Splitting methods for rotations: application to Vlasov equations

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Zoom seminar Wolfsburg,
november 12th 2020.

Motivations:

- rotation motions can be found in
  - many physical models involving magnetic field (Schrödinger, Vlasov, spin-Vlasov, ...)
  - imaging community
  - fluid models involving Coriolis force
  - ...
- efficient numerical methods are important to improve physical codes (in terms of CPU time and accuracy)

Plan

- splittings for $2D$ rotations
- application to the $1d$-$2v$ Vlasov-Maxwell equations
- conclusion

Splittings for $2D$ rotations

Splitting methods

Main goal: efficient numerical methods for

$$\partial_t u = Jx \cdot \nabla_x u, \quad x \in \mathbb{R}^2, \;\; u(t=0,x) = u^{in}(x),$$

where J is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, the exact solution is known, but when u^{in} is only known on a grid, we need a numerical method !

First natural idea: $2D$ Semi-Lagrangian method

- solve the ODE system on $[t_n, t_{n+1}]$ backward in time

$$\dot{x}(t) = -Jx(t), \quad x(t_{n+1}) = x_g$$

- the solution is constant along the characteristics:

$$u^{n+1}(x_g) = u^n(x(t_n)) = u^n(e^{\Delta t\, J} x_g)$$

Second natural idea: splitting method

Lie splitting

$$u^{n+1}(x) = u^n(e^{A_2} e^{A_1} x)$$

where

$$e^{A_1} = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1 \end{pmatrix}, \quad e^{A_2} = \begin{pmatrix} 1 & 0 \\ \Delta t & 1 \end{pmatrix},$$

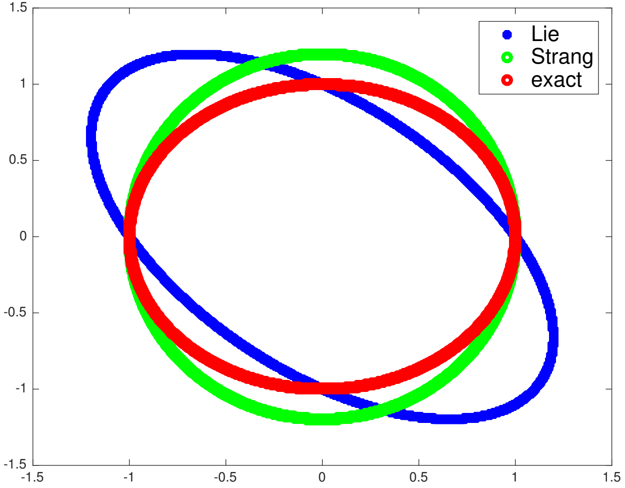
- solve $\partial_t u = x_1 \partial_{x_2} u, \quad u(0,x) = u^n(x)$ to get $u^\star(x) = u^n(x_1, x_2 + \Delta t x_1) = u^n(e^{A_2} x)$
- solve $\partial_t u = -x_2 \partial_{x_1} u, \quad u(0,x) = u^\star(x)$ to get $u^{n+1}(x) = u^\star(x_1 - \Delta t x_2, x_2) = u^\star(e^{A_1} x) = u^n(e^{A_2} e^{A_1} x)$

Strang splitting

$$u^{n+1}(x) = u^n(e^{A_1} e^{A_2} e^{A_1} x)$$

where

$$e^{A_1} = \begin{pmatrix} 1 & -\Delta t/2 \\ 0 & 1 \end{pmatrix}, \quad e^{A_2} = \begin{pmatrix} 1 & 0 \\ \Delta t & 1 \end{pmatrix},$$


Lie
Strang
exact
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

For Lie, the trajectories are ellipses

$$x_1^2 + \Delta t x_1 x_2 + x_2^2 = \mathrm{cste}.$$

For Strang, the trajectories are ellipses

$$x_1^2 + (1 - (\Delta t/2)^2) x_2^2 = \mathrm{cste}.$$

Moreover, for the two methods, the angular velocity is given by

$$\omega_{Strang}(\Delta t) = \omega_{Lie}(\Delta t) = \frac{1}{\Delta t} \arcsin(\Delta t \sqrt{1 - \Delta t^2/4}) < 1 = \omega_{ex}.$$

Two kinds of error

- trajectory
- angular velocity

Can we improve one of the two errors ? the two errors ?

