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HYPOCOERCIVITY OF THE LINEARIZED BGK EQUATION WITH STOCHASTIC COEFFICIENTS*

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Abstract. We consider an approximation of the Boltzmann equation, the Bathnagar-Gross-Krook (BGK) equation. This equation is used in many applications because it is very efficient in numerical simulations. In this paper we study the effect of randomness on a BGK-model. We prove exponential decay rate to a global equilibrium. In addition we prove the decay rate of the n-th derivative with respect to the stochastic variable of the solutions. The novelties are i.) for the first time hypocoercivity is shown for a linearized BGK model that conserves mass, momentum and energy with randomness in the collision frequency, ii.)new estimates for the decay of the derivatives of the solution with respect to the stochastic variable, which is very useful in applications.

Key words. linear BGK-equation with uncertainties, hypocoercivity, decay estimate, Lyapunov's direct method.

Mathematics Subject Classification. 35A24, 35B30, 35Q20, 82B40.

1. Introduction. In this paper, our aim is to study the decay to equilibrium of the solution of a linearized BGK model introduced in [AAC18] with a random parameter in the collision frequency. We begin by introducing the non-linear BGK model

$$\partial_t f + v \cdot \nabla_x f = \sigma(M - f). \tag{1.1}$$

Here $f(x, v, t)$ is the number density distribution of one species of gas with respect to the phase space measure $dxdv$. Here $x \in (\frac{L}{2\pi}\mathbb{T})^d$ in the $d-$dimensional torus of side length L is the position of the coordinate in phase space. $v \in \mathbb{R}^d$ is the velocity coordinate in dimension $d \in \mathbb{N}$ and $t \geq 0$ is the time. In the following, we will consider the dimension $d = 1$. The relaxation operator on the right-hand side of (1.1) involves the Maxwellian

$$M = \frac{n}{\sqrt{2\pi T}} \exp\left(-\frac{|v-u|^2}{2T}\right)$$

depending on the macroscopic quantities (density n, mean velocity u, temperature T) defined as

$$\int f(v) \begin{pmatrix} 1 \\ v \\ (v-u)^2 \end{pmatrix} dv = \begin{pmatrix} n \\ nu \\ nT \end{pmatrix}.$$

Moreover, the BGK model (1.1) contains the collision frequency σ. The purpose of the collision operator in (1.1) is to provide an approximation of the Boltzmann collision operator that is more computationally tractable, but still maintains important

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structural properties. It was first introduced in [BGG54] by Bathnagar, Gross and Krook. It has the same collision invariants as the Boltzmann operator (which lead to conservation of number of particles, momentum and energy) and it satisfies an H-Theorem.

One natural aspect of kinetic equations are uncertainties. The form of some terms (for instance of the collision frequency) in the equations are usually unjustified due to modelling errors. The blurred measurements are typically not enough to sufficiently determine all coefficients. Therefore, in this paper, we consider the collision frequency $\sigma(z)$ depending on a random parameter z. In the whole paper, we assume that this dependency is continuous.

Now, the aim of the paper is to study the regularity and the large-time behavior of f and of the derivatives $\partial_z^{(n)} f$ in dimension $d = 1$. This is based on the hypocoercivity theory which has been studied for a large variety of equations. Some considerable examples in the deterministic case are the Fokker-Plank equations [AAS15, AE14], linear kinetic equations [DMS15, FS20, NS15, BDMMS20, D06, DMS09, H06], a multi-species Boltzmann system [DJMZ16] as well as the BGK-equations [AAC16, AAC18, LP19]. Especially in [AAC16, AAC18, AE14] it was an issue to find sharp exponential decay rates. In the random case, this has been extended in many cases for example to linear kinetic equations in [LJ18, LW17, AJW20], for the multi-species Boltzmann equation in [DJL19], the Vlasov-Poisson-Fokker-Planck system [JZ2018] and equations used for traffic modelling [HI2021]. Such a study of the regularity and the large-time behavior of f and of the derivatives $\partial_z^{(n)} f$ allows to adopt the gPC framework for its possible fast convergence. To do that, one mainly needs to prove that the perturbation in the solution continuously depends on the perturbation where one chooses to perform linearization. According to the standard spectral method theory, the higher degree of continuity means the faster convergence. For example such a study is provided by [LJ18] for the Boltzmann equation.

In this paper, we want to understand the regularity and decay to equilibrium of the function f and also of its derivatives $\partial_z^{(n)} f$ for all $n \in \mathbb{N}$. We denote by $d\tilde{x} := L^{-d} dx$ the normalized Lebesque measure and consider normalized initial data

$$\int\int f^I d\tilde{x} dv = 1, \quad \int\int v f^I d\tilde{x} dv = 0, \quad \int\int v^2 f^I d\tilde{x} dv = 1 \tag{1.2}$$

Now, we linearize the BGK equation (1.1) around the unique space-homogeneous steady state

$$\mathbb{M}_1(v) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{v^2}{2}\right)$$

as it is performed in [AAC18]. For this, we consider the splitting $f(x, v, t) = \mathbb{M}_1(v) + h(x, v, t, z)$ with the macroscopic quantities of h defined as

$$\begin{aligned} \omega(x,t,z) &:= \int_{\mathbb{R}} h(x,v,t,z)\,\mathrm{d}v, \quad \mu(x,t,z) := \int_{\mathbb{R}} v h(x,v,t,z)\,\mathrm{d}v \\ \tau(x,t,z) &:= \int_{\mathbb{R}} v^2 h(x,v,t,z)\,\mathrm{d}v \end{aligned} \tag{1.3}$$

If we insert this ansatz into (1.1), do a Taylor expansion of M with respect to ω, μ, τ around 0, and take only the linear terms, one can derive similar as it is done in

[AAC18] the linearized equation

$$\partial_t h(x,v,t,z) + v\partial_x h(x,v,t,z) = \sigma(z)\,\mathcal{L}\left(h(x,v,t,z)\right) \tag{1.4}$$

with

$$\mathcal{L}(h) := \mathbb{M}_1(v)\left[\left(\frac{3}{2}-\frac{v^2}{2}\right)\omega(h) + v\mu(h) + \left(-\frac{1}{2}+\frac{v^2}{2}\right)\tau(h)\right] - h,$$

Uncertainties are also important from the point of view of numerics. Nowadays many numerical methods with the aim to address the issues related to uncertainties have been developed. Well known numerical methods are the Monte-Carlo method, the moment equation approach and the perturbation methods. In addition there are spectral-methods like the (Galerkin) generalized polynomial chaos method and the stochastic collocation method. A review of spectral type methods can be found in [Xiu10]. One thing spectral methods have in common is that they provide a higher order of accuracy if the solution has a high level of regularity. Thus it is a common procedure to check the derivatives or show boundedness or even decay in time in some reasonable norm. In this context we point out the paper by Li and Wang [LW17], where such a regularity condition has been studied for a large set of kinetic equations. Their paper contains the linear BGK-operator with constant velocity and temperature (where only mass is conserved).

