

## ON THE LOW MACH NUMBER LIMIT FOR THE COMPRESSIBLE EULER SYSTEM\*

EDUARD FEIREISL<sup>†</sup>, CHRISTIAN KLINGENBERG<sup>‡</sup>, AND SIMON MARKFELDER<sup>‡</sup>

**Abstract.** In this paper, we propose a new approach to singular limits of inviscid fluid flows based on the concept of dissipative measure-valued solutions. We show that dissipative measure-valued solutions of the compressible Euler equations converge to the smooth solution of the incompressible Euler system when the Mach number tends to zero. This holds both for well-prepared and ill-prepared initial data, where in the latter case the presence of acoustic waves causes difficulties. However this effect is eliminated on certain unbounded domains and, in particular, on the whole space, thanks to dispersion.

**Key words.** low Mach number limit, compressible Euler system, measure-valued solution

**AMS subject classifications.** 35B25, 35D99, 35Q31

**DOI.** 10.1137/17M1131799

**1. Introduction.** We propose a new approach to singular limits for inviscid fluid flows based on the concept of *measure-valued solution* for the primitive system. Specifically, we consider the barotropic compressible inviscid Euler equations in two and three space dimensions,

$$(1.1) \quad \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0,$$

$$(1.2) \quad \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0,$$

where  $\varrho = \varrho(t, x)$  represents the mass density,  $\mathbf{u} = \mathbf{u}(t, x)$  the velocity vector, and  $p = p(\varrho, t)$  the pressure. To avoid technicalities, we focus on the iconic example of the isentropic pressure-density state equation  $p = a\varrho^\gamma$ , with  $\gamma > 1$ , although more general cases can be treated as well.

One may rescale these equations by nondimensionalization. After combining terms appropriately (setting the so-called Strouhal number equal to one) one reaches the following system:

$$(1.3) \quad \partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0,$$

$$(1.4) \quad \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho_\varepsilon) = 0,$$

where  $\varepsilon$  is called the Mach number. It represents the norm of the velocity divided by the sound speed. For a more detailed derivation of this see the appendix in [3] or Klainerman and Majda [14]. We consider the asymptotic limit of solutions  $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$

\*Received by the editors May 25, 2017; accepted for publication (in revised form) February 7, 2019; published electronically April 25, 2019.

<http://www.siam.org/journals/sima/51-2/M113179.html>

**Funding:** The work of the first author was supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement 320078 and the Institute of Mathematics of the Academy of Sciences of the Czech Republic (RVO:67985840). The work of the authors was partially supported by the Simons Foundation grant 346300 and the Polish Government MNiSW 2015-2019 matching fund.

<sup>†</sup>Institute of Mathematics of the Academy of Sciences of the Czech Republic Žitná 25, CZ-115 67 Praha 1, Czech Republic (feireisl@math.cas.cz).

<sup>‡</sup>Department of Mathematics, Würzburg University, Würzburg, Germany (klingen@mathematik.uni-wuerzburg.de, simon.markfelder@mathematik.uni-wuerzburg.de).

for  $\varepsilon \rightarrow 0$ . This process represents a bridge between compressible and incompressible fluid flows. Indeed one can expand the dependent variables in terms of  $\varepsilon$ . For example, for the pressure we have

$$p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + O(\varepsilon^3).$$

We now collect terms of the same order and find that the zeroth and first order term in the pressure expansion are constant while the zeroth order term of the velocity (which we shall call  $\mathbf{v}$ ) satisfies the incompressibility condition  $\operatorname{div}_x \mathbf{v} = 0$ . The resulting limiting equations are (setting the zeroth order term of density to be constant, and now calling the second order term pressure  $p^{(2)} = \Pi$ ):

$$(1.5) \quad \operatorname{div}_x \mathbf{v} = 0,$$

$$(1.6) \quad \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0.$$

The initial data for the compressible equations for which the zeroth and first order term of the pressure are constant and the initial velocity is solenoidal are called *well-prepared*. If the the second order term of the pressure is present and the initial velocity is not required to be solenoidal, the data are called *ill-prepared*. For the well-prepared data the above formal derivation has been made rigorous by, e.g., [9], [2]. The ill-prepared data with the underlying spatial domain  $R^3$  have been treated by [14] and [22] and for the nonisentropic case in [17]. For a survey, see [20]. All of these authors assume that the solutions of the compressible flow are smooth. However, as it is well known, solutions of the compressible Euler system develop singularities in a finite time no matter how smooth and/or small the initial data are. One of the principal difficulties of this approach is therefore showing that the life span of the classical solutions is in fact independent of the Mach number.

The hypothesis of smoothness of solutions is therefore quite restrictive and even not appropriate in the context of compressible inviscid fluids. On the other hand, the limit incompressible Euler system, at least if considered in two space dimensions, admits global-in-time smooth solutions for smooth initial data. The existence of global smooth solutions for the incompressible Euler system in three space dimensions is an outstanding open problem.

To achieve global results, it is more convenient to consider the weak solutions of the compressible Euler system. Recently, the theory of convex integration produced a large number of global-in-time weak solutions basically for any regular initial data, however, most of them apparently violate the basic energy inequality associated to the system; see, e.g., Chiodaroli [4] and DeLellis and Székelyhidi [8]. In addition, there is also a nonvoid family of “wild” initial data that give rise to infinitely many weak solutions satisfying many of the conventional admissibility criteria; see Chiodaroli et al. [5], [7] and DeLellis and Székelyhidi [8]. In spite of these results, the existence of global-in-time *admissible* weak solutions for *arbitrary* (possibly smooth) initial data remains largely open for the compressible Euler system.

In this paper, we propose a new approach based on the concept of *dissipative measure-valued (DMV) solution* recently developed by Gwiazda et al. [11], [10]. Roughly speaking, they are measure-valued solutions of the compressible Euler system satisfying an appropriate form of energy inequality; see section 2.1. The energy dissipation is expressed via a *dissipation defect* that in turn dominates the concentration measures that may develop in the field equations. The main advantage of this approach can be summarized as follows:

- The DMV solutions to the barotropic Euler system exist globally in time for any finite energy initial data. Indeed they can be identified as cluster points of solutions to the Navier–Stokes system in the regime of vanishing viscosity, asymptotic limits of suitable numerical schemes, as well as limits of other suitable approximate problems; cf. Málek et al. [16].
- Although the DMV solutions are very general objects that are in general not uniquely determined by the initial data, the convergence is unconditional as soon as the limit system admits a smooth solution.
- Convergence holds for both well-prepared and ill-prepared initial data as long as the spatial domains allows dispersion of acoustic waves in the latter case; see sections 4 and 5, respectively.

Due to the low regularity of the DMV solutions, our method yields convergence in a very weak sense, specifically, in the sense of the strong topology on the space of probability measures.

The paper is organized as follows. After having introduced the necessary preliminary material in section 2, we state our main result in section 3. Section 4 is devoted to the incompressible limit for well-prepared initial data under periodic boundary conditions. Section 5 contains the proof of convergence for the ill-prepared data for the problem on the whole space  $R^N$ .

**2. Preliminaries and main result.** In this section, we collect some basic facts about DMV solutions and state our main result. The symbol  $\Omega$  will denote the spatial domain occupied by the fluid. We focus on two typical examples: periodic boundary conditions, where  $\Omega$  can be identified with the “flat” torus

$$\Omega = \mathcal{T}^N = ([-1, 1]|_{\{-1, 1\}})^N,$$

and  $\Omega = R^N$ ,  $N = 2, 3$ .

**2.1. Measure-valued solutions to the compressible Euler system.** Let

$$(2.1) \quad \mathcal{Q} = \left\{ [\varrho, \mathbf{m}] \mid \varrho \in [0, \infty), \mathbf{m} \in R^N \right\}$$

be the natural phase space associated to solutions  $[\varrho, \mathbf{m}] = [\varrho, \varrho \mathbf{u}]$  of the compressible Euler system (1.1), (1.2).

