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Global L^∞ entropy solutions to system of polytropic gas dynamics with a source

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Abstract

In this paper, we study the global L^∞ entropy solutions for the Cauchy problem of the polytropic gas dynamics system in a general nozzle with friction. First, under bounded conditions on the L^1 norm of the cross-sectional area function $A(x)$ and the friction function $\alpha(x)$, we apply the flux-approximation technique coupled with the classical viscosity method to obtain the L^∞ estimates of the viscosity-flux approximate solutions for any exponent $\gamma \geq 1$; Second, by using the compactness framework from the compensated compactness theory, we prove the convergence of the viscosity-flux approximate solutions and obtain the global existence of the L^∞ entropy solutions.

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1. Introduction

We consider the following system of isentropic gas dynamics in a general nozzle with friction

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 - \alpha(x)\rho u|u| \end{cases} \tag{1.0.1}$$

with bounded initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.0.2}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ is a slowly variable cross section area at x in the nozzle and $\alpha(x)$ denotes the coefficient function of the friction. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, where the exponent $\gamma > 1$ corresponds to the isentropic case and $\gamma = 1$ corresponds to the isothermal case. The nozzle is widely used in some types of steam turbines, rocket engine nozzles, supersonic jet engines and jet streams in astrophysics, and the friction appears due to the viscosity [39,48,15].

When $A(x) = -\frac{a'(x)}{a(x)} = 0, \alpha(x) = 0$ in (1.0.1), the study of the existence of global weak solutions for the underlying homogeneous isentropic system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \end{cases} \tag{1.0.3}$$

with the initial data (1.0.2) has a long history. For a polytropic gas in Lagrangian coordinates, the first existence theorem for large initial data of locally finite total variation was proved in [37] for $\gamma = 1$, in [38] for $\gamma \in (1, 1+\delta)$, where δ is small. The Glimm scheme [7] was used in these papers.

The ideas of compensated compactness developed in [41,36] were used in [5] to establish a global existence theorem for the Cauchy problem (1.0.3) with large initial data for $\gamma = 1 + \frac{2}{N}$, where $N \geq 5$ odd, with the use of the viscosity method. The convergence of the Lax-Friedrichs scheme and the existence of a global solution in L^∞ for large initial data with adiabatic exponent $\gamma \in (1, \frac{5}{3}]$ were proved in [3]. In [17], the global existence of a weak solution was proved for $\gamma \geq 3$ with the use of the kinetic setting in combination with the compensated compactness method. The method in [17] was finally improved in [16] to fill the gap $\gamma \in (\frac{5}{3}, 3)$, and a new proof of the existence of a global solution for all $\gamma > 1$ was given there. Later on, a new application of the method in [17] was obtained in [21] on the study of the Euler equations of one-dimensional, compressible fluid flow, where the linear combinations of weak and strong entropies were invented to replace the weak entropies. The isothermal case $\gamma = 1$ with the vacuum was studied in [11].

A global smooth solution of the Cauchy problem (1.0.3) with smooth initial data for more general pressure functions $P(\rho)$ was obtained in [22], where a sequence of nonstrictly hyperbolic systems was used to approximate system (1.0.3).

For some special inhomogeneous hyperbolic systems, the existence and qualitative behavior of global solution with initial data of small total variation were first studied in [18–20] by using Glimm random choice method. For a general inhomogeneous system of hyperbolic conservation laws, the Riemann problem was resolved in [13]. More results on inhomogeneous hyperbolic systems can be found in [6,8,9,12,31,32,34,35,40,47,10,33,46] and the references cited therein.

It is well-known that, with the help of the compensated compactness theory, the unique difficulty to prove the existence of global solution for the following inhomogeneous system

$$\begin{cases} \rho_t + (\rho u)_x + f(\rho, u) = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x + g(\rho, u) = 0 \end{cases} \tag{1.0.4}$$

is to obtain the a-priori L^∞ estimates of the approximate solutions of (1.0.4).

For systems of hyperbolic conservation laws of two equations, the well-known theory of invariant regions [2] is still a powerful tool to help us to obtain bounded estimates of solutions. Based on the invariant regions theory, under some conditions on the nonlinear functions $f(\rho, u)$ and $g(\rho, u)$, the global L^∞ entropy solutions for the Cauchy problem (1.0.4) with large initial data (1.0.2) was obtained in [4] for the usual gases $1 < \gamma \le \frac{5}{3}$.

When $A(x) \neq 0, \alpha(x) \neq 0$ in (1.0.1), in general, the theory of invariant regions can not be applied directly to obtain the a-priori L^∞ estimate of the approximate solutions because we could not find a suitable invariant region.