The aim of this article is to extend the results in [LW17] to the linearized BGK equation (1.4). In our case we have a dependency on the macroscopic quantities ω, μ, τ instead of a constant Maxwell distribution with a fixed constant mean velocity and temperature. In our case, not only the mass is conserved but also momentum and energy. We will show exponential decay in time with a rate $-\lambda$ independent of the random variable and λ strictly positive in a physical reasonable norm. To do so, we use the technique developed in [AAC16, AAC18]. The advantage of this approach is that we directly inherit the optimization strategies made in these articles. In addition to the aforementioned differences in our model, we also differ from [LW17] that we look for sharp decay rates. To achieve this, we adapt a method proposed by [AAC16] for the deterministic case. However, in contrast to the literature for sharp decay rates [AAC16] we need to find estimate which hold for every possible realization of z. This requires careful modifications of the already known approaches.

This has to be understood as kind of an a priori estimate, which means that we find sharp decay rates which serve as lower bound for all possible realizations. This means, the slowest possible decay rate which can be realized tends to be sharp in the sense of [AAC16, AAC18]. Furthermore, the resulting decay rates are directly computable. Moreover, we show that this decay rate λ also holds for the decay of the derivatives in the random space. That means, computing such a decay rate λ for the underlying BGK equation once, gives us immediately a decay rate for the derivatives in the random space.

In summary, the novelty of this article consists of showing hypocoercivity for the linearized BGK model with randomness in the collision frequency conserving mass, momentum and energy. We include new estimates for the decay of the derivatives of the solution with respect to the stochastic variable, which is very useful in applications.

In section 2, we will begin by writing the linearized BGK-model with uncertainties in one space dimension as an infinite system of ODEs similar as it is done in [AAC18]. Section 3 is divided in three parts. In the first subsection we will extend Lyapunov's direct method in infinite dimensions to equations with a random parameter in the collision frequency. This is a crucial step on our search for decay rates and directly leads to our first decay estimate presented in the second part of this section. Finally, in the third part we deal with decay estimates in z-derivatives. The main idea here is to benefit from two Gronwall-like estimate theorems presented in [LW17].

2. Transformation of the linearized BGK equation to an infinite system of ODEs. To prepare for the following proofs, we want to rewrite (1.4) into an (infinite dimensional) system of differential equations as it is done in the deterministic case in [AAC18, AAC16]. We do this by expanding $h(x,v,t,z)$ in a Fourier series in x

$$h(x,v,t,z)=\sum_{k\in\mathbb{Z}} h_k(v,t,z)\mathrm{e}^{\mathrm{i}k\frac{2\pi}{L}x}.$$

Then, we will expand $h_k(\cdot,t,z)\in L^2\left(\mathbb{R};\mathbb{M}_1^{-1}(v)\right)$ in normalized Hermite functions

$$\begin{aligned} g_m(v) &:= (\pi m!)^{-1/2} H_m(v)\exp\left(-\frac{v^2}{2}\right),\\ H_m(v) &:= (-1)^m \exp\left(\frac{v^2}{2}\right)\frac{d^m}{dv^m}\exp\left(-\frac{v^2}{2}\right) \end{aligned}$$

by writing

$$h_k(v,t,z)=\sum_{m=0}^{\infty}\hat{h}_{k,m}(t,z)g_m(v) \quad \text{with} \quad \hat{h}_{k,m}(t,z)=\langle h_k(v,\cdot,\cdot),\ g_m(v)\rangle_{L^2(\mathbb{M}_1^{-1})}.$$

For each $k\in\mathbb{Z}$ the vector $\hat{h}_k(t,z)=\left(\hat{h}_{k,0}(t,z),\ \hat{h}_{k,1}(t,z),\ \dots\right)^T\in\ell^2(\mathbb{N}_0)$ contains all Hermite coefficients of $h_k(\cdot,t,z)$. Note that the first three normalized Hermite functions are given by

$$g_0(v)=\mathbb{M}_1(v),\ \ g_1(v)=v\mathbb{M}_1(v),\ \ g_2(v)=\frac{v^2-1}{\sqrt{2}}\mathbb{M}_1(v).$$

Moreover, we have

$$\hat{h}_{k,0}(t,z)=\int_{\mathbb{R}} h_k(v,\cdot,\cdot)g_0(v)\mathbb{M}_1^{-1}(v)\,\mathrm{d}v=\omega_k(t,z) \tag{2.5}$$

$$\hat{h}_{k,1}(t,z)=\int_{\mathbb{R}} h_k(v,\cdot,\cdot)g_1(v)\mathbb{M}_1^{-1}(v)\,\mathrm{d}v=\mu_k(t,z) \tag{2.6}$$

$$\hat{h}_{k,2}(t,z)=\int_{\mathbb{R}} h_k(v,\cdot,\cdot)g_2(v)\mathbb{M}_1^{-1}(v)\,\mathrm{d}v=\frac{1}{\sqrt{2}}\left(\tau_k(t,z)-\omega_k(t,z)\right). \tag{2.7}$$

where ω_k,μ_k,τ_k are the spatial modes of the moments ω,μ,τ given by

$$\begin{aligned} \omega_k(t,z) &= \int h_k(v,t,z)\,\mathrm{d}v,\\ \mu_k(t,z) &= \int v h_k(v,t,z)\,\mathrm{d}v,\\ \tau_k(t,z) &= \int v^2 k_k(v,t,z)\,\mathrm{d}v. \end{aligned}$$

It can be shown that (1.4) is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t}h_k + \mathrm{i}k\frac{2\pi}{L}vh_k \\ = \sigma(z)\left(g_0(v)\hat{h}_{k,0} + g_1(v)\hat{h}_{k,1} + g_2(v)\hat{h}_{k,2} - h_k\right), \ k \in \mathbb{Z}; \ t \geq 0.\end{aligned} \tag{2.8}$$

For details of this derivation see [AAC18]. Since this derivation does not act on the z variable, the derivation is exactly the same as in [AAC18], so we will not repeat it here. Now, the vector of its Hermite coefficients satisfies

$$\frac{\partial}{\partial t}\hat{h}_k(t,z) + \mathrm{i}k\frac{2\pi}{L}\mathbb{L}_1\hat{h}_k(t,z) = -\sigma(z)\mathbb{L}_2\hat{h}_k(t,z), \qquad k \in \mathbb{Z}; \ t \geq 0$$

with the operators $\mathbb{L}_1$, $\mathbb{L}_2$ represented by the (infinite) matrices

$$\mathbb{L}_1 := \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}, \qquad \mathbb{L}_2 := \operatorname{diag}(0, \ 0, \ 0, \ 1, \ 1, \ \cdots)$$

Equivalently, we can also write

$$\frac{\partial}{\partial t}\hat{h}_k(t,z) = -C_k\hat{h}_k(t,z) \quad k \in \mathbb{Z}; \ t \geq 0 \quad \text{with} \quad C_k := \mathrm{i}k\frac{2\pi}{L}\mathbb{L}_1 + \sigma(z)\mathbb{L}_2. \tag{2.9}$$

We note that this model satisfies the following conservation properties.