A DMV solution to the compressible Euler system (1.1), (1.2) consists of a parameterized family of probability measures  $Y_{t,x}$ ,  $t \in (0, T)$ ,  $x \in \Omega$ ,

$$Y_{t,x} \in L_{\text{weak-}(*)}^\infty ((0, T) \times \Omega; \mathcal{P}(\mathcal{Q}))$$

and a nonnegative function  $\mathcal{D} \in L^\infty(0, T)$  called *dissipation defect* satisfying:

- **Equation of continuity.**

$$(2.2) \quad \begin{aligned} & \int_0^T \int_\Omega [\langle Y_{t,x}; \varrho \rangle \partial_t \varphi + \langle Y_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi] \, dx \, dt \\ &= - \int_\Omega \langle Y_{0,x}; \varrho \rangle \varphi(0, \cdot) \, dx - \int_0^T \int_\Omega \nabla_x \varphi \cdot d\mu_D^C \end{aligned}$$

for all  $\varphi \in C_c^\infty([0, T] \times \Omega)$  and a signed measure  $\mu_D^C \in \mathcal{M}([0, T] \times \Omega; R^N)$  called concentration defect.

- **Momentum equation.**

$$(2.3) \quad \int_0^T \int_{\Omega} \left[ \langle Y_{t,x}; \mathbf{m} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle Y_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle Y_{t,x}; p(\varrho) \rangle \operatorname{div}_x \boldsymbol{\varphi} \right] dx dt \\ = - \int_{\Omega} \langle Y_{0,x}; \mathbf{m} \rangle \cdot \boldsymbol{\varphi}(0, \cdot) dx - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : d\mu_D^M$$

for all  $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^N)$  and a signed measure  $\mu_D^M \in \mathcal{M}([0, T] \times \Omega; \mathbb{R}^{N \times N})$ .

- **Energy balance.**

The energy inequality

$$(2.4) \quad \int_{\Omega} \left\langle Y_{\tau,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right\rangle dx + \mathcal{D}(\tau) \\ \leq \int_{\Omega} \left\langle Y_{0,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right\rangle dx$$

holds for a.a.  $\tau \in (0, T)$  and a certain constant  $\bar{\varrho} > 0$ , where  $P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} dz$  is called the *pressure potential*.

- **Compatibility condition.**

$$(2.5) \quad \int_0^{\tau} \int_{\Omega} [|\mu_D^C| + |\mu_D^M|] dx dt \leq \int_0^{\tau} \xi(t) \mathcal{D}(t) dt \text{ for a.a. } \tau \in [0, T], \xi \in L^1(0, T).$$

*Remark 2.1.* Strictly speaking, the expressions containing the concentration defect in (2.2), (2.3) should be written

$$\langle \mu_D^C; \nabla_x \boldsymbol{\varphi} \rangle \text{ instead of } \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} \cdot d\mu_D^C, \text{ and } \langle \mu_D^M; \nabla_x \boldsymbol{\varphi} \rangle \text{ instead of } \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : d\mu_D^M.$$

Similarly, we should have written

$$\|\mu_D^C\|_{[0, \tau] \times \Omega} + \|\mu_D^M\|_{[0, \tau] \times \Omega} \leq \int_0^{\tau} \xi(t) \mathcal{D}(t) dt,$$

rather than

$$\int_0^{\tau} \int_{\Omega} [|\mu_D^C| + |\mu_D^M|] dx dt \leq \int_0^{\tau} \xi(t) \mathcal{D}(t) dt$$

in (2.5).

*Remark 2.2.* In contrast with the original definition introduced in [11], we prefer to work with the natural phase variable, namely, the density  $\varrho$  and the momentum  $\mathbf{m} = \varrho \mathbf{u}$ , similarly to [10].

*Remark 2.3.* The constant  $\bar{\varrho}$  in (2.4) can be taken arbitrary if  $\Omega = \mathcal{T}^N$  and becomes relevant only for  $\Omega = \mathbb{R}^N$ , where it represents the far field limit of the density,

$$\varrho \rightarrow \bar{\varrho} \text{ as } |x| \rightarrow \infty.$$

*Remark 2.4.* The functions

$$[\varrho, \mathbf{m}] \mapsto \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}, \quad [\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho}$$

are singular at the boundary of the phase space  $\mathcal{Q}$ , namely, on the vacuum zone  $\varrho = 0$ .

We set

$$\frac{|\mathbf{m}|^2}{\varrho} = \begin{cases} 0 & \text{if } \varrho \geq 0, \mathbf{m} = 0, \\ \infty & \text{if } \varrho = 0, \mathbf{m} \neq 0 \end{cases}$$

on the singular set. Accordingly, the function

$$[\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho}$$

is convex lower semi-continuous on  $\mathcal{Q}$ . Now it follows from the energy inequality (2.4) that  $[\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho}$  is integrable with respect to  $Y_{t,x}$  for a.a.  $t, x$ . In particular,

$$\text{supp}[Y_{t,x}] \cap \left\{ [\varrho, \mathbf{m}] \in \mathcal{Q} \mid \varrho = 0, \mathbf{m} \neq 0 \right\} = \emptyset.$$

In applications, the parameterized family  $Y_{t,x}$  is the Young measure generated by an oscillating sequence of approximate solutions  $[\varrho, \mathbf{m} = \varrho \mathbf{u}]$ , while the measure  $Y_0$  is determined by the initial conditions. Note, however, there are measure-valued solutions to system (1.1), (1.2) that are not generated by any sequence of weak solutions; see [6].

The measures  $\mu_D^C, \mu_D^M$  characterize the so-called concentration defect. There is a more precise characterization of these terms as soon as a measure-valued solution is identified as a suitable limit of a family of weak solutions; see Gwiazda et al. [11]. Then typically  $\mu_D^C = 0$ , while  $\mu_D^M$  is the Young measure associated to the so-called recession function corresponding to the quantity  $\sqrt{\varrho} u_i \sqrt{\varrho} u_j + p(\varrho) \delta_{i,j}$  in the sense of Alibert and Bouchitté [1]. In such a case, the concentration defect  $\mathcal{D}$  can be equally given in terms of the recession function associated to the energy  $\frac{1}{2} |\sqrt{\varrho} \mathbf{u}|^2 + P(\varrho)$ . These quantities satisfy the compatibility condition (2.5) as soon as

$$\limsup_{\varrho \rightarrow \infty} \frac{p(\varrho)}{P(\varrho)} \leq p_\infty < \infty,$$

which implies

$$(2.6) \quad p(\varrho) \leq c(\bar{\varrho}, p_\infty) [P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})]$$

for all  $\varrho$  large enough. Accordingly, the function  $\xi$  in (2.5) then can be taken constant depending only on  $p_\infty, \bar{\varrho}$ .

*Remark 2.5.* In the low Mach number limit problem studied below, the pressure takes the form  $\frac{1}{\varepsilon^2} p(\varrho)$ , while the associated pressure potential reads  $\frac{1}{\varepsilon^2} P(\varrho)$ . In accordance with (2.6), the measure-valued solutions introduced by Gwiazda et al. [11] will satisfy the compatibility condition (2.5) uniformly for  $\varepsilon \rightarrow 0$ . The same remains true in the more general setting introduced in [10] and considered in the present paper as long as the measure-valued solutions are generated by a suitable family of functions, for which the concentration defect is characterized as the difference between the weak- $(*)$  limit in the sense of measures and the biting limit of nonlinear compositions; cf. [10].

