Hybrid Multifluid Algorithms Based on the Path-Conservative Central-Upwind Scheme

Alina Chertock

joint work with

S. Chu and A. Kurganov

February 25, 2022

NC STATE UNIVERSITY

NSF

Multicomponent Flows

- Consider flow models describing the dynamics of fluids consisting of several immiscible and compressible fluids
- Assume
  - All fluid components can be described by a single velocity (u, v) and a single pressure p and the governing equations in 2-D are:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$
$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0$$
$$E_t + [u(E + p)]_x + [v(E + p)]_y = 0$$

    Here: ρ is the density of the fluid mixture and E is the total energy
  - The fluid components are separated by interfaces, and each fluid component is equipped with its own equation of state (EOS):

$$p = (\gamma - 1)\left[E - \frac{\rho}{2}(u^2 + v^2)\right] - \gamma p_\infty$$

    i.e., have different specific heat ratios γ and stiffness parameters p_∞

Multicomponent Flows

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho u \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = \mathbf{0}$$

$$\Updownarrow$$

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x + \boldsymbol{G}(\boldsymbol{U})_y = \mathbf{0}$$

EOS: $p = (\gamma - 1)\left[E - \frac{\rho}{2}(u^2 + v^2)\right] - \gamma p_\infty$

A multifluid problem with several components:

$$\gamma = \gamma_I, \quad p_\infty = p_{\infty,I}$$

$$\gamma = \gamma_{II}, \quad p_\infty = p_{\infty,II}$$

$$\gamma = \gamma_{III}, \quad p_\infty = p_{\infty,III}$$

$$\ldots$$

Multicomponent Flows

- Fluid components are usually identified by a variable ϕ that propagates with the fluid velocity:

$$\phi_t + u\phi_x + v\phi_y = 0$$

or

$$(\rho\phi)_t + (\rho u\phi)_x + (\rho v\phi)_y = 0$$

  - *Various models — various choices of* ϕ
    - a state variable, say, γ or any function of it[1]
    - the mass fraction of the fluid component in the fluid mixture[2]
    - a level-set function, whose zero level-set defines the interface between the fluid components[3]

[1] Roe, 1982; Karni, 1994.

[2] Abgrall, 1998; Abgrall, 1996; Saurel and Abgrall, 1999a; Larrouturou, 1991.

[3] Fedkiw, Aslam, Merriman, and Osher, 1999; Mulder, Osher, and Sethian, 1992.

What May Go Wrong? – 1-D Example

$$\begin{pmatrix} \rho \\ \rho u \\ E \\ \rho\phi \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \\ \rho u \phi \end{pmatrix}_x = \mathbf{0} \quad \Leftrightarrow \quad \begin{aligned} &\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \mathbf{0} \\ &\textbf{EOS:} \quad p = (\gamma - 1)\left[E - \frac{\rho}{2}u^2\right] - \gamma p_\infty \end{aligned}$$

Example: A multifluid problem with two components

$$\gamma = \gamma_I, \quad p_\infty = p_{\infty,I}$$
$$\gamma = \gamma_{II}, \quad p_\infty = p_{\infty,II}$$

Finite-Volume Framework – 1-D

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$$

- $\overline{\boldsymbol{U}}_j(t) \approx \dfrac{1}{\Delta x} \displaystyle\int_{C_j} \boldsymbol{U}(x,t)\,dx$: cell averages over $C_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$
- $\boldsymbol{U}^-_{j+\frac{1}{2}}(t)$ and $\boldsymbol{U}^+_{j+\frac{1}{2}}(t)$: reconstructed point values at $x_{j+\frac{1}{2}}$
- Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}\left(\boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\right) - \boldsymbol{H}_{j-\frac{1}{2}}\left(\boldsymbol{U}^-_{j-\frac{1}{2}}, \boldsymbol{U}^+_{j-\frac{1}{2}}\right)}{\Delta x}$$

$\boldsymbol{H}_{j\pm\frac{1}{2}} \approx \boldsymbol{F}(\boldsymbol{U}_{j\pm\frac{1}{2}}(t))$: numerical fluxes

Finite-Volume Framework – 1-D

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \left\{\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}\right\} \to \left\{\boldsymbol{H}_{j+\frac{1}{2}}\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

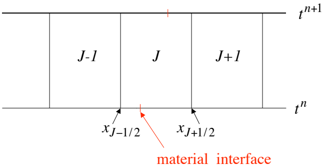
A piecewise-linear reconstruction (conservative, second-order accurate, non-oscillatory provided the slopes are computed by a nonlinear limiter):

$$\widetilde{\boldsymbol{U}}_j(x) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

$$(\boldsymbol{U}_x)_j = \text{minmod}\left(\theta\frac{\overline{\boldsymbol{U}}_j - \overline{\boldsymbol{U}}_{j-1}}{\Delta x}, \frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_{j-1}}{2\Delta x}, \theta\frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_j}{\Delta x}\right)$$

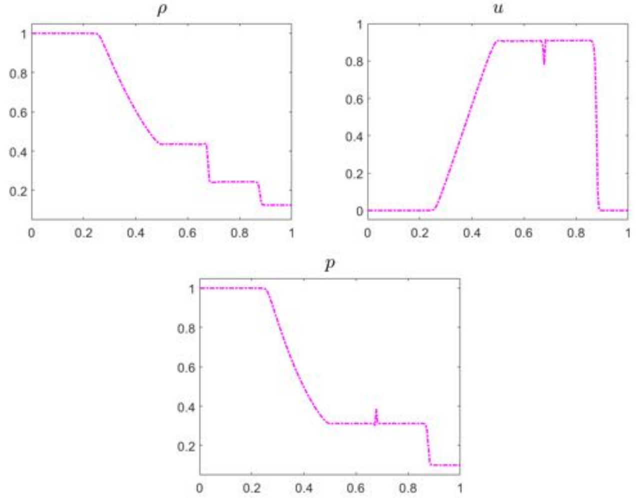
$$\boldsymbol{U}^{+}_{j+\frac{1}{2}} := \overline{\boldsymbol{U}}_j + \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$

$$\boldsymbol{U}^{-}_{j+\frac{1}{2}} := \overline{\boldsymbol{U}}_j - \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$



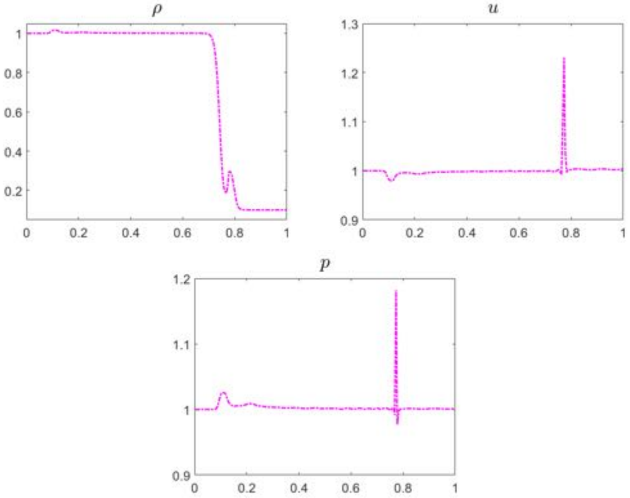
$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right) - \boldsymbol{H}_{j-\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j-\frac{1}{2}}, \boldsymbol{U}^{+}_{j-\frac{1}{2}}\right)}{\Delta x}$$