Lemma 1. *The moments $\omega_0(t,z)$, $\mu_0(t,z)$, $\tau_0(t,z)$ satisfy*

$$\omega_0(t,z) = 0, \quad \mu_0(t,z) = 0, \quad \tau_0(t,z) = 0,$$

for all $t > 0$.

This can be proven by multiplying (2.8) for $k = 0$ by $1, v, v^2$ and then integrating with respect to v. In the resulting equations one can compute the Maxwellian integrals and use the relations (2.5), (2.6). (2.7) to deduce that $\omega_0(t,z)$, $\mu_0(t,z)$ and $\tau_0(t,z)$ are constant functions in t and then equal to zero due to the assumption on the initial data (1.2).

Since in the following, we also want to find estimates for $\partial_z^{(n)}h$, we will also consider the $n-th$ derivative of equation (1.4) with respect to z, and get

$$\partial_z^{(n)}\partial_t h(x,v,t,z) + v\partial_z^{(n)}\partial_x h(x,v,t,z) = \partial_z^{(n)}\left(\sigma(z)\mathcal{L}\left(h(x,v,t,z)\right)\right) \tag{2.10}$$

With the same approach as above, this leads to

$$\begin{aligned}\frac{\partial^{(n)}}{\partial z^{(n)}}\frac{\partial}{\partial t}\hat{h}_k(t,z) \\ = -\mathrm{i}k\frac{2\pi}{L}\mathbb{L}_1\frac{\partial^{(n)}}{\partial z^{(n)}}\hat{h}_k(t,z) - \sum_{i=0}^{n}\binom{n}{i}\frac{\partial^{(i)}}{\partial z^{(i)}}\sigma(z)\mathbb{L}_2\frac{\partial^{(n-i)}}{\partial z^{(n-i)}}\hat{h}_k(t,z)\end{aligned} \tag{2.11}$$

for $k \in \mathbb{Z}$; $t \geq 0$. Alternatively, directly differentiating (2.9) n times with respect to z leads to the same result.

3. Decay rate for a linearized BGK model with uncertainties. In this section, we will study the decay to equilibrium of the function h. For this, we will follow the strategy of [AAC18]. We define the matrices P_k as

$$P_k := \left(\begin{array}{cccc|c} 1 & -\frac{\mathrm{i}\alpha}{k} & 0 & 0 & \\ \frac{\mathrm{i}\alpha}{k} & 1 & -\frac{\mathrm{i}\beta}{k} & 0 & \mathbf{0} \\ 0 & \frac{\mathrm{i}\beta}{k} & 1 & -\frac{\mathrm{i}\gamma}{k} & \\ 0 & 0 & \frac{\mathrm{i}\gamma}{k} & 1 & \\ \hline & & \mathbf{0} & & \mathbf{I} \end{array}\right) \qquad k \in \mathbb{N} \tag{3.12}$$

with I being the identity matrix and $\alpha,\ \beta,\ \gamma\ \in \mathbb{R}$ will be chosen later in an appropriate way. We start with the following lemma.

LEMMA 2. *Assume $0 < \sigma_{min} \leq \sigma(z) \leq \sigma_{max}$, $L > 0$. Choose the matrices P_k as in (3.12) and C_k from (2.9). Then there exists an $\alpha_{max} > 0$, such that with $\alpha \in (0, \alpha_{max}), \beta = \sqrt{2}\alpha, \gamma = \sqrt{3}\alpha$ the matrices P_k and $C_k^* P_k + P_k C_k$ are positive definite for all $k \in \mathbb{Z} \setminus \{0\}$ and*

$$C_k^* P_k + P_k C_k \geq 2\mu P_k$$

with a $\mu > 0$ independent of k.

The proof consists of standard algebra derivations. Therefore, we will move the proof to the appendix. Because of the structure of P_k we had to exclude the case $k = 0$ in the proof above. We want to catch up this now. This case can be deduced from lemma 1. If we insert this result into (1.4), we obtain for $k = 0$

$$\frac{\partial}{\partial t} h_0(v, t, z) = -\sigma(z)\ h_0(v, t, z). \tag{3.13}$$

Using Gronwall's lemma, this shows the decay in the case $k = 0$.

3.1. Decay estimate. Now, we continue with the decay estimate on h. For this, we define

$$\mathcal{E}(h)(t, z) := \sum_{k \in \mathbb{Z}} \langle h_k(v, z), P_k h_k(v, z) \rangle_{L^2(\mathbb{M}_1^{-1})}, \tag{3.14}$$

Here the matrices $P_0 := I$ and P_k are regarded as bounded operators on $\ell^2(\mathbb{N}_0)$ (and thus also on $L^2(\mathbb{M}_1^{-1})$).

THEOREM 3. *Let $h(t)$ be a solution of (1.4) with $0 < L$, $0 < \sigma_{min} \leq \sigma(z) \leq \sigma_{max}$ and $\mathcal{E}(h(0))(z) < \infty$, then we have*

$$\mathcal{E}\left(h(t)\right)(z) \leq \mathrm{e}^{-2\lambda t} \mathcal{E}\left(h(0)\right)(z)$$

with some $\lambda > 0$ for all z.