Finally, we remark that the *existence* of the dissipative measure-valued solutions, at least for the iconic pressure law  $p(\varrho) = a\varrho^\gamma$ ,  $\gamma \geq 1$ , can be easily established by means of an artificial/physical viscosity approximation. Neustupa [19] constructed a variant of the measure-valued solutions by considering a higher viscosity approximation to the Euler system in the spirit of the general theory of multipolar fluids developed by Nečas and Šilhavý [18]. In view of the nowadays available existence theory for the barotropic Navier–Stokes system, the measure-valued solutions of the compressible Euler can easily be identified with the cluster points for  $\delta \rightarrow 0$  of a family of weak solutions  $[\varrho_\delta, \mathbf{u}_\delta]$  of the Navier–Stokes system:

$$\begin{aligned} \partial_t \varrho_\delta + \operatorname{div}_x(\varrho_\delta \mathbf{u}_\delta) &= 0, \\ \partial_t(\varrho_\delta \mathbf{u}_\delta) + \operatorname{div}_x(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) + \nabla_x p_\delta(\varrho_\delta) &= \delta \Delta \mathbf{u}_\delta + \delta \nabla_x \operatorname{div}_x \mathbf{u}_\delta, \\ \int_\Omega \left[ \frac{1}{2} \varrho_\delta |\mathbf{u}_\delta|^2 + P_\delta(\varrho_\delta) - P'_\delta(\bar{\varrho})(\varrho_\delta - \bar{\varrho}) - P_\delta(\bar{\varrho}) \right] (\tau, \cdot) \, dx + \int_0^\tau \int_\Omega \delta [|\nabla_x \mathbf{u}_\delta|^2 \\ + |\operatorname{div}_x \mathbf{u}_\delta|^2] \, dx &\leq \int_\Omega \left[ \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P_\delta(\varrho_0) - P'_\delta(\bar{\varrho})(\varrho_0 - \bar{\varrho}) - P_\delta(\bar{\varrho}) \right] \, dx, \\ p_\delta(\varrho) &= p(\varrho) + \delta \varrho^\Gamma, \quad \delta > 0. \end{aligned}$$

Indeed the existence of the weak solutions  $[\varrho_\delta, \varrho_\delta \mathbf{u}_\delta]$  is guaranteed by the theory of Lions [15] for  $N = 1, 2, 3$ , at least if  $\Gamma \geq \Gamma(N)$ . In view of Remark 2.5, the compatibility condition (2.5) will be satisfied for a suitable constant  $\xi$  independent of  $\delta$ .

**2.2. Relative energy inequality.** For a parameterized family  $Y_{t,x}$  of probability measure defined on the phase space (2.1), we introduce *the relative energy* functional

$$\begin{aligned} (2.7) \quad & \mathcal{E}(\varrho, \mathbf{m} \mid r, \mathbf{U}) \\ &= \int_\Omega \left\langle Y_{t,x}; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{U}(t, x) \right|^2 + P(\varrho) - P'(r(t, x))(\varrho - r(t, x)) - P(r(t, x)) \right\rangle \, dx, \end{aligned}$$

where  $\mathbf{U}, r$  are continuously differentiable “test functions,”  $\mathbf{U}, r - \bar{\varrho}$  compactly supported in  $\Omega$ ,  $r > 0$ .

For all DMV solutions  $\varrho, \mathbf{m}$  of the compressible Euler system, the following relation can be deduced from (2.2)–(2.4); see [11]:

$$\begin{aligned} (2.8) \quad & \left[ \mathcal{E}(\varrho, \mathbf{m} \mid r, \mathbf{U}) \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) \\ & \leq \int_0^\tau \int_\Omega [\langle Y_{t,x}; \varrho \mathbf{U}(t, x) - \mathbf{m} \rangle \cdot \partial_t \mathbf{U} \\ & \quad + \left\langle Y_{t,x}; \varrho \mathbf{U}(t, x) - \mathbf{m} \right\rangle \otimes \frac{\mathbf{m}}{\varrho} : \nabla_x \mathbf{U} - \langle Y_{t,x}; p(\varrho) \rangle \operatorname{div}_x \mathbf{U} \Big] \, dx \, dt \\ & \quad + \int_0^\tau \int_\Omega \left[ \langle Y_{t,x}; r(t, x) - \varrho \rangle \frac{1}{r} \partial_t p(r) - \langle Y_{t,x}; \mathbf{m} \rangle \cdot \frac{1}{r} \nabla_x p(r) \right] \, dx \, dt \\ & \quad + \int_0^\tau \int_\Omega \left( \frac{1}{2} \nabla_x |\mathbf{U}|^2 - \nabla_x P'(r) \right) \cdot d\mu_D^C - \int_0^\tau \int_\Omega \nabla_x \mathbf{U} : d\mu_D^M \end{aligned}$$

for any

$$(2.9) \quad \mathbf{U}, r \in C^1([0, T] \times \Omega), \quad r > 0, \quad \text{supp}[\mathbf{U}], \quad \text{supp}[r - \bar{\varrho}] \text{ compact in } [0, T] \times \Omega.$$

*Remark 2.6.* Note that compactness of the support of the test functions claimed in (2.9) is irrelevant if  $\Omega = \mathcal{T}^N$ —a compact set.

**2.3. Solutions of the target system.** It is expected that the low Mach number limit velocity  $\mathbf{v}$  is described by the incompressible Euler system (1.5), (1.6). Our approach leans essentially on the fact that the limit field  $\mathbf{v}$  is a smooth function. Referring to the classical result of Kato [12], [13], we know that (1.5), (1.6) admits a solution  $\mathbf{v}$ , unique in the class

$$(2.10) \quad \begin{aligned} \mathbf{v} &\in C([0, T_{\max}); W^{k,2}(\Omega; R^N)), \quad \partial_t \mathbf{v}, \quad \partial_t \Pi, \quad \nabla_x \Pi \in C([0, T_{\max}); W^{k-1,2}(\Omega; R^N)) \\ &\text{for some } T_{\max} > 0, \text{ as soon as} \end{aligned}$$

$$\mathbf{v}_0 \in W^{k,2}(\Omega; R^N), \quad k > \frac{N}{2} + 1, \quad \text{div}_x \mathbf{v}_0 = 0.$$

Moreover, the solution exists globally in time, meaning  $T_{\max} = \infty$  if  $N = 2$ .

**3. Main results.** Let  $\varrho_{0,\varepsilon} = \varrho_\varepsilon(0, \cdot)$ ,  $\mathbf{u}_{0,\varepsilon} = \mathbf{u}_\varepsilon(0, \cdot)$  be the initial data for the rescaled system (1.3), (1.4). We suppose that

$$\frac{\varrho_{0,\varepsilon} - \bar{\varrho}}{\varepsilon} \rightarrow s_0, \quad \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{u}_0$$

in a certain sense specified in the forthcoming section. We say that the initial data are

- *well-prepared* if  $s_0 = 0$ ,  $\mathbf{u}_0 = \mathbf{v}_0$ ,  $\text{div}_x \mathbf{v}_0 = 0$ ;
- *ill-prepared* otherwise.

In the context of DMV solutions, where the the distribution of the initial data is determined by the measure  $Y_{0,x}^\varepsilon$ , *well-prepared initial data* translates to

$$(3.1) \quad \int_{\Omega} \left\langle Y_{0,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}_0(x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

for certain constant  $\bar{\varrho} > 0$  and a solenoidal function  $\mathbf{v}_0$ .

If the initial data are given in terms of the functions  $\varrho_{0,\varepsilon}$ ,  $\mathbf{u}_{0,\varepsilon}$ , meaning

$$Y_{0,\varepsilon}^\varepsilon = \delta_{\varrho_{0,\varepsilon}(x), [\varrho_{0,\varepsilon}(x) \mathbf{u}_{0,\varepsilon}(x)]},$$

(3.1) follows as soon as

$$\begin{aligned} \frac{\varrho_{0,\varepsilon} - \bar{\varrho}}{\varepsilon} &\text{ bounded in } L^\infty(\Omega), \quad \frac{\varrho_{0,\varepsilon} - \bar{\varrho}}{\varepsilon} \rightarrow 0 \text{ in } L^1(\Omega), \quad \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{v}_0 \text{ in } L^2(\Omega; R^N), \\ \text{div}_x \mathbf{v}_0 &= 0. \end{aligned}$$

Similarly, the initial data are *ill-prepared* if

$$(3.2) \quad \begin{aligned} \int_{\Omega} \left\langle Y_{0,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{u}_0(x) \right|^2 \right. \\ \left. + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho} + \varepsilon s_0)(\varrho - \bar{\varrho} - \varepsilon s_0) - P(\bar{\varrho} + \varepsilon s_0)) \right\rangle \, dx \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , for certain constant  $\bar{\varrho} > 0$ ,  $s_0 \in L^\infty \cap L^1(\Omega)$ , and  $\mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0$ ,  $\operatorname{div}_x \mathbf{v}_0 = 0$ . In terms of “deterministic” initial data  $\varrho_{0,\varepsilon}$ ,  $\mathbf{u}_{0,\varepsilon}$  this can be rephrased as

$$\frac{\varrho_{0,\varepsilon} - \bar{\varrho}}{\varepsilon} \text{ bounded in } L^\infty(\Omega), \quad \frac{\varrho_{0,\varepsilon} - \bar{\varrho}}{\varepsilon} \rightarrow s_0 \text{ in } L^1(\Omega),$$

$$\mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0 \text{ in } L^2(\Omega; \mathbb{R}^N), \quad \operatorname{div}_x \mathbf{v}_0 = 0.$$

**3.1. Main result for the well-prepared data.** We consider the rescaled compressible Euler system with the periodic boundary conditions  $\Omega = \mathcal{T}^N$  equipped with the well-prepared initial data.

