) being the identity matrix and \( \alpha, \beta, \gamma \in \mathbb{R} \) will be chosen later in theorem 3.1.

#### 3.1. Basic inequality estimates.

**Theorem 3.1 (Matrix inequality).** Assume \( 0 < L, 0 < \sigma_{\text{min}} \leq \sigma(z) \leq \sigma_{\text{max}} \), choose the matrices \( P_k \) as in (3.1) and \( C_k \) from (2.11), there exists a \( \alpha_{\text{max}} > 0 \), such that with \( \alpha \in (0, \alpha_{\text{max}}) \), \( \beta = \sqrt{2\alpha}, \gamma = \sqrt{3\alpha} \) the matrices \( P_k \) and \( C_k^* P_k + P_k C_k \) are positive definite for all \( k \in \mathbb{Z} \setminus \{0\} \). Moreover,

\[
C_k^* P_k + P_k C_k \geq 2\mu P_k \quad \text{uniformly in } |k| \in \mathbb{N}
\]

with \( \mu > 0 \) defined in (3.14).

**Proof.** Note first, that \( C_k^* P_k + P_k C_k \) has the form of a block-diagonal-matrix

\[
\begin{pmatrix}
D_{k, \alpha, \beta, \gamma, \sigma(z)} & 0 \\
0 & I
\end{pmatrix}
\]

with \( I \) being \( 2\sigma(z) \) times the (infinite dimensional) identity matrix and

\[
D_{k, \alpha, \beta, \gamma, \sigma(z)} :=
\begin{pmatrix}
2l\alpha & 0 & l(\sqrt{2\alpha} - \beta) & 0 & 0 \\
0 & 2l(\sqrt{2\beta} - \alpha) & 0 & l(\sqrt{3\beta} - \sqrt{2\gamma}) & 0 \\
l(\sqrt{2\alpha} - \beta) & 0 & 2l(\sqrt{3\beta} - \sqrt{2\gamma}) & -\frac{i\gamma\sigma(z)}{k} & 2l\gamma \\
0 & l(\sqrt{3\beta} - \sqrt{2\gamma}) & \frac{i\gamma\sigma(z)}{k} & 2\sigma(z) - 2l\sqrt{3\gamma} & 0 \\
0 & 0 & 2l\gamma & 2\sigma(z) & 0
\end{pmatrix}
\]

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where $I := \frac{2\pi}{L}$. Because of $0 < \sigma_{\min} < 2\sigma(z)$ the matrix $\tilde{I}$ is already positive definite, such that it only remains to show the positive definiteness of $D_{k,\alpha,\beta,\gamma,\sigma(z)}$. However, instead of seeking $\alpha$, $\beta$, $\gamma$ such that the matrix $D_{k,\alpha,\beta,\gamma,\sigma(z)}$ is positive definite for all $k \in \mathbb{Z} \setminus \{0\}$, we simplify the problem by setting $\beta = \sqrt{2\alpha}$ and $\gamma = \sqrt{3\alpha}$. Thus, we get

$$D_{k,\alpha,\sigma(z)} := \begin{pmatrix}
2l\alpha & 0 & 0 & 0 & 0 \\
0 & 2l\alpha & 0 & 0 & 0 \\
0 & 0 & 2l\alpha & -\frac{i\sqrt{3}\sigma(z)}{k} & 2\sqrt{3l\alpha} \\
0 & 0 & \frac{i\sqrt{3}\sigma(z)}{k} & 2\sigma(z) - 6l\alpha & 0 \\
0 & 0 & 2\sqrt{3l\alpha} & 0 & 2\sigma(z)
\end{pmatrix}$$

which is a way more comfortable structure to analyze. However, one has to keep in mind, that we have to pay for this with a reduction of the decay rate.

Now we will use Sylvester’s criterion to find a sufficient condition for $\alpha$, such that the matrix $D_{k,\alpha,\sigma(z)}$ is positive definite for all $k \in \mathbb{Z} \setminus \{0\}$. Therefore we define

$\delta_j(k,\alpha,\sigma(z))$ as the determinant of the lower right $j \times j$ submatrix of $D_{k,\alpha,\sigma(z)}$ with $1 \leq j \leq 5$ and search for assumptions on $\alpha$, which lead to $\delta_j(k,\alpha,\sigma(z)) > 0$ \forall $1 \leq j \leq 5$. Thus we get

- $\delta_1(k,\alpha,\sigma(z)) = 2\sigma(z)$, which is already bigger than zero because of the assumption $0 < \sigma_{\min} \leq \sigma(z) \leq \sigma_{\max}$.
- $\delta_2(k,\alpha,\sigma(z)) = 4\sigma(z) (\sigma(z) - 3l\alpha)$, which leads to the condition

$$\alpha < \frac{\sigma(z)}{3l}.$$  \hfill (3.2)

- $\delta_3(k,\alpha,\sigma(z)) = \alpha \left(72l^3\alpha^2 - \left(48l^2\sigma(z) + \frac{6\sigma(z)^3}{k^2}\right)\alpha + 8l\sigma(z)^2\right) \geq \alpha \left(72l^3\alpha^2 - \left(48l^2\sigma(z) + 6\sigma(z)^3\right)\alpha + 8l\sigma(z)^2\right) = \delta_3(1,\alpha,\sigma(z)) \hfill (3.3)$

and $\delta_3(1,\alpha,\sigma(z))$ is bigger than zero if $^4$

$$0 < \alpha < \frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^4\sigma(z)^4 + \sigma(z)^6}}{24l^3}. \hfill (3.4)$$

- $\delta_4(k,\alpha,\sigma(z)) = 2\alpha^2l \left(72l^3\alpha^2 - \left(48l^2\sigma(z) + \frac{6\sigma(z)^3}{k^2}\right)\alpha + 8l\sigma(z)^2\right) = 2\alpha l \delta_3(k,\alpha,\sigma(z))$,

thus $\delta_4(k,\alpha,\sigma(z))$ is positive, if (3.4) holds.

$^4$In order to ensure $6\alpha^2 < 1$, which we will need later in the proof, we already drop the possibility of "large" $\alpha$ at this point.

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\begin{equation}
\delta_5(k, \alpha, \sigma(z)) = 4\alpha^3 l^2 \left(72l^3 \alpha^2 - \left(48l^2 \sigma(z) + \frac{6\sigma(z)^3}{k^2}\right) \alpha + 8l \sigma(z)^2\right)
\end{equation}

which is positive, if (3.4) holds.

So, to make sure that \(D_{k, \alpha, \sigma(z)}\) is positive definite, we need to choose an \(\alpha\) such that (3.2) and (3.4) hold. However, because of

\begin{equation}
\frac{8l^2 \sigma(z) + \sigma(z)^3}{24l^3} - \sqrt{16l^2 \sigma(z)^4 + \sigma(z)^6} \leq \frac{\sigma(z)}{3l}
\end{equation}

it is sufficient to find an \(\alpha\) such that (3.4) is fulfilled. (3.5) can be shown in a straightforward computation, which we move to lemma A.1 in the appendix, in order to keep the proceeding clear.