Proof. Equation (3.13) leads to

$$\begin{aligned} \frac{\partial}{\partial t} \langle h_0(v), P_0 h_0(v) \rangle_{L^2(\mathbb{M}_1^{-1})} &= \left\langle \frac{\partial}{\partial t} h_0(v), h_0(v) \right\rangle_{L^2(\mathbb{M}_1^{-1})} + \left\langle h_0(v), \frac{\partial}{\partial t} h_0(v) \right\rangle_{L^2(\mathbb{M}_1^{-1})} \\ &= -2\sigma(z) \langle h_0(v), h_0(v) \rangle_{L^2(\mathbb{M}_1^{-1})} \\ &\leq -2\sigma_{min} \langle h_0(v), h_0(v) \rangle_{L^2(\mathbb{M}_1^{-1})} \end{aligned}$$

and thus using lemma 2 we get

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(h)(t,z) &:= \frac{\partial}{\partial t}\sum_{k\in\mathbb{Z}}\langle h_k(v,z), P_k h_k(v,z)\rangle_{L^2(\mathbb{M}_1^{-1})} \\
&= \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\partial}{\partial t}\left\langle \hat{h}_k(z), P_k\hat{h}_k(z)\right\rangle_{\ell^2} + \frac{\partial}{\partial t}\left\langle \hat{h}_0(z), P_0\hat{h}_0(z)\right\rangle_{\ell^2} \\
&\le -\sum_{k\in\mathbb{Z}\setminus\{0\}}\left\langle \hat{h}_k(z), (C_k^* P_k + P_k C_k)\hat{h}_k(z)\right\rangle_{\ell^2} - 2\sigma_{min}\left\langle \hat{h}_0(z), P_0\hat{h}_0(z)\right\rangle_{\ell^2} \\
&\le -2\mu\sum_{k\in\mathbb{Z}\setminus\{0\}}\left\langle \hat{h}_k(z), P_k\hat{h}_k(z)\right\rangle_{\ell^2} - 2\sigma_{min}\left\langle \hat{h}_0(z), P_0\hat{h}_0(z)\right\rangle_{\ell^2} \\
&= -2\mu\sum_{k\in\mathbb{Z}\setminus\{0\}}\langle h_k(v,z), P_k h_k(v,z)\rangle_{L^2(\mathbb{M}_1^{-1})} \\
&\quad - 2\sigma_{min}\langle h_0(v,z), P_0 h_0(v,z)\rangle_{L^2(\mathbb{M}_1^{-1})} \\
&\le -2\lambda\, \mathcal{E}(h)(t,z)
\end{aligned}$$

where we define $\lambda = \min\{\mu, \sigma_{min}\}$ with μ from (3.41). Applying Gronwall's lemma finishes the proof. □

3.2. Decay estimates in z-derivatives. For both, analytic and numeric reasons, one might also be interested in the decay of the n-th derivative of a solution with respect to the random variable z. For the following, we define

$$\mathcal{F}(f,g) := \sum_{k\in\mathbb{Z}}\langle f(k), P_k g(k)\rangle_{\ell^2} \quad \text{for} \quad f(k), g(k) : \mathbb{Z}\mapsto \ell^2$$

and we denote $||f||_{\mathcal{F}} := \sqrt{\mathcal{F}(f,f)}$.

3.2.1. Special case: $\sigma(z)$ linear in z. We will show that in the special case of linear random dependence, which means that $\sigma(z)$ is linear in z, the linearized BGK-equation (1.4) still follows an exponential decay with the same rate λ as in the case without z derivatives.

THEOREM 4. *Let $h(t)$ be a solution of (1.4) with $0 < L$, $0 < \sigma_{min} \le \sigma(z) \le \sigma_{max}$. Further we assume $\sigma(z)$ to be linear in z and $\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}h\right)(0,z) < \infty$ for all $n \in \mathbb{N}_0$. Then, for all $n\in\mathbb{N}_0$ and for all z, we have*

$$\sqrt{\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}h\right)(t,z)} \le \mathrm{e}^{-\lambda t}\sum_{i=0}^{n}\binom{n}{i}(\tilde{c}\,t)^i\sqrt{\mathcal{E}\left(\frac{\partial^{(n-i)}}{\partial z^{(n-i)}}h\right)(0,z)} \tag{3.15}$$

with the same positive λ as in theorem 3. Further if $\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}h\right)(0,z) \le H^{2n}$ for a constant $H > 0$ and for all $n \in \mathbb{N}_0$ we can simplify (3.15) to

$$\sqrt{\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}h\right)(t,z)} \le \mathrm{e}^{-\lambda t}(H + \tilde{c}t)^n\,. \tag{3.16}$$

Proof. We want to show the claim in two steps. First, we prove that the inequality

$$\frac{\partial}{\partial t}||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}} \le -\lambda\, ||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}} + \tilde{c}\,n\,||\hat{h}_k^{(n-1)}(t,z)||_{\mathcal{F}} \tag{3.17}$$

holds for all $n \in \mathbb{N}_0$. Then, this will imply (3.15) for all z as a direct consequence of lemma 6. To start with this we first note that because of

$$\sigma^{(n)}(z) = 0 \quad \text{for all} \quad n > 1, \quad \sigma^{(1)}(z) = c_1$$

with c_1 being a constant, equation (2.11) simplifies to

$$\begin{aligned}\frac{\partial}{\partial t}\hat{h}_k^{(n)}(t,z) &= -\mathrm{i}k\frac{2\pi}{L}\mathbb{L}_1\hat{h}_k^{(n)}(t,z) - \sigma(z)\mathbb{L}_2\hat{h}_k^{(n)}(t,z) - nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\\ &= -\left(C_k\hat{h}_k^{(n)}(t,z) + nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\right) \quad k \in \mathbb{Z};\ t \geq 0\end{aligned}$$

with C_k from (2.9). Thus, for each $k \in \mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle \hat{h}_k^{(n)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2}\\ &= \left\langle \frac{\partial}{\partial t}\hat{h}_k^{(n)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2} + \left\langle \hat{h}_k^{(n)}(t,z), P_k\frac{\partial}{\partial t}\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2}\\ &= -\left\langle C_k\hat{h}_k^{(n)}(t,z) + nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2}\\ &\quad -\left\langle \hat{h}_k^{(n)}(t,z), P_k\left(C_k\hat{h}_k^{(n)}(t,z) + nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\right)\right\rangle_{\ell^2}\\ &= -\left\langle \hat{h}_k^{(n)}(t,z), \left(C_k^*P_k + P_kC_k\right)\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2}\\ &\quad + \left\langle -nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2} + \left\langle \hat{h}_k^{(n)}(t,z), -nc_1P_k\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\right\rangle_{\ell^2}.\end{aligned}$$

Thus using theorem 3 we get

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle \hat{h}_k^{(n)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2} \leq -2\mu\left\langle \hat{h}_k^{(n)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2}\\ &+\left\langle -nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z), P_k\hat{h}_k^{(n)}(t,z)\right\rangle_{\ell^2} + \left\langle \hat{h}_k^{(n)}(t,z), -nc_1P_k\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\right\rangle_{\ell^2}.\end{aligned} \tag{3.18}$$