From the decomposition

$$u^{n+1}(x) = u^n(e^{A_1} e^{A_2} e^{A_1} x)$$

to be a directional splitting, we impose

$$e^{A_1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad e^{A_2} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

Find $a, b \in \mathbb{R}^2$ such that the two errors are improved ?

Considering $a = -\tan \frac{\Delta t}{2}$ and $b = \sin \Delta t$, we have

$$e^{A_1} e^{A_2} e^{A_1} := \begin{pmatrix} 1 & -\tan\frac{\Delta t}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\Delta t & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan\frac{\Delta t}{2} \\ 0 & 1 \end{pmatrix} = e^{\Delta t J}$$

$\Longrightarrow$ $2D$ rotation can be exactly decomposed into three shears[1]

[1] References in the image processing community: Paeth-Tanaka 86', Andres 96'. See also *Bader-Blanes, 2011*.

Full discretization

To numerically solve the PDE

$$\partial_t u = Jx \cdot \nabla_x u, \quad x \in [-R/2, R/2]^2,$$

we will use pseudo-spectral method to solve the following shears ($\alpha \in \mathbb{R}$):

$$\partial_t u = \alpha x_2 \partial_{x_1} u, \qquad \partial_t u = \alpha x_1 \partial_{x_2} u.$$

Let us consider the grid $\mathbb{G} = h \left[\!\left[-\left\lfloor \frac{N-1}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor \right]\!\right], \quad h = R/N$ and the DFT (in the first direction)

$$\mathcal{F}_1 : u \quad \mapsto \quad \mathcal{F}_1(u)_{\xi_1, g_2} := h \sum_{g_1 \in \mathbb{G}} \boldsymbol{u}_{g_1, g_2} \, e^{-i g_1 \xi_1},$$

Then, the shear operator for $\partial_t u = \alpha x_2 \partial_{x_1} u$ is

$$\mathcal{S}_1^\alpha : \begin{cases} \mathbb{C}^{\mathbb{G}^2} & \to & \mathbb{C}^{\mathbb{G}^2} \\ u & \mapsto & \mathcal{F}_1^{-1}\left[e^{i\alpha \xi_1 g_2} \mathcal{F}_1 u \right] \end{cases} \tag{1}$$

Then, the splitting can be written as (denoting $u^0 := u^{in}|_{\mathbb{G}^2}$)

$$\begin{aligned} u^n &= (\mathcal{L}_{\Delta t})^n\ u^0 := (\mathcal{S}_2^{\Delta t}\mathcal{S}_1^{-\Delta t})^n\ u^0, && \text{(Lie)} \\ u^n &= (\mathcal{T}_{\Delta t})^n\ u^0 := (\mathcal{S}_1^{-\Delta t/2}\mathcal{S}_2^{\Delta t}\mathcal{S}_1^{-\Delta t/2})^n\ u^0, && \text{(Strang)} \\ u^n &= (\mathcal{M}_{\Delta t})^n\ u^0 := (\mathcal{S}_1^{-\tan(\Delta t/2)}\mathcal{S}_2^{\sin(\Delta t)}\mathcal{S}_1^{-\tan(\Delta t/2)})^n\ u^0. && \text{(New)} \end{aligned}$$

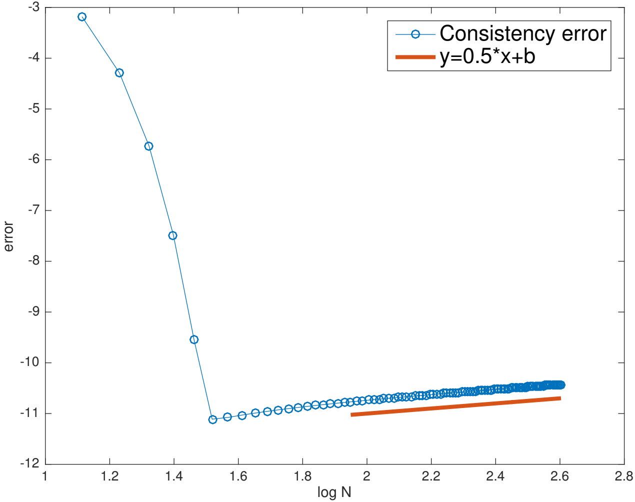
Theorem

For all $s > 0$, there exists $C > 0$ such that for all $R > 0$, $u \in \mathscr{S}(\mathbb{R}^2)$, $n \in \mathbb{N}$ and $\Delta t \in]-\pi, \pi[$, we have

$$\|(\mathcal{M}_{\Delta t})^n u^{in}_{|\mathbb{G}^2} - \left(u^{in}(e^{t_n J}x)\right)_{|\mathbb{G}^2}\|_{L^2(\mathbb{G}^2)} \leq C\ n\Delta t \frac{R^{-s} + h^s}{\sqrt{h}}\|u\|.$$