However, it still remains to show, that (3.4) makes sense. More precise, we want to show

\begin{equation}
\forall l > 0 \ \exists \alpha_{\text{max}} : 0 < \alpha_{\text{max}} \leq \frac{8l^2 \sigma(z) + \sigma(z)^3 - \sqrt{16l^2 \sigma(z)^4 + \sigma(z)^6}}{24l^3}.
\end{equation}

Therefore, we first note that

\begin{equation}
0 < \frac{1}{24l^3} \sigma(z) \left(\sqrt{64l^4 + 16l^2 \sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2 \sigma(z)^2 + \sigma(z)^4}\right) = \frac{8l^2 \sigma(z) + \sigma(z)^3 - \sqrt{16l^2 \sigma(z)^4 + \sigma(z)^6}}{24l^3} := \alpha(l, \sigma(z))
\end{equation}

because of \(\sigma(z) > 0\) and \(l > 0\). Furthermore \(\alpha(l, \sigma(z))\) is a continuous function, such that for fixed \(l\)

\begin{equation}
\alpha_{\text{max}} := \min_{\sigma(z) \in [\sigma_{\text{min}}, \sigma_{\text{max}}]} \alpha(l, \sigma(z))
\end{equation}

exists. Together with \(W(\sigma(z)) \subseteq [\sigma_{\text{min}}, \sigma_{\text{max}}]\) one gets \(\alpha_{\text{max}} \leq \alpha(l, \sigma(z))\) for arbitrary fixed \(l > 0\). This, together with (3.7) leads to (3.6). Thus we have: \(D_{k, \alpha, \sigma(z)}\) is positive definite for all \(\alpha \in (0, \alpha_{\text{max}})\) with \(\alpha_{\text{max}}\) form (3.8).

The matrices \(P_k\) are positive definite under the assumption \(|\alpha|^2 + |\beta|^2 + |\gamma|^2 < 1\).

Because we set \(\beta = \sqrt{2}\alpha\) and \(\gamma = \sqrt{3}\alpha\) this reduces to \(6\alpha^2 < 1\).

One can compute, that

\begin{equation}
\max_{l > 0, \sigma(z) > 0} \alpha(l, \sigma(z)) = \frac{4}{9\sqrt{3}}
\end{equation}

\(^5\text{Remember that }\sigma(z)\text{ is a continuous function in }z.\)

\(^6\text{\(W\) denotes the value set.}\)
Figure 3.1: The figure shows $\alpha_{\text{max}}$ for some $l$ between 1 and 5. For fixed $l$, each dot represents the value of $\alpha_{\text{max}}$ for $\sigma_{\min}, \sigma_{\max}$ chosen to be $1 - x$ respectively $1 + x$ for some $x$ between 0 and 0.9. The upper point always represents the value in the case $x = 0$ and the lowest point the case $x = 0.9$. So we see the decrease of $\alpha_{\text{max}}$ with both, the increase of the interval $[\sigma_{\min}, \sigma_{\max}]$ and the increase of $l$.

Next, we want to find a lower bound for the smallest eigenvalue of $C_k^*P_k + P_kC_k$. All eigenvalues of $I$ are $2\sigma(z)$ and because of the block diagonal structure, $D_{k,\alpha,\sigma(z)}$ has a double eigenvalue $2l\alpha$ together with the eigenvalues of its lower $3 \times 3$ submatrix

$$D^{(3)}_{k,\alpha,\sigma(z)} := \begin{pmatrix} 2l\alpha & \frac{i\sqrt{3}\alpha \sigma(z)}{k} & 2\sqrt{3l\alpha} \\ \frac{i\sqrt{3}\alpha \sigma(z)}{k} & 2\sigma(z) - 6l\alpha & 0 \\ \frac{2\sqrt{3}l\alpha}{k} & 0 & 2\sigma(z) \end{pmatrix}.$$  

Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the eigenvalues of $D^{(3)}_{k,\alpha,\sigma(z)}$ arranged in increasing order. Because $D^{(3)}_{k,\alpha,\sigma(z)}$ is positive definite for $\alpha \in (0, \alpha_{\text{max}})$, the arithmetic-geometric mean inequality

$$\sum_{i=1}^{n} \frac{x_i}{n} \geq \sqrt[n]{\prod_{i=1}^{n} x_i} \quad \forall \text{nongative } x_i \in \mathbb{R}$$

together with

$$\sum_{i=1}^{3} \lambda_i = \text{Tr} D^{(3)}_{k,\alpha,\sigma(z)}, \quad \prod_{i=1}^{3} \lambda_i = \det D^{(3)}_{k,\alpha,\sigma(z)}.$$
implies
\[
\lambda_1(k, \alpha, \sigma(z)) = \frac{\delta_3(k, \alpha, \sigma(z))}{\lambda_2 \lambda_3} \geq \frac{\delta_3(k, \alpha, \sigma(z))}{\lambda_2} \left( \frac{\lambda_2 + \lambda_3}{2} \right)^{-2} \\
\geq \delta_3(k, \alpha, \sigma(z)) \left( \frac{\text{Tr } D_{k,\alpha,\sigma(z)}^{(3)}}{2} \right)^{-2} \\
= \delta_3(k, \alpha, \sigma(z)) \frac{1}{4(\sigma(z) - \alpha l)^2} > 0.
\]

So, all in all, we need to find a lower bound of \(\min\{2\lambda_\alpha, \frac{\delta_3(k,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2}, 2\sigma(z)\}\). However, with \(\alpha \in (0, \alpha_{\text{max}})\) the following inequality holds:

\[
\min\{2\lambda_\alpha, \frac{\delta_3(k,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2}, 2\sigma(z)\} \equiv \min\{2\lambda_\alpha, \frac{\delta_3(k,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2}\} \\
\stackrel{(3.11)}{=} \min\{2\lambda_\alpha, \frac{\delta_3(1,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2}\} \\
\stackrel{(3.10)}{=} \frac{\delta_3(1,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2} := \lambda(l, \alpha, \sigma(z)).
\]

We verify (\ast) and (\ast\ast) in lemma A.2 in the appendix. To get an estimate independent of \(\omega(z)\), we define for fixed \(l > 0\) and \(\alpha \in (0, \alpha_{\text{max}})\)

\[
\lambda_{\text{min}}(l, \alpha) := \min_{\sigma(z) \in [\sigma_{\text{min}}, \sigma_{\text{max}}]} \lambda(l, \alpha, \sigma(z)) > 0.
\]

As before, because of \(W(\sigma(z)) \subseteq [\sigma_{\text{min}}, \sigma_{\text{max}}]\) one gets \(\lambda_{\text{min}}(l, \alpha) \leq \lambda(l, \alpha, \sigma(z))\). Finally we get, if \(P_k\) is chosen with some \(\alpha \in (0, \alpha_{\text{max}})\), \(\beta = \sqrt{2\alpha}\) and \(\gamma = \sqrt{3\alpha}\) uniformly \(\forall \ | k | \in \mathbb{N}\), then

\[
C_k^* P_k + P_k C_k \geq \lambda_{\text{min}}(l, \alpha) I.
\]