1-D Example – Shock Tube Problem



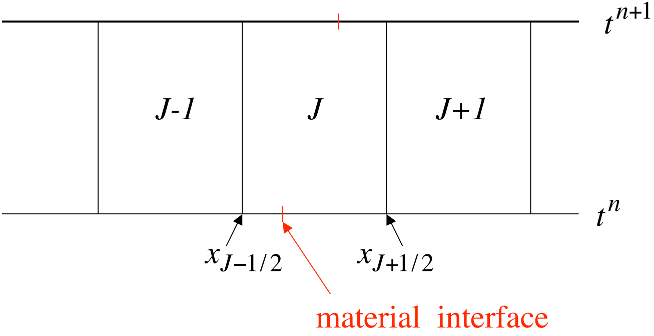
$$(\rho, u, p, \gamma, p_\infty)^T = \begin{cases} (1.000, 0, 1.0, 1.6, 0)^T, & \text{if } x < 0.25 \\ (0.125, 0, 0.1, 1.4, 0)^T, & \text{if } x > 0.25 \end{cases}$$

1-D Example – Contact Wave Problem



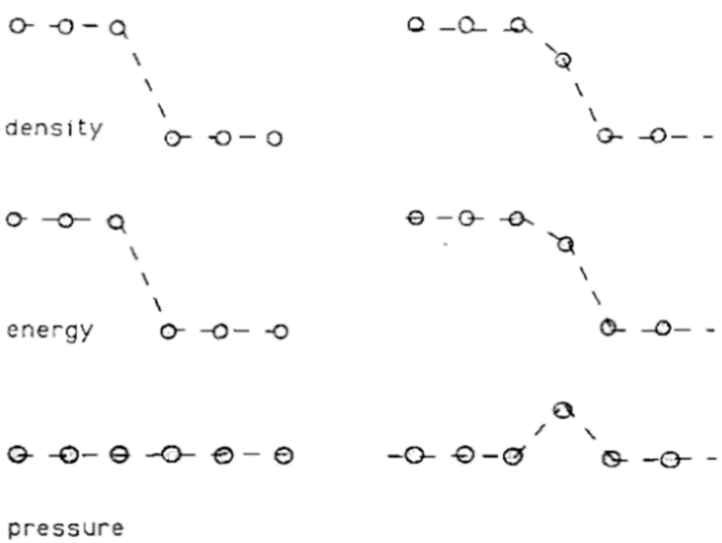
$$(\rho, u, p, \gamma, p_\infty)^T = \begin{cases} (1.0, 1.0, 1.0, 1.6, 0)^T, & \text{if} \quad x < 0.25 \\ (0.1, 1.0, 1.0, 1.4, 0)^T, & \text{if} \quad x > 0.25 \end{cases}$$

Multicomponent Flows – What may go wrong?



- Fluxes are computed using the information in the "mixed" cell.
- No valid EOS in mixed cells.

Multicomponent Flows – What may go wrong?[4]



[4] Karni, 1994.

- Front-capturing algorithms:
  - *Fluid-mixture type algorithms*[5]
  - *Five-equation model*[6]
  - *Pressure evolution method*[7]
  - *A simple fully conservative algorithm*[8]
  - *Ghost-fluid methods*[9]

- Front-tracking algorithms:
  - Moving-mesh techniques[10]
  - *Interface tracking methods*[11]

---

[5] Abgrall and Saurel, 2003; Shyue, 1998.

[6] Allaire, Clerc, and Kokh, 2002; Cheng, Zhang, and Liu, 2020.

[7] Karni, 1994.

[8] Saurel and Abgrall, 1999b.

[9] Fedkiw, Aslam, Merriman, and Osher, 1999; Abgrall and Karni, 2001.

[10] Harten and Hyman, 1983; Chertock and Kurganov, 2005.

[11] Davis, 1992; Chertock, Karni, and Kurganov, 2008; Wang and Shu, 2010.

Interface Tracking Method – 1-D[12]

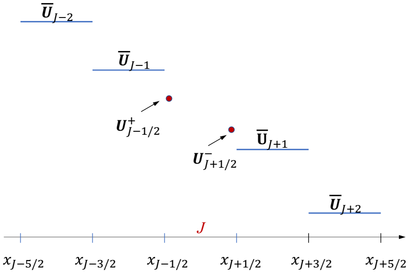
$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}\left(\boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\right) - \boldsymbol{H}_{j-\frac{1}{2}}\left(\boldsymbol{U}^-_{j-\frac{1}{2}}, \boldsymbol{U}^+_{j-\frac{1}{2}}\right)}{\Delta x}, \quad j \neq J$$

Don't use unreliable "mixed" cell data!

$$\boldsymbol{U}^+_{j-\frac{1}{2}} = \boldsymbol{\mathcal{D}}(\overline{\boldsymbol{U}}_{j-2}, \overline{\boldsymbol{U}}_{j-1}, \cancel{\overline{\boldsymbol{U}}_j})$$

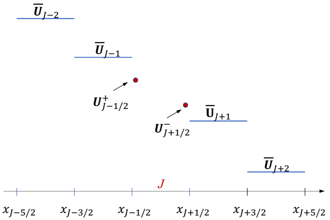
$$\boldsymbol{U}^-_{j+\frac{1}{2}} = \boldsymbol{\mathcal{D}}(\overline{\boldsymbol{U}}_{j-1}, \cancel{\overline{\boldsymbol{U}}_j}, \overline{\boldsymbol{U}}_{j+1})$$



[12]Chertock, Karni, and Kurganov, 2008.

Interface Tracking Method – Main Idea[13]

Don't use unreliable "mixed" cell data!



- Instead
  - Use the reliable single fluid data from the both sides of the interface to obtain the missing "mixed" cell information
  - Interpolate in the phase space by solving the corresponding Riemann problem using the data from both sides of the "mixed" cell

Interface tracking methods are very robust in the 1-D case, but their extensions to multi-D problems are rather cumbersome! The accuracy is restricted to the first order at the interface!

[13] Chertock, Karni, and Kurganov, 2008.

Hybrid Multifluid Algorithm – 1-D[14]

$$\begin{pmatrix}\rho\\ \rho u\\ E\\ \rho\phi\end{pmatrix}_t+\begin{pmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\\ \rho u\phi\end{pmatrix}_x=\mathbf{0}\quad\Leftrightarrow\quad \begin{array}{l}\boldsymbol{U}_t+\boldsymbol{F}(\boldsymbol{U})_x=\mathbf{0}\\ \textbf{EOS:}\quad p=(\gamma-1)\Big[E-\dfrac{\rho}{2}u^2\Big]-\gamma p_\infty\end{array}$$

Assume:

- A multifluid problem with two components

$$\gamma=\gamma_I,\quad p_\infty=p_{\infty,I}$$
$$\gamma=\gamma_{II},\quad p_\infty=p_{\infty,II}$$

- There is only one material interface and ϕ is the level-set function used to determine its position. The case of a larger, but finite number of interfaces can be treated similarly.