Numerical results

Illustration of the error $\mathcal{S}_1^\alpha u_{|\mathbb{G}^2} - u(x_1 - \alpha x_2, x_2)_{|\mathbb{G}^2}$.
$R = 15$, $\alpha = 10^{-2}$



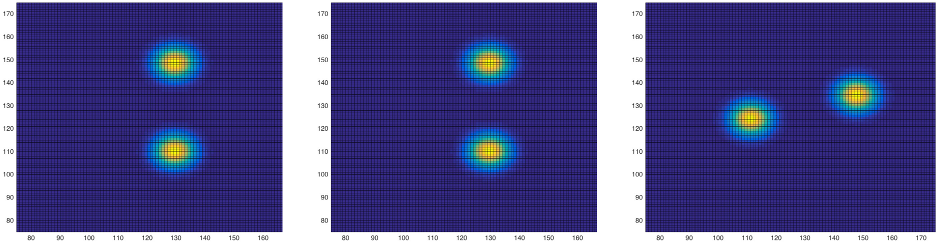


Figure: Solution $u(T = 10^5, x)$, $\Delta t \approx 0.139$, $x \in [-2, 2]^2$, $N = 243^2$.
Left: Exact solution.
Middle: Numerical solution obtained by the new splitting.
Right: Numerical solution obtained by the Strang splitting.

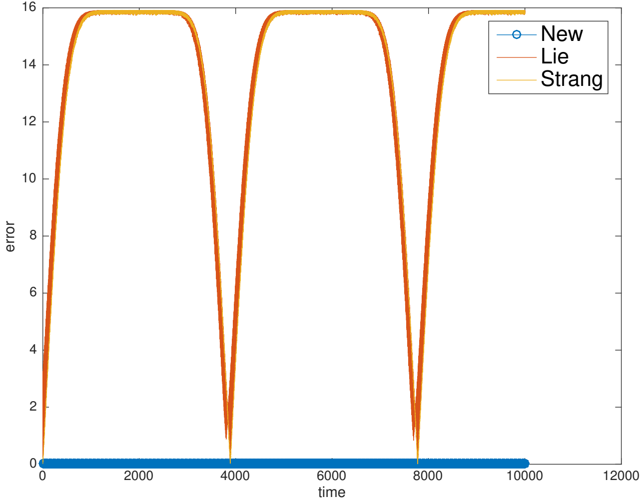


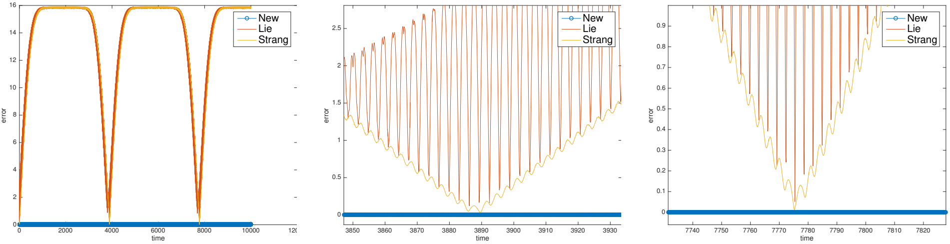
Figure: Time history of the relative L^2 errors between the exact solution and the numerical solution obtained by the different splittings.

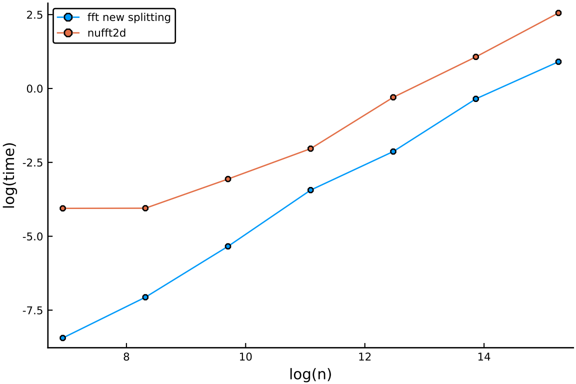
One can compute the "recurrence" time $\overline{T}$ from

$$(\omega - \omega_{Lie})\overline{T} = k\pi, \quad k \in \mathbb{Z},$$

where $\omega = 1$ and $\omega_{Lie} = \mu_{\Delta t, \Delta t} = \dfrac{\arcsin(\Delta t\sqrt{1-(\Delta t)^2/4})}{\Delta t\sqrt{1-(\Delta t)^2/4}}$.