Furthermore, a straightforward computation shows, that the eigenvalues of \(P_k\) are
\[
\{1, 1 \pm \frac{2\sqrt{3} + \sqrt{6}}{k}, 1 \pm \frac{\alpha \sqrt{3} - \sqrt{6}}{k}\}.
\]
These eigenvalues are positive \(\forall \alpha \in (0, \alpha_{\text{max}}), L > 0, k \in \mathbb{N}\) according to (3.9). Hence, uniformly in \(|k|\)

\[
\left(1 - \alpha \sqrt{3 + \sqrt{6}}\right) I \leq P_k \leq \left(1 + \alpha \sqrt{3 + \sqrt{6}}\right) I
\]

Combining (3.12) and (3.13) leads to

\[
C_k^* P_k + P_k C_k \geq \frac{\lambda_{\text{min}}(l, \alpha)}{(1 + \alpha \sqrt{3 + \sqrt{6}}) P_k} := 2t > 0
\]

which completes the proof. 

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Figure 3.2: Each subfigure shows the value $\lambda_{\text{min}}$ form (3.11) dependent on $\alpha \in (0, \alpha_{\text{max}})$ for fixed $l \in \{1, 2, 3\}$. The values have been evaluated numerically.

Remark 3.2. Unlike $l$, which is fixed by the underlying BKG equation, we are allowed to choose an $\alpha \in (0, \alpha_{\text{max}})$ to compute the decay rate $\mu$. Figure 3.3 shows the importance of choosing a suitable $\alpha$.

Remark 3.3. Following the proof of theorem 3.1 it is possible to explicitly compute a decay rate $\mu$. We want to mention, that it is not necessary to solve every optimization problem mentioned numerically, to do so. Notice for example, that the function $\alpha(l, \sigma(z))$ has a minimum at $-\frac{4}{\sqrt{3}}l$ and a maximum at $\frac{4}{\sqrt{3}}l$ for every $l$ fixed by the initial equation (2.1). Because of the polynomial structure there are no other extreme points such that $\alpha(l, \sigma(z))$ is monotonically increasing on $(0, \frac{4}{\sqrt{3}}l)$ and monotonically decreasing on $(0, \frac{4}{\sqrt{3}}l)$. Thus in the case of $\sigma_{\text{max}} \leq (0, \frac{4}{\sqrt{3}}l)$ we have $\alpha_{\text{max}} = \alpha(l, \sigma_{\text{min}})$, in the case $\sigma_{\text{min}} \geq (0, \frac{4}{\sqrt{3}}l)$ we have $\alpha_{\text{max}} = \alpha(l, \sigma_{\text{max}})$ and in all other cases one only has to compare the two boundary values $\alpha(l, \sigma_{\text{min}})$ and $\alpha(l, \sigma_{\text{max}})$ to find $\alpha_{\text{max}}$. Similar one finds out, that the function $\lambda(l, \alpha, \sigma(z))$ has a maximum at $\frac{4}{\sqrt{3}}l$ and a minimum at $3\alpha l$ for every fixed $l, \alpha$. The proof of this claim can be found in the appendix (lemma A.3).

Remark 3.4. Assume $\sigma(z)$ is a uniform scattering around 1. So we have $\sigma_{\text{min}} = 1 - \zeta$ and $\sigma_{\text{max}} = 1 + \zeta$ with some $\zeta \in [0, 1]$. According to remark 3.3 we have $\alpha_{\text{max}} = \alpha(l, \sigma_{\text{max}}) = \frac{1}{4l} (9 - 11\zeta + 3\zeta^2 - \zeta^3 - (1 - \zeta)^2 \sqrt{17 - 2\zeta + \zeta^2})$. Further we have $\frac{4}{\sqrt{3}} > \sigma_{\text{max}}$ and $3\alpha_{\text{max}} < \sigma_{\text{min}}$ for all $\zeta \in [0, 1]$. Such again following remark 3.3 we get that
Figure 3.3: Each subfigure shows the value $2\mu$ form (3.14) dependent on $\alpha \in (0, \alpha_{\text{max}})$ for fixed $l \in \{1, 2, 3\}$. The values are numerically evaluated. One can see the importance of finding a suitable $\alpha$.

Because of the structure of $P_k$ we had to exclude the case $k = 0$ in the proof above. We want to catch up this now. Therefore we want to show that $\omega_0 = 0$, $\mu_0 = 0$ and $\tau_0 = 0$. Remember that the moments of the standard-normal-distribution are given by:

$$\lambda(1, \alpha, \sigma(z))$$ is monotonically increasing on $(\sigma_{\text{min}}, \sigma_{\text{max}})$ for each $\alpha \in (0, \alpha_{\text{max}})$. Thus $\lambda_{\text{min}}(1, \alpha)$ is immediately given as

$$\lambda(1, \alpha, 1 - \zeta) = \frac{\alpha(36\alpha^2 + 3\alpha(\zeta^2 - 3\zeta^2 + 11\zeta - 9) + 4(\zeta - 1)^2)}{2(\alpha + \zeta - 1)^2}.$$ 

Finally, to get the decay rate $\mu$ as big as possible, we have to maximize $\lambda_{\text{min}}(1, \alpha)$ with respect to $\alpha \in (0, \alpha_{\text{max}})$. As we have found a formula for $\lambda_{\text{min}}(1, \alpha)$ it is possible to analyze this expression without fixing $\zeta$, but the resulting expressions are confusingly long. Thus to complete this example we fix $\zeta = 0.5$, which leads to the problem

$$\lambda_{\text{min}}(1, \alpha) = 7.71071\alpha(\alpha^2 - 0.34375\alpha + 0.0277778) \quad \text{max} \quad \frac{\alpha - 0.5)^2}{(\alpha - 0.5)^2 + \alpha + 0.428373}.$$ 

This leads to the decay rate $2\mu \approx 0.0527$ with $\alpha \approx 0.0568$. This coincides with the numerical evaluations shown in subfigure 3.3 (c) (blue line).
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Figure 3.4: The subfigures visualize the results of remark 3.3.

347 **Lemma 3.5** (Moments of the Normal-distribution). For $n \in \mathbb{N}_0$ we have

\[ \int_{-\infty}^{\infty} v^n \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = \begin{cases} \prod_{i=0, i \text{ even}}^{n} |i - 1| & n \text{ even} \\ 0 & n \text{ odd} \end{cases}. \]

With the help of lemma 3.5 we can continue the analysis of the case $k = 0$.

**Lemma 3.6** (Case $k = 0$). For $\omega_k(t, z)$, $\mu_k(t, z)$, $\tau_k(t, z)$ defined as in (2.4) we have $\omega_0(t, z) = 0$, $\mu_0(t, z) = 0$ and $\tau_0(t, z) = 0$.