[14] Chertock, Chu, and Kurganov, 2021.

Hybrid Multifluid Algorithm – 1-D

- We assume that at some time $t \geq 0$ the cell averages of the conservative variables are available:

$$\overline{\boldsymbol{U}}_j = (\overline{\rho}_j, \overline{\rho u}_j, \overline{E}_j, \overline{\rho\phi}_j)^\top$$

- We compute
  - values of primitive variables

$$\boldsymbol{V}_j = (\overline{\rho}_j, u_j, p_j, \phi_j)^\top$$

$$u_j = \frac{\overline{\rho u}_j}{\overline{\rho}_j}, \quad p_j = (\gamma_j - 1)\left[\overline{E}_j - \frac{\left(\overline{(\rho u)}_j\right)^2}{2\,\overline{\rho}_j} - \gamma_j (p_\infty)_j\right], \quad \phi_j = \frac{\overline{\rho\phi}_j}{\overline{\rho}_j}$$

  - values of γ and p_∞

$$\gamma_j = \begin{cases} \gamma_I, & \text{if } \phi_j > 0, \\ \gamma_{II}, & \text{otherwise,} \end{cases} \qquad (p_\infty)_j = \begin{cases} p_{\infty,I}, & \text{if } \phi_j > 0, \\ p_{\infty,II}, & \text{otherwise.} \end{cases}$$

- We say that cells C_J and C_{J+1} are the interface cells if

$$\phi_J(t) \cdot \phi_{J+1}(t) \leq 0$$

Hybrid Multifluid Algorithm – Main Idea

- $j \notin \{J, J+1\}$ – consider the original system

$$\begin{pmatrix} \rho \\ \rho u \\ E \\ \rho\phi \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \\ \rho u \phi \end{pmatrix}_x = \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \mathbf{0}$$

- $j \in \{J, J+1\}$ – replace the E-equation with the p-equation[15]

$$\begin{pmatrix} \rho \\ \rho u \\ p \\ \rho\phi \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ up \\ \rho u \phi \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \\ -\left[(\gamma-1)p + \gamma p_\infty\right] u_x \\ 0 \end{pmatrix} \Leftrightarrow \boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$$

[15] Karni, 1996.

Hybrid Multifluid Algorithm – Main Idea

$j \notin \{J, J+1\}$	$j \in \{J, J+1\}$
Solve the *conservative* system $\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$	Solve the *nonconservative* system $\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$
Second-Order FV Method	
Implement central-upwind (CU) scheme	Implement path-conservative central-upwind (PCCU) scheme
Fifth-Order FD Method	
Implement alternative WENO (A-WENO) scheme	Implement path-conservative A-WENO scheme

Semi-Discrete CU scheme

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}}{\Delta x}, \quad j \notin \{J, J+1\}$$

- $\boldsymbol{H}_{j\pm\frac{1}{2}}$ are CU numerical fluxes[16]:

$$\boldsymbol{H}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{F}\left(\boldsymbol{U}^-_{j+\frac{1}{2}}\right) - a^-_{j+\frac{1}{2}}\boldsymbol{F}\left(\boldsymbol{U}^+_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}\,a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\right)$$

- $a^\pm_{j+\frac{1}{2}}$ are the one-sided local speeds of propagation obtained from the largest and the smallest eigenvalues of the Jacobian $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$:

$$a^\pm_{j+\frac{1}{2}} = \begin{matrix}\max\\\min\end{matrix}\left\{u^-_{j+\frac{1}{2}} \pm c^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}} \pm c^+_{j+\frac{1}{2}}, 0\right\}, \quad c := \sqrt{\gamma(p+p_\infty)/\rho}$$

[16] Kurganov, 2016.

Semi-Discrete CU scheme

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}}{\Delta x}, \quad j \notin \{J, J+1\}$$

Important!!!

- ***Reconstruct primitive variables*** $\boldsymbol{V}$ since both the pressure and velocity are continuous across the material interface

$$\widetilde{\boldsymbol{V}}_j(x,t) = \boldsymbol{V}_j(t) + (\boldsymbol{V}_x)_j(x - x_j), \quad x \in C_j \quad \Rightarrow \quad \boldsymbol{V}^{\pm}_{j+\frac{1}{2}}(t)$$

- ***Use***

$$\widetilde{\gamma}_j(x) = \begin{cases} \gamma_I, & \text{if } x < x_{J+\frac{1}{2}}, \\ \gamma_{II}, & \text{otherwise}, \end{cases} \qquad (\widetilde{p}_\infty)_j(x) = \begin{cases} p_{\infty,I}, & \text{if } x < x_{J+\frac{1}{2}}, \\ p_{\infty,II}, & \text{otherwise} \end{cases}$$

$$\gamma^{\pm}_{j+\frac{1}{2}} = \gamma_j = \gamma_{j+1}, \qquad (p_\infty)^{\pm}_{j+\frac{1}{2}} = (p_\infty)_j = (p_\infty)_{j+1}$$

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \{\boldsymbol{V}_j\} \to \left\{\boldsymbol{V}^{\pm}_{j+\frac{1}{2}}\right\} \to \left\{\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}\right\} \to \left\{\boldsymbol{H}_{j+\frac{1}{2}}\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

Reformulating CU scheme

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0} \quad \Rightarrow \quad \frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}}{\Delta x}$$

$$\begin{gathered} \boldsymbol{U}_t + A(\boldsymbol{U})\boldsymbol{U}_x = \boldsymbol{0} \\ A(\boldsymbol{U}) = \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}} \end{gathered} \quad \Rightarrow \quad \frac{d\overline{\boldsymbol{U}}_j}{dt} = ?$$

A straightforward discretization:

$$\boldsymbol{A}_j = \frac{1}{\Delta x}\int\limits_{C_j} A(\boldsymbol{U})\boldsymbol{U}_x\,dx$$

Consistent only for smooth solution and doesn't account for the contribution of the nonconservative products at the cell interfaces.