With $\Delta t \approx 0.139$, we have $\overline{T} \approx 3888$.





10 times faster !

Extension to multi-dimensional transport equation of the form

$$\partial_t u = M\mathbf{x} \cdot \nabla u, \quad x \in \mathbb{R}^n, \quad M_{i,i} = 0. \tag{2}$$

We have the following decomposition [2, 3]

$$e^{\Delta t M\mathbf{x}\cdot\nabla} = e^{\Delta t (y^{(\ell)}\cdot\mathbf{x})\partial_{x_i}} \left(\prod_{k=1 (k\neq i)}^{n} e^{\Delta t (y^{(k)}\cdot\mathbf{x})\partial_{x_k}} \right) e^{\Delta t (y^{(r)}\cdot\mathbf{x})\partial_{x_i}}$$

with $y^{(\ell)}, y^{(k)}, y^{(r)} \in \mathbb{R}^n$ such that $y_i^{(\ell)} = y_i^{(r)} = 0$ and $y_k^{(k)} = 0$ [4]

$\Longrightarrow$ Equation (2) is split *exactly* into $(n+1)$ shears
(a Strang splitting needs $(2n-1)$ shears).

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[2] J. Bernier, Exact splitting methods for semigroups generated by inhomogeneous quadratic differential operators.

[3] J. Bernier, N. Crouseilles, Y. Li, Exact splitting methods for kinetic and Schrodinger equations, accepted in JSC

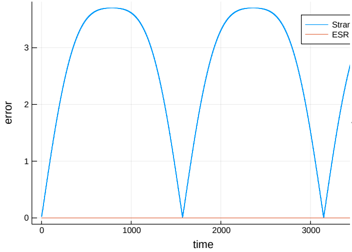
[4] The vectors $y^{(\ell)}, y^{(r)}, y^{(k)}$ are computed numerically for a given Δt.

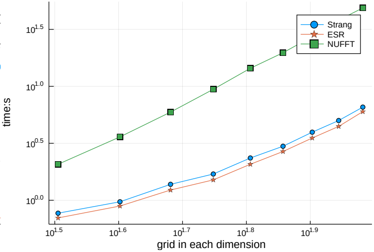
Example with $n = 3$

$$\text{Let consider } M = \begin{pmatrix} 0 & -0.36 & -0.679 \\ 0.36 & 0 & -0.758 \\ 0.679 & 0.758 & 0 \end{pmatrix}.$$

Then, we have: $e^{\Delta t M\mathbf{x}\cdot\nabla} = e^{\Delta t (y^{(\ell)}\cdot\mathbf{x})\partial_{x_3}} e^{\Delta t (y^{(2)}\cdot\mathbf{x})\partial_{x_1}} e^{\Delta t (y^{(3)}\cdot\mathbf{x})\partial_{x_2}} e^{\Delta t (y^{(r)}\cdot\mathbf{x})\partial_{x_3}}$,

with $y^{(\ell)} \simeq \begin{pmatrix} 0.345... \\ 0.379... \\ 0 \end{pmatrix}$, $y^{(2)} \simeq \begin{pmatrix} 0 \\ \text{-}0.036... \\ \text{-}0.664... \end{pmatrix}$, $y^{(3)} \simeq \begin{pmatrix} 0.036... \\ 0 \\ \text{-}0.742... \end{pmatrix}$, $y^{(r)} \simeq \begin{pmatrix} 0.339... \\ 0.384... \\ 0 \end{pmatrix}$ $(\Delta t = 0.3)$.