**Proof.** First multiplying (2.10) with $1, v, v^2$ and then integrating with respect to $v$ one gets

\[ \frac{\partial}{\partial t} \omega_0(t, z) = \sigma(z) \left( \int_{-\infty}^{\infty} M_1(v) dv \hat{h}_{0,0} + \int_{-\infty}^{\infty} v M_1(v) dv \hat{h}_{0,1} \right) \]
\[ \frac{1}{\sqrt{2}} \left( \int_{-\infty}^{\infty} v^2 M_1(v) - M_1(v) \, dv \right) \dot{h}_{0,2} - \frac{\omega_0(t, z)}{2} h_{0,0} = 0 \]  
\[ (3.15) \]

This, together with (2.3) shows that

\[ \dot{h}_{0,0} - \dot{h}_{0,0} = 0, \]

\[ (3.16) \]

Finally note, that lemma 3.6 together with (2.5) leads directly to

\[ \frac{\partial}{\partial t} h_0(v, t, z) = -\sigma(z) h_0(v, t, z). \]  
\[ (3.18) \]
Using Gronwall’s lemma, this shows the decay in the case \( k = 0 \).

### 3.2. Decay estimate.

**Theorem 3.7** (Decay estimate). Let \( h(t) \) be a normalized solution of (2.1) with \( 0 < \lambda < \sigma_{\min} \leq \sigma(z) \leq \sigma_{\max} \) and \( \mathcal{E}(h(0) + M_1)(z) < \infty \) with \( \mathcal{E} \) being an entropy functional, then \( \forall z \in \mathbb{O} \)

\[
\mathcal{E}(h(t) + M_1)(z) \leq e^{-2\lambda t} \mathcal{E}(h(0) + M_1)(z)
\]

with some \( \lambda > 0 \).

**Proof.** Let us define the entropy functional \( \mathcal{E}(\hat{f}) \) by

\[
\mathcal{E}(\hat{f})(t, z) := \sum_{k \in \mathbb{Z}} \langle h_k(v, z), P_k h_k(v, z) \rangle_{L^2(M^{-1}_1)},
\]

where \( \hat{f} := h(t) + M_1 \). Here the matrices \( P_0 := I \) and \( P_k \) are regarded as bounded operators on \( L^2(\mathbb{N}_0) \) (and thus also on \( L^2(M^{-1}_1) \)). (3.18) leads to

\[
\frac{\partial}{\partial t} \langle h_0(v), P_0 h_0(v) \rangle_{L^2(M^{-1}_1)} = \left( \frac{\partial}{\partial t} h_0(v), h_0(v) \right)_{L^2(M^{-1}_1)} + \left( h_0(v), \frac{\partial}{\partial t} h_0(v) \right)_{L^2(M^{-1}_1)}
\]

\[
= -\langle \sigma(z) h_0(v), h_0(v) \rangle_{L^2(M^{-1}_1)} - \langle h_0(v), \sigma(z) h_0(v) \rangle_{L^2(M^{-1}_1)}
\]

\[
= -2\sigma(z) \langle h_0(v), h_0(v) \rangle_{L^2(M^{-1}_1)}
\]

\[
\leq -2\sigma_{\min} \langle h_0(v), h_0(v) \rangle_{L^2(M^{-1}_1)}
\]

and thus using theorem 3.1

\[
\frac{\partial}{\partial t} \mathcal{E}(\hat{f})(t, z) := \frac{\partial}{\partial t} \sum_{k \in \mathbb{Z}} \langle h_k(v, z), P_k h_k(v, z) \rangle_{L^2(M^{-1}_1)}
\]

\[
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\partial}{\partial t} \langle h_k(z), P_k h_k(z) \rangle_{\ell^2} + \frac{\partial}{\partial t} \langle \hat{h}_0(z), P_0 \hat{h}_0(z) \rangle_{\ell^2}
\]

\[
\leq -2\mu \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle \hat{h}_k(z), P_k \hat{h}_k(z) \rangle_{\ell^2} - 2\sigma_{\min} \langle \hat{h}_0(z), P_0 \hat{h}_0(z) \rangle_{\ell^2}
\]

\[
= -2\mu \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle h_k(v, z), P_k h_k(v, z) \rangle_{L^2(M^{-1}_1)}
\]

\[
- 2\sigma_{\min} \langle h_0(v, z), P_0 h_0(v, z) \rangle_{L^2(M^{-1}_1)}
\]

\[
\leq -2 \min\{\mu, \sigma_{\min}\} \mathcal{E}(\hat{f})(t, z)
\]

with \( \mu \) from (3.14). Applying Gronwall’s lemma for each \( z \in \mathbb{O} \) finishes the proof. \( \Box \)

The decay rate \( 2\lambda \) is explicitly computable because \( \sigma_{\min} \) is given and \( \mu \) is computable as shown in theorem 3.1 and the remarks 3.3, 3.4. With the same reasoning one has to note that \( 2\lambda \) is not the exact decay rate, but gives a reasonable lower bound.
3.3. Decay estimates in $z$-derivatives. For both, analytic and numeric reasons, one might also be interested in the decay of the $n$-th derivative of a solution with respect to the random variable $z$. In accordance with theorem 3.7 we define the entropy functional as

**Definition 3.8 (Entropy functional).**

$$\mathcal{E}(f,g) := \sum_{k \in \mathbb{Z}} \langle f(k), P_k g(k) \rangle \mathcal{E} \quad \text{with} \quad f(k), g(k) : \mathbb{Z} \mapsto \ell^2$$

as well as the set:

**Definition 3.9 (The set $\varphi$).**

$$\varphi := \{ f(k) : \mathbb{Z} \mapsto \ell^2 \mid \mathcal{E}(f,f) < \infty \}.$$  

**Remark 3.10.** Note that $\varphi$ is a vector space (over $\mathbb{C}$) and $\mathcal{E}(\cdot,\cdot)$ is a scalar product on $\varphi$ itself. We will denote its induced norm with $\| \cdot \|_\mathcal{E}$. These claims can be shown in an straightforward computation, which we want to skip here.

### 3.3.1. Special case: $\sigma(z)$ linear in $z$.

We will show that in the special case of linear random dependence, which means that $\sigma(z)$ is linear in $z$, the BGK-equation (2.1) still follows kind of an exponential decay with the same rate $\lambda$ as in the case without $z$ derivatives. To keep the following proof clear, we identify the $n$-th derivative with respect to $z$ with $\sigma^{(n)}(z) := \frac{\partial^{(n)}}{\partial z^n} \sigma(z)$.

**Theorem 3.11 (Decay in derivatives (linear dependence)).** Let $h(t)$ be a solution of (2.1) with $0 < L$, $0 < \sigma_{\min} \leq \sigma(z) \leq \sigma_{\max}$ and $\mathcal{E}$ being the entropy functional defined in theorem 3.7. Further we assume $\sigma(z)$ to be linear in $z$ and $\mathcal{E} \left( \frac{\partial^{(n)}}{\partial z^n} \hat{f} \right)(0, z) < \infty \forall n \in \mathbb{N}_0$, then for all $n \in \mathbb{N}_0$ and $\forall z \in \mathbb{O}$

$$\sqrt{\mathcal{E} \left( \frac{\partial^{(n)}}{\partial z^n} \hat{f} \right)(t, z)} \leq e^{-\lambda t} \sum_{i=0}^{n} \binom{n}{i} (\hat{c} t)^i \sqrt{\mathcal{E} \left( \frac{\partial^{(n-i)}}{\partial z^{n-i}} \hat{f} \right)(0, z)}$$

with the same positive $\lambda$ as in theorem 3.7.