**THEOREM 3.1.** *Let  $p \in C^1(0, \infty) \cap C[0, \infty)$  satisfy  $p'(\varrho) > 0$  whenever  $\varrho > 0$ . Let  $\Omega = \mathcal{T}^N$ ,  $N = 2, 3$ , and let  $\{Y_{t,x}^\varepsilon\}_{t \in [0,T]; x \in \mathcal{T}^N}$ ,  $\mathcal{D}^\varepsilon$  be a family of DMV solutions of the rescaled compressible Euler system (1.3), (1.4), satisfying the compatibility condition (2.5) with  $\xi$  independent of  $\varepsilon$ . Let the initial data  $Y_{0,x}^\varepsilon$  be well-prepared, meaning (3.1) holds for  $\bar{\varrho} > 0$  and  $\mathbf{v}_0 \in W^{k,2}(\mathcal{T}^N; \mathbb{R}^N)$ ,  $k > \frac{N}{2} + 1$ ,  $\operatorname{div}_x \mathbf{v}_0 = 0$ . Finally, suppose that  $T < T_{\max}$ , where  $T_{\max}$  denotes the life span of the solution to the incompressible Euler system (1.5), (1.6) endowed with the initial data  $\mathbf{v}_0$ .*

*Then*

$$\mathcal{D}^\varepsilon \rightarrow 0 \text{ in } L^\infty(0, T),$$

$$\text{ess sup}_{t \in (0, T)} \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}(t, x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle dx \rightarrow 0$$

*as  $\varepsilon \rightarrow 0$ , where  $\mathbf{v}$  is the solution of the incompressible Euler system (1.5), (1.6) with the initial data  $\mathbf{v}_0$ .*

Theorem 3.1 asserts that the probability measures  $Y_{t,x}$  shrink to their expected value as  $\varepsilon \rightarrow 0$ , where the latter are characterized by the constant value  $\bar{\varrho}$  for the density and the solution  $\mathbf{v}$  of the incompressible system. The result is restricted to the life span of  $\mathbf{v}$  if  $N = 3$  and is global for  $N = 2$ . The required smoothness of  $\mathbf{v}_0$  could possibly be slightly relaxed. The proof of Theorem 3.1 is given in section 4 below.

**3.2. Main result for the ill-prepared data.** Convergence in the ill-prepared case is “polluted” by the presence of acoustic waves generated by the component  $s_0$ ,  $\nabla_x \Phi_0$  of the limit data. To eliminate this effect, we consider the unbounded physical space  $\Omega = \mathbb{R}^N$ , where dispersion annihilates acoustic phenomena at least on compact sets.

To simplify presentation, we also assume that the concentration defect  $\mu_D^C$  in the equation of continuity (2.2) vanishes. This is not a very sever restriction as it is always satisfied as long as the DMV solutions are obtained as a limit of a family of approximate solutions satisfying a suitable form of the energy balance.

**THEOREM 3.2.** *Let  $p \in C^1(0, \infty) \cap C[0, \infty)$  satisfy*

$$(3.3) \quad p'(\varrho) > 0 \text{ for all } \varrho > 0, \quad \limsup_{\varrho \rightarrow \infty} \frac{p(\varrho)}{P(\varrho)} = P_\infty < \infty,$$

$$\liminf_{\varrho \rightarrow \infty} \frac{p(\varrho)}{\varrho^\gamma} \geq p_\infty > 0 \text{ for some } \gamma > 1.$$

*Let  $\Omega = \mathbb{R}^N$ ,  $N = 2, 3$ , and let  $\{Y_{t,x}^\varepsilon\}_{t \in [0,T]; x \in \mathcal{T}^N}$ ,  $\mathcal{D}^\varepsilon$  be a family of DMV solutions of the rescaled compressible Euler system (1.3), (1.4), with  $\mu_D^C = 0$ , satisfying the*

compatibility condition (2.5) with  $\xi$  independent of  $\varepsilon$ . Let the initial data  $Y_{0,x}^\varepsilon$  be illprepared, meaning (3.2) holds for  $\bar{\varrho} > 0$  and  $s_0 \in W^{k,2} \cap W^{k,1}(R^N)$ ,  $\mathbf{u}_0 = \mathbf{v}_0 \in W^{k,2} \cap W^{k,1}(R^N; R^N)$ ,  $k > \frac{N}{2} + 2$ . Finally, suppose that  $T < T_{\max}$ , where  $T_{\max}$  denotes the life span of the solution to the incompressible Euler system (1.5), (1.6) endowed with the initial data  $\mathbf{v}_0 = P[\mathbf{u}_0]$ , where  $P$  denotes the standard Helmholtz projection onto the space of solenoidal functions.

Then

$$\mathcal{D}^\varepsilon \rightarrow 0 \text{ in } L^\infty(0, T),$$

$$\text{ess sup}_{t \in (\delta, T)} \int_B \left\langle Y_{t,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}(t, x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle dx \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , for any compact  $B \subset R^N$  and any  $0 < \delta < T$ , where  $\mathbf{v}$  is the solution of the incompressible Euler system (1.5), (1.6) with the initial data  $\mathbf{v}_0$ .

Note that the required regularity of the data  $s_0$ ,  $\mathbf{u}_0$  is higher than in Theorem 3.1. This is due to the required regularity properties of solutions of the corresponding acoustic equations discussed in section 5.2.1. Moreover, strong decay of  $s_0$ ,  $\mathbf{u}_0$  is necessary as  $|x| \rightarrow \infty$ . Convergence to the target system is only local, both in time and space. This is inevitable due to the presence of acoustic waves. The proof of Theorem 3.2 will be done in section 5.

**4. Incompressible limit for well-prepared initial data.** In this section, we prove Theorem 3.1. For  $Y_{t,x}^\varepsilon$ —the DMV solution of the rescaled system—we denote

$$\begin{aligned} \mathcal{E}_\varepsilon(\varrho, \mathbf{m} \mid \bar{\varrho}, \mathbf{v}) \\ = \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}(t, x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle dx \end{aligned}$$

the relative entropy associated to  $\bar{\varrho}$ ,  $\mathbf{v}$ .

**4.1. Relative energy inequality.** As the quantities  $r = \bar{\varrho}$ ,  $\mathbf{U} = \mathbf{v}$  enjoy the regularity required in (2.9), they can be used as test functions in the relative entropy inequality (2.8):

$$\begin{aligned} (4.1) \quad & \mathcal{E}_\varepsilon(\varrho, \mathbf{m} \mid \bar{\varrho}, \mathbf{v})(\tau) + \mathcal{D}^\varepsilon(\tau) \\ & \leq \int_{\mathcal{T}^N} \left\langle Y_{0,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}_0(x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle dx \\ & + \int_0^\tau \int_{\mathcal{T}^N} \left[ \langle Y_{t,x}^\varepsilon; \varrho \mathbf{v}(t, x) - \mathbf{m} \rangle \cdot \partial_t \mathbf{v} + \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{v}(t, x) - \mathbf{m}) \otimes \frac{\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{v} \right] dx dt \\ & + \int_0^\tau \int_{\mathcal{T}^N} \frac{1}{2} \nabla_x |\mathbf{v}|^2 \cdot d\mu_D^{C,\varepsilon} - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \mathbf{v} : d\mu_D^{M,\varepsilon}. \end{aligned}$$