Extension to quadratic PDEs

We consider PDEs of the form

$$\begin{cases} \partial_t u(t,\mathbf{x}) &= -p^w u(t,\mathbf{x}), \quad t \geq 0,\ \mathbf{x} \in \mathbb{R}^n \\ u(0,\mathbf{x}) &= u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \end{cases}$$

Correspondance between the operator p^w and the polynomial p

$$p^w = {}^{\mathrm{t}}\begin{pmatrix} \mathbf{x} \\ -i\nabla \end{pmatrix} Q \begin{pmatrix} \mathbf{x} \\ -i\nabla \end{pmatrix} + {}^{\mathrm{t}}Y \begin{pmatrix} \mathbf{x} \\ -i\nabla \end{pmatrix} + c \longleftrightarrow p(\mathbf{x},\boldsymbol{\xi}) = {}^{\mathrm{t}}\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{pmatrix} Q \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{pmatrix} + {}^{\mathrm{t}}Y \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{pmatrix} + c$$

where $\mathbf{x},\boldsymbol{\xi} \in \mathbb{R}^n$, $Q \in \mathcal{S}_{2n}(\mathbb{C})$, $Y \in \mathbb{C}^{2n}$ and $c \in \mathbb{C}$.

Example: Schrödinger, Fokker-Planck, Vlasov, transport, ...

$$i\frac{\partial \psi(\mathbf{x},t)}{\partial t} = -\frac{1}{2}\Delta\psi(\mathbf{x},t) - i(B\mathbf{x})\cdot\nabla\psi(\mathbf{x},t) + |\mathbf{x}|^2\psi(\mathbf{x},t),$$

We have $p(\mathbf{x},\boldsymbol{\xi}) = i\frac{|\boldsymbol{\xi}|^2}{2} + iB\mathbf{x}\cdot\boldsymbol{\xi} + i|\mathbf{x}|^2$, i.e. $Q = \frac{i}{4}\begin{pmatrix} 4\boldsymbol{I}_n & {}^{\mathrm{t}}B \\ B & 4\boldsymbol{I}_n \end{pmatrix}$, $Y = 0$, $c = 0$.

Exact splittings

Quadratic PDEs can be split *exactly* into simple operators

$$e^{\alpha \partial_{x_j}}, \; e^{i\alpha x_j}, \; e^{ia(\nabla)}, \; e^{ia(\mathbf{x})}, \; e^{\alpha x_k \partial_{x_j}}, \; e^{-b(\mathbf{x})}, \; e^{b(\nabla)}, \; e^{\gamma} \qquad (3)$$

with $\alpha \in \mathbb{R}, \gamma \in \mathbb{C}, a, b : \mathbb{R}^n \to \mathbb{R}$ are some real quadratic forms, b is nonnegative and $j, k \in [\![1, n]\!]$ and $k \neq j$.

Remark: "*simple*" means it can be solved easily using pseudo-spectral methods for instance.
More details in

- mathematical framework: J. Bernier, *Exact splitting methods for semigroups generated by inhomogeneous quadratic differential operators*.
- Numerical examples: J. Bernier, N. Crouseilles, Y. Li, *Exact splitting methods for kinetic and Schrödinger equations, accepted in JSC*.

Application to the 1d-2v Vlasov-Maxwell equations

$1d$-$2v$ Vlasov-Maxwell equations

Let consider $f(t, x_1, v_1, v_2)$, $B(t, x_1)$ and $E(t, x_1) = (E_1, E_2)(t, x_1)$ with $(x_1, v_1, v_2) \in L \times \mathbb{R}^2$, solution of

$$\begin{aligned}
&\partial_t f + v_1 \partial_{x_1} f + E \cdot \nabla_v f - BJv \cdot \nabla_v f = 0, \\
&\partial_t B = -\partial_{x_1} E_2, \\
&\partial_t E_2 = -\partial_{x_1} B - \int_{\mathbb{R}^2} v_2 f(t, x_1, v) dv + \overline{\mathcal{J}}_2(t), \\
&\partial_t E_1 = -\int_{\mathbb{R}^2} v_1 f(t, x_1, v) dv + \overline{\mathcal{J}}_1(t), \\
&\partial_{x_1} E_1 = \int_{\mathbb{R}^2} f(t, x_1, v) dv - 1, \quad \text{[Gauss relation]}
\end{aligned} \tag{4}$$

where $v = (v_1, v_2)$, $\overline{\mathcal{J}}_i(t) = 1/|L| \int_L \int_{\mathbb{R}^2} v_i f(t, x_1, v) dx_1 dv, i = 1, 2$ and J denotes

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $\vec{B} = (0, 0, B)$, the Lorentz force reduces to $BJv \cdot \nabla_v f$.