Further if $\mathcal{E} \left( \frac{\partial^{(n)}}{\partial z^n} \hat{f} \right)(0, z) \leq H^{2n} \forall n \in \mathbb{N}_0$ we can simplify (3.20) to

$$\sqrt{\mathcal{E} \left( \frac{\partial^{(n)}}{\partial z^n} \hat{f} \right)(t, z)} \leq e^{-\lambda t} (H + \hat{c} t)^n.$$  

**Proof.** We want to show the claim in two steps. First we prove that $\forall z \in \mathbb{O} \subseteq \mathbb{R}$ the inequality

$$\frac{\partial}{\partial t} \left\| \hat{h}^{(n)}_k(t,z) \right\|_\mathcal{E} \leq -\lambda \left\| \hat{h}^{(n)}_k(t,z) \right\|_\mathcal{E} + \hat{c} n \left\| \hat{h}^{(n-1)}_k(t,z) \right\|_\mathcal{E}$$

holds $\forall n \in \mathbb{N}_0$. To start with this we first note that, because of

$$\sigma^{(n)}(z) = 0 \quad \forall n > 1, \quad \sigma^{(1)}(z) = c_1$$

with $c_1$ being a constant, equation (2.13) simplifies to

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\[ \frac{\partial}{\partial t} \hat{h}_k^{(n)}(t, z) = -i k \frac{2\pi}{L} \mathbb{L}_1 \hat{h}_k^{(n)}(t, z) - \sigma(z) \mathbb{L}_2 \hat{h}_k^{(n)}(t, z) - nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \]

\[ = - \left( C_k \hat{h}_k^{(n)}(t, z) + nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \right) k \in \mathbb{Z}, t \geq 0 \]

with \( C_k \) form (2.11). Thus, for each \( k \in \mathbb{Z} \setminus \{0\} \) we have

\[ \frac{\partial}{\partial t} \left\langle \hat{h}_k^{(n)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} = \left\langle \hat{h}_k^{(n)}(t, z), P_k \frac{\partial}{\partial t} \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \]

\[ = - \left\langle C_k \hat{h}_k^{(n)}(t, z) + nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \]

\[ - \left\langle \hat{h}_k^{(n)}(t, z), P_k \left( C_k \hat{h}_k^{(n)}(t, z) + nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \right) \right\rangle_{\ell^2} \]

\[ = - \left\langle C_k \hat{h}_k^{(n)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} - \left\langle nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \]

\[ - \left\langle \hat{h}_k^{(n)}(t, z), P_k C_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} - \left\langle \hat{h}_k^{(n)}(t, z), nc_1 P_k \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \right\rangle_{\ell^2} \]

\[ = - \left\langle \hat{h}_k^{(n)}(t, z), (C_k^2 P_k + P_k C_k) \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \]

\[ + \left\langle - nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} + \left\langle \hat{h}_k^{(n)}(t, z), - nc_1 P_k \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \right\rangle_{\ell^2} \]

Thus using theorem 3.7 we get

\[ \frac{\partial}{\partial t} \left\langle \hat{h}_0^{(n)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \leq -2\mu \left\langle \hat{h}_k^{(n)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} \]

\[ + \left\langle - nc_1 \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z), P_k \hat{h}_k^{(n)}(t, z) \right\rangle_{\ell^2} + \left\langle \hat{h}_k^{(n)}(t, z), - nc_1 P_k \mathbb{L}_2 \hat{h}_k^{(n-1)}(t, z) \right\rangle_{\ell^2} \]

Now we want to get an estimate of the form (3.23) for the case \( k = 0 \). Using (3.18) we get

\[ \frac{\partial}{\partial t} \left\langle \hat{h}_0^{(n)}(v, t, z), \hat{h}_0^{(n-1)}(v, t, z) \right\rangle_{L^2(\mathbb{M}_1)} = -\sigma(z) \left\langle \hat{h}_0^{(n)}(v, t, z), \hat{h}_0^{(n-1)}(v, t, z) \right\rangle_{L^2(\mathbb{M}_1)} \]

\[ - nc_1 \left\langle \hat{h}_0^{(n-1)}(v, t, z), \hat{h}_0^{(n-1)}(v, t, z) \right\rangle_{L^2(\mathbb{M}_1)} \]

and thus with the same arguments as in the estimate \( k \neq 0 \) above
Next we want to show that $L_2 \hat{h}_k^{(n-1)}(t, z) \in \mathcal{V}$.

Obviously for every $x = (x_0, x_1, x_2, \cdots) \in \ell^2$ we have

\begin{equation}
\|L_2 x\|^2_{P_k} = (L_2 x, P_k L_2 x)_{\ell^2} = \sum_{i=3}^{\infty} |x_i|^2 \leq \sum_{i=0}^{\infty} |x_i|^2 = \langle x, x \rangle_{\ell^2}
\end{equation}

as well as

\begin{equation}
\langle x, x \rangle_{\ell^2} \leq \frac{1}{1 - \alpha \sqrt{3 + \sqrt{6}}} \langle x, P_k x \rangle_{\ell^2}
\end{equation}

for all $k \in \mathbb{Z}$ as shown in (3.13). Combining (3.25) and (3.26) leads to

\begin{equation}
\|L_2 x\|^2_{P_k} \leq \langle x, x \rangle_{\ell^2} \leq C^2 \|x\|^2_{P_k}.
\end{equation}

Summing up (3.27) with $x = L_2 \hat{h}_k^{(n-1)}(t, z)$ we have

\begin{equation}
\mathcal{E} \left( L_2 \hat{h}_k^{(n-1)}(t, z), L_2 \hat{h}_k^{(n-1)}(t, z) \right) \leq C^2 \mathcal{E} \left( \hat{h}_k^{(n-1)}(t, z), \hat{h}_k^{(n-1)}(t, z) \right)_{<\infty}
\end{equation}

which means $L_2 \hat{h}_k^{(n-1)}(t, z) \in \mathcal{V}$. Now we set $\lambda := \min\{\mu, \sigma_{\min}\}$ and remember that $\langle \cdot, \cdot \rangle_{L^2(M^{-1})} = \langle \cdot, P_0^* \rangle_{\ell^2}$ with $P_0 = I$. Thus combining (3.23) with (3.24) and summing up over all $k \in \mathbb{Z}$ leads to

\begin{equation}
\frac{\partial}{\partial t} \mathcal{E} \left( \hat{h}_k^{(n)}(t, z), \hat{h}_k^{(n)}(t, z) \right) \leq -2\lambda \mathcal{E} \left( \hat{h}_k^{(n)}(t, z), \hat{h}_k^{(n)}(t, z) \right)
\end{equation}

\begin{equation}
+ \mathcal{E} \left( \hat{h}_k(t, z), P_k \hat{h}_k^{(n)}(t, z) \right) + \mathcal{E} \left( \hat{h}_k^{(n)}(t, z), P_k \hat{h}_k(t, z) \right),
\end{equation}

where we defined

\begin{equation}
\tilde{h}_k(t, z) := \begin{cases} -nc_1 \hat{h}_0^{(n-1)}(t, z) & \text{if } k = 0, \\ -nc_1 L_2 \hat{h}_k^{(n-1)}(t, z) & \text{if } k \neq 0. \end{cases}
\end{equation}