Reformulating CU scheme

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}}{\Delta x}$$

$$\Updownarrow$$

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{1}{\Delta x}\Bigg[\underbrace{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac{1}{2}}\big)}_{\boldsymbol{D}^-_{j+\frac{1}{2}}} + \underbrace{\boldsymbol{F}\big(\boldsymbol{U}^+_{j-\frac{1}{2}}\big) - \boldsymbol{H}_{j-\frac{1}{2}}}_{\boldsymbol{D}^+_{j-\frac{1}{2}}} + \boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac{1}{2}}\big) - \boldsymbol{F}\big(\boldsymbol{U}^+_{j-\frac{1}{2}}\big)\Bigg]$$

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{1}{\Delta x}\Bigg[\boldsymbol{D}^-_{j+\frac{1}{2}} + \boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac{1}{2}}\big) - \boldsymbol{F}\big(\boldsymbol{U}^+_{j-\frac{1}{2}}\big)\Bigg]$$

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{1}{\Delta x}\Bigg[\underbrace{\boldsymbol{D}^-_{j+\frac{1}{2}} + \boldsymbol{D}^+_{j-\frac{1}{2}}}_{?} + \underbrace{\int\limits_{C_j} A\big(\widetilde{\boldsymbol{U}}_j(x)\big)\big(\widetilde{\boldsymbol{U}}_j(x)\big)_x\,dx}_{\boldsymbol{A}_j}\Bigg]$$

Recall: A piecewise-linear reconstruction:

$$\widetilde{\boldsymbol{U}}_j(x) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

Reformulating CU scheme

$$\boldsymbol{D}^-_{j+\frac12}=\boldsymbol{H}_{j+\frac12}-\boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac12}\big),\quad \boldsymbol{D}^+_{j-\frac12}=\boldsymbol{F}\big(\boldsymbol{U}^+_{j-\frac12}\big)-\boldsymbol{H}_{j-\frac12}$$

CU fluxes:

$$\boldsymbol{H}_{j+\frac12}=\frac{a^+_{j+\frac12}\boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac12}\big)-a^-_{j+\frac12}\boldsymbol{F}\big(\boldsymbol{U}^+_{j+\frac12}\big)}{a^+_{j+\frac12}-a^-_{j+\frac12}}+\frac{a^+_{j+\frac12}a^-_{j+\frac12}}{a^+_{j+\frac12}-a^-_{j+\frac12}}\Big(\boldsymbol{U}^+_{j+\frac12}-\boldsymbol{U}^-_{j+\frac12}\Big)$$

$$\Updownarrow$$

$$\boldsymbol{D}^-_{j+\frac12}=\frac{1-\alpha_{j+\frac12}}{2}\Big[\underbrace{\boldsymbol{F}\big(\boldsymbol{U}^+_{j+\frac12}\big)-\boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac12}\big)}_{?}\Big]-\frac{\beta_{j+\frac12}}{2}\Big(\boldsymbol{U}^+_{j+\frac12}-\boldsymbol{U}^-_{j+\frac12}\Big)$$

$$\alpha_{j+\frac12}=\frac{a^+_{j+\frac12}+a^-_{j+\frac12}}{a^+_{j+\frac12}-a^-_{j+\frac12}},\qquad \beta_{j+\frac12}=\frac{-2a^+_{j+\frac12}a^-_{j+\frac12}}{a^+_{j+\frac12}-a^-_{j+\frac12}}$$

Reformulating CU scheme

$$\boldsymbol{D}^-_{j+\frac{1}{2}} = \frac{1-\alpha_{j+\frac{1}{2}}}{2}\Big[\underbrace{\boldsymbol{F}\big(\boldsymbol{U}^+_{j+\frac{1}{2}}\big) - \boldsymbol{F}\big(\boldsymbol{U}^-_{j+\frac{1}{2}}\big)}_{?}\Big] - \frac{\beta_{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

Consider a sufficiently smooth path $\boldsymbol{\Psi}_{j+\frac{1}{2}}(s) := \boldsymbol{\Psi}_{j+\frac{1}{2}}\Big(s; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\Big)$:

$$\boldsymbol{\Psi} : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N,$$

$$\boldsymbol{\Psi}_{j+\frac{1}{2}}\Big(0; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\Big) = \boldsymbol{U}^-_{j+\frac{1}{2}}, \quad \boldsymbol{\Psi}_{j+\frac{1}{2}}\Big(1; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\Big) = \boldsymbol{U}^+_{j+\frac{1}{2}}$$

$$\Downarrow$$

$$\boldsymbol{D}^-_{j+\frac{1}{2}} = \frac{1-\alpha_{j+\frac{1}{2}}}{2} \underbrace{\int_0^1 A\Big(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)\Big)\Big(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)\Big)_s\,ds}_{\boldsymbol{A}_{\boldsymbol{\Psi},j+\frac{1}{2}}} - \frac{\beta_{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

Reformulating CU scheme

$$\boldsymbol{U}_t + A(\boldsymbol{U})\boldsymbol{U}_x = \boldsymbol{0} \quad \Rightarrow \quad \frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{1}{\Delta x}\Big[\boldsymbol{D}^-_{j+\frac12} + \boldsymbol{D}^+_{j-\frac12} + \boldsymbol{A}_j\Big]$$

Here:

$$\boldsymbol{D}^{\pm}_{j+\frac12} = \frac{1 \pm \alpha_{j+\frac12}}{2}\boldsymbol{A}_{\boldsymbol{\Psi},j+\frac12} \pm \frac{\beta_{j+\frac12}}{2}\Big(\boldsymbol{U}^+_{j+\frac12} - \boldsymbol{U}^-_{j+\frac12}\Big)$$

$$\alpha_{j+\frac12} = \frac{a^+_{j+\frac12} + a^-_{j+\frac12}}{a^+_{j+\frac12} - a^-_{j+\frac12}}, \qquad \beta_{j+\frac12} = \frac{-2a^+_{j+\frac12}a^-_{j+\frac12}}{a^+_{j+\frac12} - a^-_{j+\frac12}}$$

$$\boldsymbol{A}_{\boldsymbol{\Psi},j+\frac12} := \int\limits_0^1 A\Big(\boldsymbol{\Psi}_{j+\frac12}(s)\Big)\Big(\boldsymbol{\Psi}_{j+\frac12}(s)\Big)_s\,ds, \quad \boldsymbol{A}_j := \int\limits_{C_j} A\big(\widetilde{\boldsymbol{U}}_j(x)\big)\big(\widetilde{\boldsymbol{U}}_j(x)\big)_x\,dx$$

$$\boldsymbol{\Psi}_{j+\frac12}(s) := \boldsymbol{\Psi}_{j+\frac12}\Big(s;\boldsymbol{U}^-_{j+\frac12},\boldsymbol{U}^+_{j+\frac12}\Big), \quad \widetilde{\boldsymbol{U}}_j(x) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x-x_j), \quad x \in C_j$$

Hybrid Multifluid Algorithm – Main Idea

$j \notin \{J, J+1\}$	$j \in \{J, J+1\}$
Solve the *conservative* system $\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$	Solve the *nonconservative* system $\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$
Second-Order FV Method	
Implement central-upwind (CU) scheme	Implement path-conservative central-upwind (PCCU) scheme
Fifth-Order FD Method	
Implement alternative WENO (A-WENO) scheme	Implement path-conservative A-WENO scheme

Semi-Discrete PCCU scheme

$$\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$$

$$\begin{pmatrix} \rho \\ \rho u \\ p \\ \rho\phi \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ up \\ \rho u\phi \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \\ -\left[(\gamma-1)p + \gamma p_\infty\right] u_x \\ 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\left[(\gamma-1)p + \gamma p_\infty\right] u}{\rho} & \dfrac{(1-\gamma)p - \gamma p_\infty}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{B(\boldsymbol{\mathcal{U}})} \begin{pmatrix} \rho \\ \rho u \\ p \\ \rho\phi \end{pmatrix}_x$$