For the case of nozzle flow without the friction $\alpha(x) = 0$, namely the following system

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2, \end{cases} \tag{1.0.5}$$

the global solution was studied in [14,42]. Based on the flux approximation technique introduced in [23] coupled with the classical artificial viscosity, a sequence of the parabolic systems was constructed in [24]

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x)(\rho - 2\delta)u + \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x)(\rho - 2\delta)u^2 + \varepsilon(\rho u)_{xx} \end{cases} \tag{1.0.6}$$

to approximate system (1.0.5), where $\delta > 0$ denotes a regular perturbation constant, the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t)dt, \tag{1.0.7}$$

and the perturbation initial data

$$(\rho^{\delta,\varepsilon}(x, 0), u^{\delta,\varepsilon}(x, 0)) = (\rho_0(x) + 2\delta, u_0(x)). \tag{1.0.8}$$

Especially for the nozzle flow with the monotone cross section, which is corresponding to $a'(x) \geq 0$, and for the general pressure function $P(\rho)$, the new variable $v = z - B(x)$ (or $s = w - C(x)$) was introduced in [24,25] and the following inequality on v (or s)

$$v_t + a(x,t)v_x + b(x,t)v \le \varepsilon v_{xx}, \tag{1.0.9}$$

obtained from (1.0.6), where

$$z(\rho,u) = \int_l^\rho \frac{\sqrt{P'(s)}}{s} ds - u, \quad w(\rho,u) = \int_l^\rho \frac{\sqrt{P'(s)}}{s} ds + u, \tag{1.0.10}$$

are Riemann invariants of (1.0.3) and l is a constant, $B(x)$ and $C(x)$ are carefully selected non-negative bounded functions of the space variable x to control the nonlinear functions $A(x)$.

We can obtain the estimate $v \le 0$ and so the upper estimate $z(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \le B(x)$ (or $w \le C(x)$) when we apply the maximum principle to (1.0.9).

At the same time, a modified Godunov scheme was introduced to construct the approximate solutions of (1.0.5) and the global existence of weak solutions of the Cauchy problem (1.0.5)-(1.0.2) was obtained for the Laval nozzle, which is corresponding to $a'(x) \cdot x \ge 0$, in [43] and the general nozzle in [44] for the usual gases $1 < \gamma \le \frac{5}{3}$. The case for any $\gamma > 1$ was proved in [30] by using the method given in [24].

When $A(x) = 0, \alpha(x) \ne 0$ in (1.0.1), namely

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x + \alpha(x)\rho u|u| = 0, \end{cases} \tag{1.0.11}$$

the theory of invariant regions can be applied directly in [14] to obtain the a-priori L^∞ estimate of the approximate solutions if $\alpha(x) \ge 0$. However, the case of $\alpha(x) < 0$ is completely different because neither the theory of invariant regions nor the maximum principle could be applied directly. Under the condition $-M \le \alpha(x) \le 0$ and $\gamma > \frac{5}{3}$, a technique was introduced in [26] to obtain the L^∞ estimates of the approximate solutions of (1.0.11).

When $A(x) \ne 0$ and $\alpha(x)$ is a positive constant, the existence of global entropy solutions for the Cauchy problem (1.0.1) and (1.0.2) in the simplest divergent nozzle (with respect to $a'(x) \ge 0$) was first obtained in [45] for the usual gases $1 < \gamma \le \frac{5}{3}$, and later, extended in [27] to the case of $\gamma > 1$, provided that the initial data are bounded and satisfy the very special conditions $z(\rho_0(x), u_0(x)) \le 0$. When $\alpha(x) \ge 0$ and for the Laval nozzle ($a'(x) \cdot x \ge 0$), the uniform bound of solutions of (1.0.1) and (1.0.2) was proved in [26], where γ is limited in $(3, \infty)$ for a technical difficulty.

It is worthwhile to point out that, under suitable conditions among the initial data, $a(x)$ and $\alpha(x)$, the initial-boundary value problem of compressible Euler equations with friction and heating

$$\begin{cases} (a(x)\rho)_t + (a(x)\rho u)_x = 0, \\ (a(x)\rho u)_t + (a(x)\rho u^2 + a(x)P)_x = a'(x)P - \alpha\sqrt{a(x)}\rho u|u|, \\ (a(x)E)_t + (a(x)u(E+P))_x = \beta a(x)q(x) - \alpha\sqrt{a(x)}\rho u^2|u|, \end{cases} \tag{1.0.12}$$

was studied in [1] for $1 < \gamma \le \frac{5}{3}$, by using a new version of a generalized Glimm scheme, where ρ, u, E are, respectively, the density, velocity, total energy and pressure of the gas, α is the coefficient of friction, $q(x)$ is a given function representing the heating effect from the force outside the nozzle.