Now we want to get an estimate of the form (3.18) for the case $k = 0$. Using (3.13) we get

$$\begin{aligned}\frac{\partial}{\partial t}h_0^{(n)}(v,t,z) &= \frac{\partial^{(n)}}{\partial z^{(n)}}\big(-\sigma(z)h_0(v,t,z)\big) = \sum_{i=0}^{n}\binom{n}{i} - \sigma^{(i)}(z)\ h_0^{(n-i)}(v,t,z)\\ &= -\sigma(z)h_0^{(n)}(v,t,z) - nc_1h_0^{(n-1)}(v,t,z)\end{aligned}$$

and thus with the same arguments as in the estimate $k \neq 0$ above

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle h_0^{(n)}(v,z), P_0h_0^{(n)}(v,z)\right\rangle_{L^2(\mathbb{M}_1^{-1})} = -2\sigma(z)\left\langle h_0^{(n)}(v,z), h_0^{(n)}(v,z)\right\rangle_{L^2(\mathbb{M}_1^{-1})}\\ &+\left\langle -nc_1h_0^{(n-1)}(v,z), h_0^{(n)}(v,z)\right\rangle_{L^2(\mathbb{M}_1^{-1})} + \left\langle h_0^{(n)}(v,z), -nc_1h_0^{(n-1)}(v,z)\right\rangle_{L^2(\mathbb{M}_1^{-1})}\end{aligned} \tag{3.19}$$

Now we set $\lambda := \min\{\mu, \sigma_{min}\}$ and remember that $\langle\cdot,\cdot\rangle_{L^2(\mathbb{M}_1^{-1})} = \langle\hat{\cdot}, P_0\hat{\cdot}\rangle_{\ell^2}$ with $P_0 = I$. Thus combining (3.18) with (3.19) and summing up over all $k \in \mathbb{Z}$ leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\mathcal{F}\left(\hat{h}_k^{(n)}(t,z), \hat{h}_k^{(n)}(t,z)\right) \leq -2\lambda\,\mathcal{F}\left(\hat{h}_k^{(n)}(t,z), \hat{h}_k^{(n)}(t,z)\right)\\ &+\mathcal{F}\left(\tilde{h}_k(t,z), P_k\hat{h}_k^{(n)}(t,z)\right) + \mathcal{F}\left(\hat{h}_k^{(n)}(t,z), P_k\tilde{h}_k(t,z)\right),\end{aligned} \tag{3.20}$$

where we defined

$$\tilde{h}_k(t,z) := \begin{cases} -nc_1\hat{h}_0^{(n-1)}(t,z) & \text{if} \quad k=0 \\ -nc_1\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z) & \text{if} \quad k\neq 0. \end{cases}$$

More precise, the only difference between $\tilde{h}_k(t,z)$ and $\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)$ is the first summand (this is the case $k=0$). So, continuing estimate (3.20):

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),\hat{h}_k^{(n)}(t,z)\right) &\leq -2\lambda\,\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),\hat{h}_k^{(n)}(t,z)\right) \\ &\quad + \left|\mathcal{F}\left(\tilde{h}_k(t,z),P_k\hat{h}_k^{(n)}(t,z)\right) + \mathcal{F}\left(\hat{h}_k^{(n)}(t,z),P_k\tilde{h}_k(t,z)\right)\right| \\ &\leq -2\lambda\,\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),\hat{h}_k^{(n)}(t,z)\right) \\ &\quad + \left|\mathcal{F}\left(\tilde{h}_k(t,z),P_k\hat{h}_k^{(n)}(t,z)\right)\right| + \left|\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),P_k\tilde{h}_k(t,z)\right)\right|. \end{aligned}$$

Using the Cauchy–Schwartz inequality leads to

$$\begin{aligned} &\frac{\partial}{\partial t}\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),\hat{h}_k^{(n)}(t,z)\right) \\ &\leq -2\lambda\,\mathcal{F}\left(\hat{h}_k^{(n)}(t,z),\hat{h}_k^{(n)}(t,z)\right) \\ &\quad + ||\tilde{h}_k(t,z)||_{\mathcal{F}}\;||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}} + ||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}\;||\tilde{h}_k(t,z))||_{\mathcal{F}}. \end{aligned} \tag{3.21}$$

We have the following relation

$$\begin{aligned} &\mathcal{F}\left(\tilde{h}_k(t,z),\tilde{h}_k(t,z)\right) \\ &= (nc_1)^2\left(\left\langle \hat{h}_0^{(n-1)}(t,z),P_0\hat{h}_0^{(n-1)}(t,z)\right\rangle_{\ell^2}\right. \\ &\qquad \left. + \sum_{k\in\mathbb{Z}\setminus\{0\}}\left\langle \mathbb{L}_2\hat{h}_k^{(n-1)}(t,z),P_k\mathbb{L}_2\hat{h}_k^{(n-1)}(t,z)\right\rangle_{\ell^2}\right) \\ &\leq \left(nc_1\tilde{C}\right)^2\mathcal{F}\left(\hat{h}_k^{(n-1)}(t,z),\hat{h}_k^{(n-1)}(t,z)\right). \end{aligned}$$

In the last inequality, we used the definition of $\mathbb{L}_2$. Now taking the roots, define $\tilde{c} := |c_1|\tilde{C}$ and inserting into (3.21) leads to

$$\frac{\partial}{\partial t}||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 \leq -2\lambda\;||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 + 2n\tilde{c}||\hat{h}_k^{(n-1)}(t,z)||_{\mathcal{F}}\;||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}.$$

Dividing by $2||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}$ gives (3.17).

Now, we can deduce (3.15) as it is described in the beginning of the proof. Finally inserting $\sqrt{\mathcal{E}\left(\tilde{h}^{(n)}\right)(0,z)} \leq H^n$ for all $n\in\mathbb{N}_0$ in (3.15) and using the binomial theorem leads directly to (3.16). This finishes the proof. □

3.2.2. General case under the assumption $\left|\frac{1}{n!}\frac{\partial^{(n)}}{\partial z^{(n)}}\sigma(z)\right| < C$. The assumption that $\sigma(z)$ is linear in z, is very restrictive, so that our next goal is to loosen this condition. Therefore, from now on, the z-dependence of $\sigma(z)$ can be arbitrary, as long as $\left|\frac{1}{n!}\frac{\partial^{(n)}}{\partial z^{(n)}}\sigma(z)\right| < C$ for all $n \in \mathbb{N}_0$, where C is a constant independent of n. Actually, this is a very weak constraint. It does not require that all derivatives have to be bounded by the same constant, the bound can grow with $n!$. Further, we want to simplify the notation and set

$$\hat{h}_k^{(n)}(t,z) := \frac{\partial^{(n)}}{\partial z^{(n)}}\hat{h}_k(t,z) \qquad \tilde{h}_k^{(n)}(t,z) := \frac{\hat{h}_k^{(n)}(t,z)}{n!}$$
$$\sigma^{(n)}(z) := \frac{\partial^{(n)}}{\partial z^{(n)}}\sigma(z) \qquad \eta_k^{(n)}(t,z) := \mathbf{e}^{\lambda t}||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}}.$$