Semi-Discrete PCCU Scheme

$$\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{U}}_t + \mathcal{A}(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x = \mathbf{0}, \quad \mathcal{A}(\boldsymbol{\mathcal{U}}) := \frac{\partial \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})}{\partial \boldsymbol{\mathcal{U}}} - B(\boldsymbol{\mathcal{U}})$$

$$\Downarrow$$

$$\frac{d\overline{\boldsymbol{U}}_j}{dt} = -\frac{1}{\Delta x}\left[\boldsymbol{D}^-_{j+\frac{1}{2}} + \boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{A}_j\right], \quad j \in \{J, J+1\}$$

$$\Downarrow$$

$$\frac{\mathrm{d}\boldsymbol{\mathcal{U}}_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\Bigg[\underbrace{\boldsymbol{\mathcal{H}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{H}}_{j-\frac{1}{2}}}_{\text{CU fluxes}}$$

$$\underbrace{-\boldsymbol{B}_j - \frac{a^+_{j-\frac{1}{2}}}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\boldsymbol{B}_{\Psi,j-\frac{1}{2}} + \frac{a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\boldsymbol{B}_{\Psi,j+\frac{1}{2}}}_{\text{discretization of the nonconservative terms}}\Bigg]$$

Semi-Discrete PCCU scheme

$$\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$$

$$\frac{\mathrm{d}\boldsymbol{\mathcal{U}}_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\Bigg[\boldsymbol{\mathcal{H}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{H}}_{j-\frac{1}{2}} - \boldsymbol{B}_j - \frac{a^+_{j-\frac{1}{2}}}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\boldsymbol{B}_{\Psi,j-\frac{1}{2}} + \frac{a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\boldsymbol{B}_{\Psi,j+\frac{1}{2}}\Bigg], \quad j \in \{J, J+1\}$$

We use a piecewise linear reconstruction and a linear path

$$\boldsymbol{\mathcal{H}}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{\mathcal{F}}\big(\boldsymbol{\mathcal{U}}^-_{j+\frac{1}{2}}\big) - a^-_{j+\frac{1}{2}}\boldsymbol{\mathcal{F}}\big(\boldsymbol{\mathcal{U}}^+_{j+\frac{1}{2}}\big)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\Big(\boldsymbol{\mathcal{U}}^+_{j+\frac{1}{2}} - \boldsymbol{\mathcal{U}}^-_{j+\frac{1}{2}}\Big)$$

$$\boldsymbol{B}_j = \left(0, 0, -\int\limits_{C_j}\left[(\widetilde{\gamma}_j(x) - 1)\widetilde{p}_j(x) + \widetilde{\gamma}_j(x)(\widetilde{p}_\infty)_j(x)\right](u_x)_j\,dx, 0\right)^\top$$

$$\boldsymbol{B}_{\Psi,j+\frac{1}{2}} = \left(0, 0, \int\limits_0^1 B(p_{j+\frac{1}{2}}(s))(u_{j+\frac{1}{2}})_s\,ds, 0\right)^\top$$

Hybrid Multifluid Algorithm – Main Idea

$j \notin \{J, J+1\}$	$j \in \{J, J+1\}$
Solve the *conservative* system $\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$	Solve the *nonconservative* system $\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x$
Second-Order FV Method	
Implement central-upwind (CU) scheme	Implement path-conservative central-upwind (PCCU) scheme
Fifth-Order FD Method	
Implement alternative WENO (A-WENO) scheme	Implement path-conservative A-WENO scheme

Fifth-Order A-WENO Scheme[18]

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0} \quad \Rightarrow \quad \frac{d\boldsymbol{U}_j}{dt} = -\frac{\boldsymbol{\mathfrak{H}}_{j+\frac{1}{2}} - \boldsymbol{\mathfrak{H}}_{j-\frac{1}{2}}}{\Delta x}, \quad j \neq \{J, J+1\}$$

- $\boldsymbol{U}_j(t) \approx \boldsymbol{U}(x_j, t)$
- $\boldsymbol{\mathfrak{H}}_{j+\frac{1}{2}}$ is the fifth-order numerical flux:

$$\boldsymbol{\mathfrak{H}}_{j+\frac{1}{2}} = \boldsymbol{H}_{j+\frac{1}{2}} - \frac{1}{24}(\Delta x)^2(\boldsymbol{F}_{xx})_{j+\frac{1}{2}} + \frac{7}{5760}(\Delta x)^4(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2}}$$

  - $\boldsymbol{H}_{j+\frac{1}{2}} = \boldsymbol{H}(\boldsymbol{U}^{\pm}_{j+\frac{1}{2}})$ is the FV numerical flux with a fifth-order accurate (WENO-Z)[17] reconstruction applied, as before, to primitive variables $\boldsymbol{V}_j$
  - $(\boldsymbol{F}_{xx})_{j+\frac{1}{2}}$ and $(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2}}$ are computed using the second- and fourth-order finite differences

[17] Wang, Li, Gao, and Don, 2018.

[18] Wang, Don, Garg, and Kurganov, 2020.

Fifth-Order Path-Conservative A-WENO Scheme[19]

$$\boldsymbol{\mathcal{U}}_t + \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}})_x = B(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}}_x\,, \qquad j \in \{J, J+1\}$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\mathcal{U}}_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\Bigg[&\boldsymbol{\mathcal{H}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{H}}_{j-\frac{1}{2}} \\ &- \boldsymbol{B}_j - \frac{a^+_{j-\frac{1}{2}}}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\boldsymbol{B}_{\Psi,j-\frac{1}{2}} + \frac{a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\boldsymbol{B}_{\Psi,j+\frac{1}{2}} \\ &+ \frac{\Delta x}{24}\left[(\boldsymbol{K}_{xx})_{j+\frac{1}{2}} - (\boldsymbol{K}_{xx})_{j-\frac{1}{2}}\right] - \frac{7}{5760}(\Delta x)^3\left[(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2}} - (\boldsymbol{K}_{xxxx})_{j-\frac{1}{2}}\right]\Bigg]\end{aligned}$$

where

$$\boldsymbol{K}(\mathcal{U}(\cdot,t)) = \mathcal{F}(\mathcal{U}(x,t)) - \int_{-\infty}^{x} B(\mathcal{U}(\xi,t))\mathcal{U}_x(\xi,t)\,\mathrm{d}\xi$$

[19] Chu, Kurganov, and Na, 2021.