In this paper, we apply our flux approximate method introduced in [23] to study the global existence of the entropy solutions for the general nozzle ($A(x) \in L^1$), the general coefficient function of friction ($\alpha(x) \in L^1$) and the arbitrary exponent $\gamma \geq 1$.

We consider the viscosity-flux approximate solutions $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ of the Cauchy problem

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x)(\rho - 2\delta)u + \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = A(x)(\rho - 2\delta)u^2 - \alpha(x)\rho u|u| + \varepsilon(\rho u)_{xx} \end{cases} \tag{1.0.13}$$

with the initial data (1.0.8).

An obvious advantage of this kind of approximation on the flux functions is to obtain the positive lower bound $\rho \geq 2\delta > 0$, directly from the first equation in (1.0.13), which grantees that the term $\rho u^2 = \frac{m^2}{\rho}$ is regular. Moreover, as proved in [23], both systems (1.0.1) and (1.0.13) have the same Riemann invariants and the entropy equation. With the help of these special behaviors of system (1.0.13), for any weak entropy-entropy flux pair $(\eta(\rho,m), q(\rho,m))$ of system (1.0.1) and for a general pressure function $P(\rho)$, we can easily prove that

$$\eta(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})_t + q(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})_x \quad \text{are compact in} \quad H^{-1}_{loc}(R \times R^+),$$

with respect to the viscosity solutions $(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})$, and do not need to introduce the viscous periodic solutions with respect to the spatial variable x to derive the auxiliary estimate (see $(I.53)$ in [16]),

$$\int\int_{K_1} \varepsilon^2 (\rho_x)^2 dxdt \leq C\delta^2$$

for the special pressure $P(\rho) = \frac{1}{\gamma}\rho^\gamma$ and $\gamma > 2$.

Mainly, we have the following theorems.

Theorem 1. *Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma \geq 3$. If there exist a positive constant M and a nonnegative function $\beta(x)$ such that*

$$\theta M|A(x)| + \frac{3}{2}M|\alpha(x)| < \beta(x), \quad \int_{-\infty}^{\infty} \beta(s)ds < \frac{M}{2}, \tag{1.0.14}$$

then we have

$$z(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{(\rho^{\delta,\varepsilon}(x,t))^\theta}{\theta} - u^{\delta,\varepsilon}(x,t) \leq M - \int_{-\infty}^{x} \beta(s)ds \tag{1.0.15}$$

and

$$w(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))=\frac{(\rho^{\delta,\varepsilon}(x,t))^{\theta}}{\theta}+u^{\delta,\varepsilon}(x,t)\le M+\int_{-\infty}^{x}\beta(s)ds \tag{1.0.16}$$

if the initial data

$$z(\rho^{\delta,\varepsilon}(x,0),u^{\delta,\varepsilon}(x,0))<M-\int_{-\infty}^{x}\beta(s)ds \tag{1.0.17}$$

and

$$w(\rho^{\delta,\varepsilon}(x,0),u^{\delta,\varepsilon}(x,0))<M+\int_{-\infty}^{x}\beta(s)ds, \tag{1.0.18}$$

where $\theta=\frac{\gamma-1}{2}$ *and* $(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))$ *are the solutions of the Cauchy problem (1.0.13) and (1.0.18).*

Theorem 2. *Let* $P(\rho)=\frac{1}{\gamma}\rho^{\gamma}, 1<\gamma<3$. *If there exist a positive constant* M *and a nonnegative function* $\beta(x)$ *such that*

$$\theta M|A(x)|<\frac{3-\gamma}{3+\gamma}\beta(x),\qquad \int_{-\infty}^{\infty}\beta(s)ds<\frac{\gamma-1}{4}M,\qquad |\alpha(x)|M<\frac{3-\gamma}{2(\gamma+1)}\beta(x), \tag{1.0.19}$$

then we have the same estimates given in (1.0.15) and (1.0.16), if the initial data satisfy (1.0.17) and (1.0.18).