Then the following theorem, with the same explicit computable λ as in theorem 4, holds:

THEOREM 5. *Let $h(t)$ be a solution of (1.4) with $0 < L$, $0 < \sigma_{min} \leq \sigma(z) \leq \sigma_{max}$.Further we assume $\left|\frac{1}{n!}\sigma^{(n)}(z)\right| < C$ as well as $\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}\tilde{f}\right)(0,z) \leq H^{2n}$ for all $n \in \mathbb{N}_0$ for the initial data, then, we have*

$$\sqrt{\mathcal{E}\left(\frac{\partial^{(n)}}{\partial z^{(n)}}h\right)(t,z)} \leq \mathbf{e}^{-\lambda t}H^n + n!(1+H)^{n+1}\min\left\{\mathbf{e}^{-\lambda t}(1+\hat{C}t)^n,\ \mathbf{e}^{(\hat{C}-\lambda)t}2^{n-1}\right\}$$

for all $n \in \mathbb{N}$ with the same positive λ as in theorem 3 and a positive constant $\hat{C}$.

Proof. Repeating the same arguments as presented in the proof of theorem 4 leads to

$$\frac{\partial}{\partial t}||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 \leq -2||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 + 2\tilde{C}\sum_{i=1}^{n}||\binom{n}{i}\sigma^{(i)}\hat{h}_k^{(n-i)}||_{\mathcal{F}}||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}.$$

Now we will use arguments presented in [LW17]. They are presented in lemma 7. We first prove that all requirements are satisfied to use it. Therefore we first use $\left|\frac{1}{n!}\sigma^{(n)}(z)\right| < C$ for all $n \in \mathbb{N}_0$ to estimate further:

$$\frac{\partial}{\partial t}||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 \leq -2\lambda||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 + 2\tilde{C}C||\hat{h}_k^{(n)}(t,z)||_{\mathcal{F}}\sum_{i=1}^{n}\frac{n!}{(n-i)!}||\hat{h}_k^{(n-i)}(t,z)||_{\mathcal{F}}.$$

We denote $\hat{C} := \tilde{C}C$, shift the index in the sum, and divide by $(n!)^2$ on both sides, we have

$$\frac{\partial}{\partial t}||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 \leq -2\lambda||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}}^2 + 2\hat{C}||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}}\sum_{i=0}^{n-1}||\tilde{h}_k^{(i)}(t,z)||_{\mathcal{F}}$$

and dividing by $2||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}}$ leads to

$$\frac{\partial}{\partial t}||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}} \leq -\lambda||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}} + \hat{C}\sum_{i=0}^{n-1}||\tilde{h}_k^{(i)}(t,z)||_{\mathcal{F}}. \tag{3.22}$$

Then,we obtain for $\eta_k^{(n)}$

$$\frac{\partial}{\partial t}\eta_k^{(n)}(t,z) \leq \hat{C}\sum_{i=0}^{n-1}\eta_k^{(i)}(t,z).$$

Because of $\left(\frac{\partial^{(n)}}{\partial z^{(n)}} h\right)(0,z) \leq H^{2n}$ we have $\eta_k^{(n)}(0,z) \leq \frac{H^n}{n!}$, so that we can use lemma 7 point-wise in z to get

$$\eta_k^{(n)}(t,z) \leq \frac{H^n}{n!} + (1+H)^{n+1} \min\left\{(1+\hat{C}t)^n,\ \mathbf{e}^{\hat{C}t} 2^{n-1}\right\}. \tag{3.23}$$

Now we multiply (3.23) with $\mathbf{e}^{-\lambda t}$ to reach

$$||\tilde{h}_k^{(n)}(t,z)||_{\mathcal{F}} \leq \mathbf{e}^{-\lambda t} \frac{H^n}{n!} + (1+H)^{n+1} \min\left\{\mathbf{e}^{-\lambda t}(1+\hat{C}t)^n,\ \mathbf{e}^{(\hat{C}-\lambda)t} 2^{n-1}\right\}.$$

Multiplying with $n!$ finishes the proof. □

Appendix.

3.3. Inequalities and estimates from the literature. The following two inequalities had first been introduced in [LW17]. Even so we use a slightly different notation in our article, the proofs can be taken from their article.

LEMMA 6. *Assume $J = [0,\infty)$, $n \in \mathbb{N}_0$ and a sequence $f_{(l)} \in C^1(J,\mathbb{R})$ for all $l \in \{0,\cdots,n\}$. If further the system of inequalities*

$$\frac{\partial}{\partial t} f_{(l)} \leq -\lambda f_{(l)} + Cl f_{(l-1)}, \qquad l \in \{0,\cdots,n\} \tag{3.24}$$

with constants $\lambda,\ C > 0$ holds, then

$$f_{(n)}(t) \leq \mathbf{e}^{-\lambda t} \sum_{i=0}^{n} \binom{n}{i} (Ct)^i f_{(n-i)}(0), \tag{3.25}$$

where we set $f_{(-1)}$ to zero.

LEMMA 7. *Assume $J = [0,\infty)$, $n \in \mathbb{N}_0$ and a sequence $f_{(l)} \in C^1(J,\mathbb{R}_+)$ for all $l \in \{0,\cdots,n\}$.*

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{f}_{(l)}(t) &\leq C \sum_{k=0}^{l-1} \tilde{f}_{(k)}(t) \\ \tilde{f}_{(l)}(0) &\leq \frac{H^l}{l!} \end{aligned} \tag{3.26}$$

with constants $\lambda,\ C > 0$, $H \geq 0$ and $\tilde{f}_{(l)}(t) := \mathbf{e}^{\lambda t} f_{(l)}(t)$ hold for all $l \in \{0,\cdots,n\}$, then

$$\tilde{f}_{(n)}(t) \leq \frac{H^n}{n!} + (1+H)^{n+1} \sum_{k=1}^{n} \frac{(Ct)^k}{k!(k-1)!} \frac{(n-1)!}{(n-k)!} \tag{3.27}$$

and (3.27) can further be relaxed to

$$\tilde{f}_{(n)}(t) \leq \frac{H^n}{n!} + (1+H)^{n+1} \min\left\{(1+Ct)^n,\ \mathbf{e}^{Ct} 2^{n-1}\right\}. \tag{3.28}$$