Mixed-Order Approach

Algorithm

Reconstruct all of the required point values using a WENO-Z interpolant

Check monotonicity for the following sequences:

$$\left(\rho_j,\rho^-_{j+\frac{1}{2}},\rho^+_{j+\frac{1}{2}},\rho_{j+1}\right),\ \left(u_j,u^-_{j+\frac{1}{2}},u^+_{j+\frac{1}{2}},u_{j+1}\right),\ \left(p_j,p^-_{j+\frac{1}{2}},p^+_{j+\frac{1}{2}},p_{j+1}\right)$$

Check if the pressure profile is locally smooth, namely,

$$\frac{|p^+_{j+\frac{1}{2}}-p^-_{j+\frac{1}{2}}|}{\max\left\{p^+_{j+\frac{1}{2}},p^-_{j+\frac{1}{2}}\right\}}<\mathtt{C}(\Delta x)^2$$

if *the monotonicity and smoothness conditions are satisfied at* $x=x_{j+\frac{1}{2}}$

then

use the fifth-order A-WENO fluxes

else

switch locally to the second-order CU fluxes

Numerical Examples

- 1-D examples
  - Shock-tube problem
  - Stiff shock-tube problem
  - Water-air model problem

$$\phi(x,0) = \begin{cases} 1, & x \in \Omega_I, \\ -1, & x \in \Omega_{II} \end{cases}$$

- 2-D examples
  - Helium bubble problem
  - R22 bubble problem

$$\phi(x,y,0) = \begin{cases} 1, & (x,y) \in \Omega_I, \\ -1, & (x,y) \in \Omega_{II} \end{cases}$$

Time evolution: three-stage third-order strong SSP Runge-Kutta method[20]; CFL=0.3

smoothness constant: $\mathtt{C} = 1$ (1-D) and $\mathtt{C} = 5$ (2-D)

[20] Gottlieb, Shu, and Tadmor, 2001.

Shock-Tube Problem

Consider

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + p)_x = 0 \\ &E_t + [u(E+p)]_x = 0 \\ &(\rho\phi)_t + (\rho u \phi)_x = 0 \end{aligned}$$

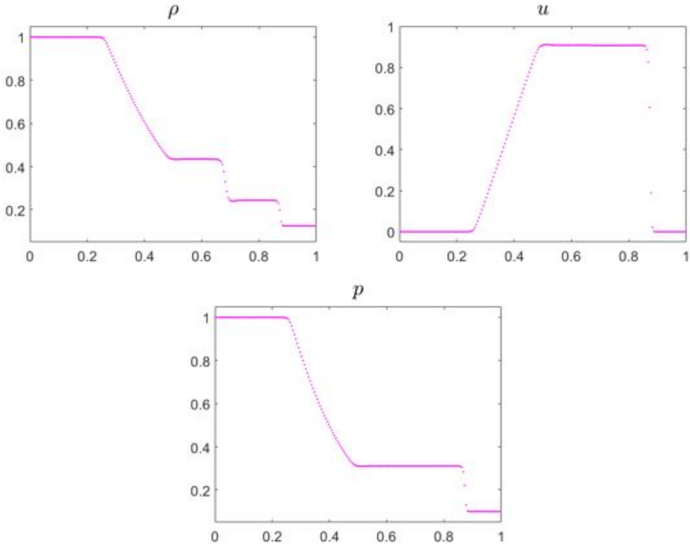
combined with the EOS

$$p = (\gamma - 1)\Big[E - \frac{1}{2}\rho u^2\Big] - \gamma p_\infty$$

Initial condition:

$$(\rho, u, p; \gamma, p_\infty) = \begin{cases} (1.000, 0, 1.0; 1.4, 0), & x < 0.5 \\ (0.125, 0, 0.1; 1.6, 0), & x > 0.5 \end{cases}$$

Shock-Tube Problem



Mixed-order A-WENO scheme: $\Delta x = 1/200, t = 0.2$

Shock-Tube Problem

Relative conservation error in the computation of the total energy E:

$$\text{err} := \frac{\sum_j \overline{E}_j(t) - \sum_j \overline{E}_j(0)}{\sum_j \overline{E}_j(0)}$$

Δx	err	rate
1/200	8.35e-4	–
1/400	5.84e-4	0.52
1/800	4.58e-4	0.35
1/1600	2.70e-4	0.76
1/3200	1.62e-4	0.74

Stiff Shock-Tube Problem

Consider

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + [u(E+p)]_x = 0$$
$$(\rho\phi)_t + (\rho u\phi)_x = 0$$

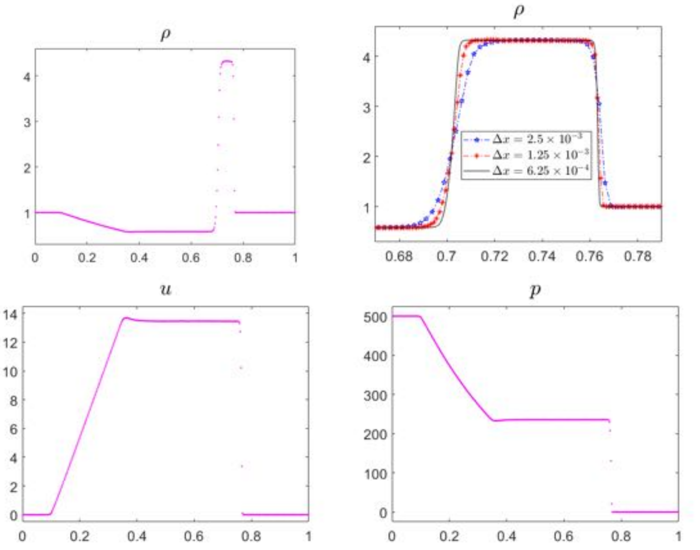
combined with the EOS

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right] - \gamma p_\infty$$

Initial condition:

$$(\rho, u, p; \gamma, p_\infty) = \begin{cases} (1, 0, 500; 1.4, 0), & x < 0.5 \\ (1, 0, 0.2; 1.6, 0), & x > 0.5 \end{cases}$$

Stiff Shock-Tube Problem



Mixed-order A-WENO scheme: $\Delta x = 1/400, t = 0.015$

Shock-Tube Problem

Relative conservation error in the computation of the total energy E:

$$\text{err} := \frac{\sum_j \overline{E}_j(t) - \sum_j \overline{E}_j(0)}{\sum_j \overline{E}_j(0)}$$

Δx	err	rate
1/400	1.82e-3	–
1/800	8.12e-4	1.16
1/1600	5.27e-4	0.62
1/3200	3.74e-4	0.49
1/6400	2.50e-4	0.58

Water-Air Model with the Stiff Equation of State

Consider

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + [u(E+p)]_x = 0$$
$$(\rho\phi)_t + (\rho u \phi)_x = 0$$

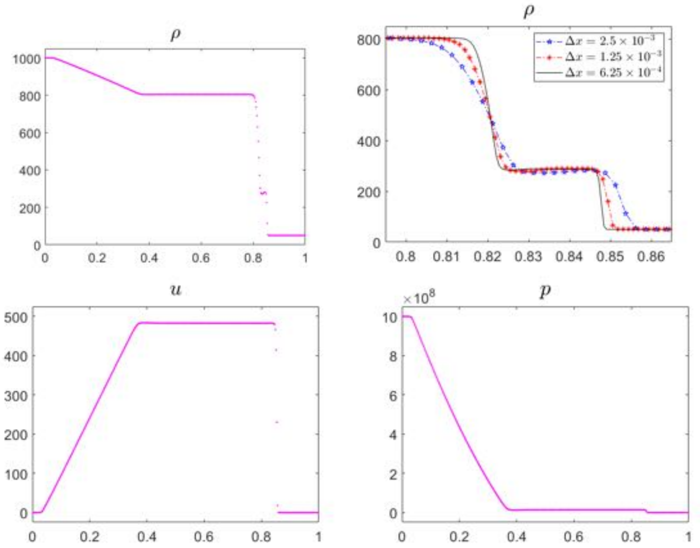
combined with the EOS

$$p = (\gamma - 1)\Big[E - \frac{1}{2}\rho u^2\Big] - \gamma p_\infty$$

Initial condition:

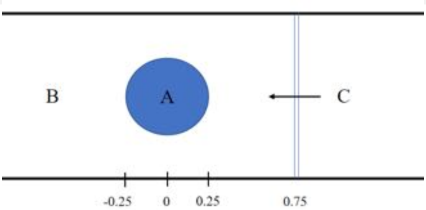
$$(\rho, u, p; \gamma, p_\infty) = \begin{cases} (1000, 0, 10^9; 4.4, 6\cdot 10^8), & x < 0.7 \\ (50, 0, 10^5; 1.4, 0), & x > 0.7 \end{cases}$$

Water-Air Model with the Stiff Equation of State



Mixed-order A-WENO scheme: $\Delta x = 1/400, t = 0.00025$

2-D Examples



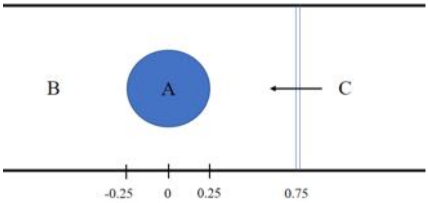
Consider

$$
\begin{aligned}
&\rho_t + (\rho u)_x + (\rho v)_y = 0 \\
&(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0 \\
&(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0 \\
&E_t + [u(E + p)]_x + [v(E + p)]_y = 0 \\
&(\rho \phi)_t + (\rho u \phi)_x + (\rho v \phi)_y = 0
\end{aligned}
$$

combined with the EOS

$$
p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right] - \gamma p_\infty
$$

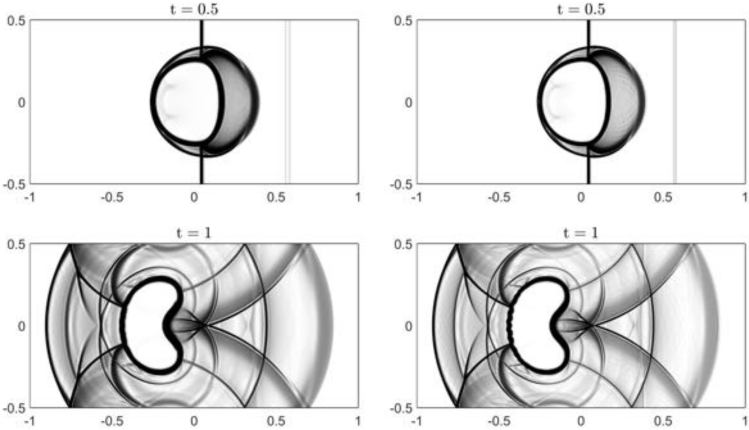
Helium Bubble



$$(\rho, u, p; \gamma, p_\infty) = \begin{cases} (4/29, 0, 0, 1; 5/3, 0), & \text{in region A} \\ (1, 0, 0, 1; 1.4, 0), & \text{in region B} \\ (4/3, -0.3535, 0, 1.5; 1.4, 0), & \text{in region C} \end{cases}$$

The top and bottom boundaries are solid walls, while the left and right boundaries are open

Helium Bubble

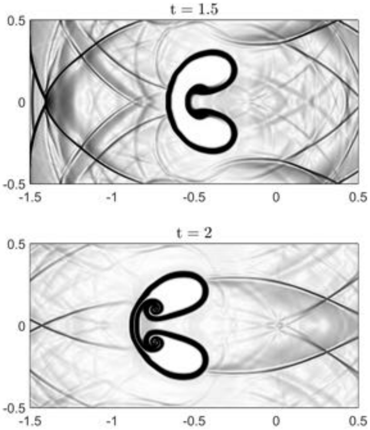


second-order CU scheme

mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

Helium Bubble

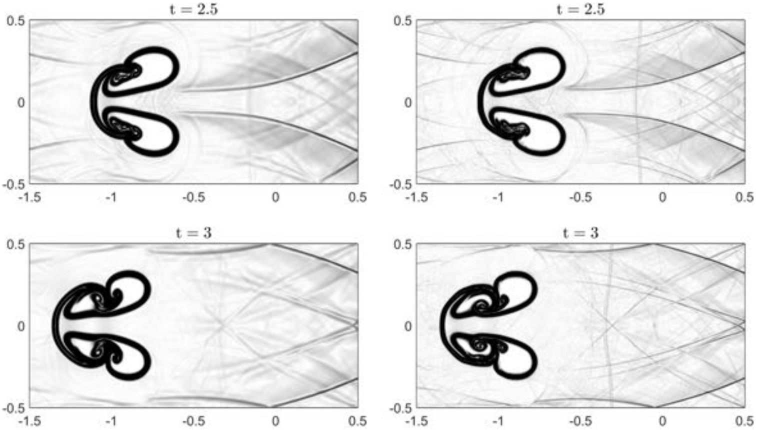


second-order CU scheme

mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

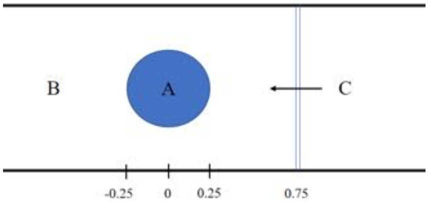
Helium Bubble



second-order CU scheme   mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

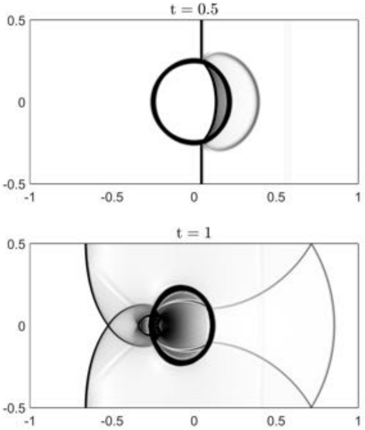
R22 Bubble



$$(\rho, u, p; \gamma, p_\infty) = \begin{cases} (3.1538, 0, 0, 1; 1.249, 0), & \text{in region A} \\ (1, 0, 0, 1; 1.4, 0), & \text{in region B} \\ (4/3, -0.3535, 0, 1.5; 1.4, 0), & \text{in region C} \end{cases}$$

The top and bottom boundaries are solid walls, while the left and right boundaries are open