Theorem 3. *For such functions* $A(x),\alpha(x)$ *and the initial data satisfying the conditions in Theorems 1-2, there exists a subsequence of* $(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))$*, which converges pointwisely to a pair of bounded functions* $(\rho(x,t),u(x,t))$ *as* δ,ε *tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.0.1)-(1.0.2).*

Definition 1. A pair of bounded functions $(\rho(x,t),u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.0.1)-(1.0.2) if

$$\begin{cases}\int_0^{\infty}\int_{-\infty}^{\infty}\rho\phi_t+(\rho u)\phi_x-\frac{a'(x)}{a(x)}(\rho u)\phi dxdt+\int_{-\infty}^{\infty}\rho_0(x)\phi(x,0)dx=0,\\ \int_0^{\infty}\int_{-\infty}^{\infty}\rho u\phi_t+(\rho u^2+P(\rho))\phi_x-(\frac{a'(x)}{a(x)}\rho u^2+\alpha(x)\rho u|u|)\phi dxdt\\ +\int_{-\infty}^{\infty}\rho_0(x)u_0(x)\phi(x,0)dx=0\end{cases} \tag{1.0.20}$$

holds for all test function $\phi\in C_0^1(R\times R^+)$ and

$$\begin{aligned}&\int_0^{\infty}\int_{-\infty}^{\infty}\eta(\rho,m)\phi_t+q(\rho,m)\phi_x-(\frac{a'(x)}{a(x)}\rho u+\alpha(x)\rho u|u|)\eta(\rho,m)_{\rho}\\&-\frac{a'(x)}{a(x)}\rho u^2\eta(\rho,m)_m\phi dxdt\ge 0\end{aligned} \tag{1.0.21}$$

holds for any non-negative test function $\phi \in C_0^\infty(R \times R^+ - \{t=0\})$, where $m = \rho u$ and (η, q) is a pair of convex entropy-entropy flux of system (1.0.1).

Finally, about the isentropic isothermal system, which is corresponding to the case of $\gamma = 1$, we improve the existence result of the bounded entropy solutions, given in [28], where the L^∞ bound depends on the time t, and obtain the following

Theorem 4. *Let $P(\rho) = \rho$ and $a(x) > 0$ be a continuous function in R, $A(x) = -\frac{a'(x)}{a(x)} \in L^1(R)$ and $\alpha(x) \in L^1(R)$. Moreover, if*

$$|A(x)|_{L^1(R)} \le \frac{1}{12}, \quad |\alpha(x)|_{L^1(R)} \le \frac{1}{12} \tag{1.0.22}$$

and the bounded initial data satisfy

$$\begin{cases} \ln(\rho_0(x)a(x)) - u_0(x) < M - 3(|A(x)|_{L^1(R)} + |\alpha(x)|_{L^1(R)}), \\ \ln(\rho_0(x)a(x)) + u_0(x) < M, \end{cases} \tag{1.0.23}$$

where $M > 1$ is a constant, then the Cauchy problem (1.0.1)-(1.0.2) has a bounded weak solution (ρ, u), which has the following uniform bound

$$\begin{cases} \ln(\rho a(x)) - u \le M, \\ \ln(\rho a(x)) + u \le M + 3(|A(x)|_{L^1(R)} + |\alpha(x)|_{L^1(R)}). \end{cases} \tag{1.0.24}$$

The arrangement of this paper is as follows: in Section 2, we introduce our main ideas how to apply the maximum principle coupled with the flux-viscosity approximation to obtain the L^∞ estimates of approximate solutions. In Sections 3-4, we give the details to deduce the inequalities (2.0.12) in four different regions G_i of (x,t): where $G_1 = \{(x,t) : \alpha(x) \ge 0, A(x) \ge 0\}$, $G_2 = \{(x,t) : \alpha(x) \ge 0, A(x) \le 0\}$, $G_3 = \{(x,t) : \alpha(x) \le 0, A(x) \ge 0\}$ and $G_4 = \{(x,t) : \alpha(x) \le 0, A(x) \le 0\}$ respectively. In Section 5, we prove the pointwise convergence of the approximation solutions by using the compactness framework given in [16,17] from the compensated compactness theory and obtain the proof of Theorem 3.

Finally we give the proof of Theorem 4 in Section 6. Since the case of $\gamma = 1$ is different from that of $\gamma > 1$, in this section, we still adopt the technique given in [28] and rewrite (1.0.1) as follows

$$\begin{cases} v_t + (vu)_x = 0 \\ (vu)_t + (vu^2 + v)_x + A(x)v + \alpha(x)vu|u| = 0, \end{cases} \tag{1.0.25}$$

where the new variable $v = \rho a(x)$. By introducing the viscosity parameter $\varepsilon > 0$ and flux-approximation parameter $\delta > 0$ to System (1.0.25), we study the following parabolic system

$$\begin{cases} v_t + ((v - 2\delta)u)_x = \varepsilon v_{xx} \\ (vu)_t + ((v-\delta)u^2 + v - 2\delta \ln v)_x + A^\delta(x)v + \alpha^\delta(x)vu|u| = \varepsilon(vu)_{xx}. \end{cases} \tag{1.0.26}$$

The technique to prove Theorems 1-2 can be similarly applied to obtain the uniformly, time-independent L^∞ estimates (1.0.24) of the viscosity solutions, and the convergence framework from the compensated compactness theory deduces the proof of Theorem 4.