Note that $\tilde{h}_k(t, z) \in \mathcal{V}$ because $L_2 \hat{h}_k^{(n-1)}(t, z) \in \mathcal{V}$ as shown above. More precise, the only difference between $\tilde{h}_k(t, z)$ and $L_2 \hat{h}_k^{(n-1)}(t, z)$ is the first summand (this is the case $k = 0$). This term however, was already included in estimate (3.27).

Now, because of

\begin{equation}
\frac{\partial}{\partial t} \mathcal{E} \left( \hat{h}_k^{(n)}(t, z), \hat{h}_k^{(n)}(t, z) \right) \in \mathbb{R}; \quad \mathcal{E} \left( \hat{h}_k^{(n)}(t, z), \hat{h}_k^{(n)}(t, z) \right) \in \mathbb{R}
\end{equation}

we have
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\[ E \left( \tilde{h}_k(t, z), P_k \hat{h}^{(n)}_k(t, z) \right) + E \left( \hat{h}^{(n)}_k(t, z), P_k \tilde{h}_k(t, z) \right) \in \mathbb{R}, \]

too. So, continuing estimate (3.28):

\[
\frac{\partial}{\partial t} E \left( \hat{h}^{(n)}_k(t, z), \hat{h}^{(n)}_k(t, z) \right)
\leq -2\lambda E \left( \hat{h}^{(n)}_k(t, z), \hat{h}^{(n)}_k(t, z) \right)
+ \left| E \left( \hat{h}_k(t, z), P_k \hat{h}^{(n)}_k(t, z) \right) + E \left( \hat{h}^{(n)}_k(t, z), P_k \tilde{h}_k(t, z) \right) \right|
\leq -2\lambda E \left( \hat{h}^{(n)}_k(t, z), \hat{h}^{(n)}_k(t, z) \right)
+ \left| E \left( \hat{h}_k(t, z), P_k \hat{h}^{(n)}_k(t, z) \right) \right| + \left| E \left( \hat{h}^{(n)}_k(t, z), P_k \tilde{h}_k(t, z) \right) \right|.
\]

Now let \( \| \cdot \|_\varepsilon \) be the Norm induced by \( E(\cdot, \cdot) \). Using the Cauchy–Schwarz inequality shows

\[ \frac{\partial}{\partial t} E \left( \hat{h}^{(n)}_k(t, z), \hat{h}^{(n)}_k(t, z) \right) \leq -2\lambda E \left( \hat{h}^{(n)}_k(t, z), \hat{h}^{(n)}_k(t, z) \right) + \left\| \hat{h}_k(t, z) \right\|_\varepsilon \left\| \hat{h}^{(n)}_k(t, z) \right\|_\varepsilon + \left\| \hat{h}^{(n)}_k(t, z) \right\|_\varepsilon \left\| \tilde{h}_k(t, z) \right\|_\varepsilon.
\]

Because of (3.27) we have the following relation between \( E \left( \hat{h}_k(t, z), \tilde{h}_k(t, z) \right) \) and \( E \left( \hat{h}^{(n-1)}_k(t, z), \tilde{h}^{(n-1)}_k(t, z) \right) \):

\[
E \left( \hat{h}^{(n-1)}_k(t, z), \tilde{h}^{(n-1)}_k(t, z) \right) = (nc_1)^2 \left( \hat{h}^{(n-1)}_0(t, z), P_0 \hat{h}^{(n-1)}_0(t, z) \right)_{\ell^2}
+ \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \hat{h}^{(n-1)}_k(t, z), P_k \hat{h}^{(n-1)}_k(t, z) \right)_{\ell^2}
\leq C^2 \left( \hat{h}^{(n-1)}_k(t, z), P_k \hat{h}^{(n-1)}_k(t, z) \right)_{\ell^2}
\leq C^2 \left( \hat{h}^{(n-1)}_k(t, z), \tilde{h}^{(n-1)}_k(t, z) \right)
\leq \left( nc_1 \tilde{C} \right)^2 E \left( \hat{h}^{(n-1)}_k(t, z), \tilde{h}^{(n-1)}_k(t, z) \right).
\]

Now taking the roots, define \( \tilde{c} := |c_1| \tilde{C} \) and inserting into (3.29) leads to

\[
\frac{\partial}{\partial t} \left\| \hat{h}^{(n)}_k(t, z) \right\|_\varepsilon^2 \leq -2\lambda \left\| \hat{h}^{(n)}_k(t, z) \right\|_\varepsilon^2 + 2nc \left\| \hat{h}^{(n-1)}_k(t, z) \right\|_\varepsilon \left\| \tilde{h}^{(n)}_k(t, z) \right\|_\varepsilon.
\]

Dividing by \( 2 \left\| \hat{h}^{(n)}_k(t, z) \right\|_\varepsilon \) gives (3.22).

In a second step we want to show that (3.22) implies
This however is a direct consequence of lemma A.4 which can be found in the appendix. Thus (3.20) is proven. Finally inserting \( \sqrt{E(f^{(n)})(0,z)} \leq H^n \forall n \in \mathbb{N}_0 \) in (3.20) and using the binomial theorem leads directly to (3.21). This finishes the proof.

**Remark 3.12.** A uniform scattering as mentioned in remark 3.4 can be modeled as \( \sigma(z) = 1 + z \) with \( z \) being a uniform distributed random variable on \([-\zeta, \zeta]\). Thus in the case \( \zeta = 0.5 \) a explicit computed decay rate of theorem 3.11 is given by \( \lambda = \frac{0.6527}{2} \) as already computed in remark 3.4.