3.4. Proof of lemma 2.

Proof of lemma 2. Note first that $C_k^* P_k + P_k C_k$ has the form of a block-diagonal-matrix

$$\begin{pmatrix} D_{k,\alpha,\beta,\gamma,\sigma(z)} & 0 \\ 0 & \tilde{I} \end{pmatrix}$$

with $\tilde{I}$ being $2\sigma(z)$ times the (infinite dimensional) identity matrix and

$$D_{k,\alpha,\beta,\gamma,\sigma(z)} := \begin{pmatrix} 2l\alpha & 0 & l\left(\sqrt{2}\alpha - \beta\right) & 0 & 0 \\ 0 & 2l\left(\sqrt{2}\beta - \alpha\right) & 0 & l\left(\sqrt{3}\beta - \sqrt{2}\gamma\right) & 0 \\ l\left(\sqrt{2}\alpha - \beta\right) & 0 & 2l\left(\sqrt{3}\gamma - \sqrt{2}\beta\right) & -\frac{i\gamma\sigma(z)}{k} & 2l\gamma \\ 0 & l\left(\sqrt{3}\beta - \sqrt{2}\gamma\right) & \frac{i\gamma\sigma(z)}{k} & 2\sigma(z) - 2l\sqrt{3}\gamma & 0 \\ 0 & 0 & 2l\gamma & 0 & 2\sigma(z) \end{pmatrix}$$

where $l := \frac{2\pi}{L}$. Because of $0 < 2\sigma_{min} < 2\sigma(z)$ the matrix $\tilde{I}$ is already positive definite, such that it only remains to show the positive definiteness of $D_{k,\alpha,\beta,\gamma,\sigma(z)}$. However, instead of seeking $\alpha,\ \beta\ ,\gamma$ such that the matrix $D_{k,\alpha,\beta,\gamma,\sigma(z)}$ is positive definite for all $k \in \mathbb{Z}\setminus\{0\}$, we simplify the problem by setting $\beta = \sqrt{2}\alpha$ and $\gamma = \sqrt{3}\alpha$. Thus, we get

$$D_{k,\alpha,\sigma(z)} := \begin{pmatrix} 2l\alpha & 0 & 0 & 0 & 0 \\ 0 & 2l\alpha & 0 & 0 & 0 \\ 0 & 0 & 2l\alpha & -\frac{i\sqrt{3}\alpha\sigma(z)}{k} & 2\sqrt{3}l\alpha \\ 0 & 0 & \frac{i\sqrt{3}\alpha\sigma(z)}{k} & 2\sigma(z) - 6l\alpha & 0 \\ 0 & 0 & 2\sqrt{3}l\alpha & 0 & 2\sigma(z) \end{pmatrix}$$

which will be an easier structure to analyze. However, we note that we have to pay for this with a reduction of the decay rate. Now we will use Sylvester's criterion to find a sufficient condition for α, such that the matrix $D_{k,\alpha,\sigma(z)}$ is positive definite for all $k \in \mathbb{Z}\setminus\{0\}$ Therefore we define $\delta_j\left(k,\alpha,\sigma(z)\right)$ as the determinant of the lower right $j \times j$ submatrix of $D_{k,\alpha,\sigma(z)}$ with $1 \leq j \leq 5$ and search for assumptions on α, which lead to $\delta_j\left(k,\alpha,\sigma(z)\right) > 0$ for all $1 \leq j \leq 5$. Thus we get

$$\begin{aligned} \delta_1\left(k,\alpha,\sigma(z)\right) &= 2\sigma(z) \\ \delta_2\left(k,\alpha,\sigma(z)\right) &= 4\sigma(z)\left(\sigma(z) - 3l\alpha\right) \end{aligned}$$

$$\begin{aligned} \delta_3\left(k,\alpha,\sigma(z)\right) &= \alpha\left(72l^3\alpha^2 - \left(48l^2\sigma(z) + \frac{6\sigma(z)^3}{k^2}\right)\alpha + 8l\sigma(z)^2\right) \\ &\geq \alpha\left(72l^3\alpha^2 - \left(48l^2\sigma(z) + 6\sigma(z)^3\right)\alpha + 8l\sigma(z)^2\right) \\ &= \delta_3\left(1,\alpha,\sigma(z)\right) \end{aligned} \tag{3.29}$$

$$\begin{aligned} \delta_4\left(k,\alpha,\sigma(z)\right) &= 2\alpha l\ \delta_3\left(k,\alpha,\sigma(z)\right) \\ \delta_5\left(k,\alpha,\sigma(z)\right) &= 4\alpha^2 l^2\ \delta_3\left(k,\alpha,\sigma(z)\right) \end{aligned}$$

The first determinant δ_1 is positive because of the assumption $0 < \sigma(z)$. The second determinant δ_2 is positive if we have

$$\alpha < \frac{\sigma(z)}{3l}, \tag{3.30}$$

whereas δ_3, δ_4 and δ_5 are positive if

$$0 < \alpha < \frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3}. \tag{3.31}$$

So, to make sure that $D_{k,\alpha,\sigma(z)}$ is positive definite, we need to choose an α such that (3.30) and (3.31) hold. However, because of

$$\frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3} \leq \frac{\sigma(z)}{3l} \tag{3.32}$$

it is sufficient to find an α such that (3.31) is fulfilled. Equation (3.32) is true since we have $\sqrt{16l^2\sigma(z)^4 + \sigma(z)^6} > \sigma(z)^3$. However, it still remains to show that (3.31) can be fullfilled. So, we want to show that there exists an α_{max} such that

$$0 < \alpha_{max} \leq \frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3}. \tag{3.33}$$

Then (3.31) would be true for all $\alpha \in (0, \alpha_{max})$. For proving this, we first note that

$$\begin{aligned} 0 &< \frac{1}{24l^3}\,\sigma(z)\,\left(\sqrt{64l^4 + 16l^2\sigma(z)^2 + \sigma(z)^4} - \sqrt{16l^2\sigma(z)^2 + \sigma(z)^4}\right) \\ &= \frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3} := \alpha(l, \sigma(z)) \end{aligned} \tag{3.34}$$

because of $\sigma(z) > 0$ and $l > 0$. Furthermore, $\alpha(l, \sigma(z))$ is a continuous function, such that if we take

$$\alpha_{max} := \min_{\sigma(z) \in [\sigma_{min}, \sigma_{max}]} \alpha(l, \sigma(z)) \tag{3.35}$$

one gets $\alpha_{max} \leq \alpha(l, \sigma(z))$ for arbitrary fixed $l > 0$. This, together with (3.34) leads to (3.33). Therefore, we get that $D_{k,\alpha,\sigma(z)}$ is positive definite for all $\alpha \in (0, \alpha_{max})$ with α_{max} given by (3.35).