Helium Bubble

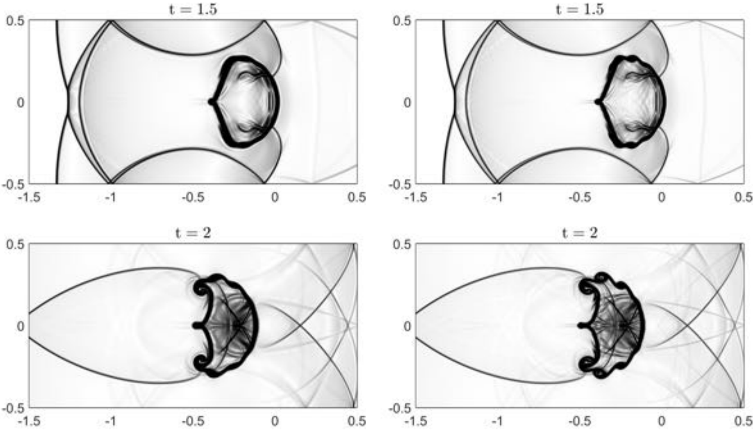


second-order CU scheme

mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

Helium Bubble

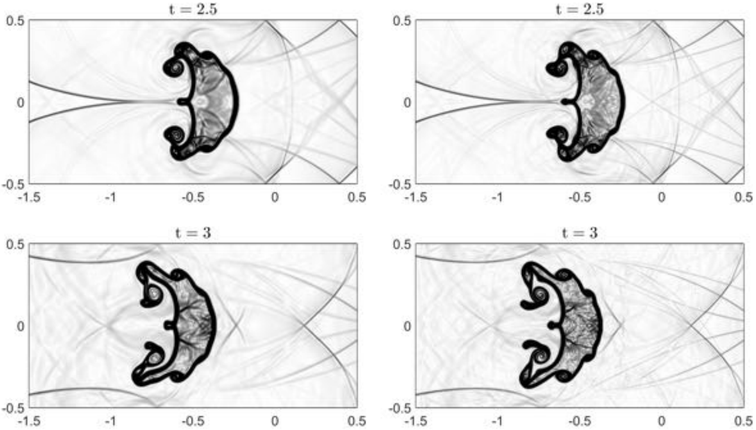


second-order CU scheme

mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

Helium Bubble



second-order CU scheme

mixed-order A-WENO scheme

$$\Delta x = \Delta y = 1/500$$

Thank you!

References

- Abgrall, R. (1996). "How to prevent pressure oscillations in multicomponent flow calculations: a quasi-conservative approach". In: *J. Comput. Phys.* 125.1, pp. 150–160.
- – (1998). "Generalization of the ROE scheme for the computation of mixture of perfect gases". In: *Rech. Aérospat.* 6, pp. 31–43.
- Abgrall, R. and S. Karni (2001). "Ghost-fluids for the poor: a single fluid algorithm for multifluids". In: *Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000).* Vol. 141. Internat. Ser. Numer. Math., 140. Basel: Birkhäuser, pp. 1–10.
- Abgrall, R. and R. Saurel (2003). "Discrete equations for physical and numerical compressible multiphase flow mixtures". In: *J. Comput. Phys.* 186.2, pp. 361–396.
- Allaire, G., S. Clerc, and S. Kokh (2002). "A five-equation model for the simulation of interfaces between compressible fluids". In: *J. Comput. Phys.* 181.2, pp. 577–616.
- Cheng, J., F. Zhang, and T. G. Liu (2020). "A quasi-conservative discontinuous Galerkin method for solving five equation model of compressible two-medium flows". In: *J. Sci. Comput.* 85, pp. 12–35.
- Chertock, A., S. Chu, and A. Kurganov (2021). "Hybrid multifluid algorithms based on the path-conservative central-upwind scheme". In: *J. Sci. Comput.* 89.2, Paper No. 48, 24.
- Chertock, A., S. Karni, and A. Kurganov (2008). "Interface tracking method for compressible multifluids". In: *M2AN Math. Model. Numer. Anal.* 42, pp. 991–1019.
- Chertock, A. and A. Kurganov (2005). "Conservative locally moving mesh method for multifluid flows". In: *Finite Volumes for Complex Applications IV.* Ed. by F. Benkhaldoun, D. Ouazar, and S. Raghay. Hermes Science, pp. 273–284.
- Chu, S., A. Kurganov, and M. Na (2021). "Fifth-order A-WENO schemes based on the path-conservative central-upwind method". In: submitted.
- Davis, S. F. (1992). "An interface tracking method for hyperbolic systems of conservation laws". In: *Appl. Numer. Math.* 10, pp. 447–472.
- Fedkiw, R. P., T. Aslam, B. Merriman, and S. Osher (1999). "A non-oscillatory Eulerian approach to interfaces in multimaterial flows (the ghost fluid method)". In: *J. Comput. Phys.* 152.2, pp. 457–492.
- Gottlieb, S., C.-W. Shu, and E. Tadmor (2001). "Strong stability-preserving high-order time discretization methods". In: *SIAM Rev.* 43.1, pp. 89–112.
- Harten, A. and J. M. Hyman (1983). "Self-adjusting grid methods for one-dimensional hyperbolic conservation laws". In: *J. Comput. Phys.* 50.2, pp. 235–269.
- Karni, S. (1994). "Multicomponent flow calculations by a consistent primitive algorithm". In: *J. Comput. Phys.* 112.1, pp. 31–43.
- – (1996). "Hybrid multifluid algorithms". In: *SIAM J. Sci. Comput.* 17.5, pp. 1019–1039.
- Kurganov, A. (2016). "Central schemes: A powerful black-box solver for nonlinear hyperbolic PDEs". In: *Handbook of numerical methods for hyperbolic problems.* Ed. by R. Abgrall and C.-W. Shu. Vol. 17. Handb. Numer. Anal. Amsterdam: Elsevier/North-Holland, pp. 525–548.
- Larrouturou, B. (1991). "How to preserve the mass fractions positivity when computing compressible multi-component flows". In: *J. Comput. Phys.* 95.1, pp. 59–84.
- Mulder, W., S. Osher, and J. A. Sethian (1992). "Computing interface motion in compressible gas dynamics". In: *J. Comput. Phys.* 100.2, pp. 209–228.
- Roe, P. L. (1982). "Fluctuations and signals — A framework for numerical evolution problems". In: *Numerical Methods for Fluid Dynamics.* Ed. by K. Morton and M. Baines. Academic Press, New York, pp. 219–257.
- Saurel, R. and R. Abgrall (1999a). "A multiphase Godunov method for compressible multifluid and multiphase flows". In: *J. Comput. Phys.* 150.2, pp. 425–467.
- – (1999b). "A simple method for compressible multifluid flows". In: *SIAM J. Sci. Comput.* 21.3, pp. 1115–1145.
- Shyue, K. M. (1998). "An efficient shock-capturing algorithm for compressible multicomponent problems". In: *J. Comput. Phys.* 142.1, pp. 208–242.
- Wang, B.-S., W. S. Don, N. K. Garg, and A. Kurganov (2020). "Fifth-order A-WENO finite-difference schemes based on a new adaptive diffusion central numerical flux". In: *SIAM J. Sci. Comput.* 42.6, A3932–A3956.
- Wang, B.-S., P. Li, Z. Gao, and W. S. Don (2018). "An improved fifth order alternative WENO-Z finite difference scheme for hyperbolic conservation laws". In: *J. Comput. Phys.* 374.1, pp. 469–477.
- Wang, C. and C.-W. Shu (2010). "An interface treating technique for compressible multi-medium flow with Runge-Kutta discontinuous Galerkin method". In: *J. Comput. Phys.* 229.23, pp. 8823–8843.