**3.3.2. General case with** \( \left| \frac{1}{n!} \frac{\partial^{(n)}}{\partial z^{(n)}} \sigma(z) \right| < C \). The assumption, that \( \sigma(z) \) is linear in \( z \) is very restrictive, so that our next goal is to loosen this condition. Therefore, from now on, the \( z \)-dependence of \( \sigma(z) \) can be arbitrary, as long as \( \left| \frac{1}{n!} \frac{\partial^{(n)}}{\partial z^{(n)}} \sigma(z) \right| < C \) for all \( n \in \mathbb{N}_0 \), where \( C \) is a constant independent of \( n \). Further, we want to simplify the notation and set

\[
\hat{h}_k^{(n)}(t,z) := \frac{\partial^{(n)}}{\partial z^{(n)}} \hat{h}_k(t,z) \quad \quad \hat{h}_k^{(n)}(t,z) := \frac{\hat{h}_k^{(n)}(t,z)}{n!}
\]

\[
\sigma^{(n)}(z) := \frac{\partial^{(n)}}{\partial z^{(n)}} \sigma(z) \quad \quad \eta_k^{(n)}(t,z) := e^{\lambda t} \left\| \hat{h}_k^{(n)}(t,z) \right\|_E.
\]

Then the following theorem, with the same explicit computable \( \lambda \) as in theorem 3.11, holds:

**Theorem 3.13** (Decay in derivatives (more) general case). Let \( h(t) \) be a solution of (2.1) with \( 0 < L, 0 < \sigma_{\min} \leq \sigma(z) \leq \sigma_{\max} \) and \( E \) being a entropy functional defined in theorem 3.7. Further we assume \( \left| \frac{1}{n!} \frac{\partial^{(n)}}{\partial z^{(n)}} \sigma(z) \right| < C \) as well as \( E \left( \frac{\partial^{(n)}}{\partial z^{(n)}} \right) (0,z) \leq H^{2n} \forall n \in \mathbb{N}_0 \) for the initial data, then for all \( n \in \mathbb{N}_0 \) and uniform in all \( z \in \mathbb{O} \)

\[
\sqrt{E \left( \frac{\partial^{(n)}}{\partial z^{(n)}} f \right)(t,z)} \leq e^{-\lambda t} + n!(1 + H)^{n+1} \min \left\{ e^{-\lambda t} (1 + \hat{C}t)^n, e^{(\hat{C} - \lambda)t} 2^{n-1} \right\}
\]

with the same positive \( \lambda \) as in theorem 3.7.

**Proof.** Repeating the same arguments as presented in the proof of theorem 3.11 leads to

\[
\frac{\partial}{\partial t} \left\| \hat{h}_k^{(n)}(t,z) \right\|_E \leq -2\lambda \left\| \hat{h}_k^{(n)}(t,z) \right\|_E^2 + 2\hat{C} \sum_{i=1}^{n} \left\| \left( \frac{n}{i} \right) \sigma^{(i)}(z) \hat{h}_k^{(n-i)}(t,z) \right\|_E \left\| \hat{h}_k^{(n)}(t,z) \right\|_E
\]

Now we will use some argumentation first presented in [LW17] to bring the above inequality in a shape, that allows the use of lemma A.5 in the appendix. Therefor we first use \( \left| \frac{1}{n!} \sigma^{(n)}(z) \right| < C \) for all \( n \in \mathbb{N}_0 \) to estimate further:
\[
\frac{\partial}{\partial t} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \leq -2\lambda \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} + 2 \tilde{C} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \sum_{i=0}^{n-1} \left\| \tilde{h}_k^{(i)}(t, z) \right\|_\mathcal{E}.
\]

Dividing by \((n!)^2\) on both sides, we have

\[
\frac{\partial}{\partial t} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \leq -2\lambda \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} + 2 \tilde{C} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \sum_{i=0}^{n-1} \frac{n!}{i!} \left\| \tilde{h}_k^{(i)}(t, z) \right\|_\mathcal{E}.
\]

and dividing by \(2 \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E}\) leads to

\[
(3.31) \quad \frac{\partial}{\partial t} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \leq -\lambda \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} + \tilde{C} \sum_{i=0}^{n-1} \left\| \tilde{h}_k^{(i)}(t, z) \right\|_\mathcal{E}.
\]

Note that

\[
\frac{\partial}{\partial t} h_k^{(n)}(t, z) = e^{\lambda t} \left( \frac{\partial}{\partial t} \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} + \lambda \left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \right).
\]

Thus, multiplying (3.31) with \(e^{\lambda t}\) results in

\[
\frac{\partial}{\partial t} h_k^{(n)}(t, z) \leq \tilde{C} \sum_{i=0}^{n-1} h_k^{(i)}(t, z).
\]

Because of \(\frac{\partial}{\partial t} \left( \frac{\partial}{\partial z} \tilde{f}(0, z) \right) \leq H^{2n}\) we have \(h_k^{(n)}(0, z) \leq \frac{H^n}{n!}\), so that we can use

\[
(3.32) \quad h_k^{(n)}(t, z) \leq \frac{H^n}{n!} + (1 + H)^{n+1} \min \left\{ (1 + \tilde{C}t)^n, e^{\tilde{C}t^2n^{-1}} \right\}.
\]

Now we multiply (3.32) with \(e^{-\lambda t}\) to reach

\[
\left\| \tilde{h}_k^{(n)}(t, z) \right\|_\mathcal{E} \leq e^{-\lambda t} \frac{H^n}{n!} + (1 + H)^{n+1} \min \left\{ e^{-\lambda t}(1 + \tilde{C}t)^n, e^{(\tilde{C}-\lambda)t^2n^{-1}} \right\}.
\]

Multiplying with \(n!\) finishes the proof. \(\Box\)

4. Conclusion. In [AAC18] the authors developed an estimate for the decay rate of the linearized BGK equation (2.1) for the deterministic case. In [LW17] decay rates had been shown for some kinetic equations under the influence of stochastic uncertainties. We have shown how to compute a lower bound of the decay rate independent of the stochastic influence for a stochastic version of (2.1) in a physically obvious norm\(^7\). Finally we want to mention that the method used in this article

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may be also used to calculate an expected lower bound of the decay rate, if the
distribution of \(\sigma(z)\) is known or can be estimated. Therefore one has to note, that the
lowest eigenvalue computed in the proof of theorem 3.1 is a random variable itself.
Especially in the case of a high variance we tend to underestimate the real decay rate
in many cases, however this is necessary to get a bound which is valid for all \(z\).

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Appendix A. Used inequalities and estimates.

A.1. Estimates of theorem 3.1. We want to prove the inequalities used in
the proof of theorem 3.1.

**Lemma A.1.** The inequality (3.5) holds.

**Proof.** We have

\[
\frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3} = \frac{1}{24l^3} \sigma(z) \left( \sqrt{64l^4 + 16l^2\sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4} \right) = \frac{\sigma(z)}{3l} \frac{1}{8l^2} \left( \sqrt{64l^4 + 16l^2\sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4} \right).
\]

Thus (3.5) is true if

\[
\frac{1}{8l^2} \left( \sqrt{64l^4 + 16l^2\sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4} \right) \leq 1.
\]

To show this, one computes

\[
\frac{1}{8l^2} \left( \sqrt{64l^4 + 16l^2\sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4} \right) \leq 1
\Rightarrow \frac{1}{8l^2} \left( 8l^2 + \sigma(z)^2 \right) \leq 1 + \frac{1}{8l^2} \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4}
\Rightarrow 1 + \frac{\sigma(z)^2}{8l^2} \leq 1 + \frac{\sigma(z)^2}{8l^2} \sqrt{\frac{16l^2}{\sigma(z)^2} + 1}.
\]

This shows (3.5).