2. Preliminary

In this section, we introduce the main ideas to prove Theorems 1-2.

Multiplying (1.0.13) by $(\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial m})$ and $(\frac{\partial z}{\partial \rho}, \frac{\partial z}{\partial m})$, respectively, where (w, z) are given in (1.0.10), we obtain

$$\begin{aligned} & w_t + \lambda_2^\delta w_x \\ & = \varepsilon w_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x w_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ & + A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \end{aligned} \tag{2.0.1}$$

and

$$\begin{aligned} & z_t + \lambda_1^\delta z_x \\ & = \varepsilon z_{xx} + \tfrac{2\varepsilon}{\rho}\rho_x z_x - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ & + A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u|, \end{aligned} \tag{2.0.2}$$

where $\lambda_1^\delta, \lambda_2^\delta$ are two eigenvalues of (1.0.13)

$$\lambda_1^\delta = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2^\delta = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}. \tag{2.0.3}$$

Letting $z = B(x) + v$ in (2.0.2), where $B(x) = M - \int_{-\infty}^x \beta(s)ds$, we have

$$\begin{aligned} & v_t + (u - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(v_x + B'(x)) - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\ & = \varepsilon v_{xx} + \varepsilon B''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_x + \tfrac{2\varepsilon}{\rho}\rho_x B'(x) - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \end{aligned} \tag{2.0.4}$$

or

$$\begin{aligned} & v_t + (u - \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})v_x - B'(x)(B(x) + v - \textstyle\int_l^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho) \\ & - B'(x)\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - A(x)(\rho - 2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\ & = \varepsilon v_{xx} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \tfrac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x B'(x) + (\tfrac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x))^2] \\ & + \varepsilon B''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_x + \tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 \end{aligned} \tag{2.0.5}$$

or

$$
\begin{aligned}
&v_t + a(x,t)v_x + b(x,t)v + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_1 B(x)B'(x)] \leq \varepsilon v_{xx} \\
&-\textstyle\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) + (1-\varepsilon_1)B(x)B'(x) + B'(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
&+A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u|,
\end{aligned}
\tag{2.0.6}
$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a(x,t) = u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b(x,t) = -B'(x)$.

Similarly, letting $w = C(x) + v_1$ in (2.0.1), where $C(x) = M + \int_{-\infty}^x \beta(s)ds$, we have

$$
\begin{aligned}
&v_{1t} + (u + \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(v_{1x} + C'(x)) - A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= \varepsilon v_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_{1x} + \tfrac{2\varepsilon}{\rho}\rho_x C'(x) - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2
\end{aligned}
\tag{2.0.7}
$$

or

$$
\begin{aligned}
&v_{1t} + (u + \tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})v_{1x} + C'(x)(C(x) + v_1 - \textstyle\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho) \\
&+C'(x)\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= \varepsilon v_{1xx} - \tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \tfrac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x C'(x) + (\tfrac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x))^2] \\
&+\varepsilon C''(x) + \tfrac{2\varepsilon}{\rho}\rho_x v_{1x} + \tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2
\end{aligned}
\tag{2.0.8}
$$

or

$$
\begin{aligned}
&v_{1t} + a_1(x,t)v_{1x} + b_1(x,t)v_1 + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x)] \\
&\leq \varepsilon v_{1xx} + \textstyle\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x) - (1-\varepsilon_1)C(x)C'(x) - C'(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
&+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u|,
\end{aligned}
\tag{2.0.9}
$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a_1(x,t) = u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b_1(x,t) = C'(x)$.