As the initial data are well-prepared, we get

$$(4.2) \quad \int_{\mathcal{T}^N} \left\langle Y_{0,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}_0(x) \right|^2 + \frac{1}{\varepsilon^2} (P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho})) \right\rangle dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

In addition, since the compatibility condition (2.5) is satisfied uniformly with respect to  $\varepsilon$ , we deduce

$$(4.3) \quad \int_0^\tau \int_{\mathcal{T}^N} \frac{1}{2} \nabla_x |\mathbf{v}|^2 \cdot d\mu_D^{C,\varepsilon} - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \mathbf{v} : d\mu_D^{M,\varepsilon} \leq c(\|\mathbf{v}_0\|_{W^{k,2}}) \int_0^\tau \xi \mathcal{D}^\varepsilon dt.$$

In view of (4.2), (4.3), the conclusion of Theorem 3.1 follows by Gronwall's lemma as soon as we show

$$(4.4) \quad \begin{aligned} & \int_0^\tau \int_{\mathcal{T}^N} \left[ \langle Y_{t,x}^\varepsilon; \varrho \mathbf{v}(t,x) - \mathbf{m} \rangle \cdot \partial_t \mathbf{v} + \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{v}(t,x) - \mathbf{m}) \otimes \frac{\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{v} \right] dx dt \\ & \leq \omega(\varepsilon) + c \int_0^\tau (1 + \xi) \left( \mathcal{E}_\varepsilon(\varrho, \mathbf{m} \mid \bar{\varrho}, \mathbf{v}) + \mathcal{D}^\varepsilon \right) dt, \quad \omega(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

**4.2. Estimates.** Our goal is to show (4.4).

**4.2.1. Step 1—convective term.** We start by writing

$$\begin{aligned} & \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{v}(t,x) - \mathbf{m}) \otimes \frac{\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{v} dx \\ & = \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{v}(t,x) - \mathbf{m}) \otimes \frac{\mathbf{m} - \varrho \mathbf{v}}{\varrho} \right\rangle : \nabla_x \mathbf{v} dx \\ & \quad + \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho \mathbf{v}(t,x) - \mathbf{m} \rangle \cdot \mathbf{v} \cdot \nabla_x \mathbf{v} dx, \end{aligned}$$

where, obviously,

$$\int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{v}(t,x) - \mathbf{m}) \otimes \frac{\mathbf{m} - \varrho \mathbf{v}}{\varrho} \right\rangle : \nabla_x \mathbf{v} dx \leq c(\|\mathbf{v}_0\|_{W^{k,2}}) \mathcal{E}_\varepsilon(\varrho, \mathbf{m} \mid \bar{\varrho}, \mathbf{v}).$$

Moreover, as  $\mathbf{v}$  fulfills (1.6), we may go back to (4.4) to deduce that (4.4) reduces to showing

$$(4.5) \quad \begin{aligned} & \int_0^\tau \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} - \varrho \mathbf{v}(t,x) \rangle \cdot \nabla_x \Pi dx dt \leq \omega(\varepsilon) \\ & \quad + c \int_0^\tau (1 + \xi) \left( \mathcal{E}_\varepsilon(\varrho, \mathbf{m} \mid \bar{\varrho}, \mathbf{v}) + \mathcal{D}^\varepsilon \right) dt. \end{aligned}$$

**4.2.2. Step 2—pressure estimates.** To see (4.5), we deduce from (2.2) that

$$(4.6) \quad \begin{aligned} & \int_0^\tau \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot \nabla_x \Pi dx dt \\ & = - \int_0^\tau \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho \rangle \partial_t \Pi dx dt + \left[ \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho \rangle \Pi dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \Pi \cdot d\mu_D^{C,\varepsilon} \\ & = - \int_0^\tau \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho - \bar{\varrho} \rangle \partial_t \Pi dx dt \\ & \quad + \left[ \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho - \bar{\varrho} \rangle \Pi dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \Pi \cdot d\mu_D^{C,\varepsilon} \\ & = -\varepsilon \int_0^\tau \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \partial_t \Pi dx dt + \varepsilon \left[ \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \Pi dx \right]_{t=0}^{t=\tau} \\ & \quad - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \Pi \cdot d\mu_D^{C,\varepsilon}. \end{aligned}$$

Similarly, we may use the incompressibility condition  $\operatorname{div}_x \mathbf{v} = 0$  to obtain

$$(4.7) \quad \int_0^\tau \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; \varrho \mathbf{v}(t, x) \rangle \cdot \nabla_x \Pi \, dx \, dt = \varepsilon \int_0^\tau \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \mathbf{v} \cdot \nabla_x \Pi \, dx \, dt.$$

Now observe that the rightmost integral in (4.6) can be controlled by the dissipation defect  $\mathcal{D}^\varepsilon$ . Consequently, as the pressure  $\Pi$  belongs to the regularity class (2.10), in particular  $\Pi, \partial_t \Pi$  and  $\nabla_x \Pi$  are bounded continuous in  $[0, T] \times \mathcal{T}^N$ , it is enough to establish a uniform bound

$$(4.8) \quad \int_{\mathcal{T}^N} \left\langle Y_{t,x}^\varepsilon; \left| \frac{\varrho - \bar{\varrho}}{\varepsilon} \right| \right\rangle \, dx \leq c.$$

**4.2.3. Step 3—energy estimates.** As the DMV solutions satisfy the energy inequality (2.4), we deduce from (3.1) that

$$(4.9) \quad \frac{1}{\varepsilon^2} \int_{\mathcal{T}^N} \langle Y_{t,x}^\varepsilon; P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \rangle \, dx \leq c \text{ uniformly as } \varepsilon \rightarrow 0.$$

Since

$$P''(\varrho) = \frac{p'(\varrho)}{\varrho} \text{ for } \varrho > 0,$$

the function  $P$  is strictly convex, and, consequently

$$(4.10) \quad |\varrho - \bar{\varrho}|^2 \leq c(\delta) \left( P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right) \text{ whenever } 0 < \delta \leq \varrho, \bar{\varrho} \leq \frac{1}{\delta}, \delta > 0,$$

and

$$(4.11) \quad \begin{aligned} 1 + |\varrho - \bar{\varrho}| + P(\varrho) &\leq c(\delta) \left( P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right) \\ \text{if } 0 < 2\delta < \bar{\varrho} &< \frac{1}{2\delta}, \varrho \in [0, \delta) \cup [\frac{1}{\delta}, \infty), \delta > 0. \end{aligned}$$

Combining (4.10), (4.11) with (4.9) we obtain (4.8). Theorem 3.1 has been proved.

**5. Incompressible limit for ill-prepared initial data.** Our goal is to prove Theorem 3.2. To begin, we introduce a function  $\chi = \chi(\varrho)$  such that

$$\chi(\varrho) \in C_c^\infty(0, \infty), 0 \leq \chi \leq 1, \chi(\varrho) = 1 \text{ if } \frac{\bar{\varrho}}{2} \leq \varrho \leq 2\bar{\varrho}.$$

For a function  $H = H(\varrho, \mathbf{m})$  we set

$$H_{\text{ess}}(\varrho, \mathbf{m}) = \chi(\varrho)H(\varrho, \mathbf{m}), H_{\text{res}}(\varrho, \mathbf{m}) = (1 - \chi(\varrho))H(\varrho, \mathbf{m}).$$

**5.1. Energy bounds.** As the initial distribution  $Y_{0,x}^\varepsilon$  is ill-prepared, meaning satisfies (3.2), and the functions  $s_0, \mathbf{u}_0$  belong to  $L^\infty \cap L^1(R^N)$ , the initial energy

$$\int_{R^N} \left\langle Y_{0,x}^\varepsilon; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\varepsilon^2} \left( P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right) \right\rangle \, dx \leq E_0$$

is bounded uniformly for  $\varepsilon \rightarrow 0$ . In accordance with the energy inequality we obtain

$$(5.1) \quad \text{ess sup}_{t \in (0, T)} \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\varepsilon^2} \left( P(\varrho) - P'(\bar{\varrho})(\varrho - \bar{\varrho}) - P(\bar{\varrho}) \right) \right\rangle \, dx \leq E_0.$$