Splitting for VM[5]

The following decomposition will be used

$$\partial_t \begin{pmatrix} f \\ E_1 \\ E_2 \\ B \end{pmatrix} = -\begin{pmatrix} v_1 \partial_{x_1} f \\ \int_{\mathbb{R}^2} v_1 f \, \mathrm{d}v - \overline{\mathcal{J}}_1 \\ \int_{\mathbb{R}^2} v_2 f \, \mathrm{d}v - \overline{\mathcal{J}}_2 \\ 0 \end{pmatrix} - \begin{pmatrix} E \cdot \nabla_v f \\ 0 \\ 0 \\ \partial_{x_1} E_2 \end{pmatrix} + \begin{pmatrix} BJv \cdot \nabla_v f \\ 0 \\ -\partial_{x_1} B \\ 0 \end{pmatrix}.$$

Denoting $\mathcal{Z} = (f, E_1, E_2, B)$, we rewrite the VM system as

$$\partial_t \mathcal{Z} + \mathcal{H}_f(\mathcal{Z}) + \mathcal{H}_E(\mathcal{Z}) + \mathcal{H}_B(\mathcal{Z}) = 0,$$

which suggests a first order splitting method

$$\chi_{\Delta t} = \varphi_{\Delta t}^{[\mathcal{H}_E]} \circ \varphi_{\Delta t}^{[\mathcal{H}_f]} \circ \varphi_{\Delta t}^{[\mathcal{H}_B]}$$

where $\varphi_{\Delta t}^{[\mathcal{H}_{f,E,B}]}$ denotes the *exact* solution of each subpart.

[5] C., Einkemmer, Faou, JCP 2015.
See also Li et al, JCP 2019 and Krauss et al, JPP 2017.

Each step can be solved exactly in time.

In particular, for $\varphi_{\Delta t}^{[\mathcal{H}_B]}$, we have

$$\partial_t \begin{pmatrix} f \\ E_1 \\ E_2 \\ B \end{pmatrix} = \begin{pmatrix} BJv \cdot \nabla_v f \\ 0 \\ -\partial_{x_1} B \\ 0 \end{pmatrix}$$

with the IC: $(f(0), E_1(0), E_2(0), B(0))$.

We can compute the solution exactly in time

- $B(\Delta t, x_1) = B(0, x_1)$ and $E_1(\Delta t, x_1) = E_1(0, x_1)$
- $E_2(\Delta t, x_1) = E_2(0, x_1) - \Delta t \partial_{x_1} B(0, x_1)$
- use the new splitting for *rotation* part since B is frozen

Remark: Strang splitting can be also used !

High order splittings for systems split into three parts

Instead of using composition of exact flows, we shall consider composition of

$$\chi_{\Delta t} := \varphi_{\Delta t}^{[\mathcal{H}_E]} \circ \varphi_{\Delta t}^{[\mathcal{H}_f]} \circ \varphi_{\Delta t}^{[\mathcal{H}_B]} \quad \text{and} \quad \chi_{\Delta t}^{\star} := \varphi_{\Delta t}^{[\mathcal{H}_B]} \circ \varphi_{\Delta t}^{[\mathcal{H}_f]} \circ \varphi_{\Delta t}^{[\mathcal{H}_E]}$$

More specifically, we construct integrators within the family

$$\begin{aligned} \psi_{\Delta t}^{[s]} &= \Pi_{i=1}^{s} \left(\chi_{\alpha_{2i-1}\Delta t} \circ \chi_{\alpha_{2i}\Delta t}^{\star} \right) \\ &= \chi_{\alpha_1 \Delta t} \circ \chi_{\alpha_2 \Delta t}^{\star} \circ \cdots \circ \chi_{\alpha_{2s-1}\Delta t} \circ \chi_{\alpha_{2s}\Delta t}^{\star}, \end{aligned}$$

with $\alpha_{2s+1-i} = \alpha_i, \;\; i = 1, \ldots, s$ to ensure time-symmetry.

Some remarks

- $\psi_{\Delta t}^{[s]}$ can be of order p even if it only involves first-order approximations to the flows $\varphi_{\Delta t}^{[\mathcal{H}_E]}$, $\varphi_{\Delta t}^{[\mathcal{H}_f]}$, and $\varphi_{\Delta t}^{[\mathcal{H}_B]}$
- one needs to construct its adjoint $\chi_{\Delta t}^{\star}$ (easy when flows are exact in time)
- methods involving the minimum number of maps (or *stages*) do not usually provide the best efficiency.