Using the first equation in (1.0.13), we have the a priori estimate $\rho \geq 2\delta$. Since $B(x)$ is strictly positive, we can choose $\beta(x)$ to be smooth enough, $\varepsilon = o(\delta)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of (2.0.6) and (2.0.9)

$$
-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_1 B(x)B'(x) \geq 0 \tag{2.0.10}
$$

and

$$
-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) \geq 0. \tag{2.0.11}
$$

With the help of the special structures of $B(x) = M - \int_{-\infty}^{x}\beta(s)ds, C(x) = M + \int_{-\infty}^{x}\beta(s)ds$, by carefully analyzing the relations among $A(x), \alpha(x)$ and $\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho$, we obtain in different regions $G_i, i = 1, 2, 3, 4$, the following inequalities on v and v_1 respectively

$$\begin{cases} v_t + a(x,t)v_x + b(x,t)v + c(x,t)v_1 \le \varepsilon v_{xx}, \\ v_{1t} + a_1(x,t)v_{1x} + b_1(x,t)v_1 + c_1(x,t)v \le \varepsilon v_{1xx}, \end{cases} \tag{2.0.12}$$

where the coefficient functions $b(x,t), c(x,t), b_1(x,t), c_1(x,t)$ could be different in different regions G_i, but there's always $c(x,t) \le 0, c_1(x,t) \le 0$; and the regions $G_1 = \{(x,t) : \alpha(x) \ge 0, A(x) \ge 0\}$, $G_2 = \{(x,t) : \alpha(x) \ge 0, A(x) \le 0\}$, $G_3 = \{(x,t) : \alpha(x) \le 0, A(x) \ge 0\}$ and $G_4 = \{(x,t) : \alpha(x) \le 0, A(x) \le 0\}$ respectively. So the maximum principle (see [22] or [14] for the details) on nonlinear coupled parabolic inequalities (2.0.12) gives us the estimates $v(x,t) \le 0, v_1(x,t) \le 0$ and the upper bounds of z and w. This could deduce the Proofs of Theorems 1-2.

The details will be given in the following several sections.

3. Proof of Theorem 1: the case of $\gamma \ge 3$

In this section, we shall prove Theorem 1. As introduced in Section 2, we will give the details how to obtain the coupled inequalities in (2.0.12).

When $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma \ge 3$, we choose $l = 2\delta$, then by using

$$\frac{1}{\theta}(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \le \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds \le (\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \quad \text{for} \quad \gamma \ge 3, \tag{3.0.1}$$

we have the following estimates on the terms in (2.0.6) and in (2.0.9) respectively

$$\begin{aligned} L &= -\int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) + (1-\varepsilon_1)B(x)B'(x) \\ &+ B'(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\ &\le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \end{aligned} \tag{3.0.2}$$

and

$$\begin{aligned} L_1 &= \int_l^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x) - (1-\varepsilon_1)C(x)C'(x) - C'(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\ &+ A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\ &\le -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u|. \end{aligned} \tag{3.0.3}$$

3.1. The case of $\alpha(x) \geq 0$; $A(x) \geq 0$

Let $\alpha(x) \geq 0$, $A(x) \geq 0$. First, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x} \beta(s)ds \leq 0$, we have from (3.0.2) that

$$\begin{aligned}
&L \leq (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(w-z) + \tfrac{1}{4}\alpha(x)(w-z)|w-z| \\
&= A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1 - v) + (1-\varepsilon_1)B(x)B'(x) \\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(C(x) - B(x)) \\
&+\tfrac{1}{4}\alpha(x)(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds| \\
&\leq -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds - \frac{1}{4}\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&\leq -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds - \frac{1}{4}\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&= -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(w+z) - \frac{1}{4}\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&= -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(v_1 + v + C(x) + B(x)) - \frac{1}{4}\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&= (\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v_1 \\
&+(\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho} - \frac{1}{4}\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|)v \\
&-((1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) - \theta M A(x)\int_{-\infty}^{x}\beta(s)ds),
\end{aligned} \tag{3.1.1}$$

where

$$\begin{aligned}
&-((1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) - \theta M A(x)\int_{-\infty}^{x}\beta(s)ds) \\
&< -(\tfrac{M}{2}\beta(x) - \theta M A(x)|\beta(x)|_{L^1(R)}) \leq 0,
\end{aligned} \tag{3.1.2}$$

for a small ε_1, due to the conditions $|\beta(x)|_{L^1(R)} < \frac{M}{2}$ and $\theta M A(x) < \beta(x)$ in Theorem 1. Therefore we obtain the following inequality from (2.0.6), (2.0.10), (3.0.2), (3.1.1) and (3.1.2)

$$v_t + a(x,t)v_x + l_1(x,t)v + l_2(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.1.3}$$

where $l_1(x,t), l_2(x,t) \le 0$ are suitable functions.