Thus, using estimates (4.10), (4.11), we may infer that

$$(5.2) \quad \text{ess sup}_{t \in (0, T)} \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \left[ \left[ \frac{\varrho - \bar{\varrho}}{\varepsilon} \right]_{\text{ess}} \right]^2 \right\rangle dx + \text{ess sup}_{t \in (0, T)} \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \left[ \frac{P(\varrho) + 1}{\varepsilon^2} \right]_{\text{res}} \right\rangle dx \leq c.$$

Furthermore, we get

$$(5.3) \quad \int_{R^N} \left\langle Y_{t,x}^\varepsilon; [\mathbf{m}]^2 \right\rangle dx \leq c \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \left[ \varrho \left| \frac{\mathbf{m}}{\varrho} \right|^2 \right]_{\text{ess}} \right\rangle dx \leq E_0.$$

Seeing that

$$[\mathbf{m}]_{\text{res}} \leq \frac{|\mathbf{m}|}{\sqrt{\varrho}} [\sqrt{\varrho}]_{\text{res}},$$

we deduce

$$\left[ |\mathbf{m}|^{\frac{2\gamma}{\gamma+1}} \right]_{\text{res}} \leq c \left( \varepsilon \varrho \frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\varepsilon} [\varrho^\gamma]_{\text{res}} \right);$$

whence

$$(5.4) \quad \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \left[ |\mathbf{m}|^{\frac{2\gamma}{\gamma+1}} \right]_{\text{res}} \right\rangle dx \leq \varepsilon E_0.$$

Finally, we recall Jensen's inequality

$$(5.5) \quad \left\langle Y_{t,x}^\varepsilon; |\mathbf{F}| \right\rangle^q \leq \left\langle Y_{t,x}^\varepsilon; |\mathbf{F}|^q \right\rangle, \quad q \geq 1.$$

Consequently, the estimates (5.3)–(5.4) give rise to

$$(5.6) \quad \begin{aligned} \left\langle Y_{t,x}^\varepsilon; \mathbf{m} \right\rangle &\text{ bounded in } \left[ L^2 + L^{\frac{2\gamma}{\gamma+1}} \right] (R^N, R^N), \\ \left\langle Y_{t,x}^\varepsilon; \left[ \frac{\varrho - \bar{\varrho}}{\varepsilon} \right]_{\text{ess}} \right\rangle &\text{ bounded in } L^2(R^N), \\ \varepsilon^{-\frac{2}{\gamma}} \left\langle Y_{t,x}^\varepsilon; [\varrho]_{\text{res}} \right\rangle &\text{ bounded in } L^\gamma(R^N). \end{aligned}$$

## 5.2. Acoustic equation.

Write

$$\mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0, \quad \mathbf{v}_0 = P[\mathbf{u}_0].$$

The evolution of acoustic waves is described by the *acoustic equation*

$$(5.7) \quad \varepsilon \partial_t s_\varepsilon + \text{div}_x (\bar{\varrho} \nabla_x \Phi_\varepsilon) = 0,$$

$$(5.8) \quad \varepsilon \partial_t \nabla_x \Phi_\varepsilon + \frac{p'(\bar{\varrho})}{\bar{\varrho}} \nabla_x s_\varepsilon = 0,$$

$$s(0, \cdot) = s_0, \quad \nabla_x \Phi_\varepsilon(0, \cdot) = \nabla_x \Phi_0$$

considered in the whole space  $R^N$ ,  $N = 2, 3$ .

**5.2.1. Acoustic energy.** Solutions of (5.7), (5.8) conserve the total (acoustic) energy, specifically,

$$(5.9) \quad \frac{d}{dt} \int_{R^N} [p'(\bar{\varrho}) s_\varepsilon^2 + \bar{\varrho}^2 |\nabla_x \Phi_\varepsilon|^2] dx = 0.$$

Differentiating the (linear) system (5.7), (5.8) we easily extend (5.9) to

$$(5.10) \quad \begin{aligned} \|s_\varepsilon(\tau, \cdot)\|_{W^{k,2}(R^N)}^2 + \|\nabla_x \Phi_\varepsilon(\tau, \cdot)\|_{W^{k,2}(R^N; R^N)}^2 &\leq c \left[ \|s_0\|_{W^{k,2}(R^N)}^2 \right. \\ &\quad \left. + \|\nabla_x \Phi_0\|_{W^{k,2}(R^N; R^N)}^2 \right] \end{aligned}$$

for any  $\tau \geq 0$ ,  $k \geq 0$ .

**5.2.2. Dispersion.** Finally, we report the dispersive estimates

$$(5.11) \quad \begin{aligned} \|s_\varepsilon(\tau, \cdot)\|_{L^p(R^3)}^2 + \|\nabla_x \Phi_\varepsilon(\tau, \cdot)\|_{L^p(R^3; R^3)}^2 \\ \leq c \left(1 + \frac{\tau}{\varepsilon}\right)^{(N-1)(\frac{1}{p} - \frac{1}{q})} \left[ \|s_0\|_{W^{k,q}(R^3)}^2 + \|\nabla_x \Phi_0\|_{W^{k,q}(R^3; R^3)}^2 \right], \end{aligned}$$

$k \geq N(\frac{1}{q} - \frac{1}{p})$ ,  $2 \leq p \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ; see Strichartz [21].

**5.3. Relative energy inequality.** The first observation is that

$$r = \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{U} = \mathbf{v} + \nabla_x \Phi_\varepsilon,$$

where  $\mathbf{v}$  is the solution of the incompressible Euler system (1.5), (1.6), and  $s_\varepsilon$ ,  $\Phi_\varepsilon$  solve the acoustic system (5.7), (5.8) can be taken as test functions in the relative energy inequality (2.8). Note that, strictly speaking, these functions do not belong to the class (2.9) but decay sufficiently fast to their far field limit. Validity of (2.8) can be verified by a density argument. Note the the most problematic term containing the pressure can be handled as follows:

$$\begin{aligned} \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) \rangle \operatorname{div}_x \mathbf{U} \, dx &= \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \operatorname{div}_x \mathbf{U} \, dx \\ &= \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \Delta_x \Phi \, dx. \end{aligned}$$

Writing

$$\begin{aligned} \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) \\ = \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \mathbf{v}(t, x) - \nabla_x \Phi_\varepsilon(t, x) \right|^2 \right\rangle \, dx \\ + \frac{1}{\varepsilon^2} \int_{R^N} \langle Y_{t,x}^\varepsilon; P(\varrho) - P'(\bar{\varrho} + \varepsilon s_\varepsilon(t, x))(\varrho - \bar{\varrho} - \varepsilon s_\varepsilon(t, x)) - P(\bar{\varrho} + \varepsilon s_\varepsilon(t, x)) \rangle \, dx, \end{aligned}$$

we obtain the relative energy inequality in the form

$$(5.12) \quad \begin{aligned} &\left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) \right]_{t=0}^{t=\tau} + \mathcal{D}^\varepsilon(\tau) \leq \omega(\varepsilon) \\ &+ \int_0^\tau \int_{R^N} \left[ \langle Y_{t,x}^\varepsilon; \varrho \mathbf{U}(t, x) - \mathbf{m} \rangle \cdot \partial_t \mathbf{U} + \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{U}(t, x) - \mathbf{m}) \otimes \frac{\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} \right] \, dx \, dt \\ &- \frac{1}{\varepsilon^2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \Delta \Phi_\varepsilon \, dx \, dt \\ &+ \frac{1}{\varepsilon} \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; r(t, x) - \varrho \rangle P''(r) \partial_t s_\varepsilon - \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot P''(r) \nabla_x s_\varepsilon] \, dx \, dt \\ &- \int_0^\tau \int_{R^N} \nabla_x \mathbf{U} : d\mu_D^{M,\varepsilon}, \quad \omega(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

In view of the dispersive estimates (5.11), the conclusion of Theorem 3.2 follows as soon as we show that the expression on the right-hand side of (5.12) vanishes for  $\varepsilon \rightarrow 0$ . Similarly to the previous section, we use Gronwall type arguments proceeding in several steps.