Considering additional stages $\Longrightarrow$ some free parameters
How to fix the free parameters ?
To determine the coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{2s}) \in \mathbb{R}^{2s}$), we decide to minimize the following objective functions

$$\mathcal{E}_1(\boldsymbol{\alpha}) = \sum_{i=1}^{2s} |\alpha_i| \qquad \text{and} \qquad \mathcal{E}_2(\boldsymbol{\alpha}) = 2s \left| \sum_{i=1}^{2s} \alpha_i^5 \right|^{1/4}.$$

$\mathcal{E}_1$ has an influence on the CFL condition,
$\mathcal{E}_2$ is usually the dominant error term for a number of problems.

Some examples

The integrator with $s = 3$ reads

$$\psi_{\Delta t}^{[3]} = \chi_{\alpha_1 \Delta t} \circ \chi^{\star}_{\alpha_2 \Delta t} \circ \chi_{\alpha_3 \Delta t} \circ \chi^{\star}_{\alpha_3 \Delta t} \circ \chi_{\alpha_2 \Delta t} \circ \chi^{\star}_{\alpha_1 \Delta t}$$

and the unique (real) solution to the order conditions $w_1 = 1$, $w_3 = w_{12} = 0$ is given by

$$\alpha_1 = \alpha_2 = \frac{1}{2(2 - 2^{1/3})}, \qquad \alpha_3 = \frac{1}{2} - 2\alpha_1.$$

If $\chi_{\Delta t} = \varphi_{\Delta t}^{[\mathcal{H}_E]} \circ \varphi_{\Delta t}^{[\mathcal{H}_f]} \circ \varphi_{\Delta t}^{[\mathcal{H}_B]}$, then it involves 13 maps (the minimum number), and the values of the objective functions are

$$\mathcal{E}_1(\boldsymbol{\alpha}) = 4.40483, \qquad \mathcal{E}_2(\boldsymbol{\alpha}) = 4.55004.$$

This is the Yoshida method[6]

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[6] Yoshida 90'

Fourth order methods can be designed in this spirit by increasing the number of stages s

- $s = 4$ (17 maps), the composition is

$$\psi_{\Delta t}^{[4]} = \chi_{\alpha_1 \Delta t} \circ \chi^{\star}_{\alpha_2 \Delta t} \circ \chi_{\alpha_3 \Delta t} \circ \chi^{\star}_{\alpha_4 \Delta t} \circ \chi_{\alpha_4 \Delta t} \circ \chi^{\star}_{\alpha_3 \Delta t} \circ \chi_{\alpha_2 \Delta t} \circ \chi^{\star}_{\alpha_1 \Delta t},$$

$$\mathcal{E}_1(\boldsymbol{\alpha}) = 2.9084, \qquad \mathcal{E}_2(\boldsymbol{\alpha}) = 3.1527.$$

- $s = 5$ (21 maps), the composition is

$$\psi_{\Delta t}^{[5]} = \chi_{\alpha_1 \Delta t} \circ \chi^{\star}_{\alpha_2 \Delta t} \circ \chi_{\alpha_3 \Delta t} \circ \chi^{\star}_{\alpha_4 \Delta t} \circ \chi_{\alpha_5 \Delta t} \circ \chi^{\star}_{\alpha_5 \Delta t} \cdots \circ \chi_{\alpha_2 \Delta t} \circ \chi^{\star}_{\alpha_1 \Delta t}$$

$$\mathcal{E}_1(\boldsymbol{\alpha}) = 2.3159, \mathcal{E}_2(\boldsymbol{\alpha}) = 2.6111.$$

- $s = 6$ (25 maps), the composition is

$$\psi_{\Delta t}^{[6]} = \chi_{\alpha_1 \Delta t} \circ \chi^{\star}_{\alpha_2 \Delta t} \circ \cdots \circ \chi^{\star}_{\alpha_6 \Delta t} \circ \chi_{\alpha_6 \Delta t} \circ \ldots \chi_{\alpha_2 \Delta t} \circ \chi^{\star}_{\alpha_1 \Delta t}.$$

$$\mathcal{E}_1(\boldsymbol{\alpha}) = 2.0513, \qquad \mathcal{E}_2(\boldsymbol{\alpha}) = 2.4078.$$

Numerical results

To do so, we consider the following initial condition for VM

$$f(0,x_1,v_1,v_2)=\frac{1}{\pi v_{\mathrm{th}^2}\sqrt{T_r}}e^{-(v_1^2+v_2^2/T_r)/v_{\mathrm{th}}}(1+\alpha\cos(kx_1)),$$

and $B(0,x_1)=10+3\cos(kx_1),\quad E_2(0,x_1)=0.$

- $\alpha=10^{-4}$, $k=0.4$, $v_{th}=0.02$, $k=0.4$ and $T_r=12$.
- $N_x=32$ and $N_v=513^2$
- final time $T=2$
- different values of Δt between 10^{-3} to 0.4.