Now we can use lemma A.1 to prove

**Lemma A.2.** The inequalities (*) and (**) in (3.10) hold.

**Proof.** In order to show (*) we have to verify

\[
2l\alpha \leq 2\sigma(z)
\]

with \(\alpha \in (0, \alpha_{\text{max}})\). Note the definition of \(\alpha_{\text{max}}\) in (3.8) and (3.7), so multiplying
(3.5) with \(2l\) on gets

\[
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This is (\ref{eq:cond}). To prove (\ref{eq:cond_e}) we have to show
\begin{equation}
\alpha \left(72\alpha^3 - (48\alpha^2\sigma(z) + 6\sigma(z)^3) + 8\sigma(z)^2\right) 
\leq 2l\alpha
\end{equation}
with \(\alpha \in (0, \alpha_{\text{max}})\). Therefore we divide both sides by \(2l\alpha\) to obtain
\begin{equation}
\frac{36\alpha^2 l^3 - 24\alpha^2 \sigma(z) + 4\sigma(z)^2 - 3\alpha \sigma(z)^3}{4l\sigma(z)^2 - 8l^2 \sigma(z) + 4\alpha^2 l^3} \leq 1.
\end{equation}
This is true, because of
\begin{equation}
\frac{36\alpha^2 l^3 - 24\alpha^2 \sigma(z) + 4\sigma(z)^2 - 3\alpha \sigma(z)^3}{4l\sigma(z)^2 - 8l^2 \sigma(z) + 4\alpha^2 l^3} 
\leq 0.
\end{equation}
Thus everything is proven. \(\square\)

\subsection*{A.2. Claim of remark 3.3.}

\textbf{Lemma A.3.} For each fixed \(l, \alpha\), the function \(\lambda(l, \alpha, \sigma(z))\) (defined in (3.10)) has a (local) maximum at \(\frac{1}{\sqrt{3}} l\) and a (local) minimum at \(3\alpha l\) (if these points are in \((0, \alpha_{\text{max}})\)).

\textit{Proof.} First note that \(\alpha \neq 0\) because of \(\alpha \in (0, \alpha_{\text{max}})\). Thus for fixed \(l\) and \(\alpha\) the derivative of \(\lambda(l, \alpha, \sigma)\) (with respect to \(\sigma\)) is zero at the points
\begin{align*}
\sigma_1 &= \frac{-4}{\sqrt{3}} l; & \sigma_2 &= \frac{4}{\sqrt{3}} l; & \sigma_3 &= 3\alpha l.
\end{align*}
We always assume \(0 < \sigma_{\text{min}} \leq \sigma\), so that we can neglect the first solution. So it remains to check if the remaining two points are maxima or minima. Therefore we compute the second derivative at the point \(\sigma_2\)
\begin{equation}
\lambda_{\sigma, \sigma}(l, \alpha, \sigma_2) = -\frac{108\alpha^2 (9\sqrt{3}\alpha^2 - 48\alpha + 16\sqrt{3})}{(4\sqrt{3} - 3\alpha)^4 l} \begin{cases} > 0 & \text{if } 9\sqrt{3}\alpha^2 - 48\alpha + 16\sqrt{3} < 0 \; \text{,} \\ = 0 & \text{if } 9\sqrt{3}\alpha^2 - 48\alpha + 16\sqrt{3} = 0 \; \text{,} \\ < 0 & \text{if } 9\sqrt{3}\alpha^2 - 48\alpha + 16\sqrt{3} > 0 \; \text{.} \end{cases}
\end{equation}
because of \(l, \alpha > 0\). The polynomial \(p(\alpha) := 9\sqrt{3}\alpha^2 - 48\alpha + 16\sqrt{3}\) has the two roots
\begin{align*}
\alpha_1 &= \frac{4}{3\sqrt{3}}; & \alpha_2 &= \frac{4}{\sqrt{3}}.
\end{align*}
Thus we know
\[
\begin{cases}
p(\alpha) > 0 & \alpha \in (-\infty, \frac{4}{3\sqrt{3}}) \cup (\frac{4}{3\sqrt{3}}, \infty) \\
p(\alpha) = 0 & \alpha \in \{\frac{4}{3\sqrt{3}}, \frac{4}{\sqrt{3}}\} \\
p(\alpha) < 0 & \alpha \in (\frac{4}{3\sqrt{3}}, \frac{4}{\sqrt{3}})
\end{cases}
\]

From (3.9) we know that \(\alpha_{\text{max}} \leq \frac{4}{9\sqrt{3}}\), such that \(p(\alpha) > 0\) for every \(\alpha \in (0, \alpha_{\text{max}})\). This shows that \(\sigma_2\) is a maximum. The same can be done for the point \(\sigma_3\). Keeping in mind that \(l, \alpha > 0\) this which shows

\[
\lambda_{\sigma,\sigma}(l, \alpha, \sigma_3) = \frac{1}{\alpha l} - \frac{27\alpha}{16l}
\]

Again because of \(\alpha_{\text{max}} \leq \frac{4}{9\sqrt{3}}\) the point \(\sigma_3\) needs to be a minimum for all \(\alpha \in (0, \alpha_{\text{max}})\).

### A.3. Inequalities we use.

The following two inequalities had first been introduced in [LW17]. Even so we use a slightly different notation in our article, the proofs can be taken from their article.

**Lemma A.4.** Assume \(J = [0, \infty), n \in \mathbb{N}_0\) and \(f(l) \in C^1(J, \mathbb{R}) \forall l \in \{0, \cdots, n\}\). If further the system of inequalities

\[
\frac{\partial}{\partial t} f(l)(t) \leq -\lambda f(l) + C l f(l-1), \quad l \in \{0, \cdots, n\}
\]

with constants \(\lambda, C > 0\) holds, then

\[
 f(n)(t) \leq e^{-\lambda t} \sum_{i=0}^{n} \binom{n}{i} (Ct)^i f(n-i)(0),
\]

where we set \(f(-1)\) to zero.

**Lemma A.5.** Assume \(J = [0, \infty), n \in \mathbb{N}_0\) and \(f(l) \in C^1(J, \mathbb{R}_+) \forall l \in \{0, \cdots, n\}\). If further the inequalities

\[
\frac{\partial}{\partial t} \hat{f}(l)(t) \leq C \sum_{k=0}^{l-1} \hat{f}(k)(t)
\]

with constants \(\lambda, C > 0, H \geq 0\) and \(\hat{f}(l)(t) := e^{\lambda t} f(l)(t)\) hold for all \(l \in \{0, \cdots, n\}\), then

---

\(^8\)Here with \(\mathbb{R}_+\) we denote all non negative real numbers.
\[ \tilde{f}(n)(t) \leq \frac{H^n}{n!} + (1 + H)^{n+1} \sum_{k=1}^{n} \frac{(Ct)^k}{k!(k-1)!(n-k)!} (n-1)! \]

and (A.5) can further be relaxed to

\[ \tilde{f}(n)(t) \leq \frac{H^n}{n!} + (1 + H)^{n+1} \min \left\{ (1 + Ct)^n, e^{Ct2^{n-1}} \right\}. \]

REFERENCES