**5.3.1. Step 1—convective term I.** Similarly to section 4, one may use the compatibility condition (2.5) to control the error term

$$\int_0^\tau \int_{R^N} \nabla_x \mathbf{U} : d\mu_D^{M,\varepsilon} \leq \|\nabla_x \mathbf{U}\|_{L^\infty} \int_0^\tau \xi(t) \mathcal{D}^\varepsilon(t) dt.$$

Next, exactly as in the well-prepared case, we write

$$\begin{aligned} & \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{U}(t, x) - \mathbf{m}) \otimes \frac{\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} \\ &= \left\langle Y_{t,x}^\varepsilon; (\varrho \mathbf{U}(t, x) - \mathbf{m}) \otimes \left( \frac{\mathbf{m}}{\varrho} - \mathbf{U} \right) \right\rangle : \nabla_x \mathbf{U} + \langle Y_{t,x}^\varepsilon; \varrho \mathbf{U}(t, x) - \mathbf{m} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \end{aligned}$$

to deduce that (5.12) reduces to

$$\begin{aligned} (5.13) \quad & \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) (\tau) + \mathcal{D}^\varepsilon(\tau) \leq \omega(\varepsilon) \\ &+ \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U})] dx dt \\ &- \frac{1}{\varepsilon^2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \Delta \Phi_\varepsilon dx dt \\ &+ \frac{1}{\varepsilon} \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; r - \varrho \rangle P''(r) \partial_t s_\varepsilon - \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot P''(r) \nabla_x s_\varepsilon] dx dt \\ &+ c \int_0^\tau (1 + \xi(t)) \left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) + \mathcal{D}^\varepsilon \right] dt. \end{aligned}$$

**5.3.2. Step 2—convective term II.** Next, we rewrite

$$\begin{aligned} & \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U})] dx dt \\ &= \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v})] dx dt \\ &+ \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \nabla_x \Phi_\varepsilon)] dx dt \\ &+ \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot \nabla_x \Phi_\varepsilon \cdot \nabla_x \mathbf{v}] dx dt \\ &+ \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \otimes \mathbf{v}] : \nabla_x^2 \Phi_\varepsilon dx dt \\ &+ \frac{1}{2} \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot \nabla_x |\nabla_x \Phi_\varepsilon|^2] dx dt. \end{aligned}$$

First observe that the last three integrals can be controlled in terms of

$$\int_0^\tau \|\nabla_x \Phi_\varepsilon\|_{W^{1,p}(R^3; R^3)} dt \quad \text{for some } p > 2 \text{ sufficiently large,}$$

and, consequently, in accordance with the dispersive estimates (5.11) vanish in the asymptotic limit  $\varepsilon \rightarrow 0$ . Indeed the desired estimates on  $\langle Y; \varrho \rangle$ ,  $\langle Y; \mathbf{m} \rangle$  follow from (5.6), while  $\mathbf{v}$  is bounded being a smooth solution of the incompressible Euler system.

Next, we have

$$\begin{aligned} & \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v})] \, dx \, dt \\ &= \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot \nabla_x \Pi \, dx \, dt - \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \varrho \rangle \mathbf{U} \cdot \nabla_x \Pi \, dx \, dt, \end{aligned}$$

where the former term on the right-hand side may be handled exactly as in (4.6). As for the latter, we get

$$\begin{aligned} & \left| \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \varrho \rangle \mathbf{U} \cdot \nabla_x \Pi \, dx \, dt \right| \\ & \leq \varepsilon \left| \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \mathbf{U} \cdot \nabla_x \Pi \, dx \, dt \right| + \bar{\varrho} \left| \int_0^\tau \int_{R^N} \nabla_x \Phi_\varepsilon \cdot \nabla_x \Pi \, dx \, dt \right|, \end{aligned}$$

where the first term is small because of (5.6), while the second one vanishes for  $\varepsilon \rightarrow 0$  because of dispersive estimates. Indeed the pressure  $\Pi$  may be computed by means of (1.6) as

$$\Pi = -\Delta_x^{-1} \operatorname{div}_x \operatorname{div}_x(\mathbf{v} \otimes \mathbf{v});$$

whence it is uniformly bounded in  $W^{1,q}(R^N)$  for any  $1 < q < \infty$ .

In accordance with (5.7),

$$\begin{aligned} & \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot (\partial_t \nabla_x \Phi_\varepsilon)] \, dx \, dt = - \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt \\ & + \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon, \varrho \rangle \mathbf{v} \cdot \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon, \varrho \rangle \cdot \partial_t |\nabla_x \Phi_\varepsilon|^2 \, dx \, dt, \end{aligned}$$

where, as  $\mathbf{v}$  is solenoidal,

$$\begin{aligned} & \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon, \varrho \rangle \mathbf{v} \cdot \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt \\ &= \varepsilon \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon, \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \mathbf{v} \cdot \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt \\ &= -\frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon, \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \mathbf{v} \cdot \nabla_x s_\varepsilon \, dx \, dt. \end{aligned}$$

The rightmost expression tends to zero because of the dispersive estimates for  $s_\varepsilon$ .

Similarly,

$$\begin{aligned} & \frac{1}{2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon, \varrho \rangle \cdot \partial_t |\nabla_x \Phi_\varepsilon|^2 \, dx \, dt \\ &= \varepsilon \frac{1}{2} \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon, \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \cdot \partial_t |\nabla_x \Phi_\varepsilon|^2 \, dx \, dt + \frac{\bar{\varrho}}{2} \left[ \int_{R^N} |\nabla_x \Phi_\varepsilon|^2 \, dx \right]_{t=0}^{t=\tau} \\ &= -\frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon, \frac{\varrho - \bar{\varrho}}{\varepsilon} \right\rangle \cdot \nabla_x \Phi_\varepsilon \cdot \nabla_x s_\varepsilon \, dx \, dt + \frac{\bar{\varrho}}{2} \left[ \int_{\Omega} |\nabla_x \Phi_\varepsilon|^2 \, dx \right]_{t=0}^{t=\tau}, \end{aligned}$$

where the first term on the right-hand side vanishes for  $\varepsilon \rightarrow 0$  because of the dispersive estimates.

Consequently, the inequality (5.13) can be recast in the form

$$\begin{aligned}
& \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) (\tau) + \mathcal{D}^\varepsilon(\tau) \leq \omega(\varepsilon) \\
& - \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt + \frac{\bar{\varrho}}{2} \left[ \int_{R^N} |\nabla_x \Phi_\varepsilon|^2 \, dx \right]_{t=0}^{t=\tau} \\
(5.14) \quad & - \frac{1}{\varepsilon^2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \Delta \Phi_\varepsilon \, dx \, dt \\
& + \frac{1}{\varepsilon} \int_0^\tau \int_{R^N} [\langle Y_{t,x}^\varepsilon; r - \varrho \rangle P''(r) \partial_t s_\varepsilon - \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot P''(r) \nabla_x s_\varepsilon] \, dx \, dt \\
& + c \int_0^\tau (1 + \xi(t)) \left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) + \mathcal{D}^\varepsilon \right] \, dt,
\end{aligned}$$

where  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

### 5.3.3. Step 3—pressure estimates I.

We have

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle P''(r) \nabla_x s_\varepsilon \, dx \, dt \\
& = - \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \frac{P''(\bar{\varrho} + \varepsilon s_\varepsilon) - P''(\bar{\varrho})}{\varepsilon} \nabla_x s_\varepsilon \, dx \, dt \\
& \quad - \frac{1}{\varepsilon} \frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot \nabla_x s_\varepsilon \, dx \, dt \\
& = - \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \frac{P''(\bar{\varrho} + \varepsilon s_\varepsilon) - P''(\bar{\varrho})}{\varepsilon} \nabla_x s_\varepsilon \, dx \, dt \\
& \quad + \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; \mathbf{m} \rangle \cdot \partial_t \nabla_x \Phi_\varepsilon \, dx \, dt,
\end{aligned}$$

where the first integral on the right-hand side vanished for  $\varepsilon \rightarrow 0$  by virtue of the dispersive estimates.