We look at the error

$$\mathrm{err}(\Delta t):=\max_{t\in[0,T]}\left|\frac{\mathcal{H}_{\Delta t}(t)-\mathcal{H}(0)}{\mathcal{H}(0)}\right|.$$

with

$$\mathcal{H}_{\Delta t}(t)\approx\int_0^L|E(t,x)|^2dx+\int_0^L|B(t,x)|^2dx+\int_{[0,L]\times\mathbb{R}^2}|v|^2f(t,x,v)dvdx$$

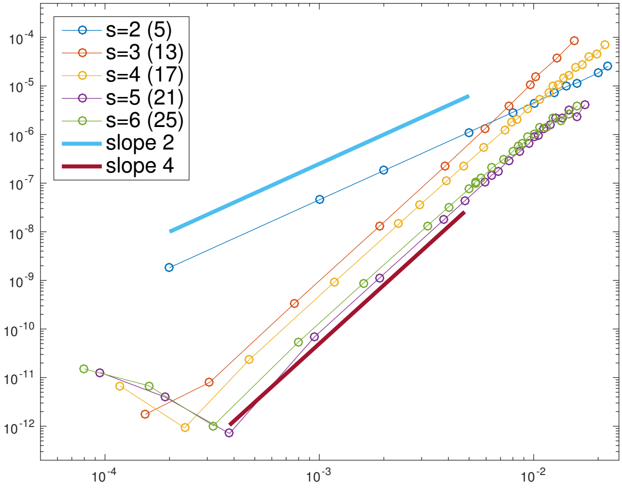

Figure: Efficiency diagrams for the different composition methods $\psi_{\Delta t}^{[s]}, s = 2, 3, 4, 5, 6$. The number of maps for each method is indicated into parenthesis.

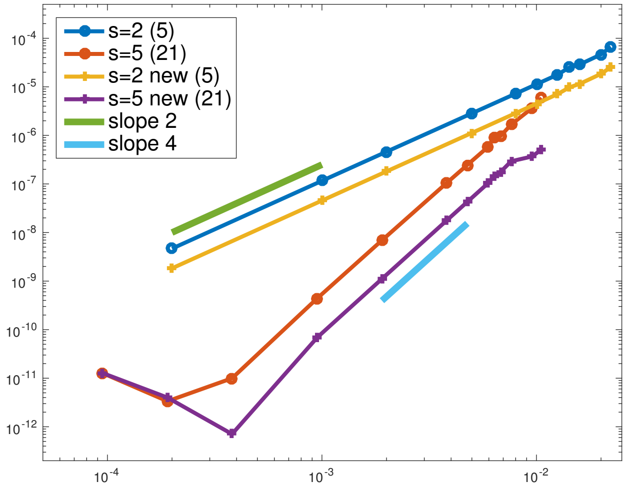

Figure: Efficiency diagrams for (i) $\tilde{\psi}^{[2]}_{\Delta t}$ and $\psi^{[2]}_{\Delta t}$; (ii) $\tilde{\psi}^{[5]}_{\Delta t}$ and $\psi^{[5]}_{\Delta t}$.

Conclusions

- exact splitting for 2D rotations
- application to Vlasov-Maxwell equations : construction of new high order splitting methods
- extension to nD transport equations

$$\partial_t f + Mx \cdot \nabla f = 0, \quad x \in \mathbb{R}^n, \ \ M_{i,i} = 0$$

In particular, 3D rotations can be decomposed into *four* 1D linear advections of the form

$$\partial_t f - (bv_x + av_z)\partial_{v_y} f = 0.$$

Perspectives

- spin-Vlasov models $f(t, \mathbf{x}, \mathbf{v}, \mathbf{s}), \mathbf{s} \in \mathbb{S}^2$ or $\mathbf{f}(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}^4$
- magnetic Schrödinger equation