Second, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \ge 0$,

$$\begin{aligned}
&L \le A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1-v) + (1-\varepsilon_1)B(x)B'(x)\\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(C(x)-B(x))\\
&+\tfrac{1}{4}\alpha(x)(v_1-v+2\int_{-\infty}^{x}\beta(s)ds)|v_1-v+2\int_{-\infty}^{x}\beta(s)ds|\\
&\le A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1-v) - (1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^{x}\beta(s)ds)\\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_{-\infty}^{x}\beta(s)ds - \tfrac{1}{4}\alpha(x)|v_1-v+2\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1+2\int_{-\infty}^{x}\beta(s)ds)(v_1+2\int_{-\infty}^{x}\beta(s)ds+|v|)\\
&\le l_3(x,t)v + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}v_1 - (1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^{x}\beta(s)ds)\\
&+\theta A(x)\int_{-\infty}^{x}\beta(s)ds\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds + \tfrac{1}{4}\alpha(x)(v_1+2\int_{-\infty}^{x}\beta(s)ds)^2\\
&= l_3(x,t)v + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}v_1 - (1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{1}{2}\theta A(x)\int_{-\infty}^{x}\beta(s)ds(v+v_1+2M) + \alpha(x)(\tfrac{1}{4}v_1+\int_{-\infty}^{x}\beta(s)ds)v_1 + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&= (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho} + \tfrac{1}{2}\theta A(x)\int_{-\infty}^{x}\beta(s)ds + \tfrac{1}{4}\alpha(x)v_1 + \alpha(x)\int_{-\infty}^{x}\beta(s)ds)v_1\\
&+l_4(x,t)v - (1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^{x}\beta(s)ds) + \theta MA(x)\int_{-\infty}^{x}\beta(s)ds + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2,
\end{aligned} \tag{3.1.4}$$

where $l_3(x,t), l_4(x,t)$ are suitable functions. Since $v_1 \ge -2\int_{-\infty}^{x}\beta(s)ds$, we know that the coefficient before v_1 is nonnegative and

$$\begin{aligned}
&-(1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^{x}\beta(s)ds) + \theta MA(x)\int_{-\infty}^{x}\beta(s)ds + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&< -\frac{M}{2}\beta(x) + \theta MA(x)|\beta(x)|_{L^1(R)} + \tfrac{1}{2}M\alpha(x)|\beta(x)|_{L^1(R)} \le 0
\end{aligned} \tag{3.1.5}$$

due to the conditions

$$|\beta(x)|_{L^1(R)} < \frac{M}{2}, \quad \theta MA(x) + \tfrac{1}{2}M\alpha(x) < \beta(x) \tag{3.1.6}$$

in Theorem 1. Therefore we also obtain the inequality from (2.0.6), (2.0.10), (3.0.2), (3.1.4) and (3.1.5)

$$v_t + a(x,t)v_x + l_5(x,t)v + l_6(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.1.7}$$

where $l_5(x,t), l_6(x,t) \le 0$ are suitable functions.

Similarly, at the points (x,t), where $\alpha(x) \geq 0$ and $A(x) \geq 0$, we have

$$
\begin{aligned}
&L_1 \leq -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\
&= -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(w - \textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho) - \alpha(x)|u|\frac{1}{2}(w-z) \\
&\leq -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1 + C(x)) \\
&-\tfrac{1}{2}\alpha(x)|u|(v_1 - v + C(x) - B(x)) \\
&\leq -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 \\
&+\theta A(x)C(x)\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho - \frac{1}{2}\alpha(x)|u|(v_1 - v) \\
&= -(1-\varepsilon_1)C(x)\beta(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 \\
&+\tfrac{1}{2}\theta A(x)C(x)(w+z) - \tfrac{1}{2}\alpha(x)|u|(v_1 - v) \\
&= -(1-\varepsilon_1)C(x)\beta(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 \\
&+\tfrac{1}{2}\theta A(x)C(x)(v + v_1 + C(x) + B(x)) - \tfrac{1}{2}\alpha(x)|u|(v_1 - v) \\
&= (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac{1}{2}\theta A(x)C(x) - \tfrac{1}{2}\alpha(x)|u|)v_1 \\
&+(\tfrac{1}{2}\theta A(x)C(x) + \tfrac{1}{2}\alpha(x)|u|)v - ((1-\varepsilon_1)\beta(x) - \theta M A(x))C(x) \\
&\leq (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac{1}{2}\theta A(x)C(x) - \tfrac{1}{2}\alpha(x)|u|)v_1 \\
&+(\tfrac{1}{2}\theta A(x)C(x) + \tfrac{1}{2}\alpha(x)|u|)v
\end{aligned}
\tag{3.1.8}
$$

due to $\theta M A(x) < \beta(x)$. So, we have the following inequality from (2.0.9), (2.0.11), (3.0.3) and (3.1.8)

$$v_{1t} + a_1(x,t)v_{1x} + l_7(x,t)v + l_8(x,t)v_1 \leq \varepsilon v_{1xx}, \tag{3.1.9}$$

where $l_7(x,t) \leq 0$, $l_8(x,t)$ are suitable functions.