It remains to prove that the matrices P_k are positive definite for this choice. In [AAC18] it is proven that P_k is positive definite if $\mid \alpha \mid^2 + \mid \beta \mid^2 + \mid \gamma \mid^2 < 1$. Since we set $\beta = \sqrt{2}\alpha$ and $\gamma = \sqrt{3}\alpha$ this reduces to $6\alpha^2 < 1$.

One can compute that $\alpha(l, \sigma(z))$ takes its maximum at $l = \frac{\sqrt{3}}{4}\sigma(z)$ and we get

$$\alpha^2 \leq \alpha_{max}^2 < \left(\frac{8l^2\sigma(z) + \sigma(z)^3 - \sqrt{16l^2\sigma(z)^4 + \sigma(z)^6}}{24l^3}\right)^2 < \frac{4}{9\sqrt{3}} \tag{3.36}$$

if $l < \frac{\sigma_{max}}{2\sqrt{2}}$, so that the matrices P_k are also positive definite for all $\alpha \in (0, \alpha_{max})$ with α_{max} from (3.35) for $l < \frac{\sigma_{max}}{2\sqrt{2}}$.

Next, we want to find a lower bound for the smallest eigenvalue of $C_k^* P_k + P_k C_k$. All eigenvalues of $\tilde{I}$ are $2\sigma(z)$ and because of the block diagonal structure, $D_{k,\alpha,\sigma(z)}$ has a double eigenvalue $2l\alpha$ together with the eigenvalues of its lower 3×3 submatix

$$D^{(3)}_{k,\alpha,\sigma(z)} := \begin{pmatrix} 2l\alpha & -\frac{i\sqrt{3}\alpha\sigma(z)}{k} & 2\sqrt{3}l\alpha \\ \frac{i\sqrt{3}\alpha\sigma(z)}{k} & 2\sigma(z) - 6l\alpha & 0 \\ 2\sqrt{3}l\alpha & 0 & 2\sigma(z) \end{pmatrix}.$$

Let $\{\lambda_1,\ \lambda_2,\ \lambda_3\}$ be the eigenvalues of $D^{(3)}_{k,\alpha,\sigma(z)}$ arranged in increasing order. So our aim is to find the minimum of the eigenvalues $2\sigma(z), 2l\alpha$ and λ_1. We can estimate λ_1 from below by using the inequality of the arithmetic-geometric mean and get

$$\begin{aligned}\lambda_1(k,\alpha,\sigma(z)) &= \frac{\delta_3(k,\alpha,\sigma(z))}{\lambda_2\lambda_3} \geq \delta_3(k,\alpha,\sigma(z))\ \left(\frac{\lambda_2+\lambda_3}{2}\right)^{-2} \\ &\geq \delta_3(k,\alpha,\sigma(z))\ \left(\frac{\mathrm{Tr}\ D^{(3)}_{k,\alpha,\sigma(z)}}{2}\right)^{-2} = \delta_3(k,\alpha,\sigma(z))\ \frac{1}{4\left(\sigma(z)-\alpha l\right)^2} > 0,\end{aligned}$$

since $D^{(3)}_{k,\alpha,\sigma(z)}$ is positive definite for $\alpha \in (0,\alpha_{max})$. So, all in all, we need to find a lower bound of $\min\{2l\alpha,\ \frac{\delta_3(k,\alpha,\sigma(z))}{4(\sigma(z)-\alpha l)^2},\ 2\sigma(z)\}$. However, with $\alpha \in (0,\alpha_{max})$ the following holds:

$$\begin{aligned}\min\{2l\alpha,\ \frac{\delta_3(k,\alpha,\sigma(z))}{4\left(\sigma(z)-\alpha l\right)^2},\ 2\sigma(z)\} &= \min\{2l\alpha,\ \frac{\delta_3(k,\alpha,\sigma(z))}{4\left(\sigma(z)-\alpha l\right)^2}\} \\ &\geq \min\{2l\alpha,\ \frac{\delta_3(1,\alpha,\sigma(z))}{4\left(\sigma(z)-\alpha l\right)^2}\} \\ &= \frac{\delta_3(1,\alpha,\sigma(z))}{4\left(\sigma(z)-\alpha l\right)^2} := \lambda(l,\alpha,\sigma(z)). \qquad (3.37)\end{aligned}$$

The first equality is true due to (3.30), the inequality follows from (3.29). For the last equality we have to show

$$\frac{\alpha\left(72l^3\alpha^2-\left(48l^2\sigma(z)+6\sigma(z)^3\right)\alpha+8l\sigma(z)^2\right)}{4\left(\sigma(z)-\alpha l\right)^2} \leq 2l\alpha$$

with $\alpha \in (0,\alpha_{max})$. One can compute that this is equivalent to

$$\alpha\bigg(16l^2\left(2l\alpha-\sigma\right)-3\sigma^3\bigg) \leq 0,$$

which is true due to (3.30). To get an estimate independent of $\sigma(z)$, we define for fixed $l > 0$ and $\alpha \in (0,\alpha_{max})$

$$\lambda_{min}(l,\alpha) := \min_{\sigma(z)\in[\sigma_{min},\sigma_{max}]} \lambda(l,\alpha,\sigma(z)) > 0. \qquad (3.38)$$

Then we get

$$C_k^* P_k + P_k C_k \geq \lambda_{min}(l,\alpha)\ I. \qquad (3.39)$$

Furthermore, a straight forward computation shows that the eigenvalues of P_k are $\{1,\ \ 1 \pm \frac{\alpha\sqrt{3+\sqrt{6}}}{k},\ \ 1 \pm \frac{\alpha\sqrt{3-\sqrt{6}}}{k}\}$. These eigenvalues are positive for all $\alpha \in (0,\alpha_{max}),\ L > 0,\ k \in \mathbb{N}$ according to (3.36). Hence

$$\left(1-\alpha\sqrt{3+\sqrt{6}}\right)\ I \leq P_k \leq \left(1+\alpha\sqrt{3+\sqrt{6}}\right)\ I \qquad (3.40)$$

Combining (3.39) and (3.40) leads to

$$C_k^* P_k + P_k C_k \geq 2\mu\ P_k \qquad (3.41)$$

with $\mu = \frac{1}{2}\frac{\lambda_{min}(l,\alpha)}{\left(1+\alpha\sqrt{3+\sqrt{6}}\right)} > 0$, which completes the proof. □

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