Next,

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; r - \varrho \rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& = \frac{1}{2} \int_0^\tau \int_{R^N} P''(r) \partial_t |s_\varepsilon|^2 \, dx \, dt + \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& = \left[ \frac{1}{2} \frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_{R^N} |s_\varepsilon|^2 \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& \quad + \frac{\varepsilon}{2} \int_0^\tau \int_{R^N} \frac{P''(r) - P''(\bar{\varrho})}{\varepsilon} \partial_t |s_\varepsilon|^2 \, dx \, dt \\
& = \left[ \frac{1}{2} \frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_{R^N} |s_\varepsilon|^2 \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& \quad - \bar{\varrho} \int_0^\tau \int_{R^N} \frac{P''(r) - P''(\bar{\varrho})}{\varepsilon} s_\varepsilon \Delta \Phi_\varepsilon \, dx \, dt.
\end{aligned}$$

Similarly to the above, the last integral is small in view of the dispersive estimates.

Summing up the previous observations with (5.14) and using the acoustic energy balance (5.9), we deduce from (5.14) that

$$\begin{aligned}
& (5.15) \\
& \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) (\tau) + \mathcal{D}^\varepsilon (\tau) \leq \omega(\varepsilon) \\
& - \frac{1}{\varepsilon^2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; p(\varrho) - p(\bar{\varrho}) \rangle \Delta \Phi_\varepsilon \, dx \, dt + \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& + c \int_0^\tau (1 + \xi(t)) \left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) + \mathcal{D}^\varepsilon \right] \, dt.
\end{aligned}$$

### 5.3.4. Step 4—pressure estimates II.

We have

$$\begin{aligned}
& \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle P''(r) \partial_t s_\varepsilon \, dx \, dt \\
& = \frac{p'(\bar{\varrho})}{\bar{\varrho}} \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle \partial_t s_\varepsilon \, dx \, dt \\
& + \varepsilon \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle \frac{P''(r) - P''(\bar{\varrho})}{\varepsilon} \partial_t s_\varepsilon \, dx \, dt \\
& = p'(\bar{\varrho}) \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\varrho - \bar{\varrho}}{\varepsilon^2} \right\rangle \Delta \Phi_\varepsilon \, dx \, dt \\
& - \bar{\varrho} \int_0^\tau \int_{R^N} \left\langle Y_{t,x}^\varepsilon; \frac{\bar{\varrho} - \varrho}{\varepsilon} \right\rangle \frac{P''(r) - P''(\bar{\varrho})}{\varepsilon} \Delta \Phi_\varepsilon \, dx \, dt,
\end{aligned}$$

where the last integral vanishes for  $\varepsilon \rightarrow 0$ . Thus we obtain that

$$\begin{aligned}
& \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) (\tau) + \mathcal{D}^\varepsilon (\tau) \leq \omega(\varepsilon) \\
& - \frac{1}{\varepsilon^2} \int_0^\tau \int_{R^N} \langle Y_{t,x}^\varepsilon; [p(\varrho) - p'(\bar{\varrho})(\varrho - \bar{\varrho}) - p(\bar{\varrho})] \rangle \Delta_x \Phi_\varepsilon \, dx \, dt \\
& + c \int_0^\tau (1 + \xi(t)) \left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) + \mathcal{D}^\varepsilon \right] \, dt.
\end{aligned}$$

Consequently, using again the dispersive estimates (5.11), together with the energy bounds, we obtain the desired conclusion

$$\begin{aligned}
& (5.16) \quad \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) (\tau) + \mathcal{D}^\varepsilon (\tau) \leq \omega(\varepsilon) \\
& + c \int_0^\tau (1 + \xi(t)) \left[ \mathcal{E}_\varepsilon \left( \varrho, \mathbf{m} \mid \bar{\varrho} + \varepsilon s_\varepsilon, \mathbf{v} + \nabla_x \Phi_\varepsilon \right) + \mathcal{D}^\varepsilon \right] \, dt.
\end{aligned}$$

where  $\xi \in L^1(0, T)$ , and  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Thus a direct application of Gronwall's lemma completes the proof of Theorem 3.2.

## REFERENCES

- [1] J. J. ALIBERT AND G. BOUCHITTÉ, *Non-uniform integrability and generalized Young measures*, J. Convex Anal., 4 (1997), pp. 129–147.
- [2] K. ASANO, *On the incompressible limit of the compressible Euler equation*, Jpn. J. Appl. Math., 4 (1987), pp. 455–488.
- [3] W. BARSUKOW, P. EDELMANN, CH. KLINGENBERG, F. MICZEK, AND F. K. ROEPKE, *A numerical scheme for the compressible low-Mach number regime of ideal fluid dynamics*, J. Sci. Comput., 2017, to appear.
- [4] E. CHIODAROLI, *A counterexample to well-posedness of entropy solutions to the compressible Euler system*, J. Hyperbolic Differ. Equ., 11 (2014), pp. 493–519.

- [5] E. CHIODAROLI, C. DE LELLIS, AND O. KREML, *Global ill-posedness of the isentropic system of gas dynamics*, Comm. Pure Appl. Math., 68 (2015), pp. 1157–1190.
- [6] E. CHIODAROLI, E. FEIREISL, O. KREML, AND E. WIEDEMANN, *A-free rigidity and applications to the compressible Euler system*, Ann. Mat. Pura Appl. (4), 196 (2017), pp. 1557–1572.
- [7] E. CHIODAROLI AND O. KREML, *On the energy dissipation rate of solutions to the compressible isentropic Euler system*, Arch. Ration. Mech. Anal., 214 (2014), pp. 1019–1049.
- [8] C. DE LELLIS AND L. SZÉKELYHIDI, JR., *On admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal., 195 (2010), pp. 225–260.
- [9] D. B. EBIN, *The motion of slightly compressible fluids viewed as a motion with strong constraining force*, Ann. Math., 105 (1977), pp. 141–200.
- [10] E. FEIREISL, P. GWIAZDA, A. ŚWIERCZEWSKA-GWIAZDA, AND E. WIEDEMANN, *Dissipative measure-valued solutions to the compressible Navier-Stokes system*, Calc. Var. Partial Differential Equations, 55 (2016), 141, 20 pp.
- [11] P. GWIAZDA, A. ŚWIERCZEWSKA-GWIAZDA, AND E. WIEDEMANN, *Weak-strong uniqueness for measure-valued solutions of some compressible fluid models*, Nonlinearity, 28 (2015), pp. 3873–3890.
- [12] T. KATO, *Nonstationary flows of viscous and ideal fluids in  $r^3$* , J. Funct. Anal., 9 (1972), pp. 296–305.
- [13] T. KATO AND C.Y. LAI, *Nonlinear evolution equations and the Euler flow*, J. Funct. Anal., 56 (1984), pp. 15–28.
- [14] S. KLAINERMAN AND A. MAJDA, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math., 34 (1981), pp. 481–524.
- [15] P.-L. LIONS, *Mathematical topics in fluid dynamics, Compressible models*, Vol.2, Oxford Science Publication, Oxford, 1998.
- [16] J. MÁLEK, J. NEČAS, M. ROKYTA, AND M. RŮŽIČKA, *Weak and Measure-Valued Solutions to Evolutionary PDE's*, Chapman and Hall, London, 1996.
- [17] G. MÉTIVIER AND S. SCHOCHET, *The incompressible limit of the non-isentropic Euler equations*, Arch. Rational Mech. Anal., 158 (2001), pp. 61–90.
- [18] J. NEČAS AND M. ŠILHÁVÝ, *Viscous multipolar fluids*, Quart. Appl. Math., 49 (1991), pp. 247–266.
- [19] J. NEUSTUPA, *Measure-valued solutions of the Euler and Navier-Stokes equations for compressible barotropic fluids*, Math. Nachr., 163 (1993), pp. 217–227.
- [20] S. SCHOCHET, *The mathematical theory of low Mach number flows*, Model Numer. Anal., 39 (2005), pp. 441–458.
- [21] R. S. STRICHARTZ, *A priori estimates for the wave equation and some applications*, J. Functional Analysis, 5 (1970), pp. 218–235.
- [22] S. UKAI, *The incompressible limit and the initial layer of the compressible Euler equation*, J. Math. Kyoto Univ., 26 (1986), pp. 323–331.