3.2. *The case of* $\alpha(x) \geq 0$; $A(x) \leq 0$

Let $\alpha(x) \geq 0$, $A(x) \leq 0$. First, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \geq 0$, we have from (3.0.2) that

$$
\begin{aligned}
&L \le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}\\
&+\tfrac{1}{4}\alpha(x)(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|\\
&\le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}(\textstyle\int_{2\delta}^{\rho}\tfrac{\sqrt{P'(s)}}{s}ds - z)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}(v + B(x))\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}v - \theta A(x)B(x)\textstyle\int_{2\delta}^{\rho}\tfrac{\sqrt{P'(s)}}{s}ds\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&= (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}v - \tfrac{1}{2}\theta A(x)B(x)(w+z)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&= -(1-\varepsilon_1)\beta(x)B(x) - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}v - \tfrac{1}{2}\theta A(x)B(x)(v + v_1 + 2M)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v + \tfrac{1}{4}\alpha(x)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds)|v|\\
&+(\tfrac{1}{4}v_1 + \textstyle\int_{-\infty}^{x}\beta(s)ds)\alpha(x)v_1 + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&= -l_9(x,t)v - l_{10}(x,t)v_1 - ((1-\varepsilon_1)\beta(x) + \theta MA(x))B(x) + \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2
\end{aligned}
\tag{3.2.1}
$$

where

$$
\begin{aligned}
l_{10}(x,t) &= \tfrac{1}{2}\theta A(x)B(x) - (\tfrac{1}{4}v_1 + \textstyle\int_{-\infty}^{x}\beta(s)ds)\alpha(x)\\
&= \tfrac{1}{2}\theta A(x)B(x) - \tfrac{1}{4}(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)\alpha(x) - \tfrac{1}{2}\alpha(x)\int_{-\infty}^{x}\beta(s)ds \le 0
\end{aligned}
\tag{3.2.2}
$$

and

$$
\begin{aligned}
&-((1-\varepsilon_1)\beta(x) + \theta MA(x))B(x) + \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&\le -((1-\varepsilon_1)\beta(x) + \theta MA(x))(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{1}{2}M\alpha(x)|\beta(x)|_{L^1(R)}\\
&\le -\tfrac{M}{2}((1-\varepsilon_1)\beta(x) + \theta MA(x) + \tfrac{1}{2}M\alpha(x)) \le 0
\end{aligned}
\tag{3.2.3}
$$

due to the conditions

$$
|\beta(x)|_{L^1(R)} < \frac{M}{2}, \quad \theta M|A(x)| + \frac{1}{2}M\alpha(x) < \beta(x).
$$

Therefore we have

$$v_t + a(x,t)v_x + l_9(x,t)v + l_{10}(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.2.4}$$

where $l_9(x,t), l_{10}(x,t) \le 0$ are suitable functions.

Second, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \le 0$, we have from (3.0.2) that

$$\begin{aligned}
&L \le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds - z) + \tfrac14\alpha(x)(w-z)|w-z| \\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v+B(x)) + \tfrac14\alpha(x)(w-z)|w-z| \\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \theta A(x)B(x)\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds \\
&+\tfrac14\alpha(x)(v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds| \\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \tfrac12\theta A(x)B(x)(w+z) \\
&-\tfrac14\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&= (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \tfrac12\theta A(x)B(x)(v+v_1+2M) \\
&-\tfrac14\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&= -(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac12\theta A(x)B(x))v \\
&-\tfrac12\theta A(x)B(x)v_1 - ((1-\varepsilon_1)\beta(x) + \theta M A(x))B(x) \\
&-\tfrac14\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v \\
&\le -(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac12\theta A(x)B(x))v \\
&-\tfrac14\alpha(x)|v_1 - v + 2\int_{-\infty}^{x}\beta(s)ds|v - \tfrac12\theta A(x)B(x)v_1.
\end{aligned} \tag{3.2.5}$$

Therefore we have also a similar inequality

$$v_t + a(x,t)v_x + l_{11}(x,t)v + l_{12}(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.2.6}$$

where $l_{11}(x,t), l_{12}(x,t) \le 0$ are suitable functions.

Similarly, at the points (x,t), where $\alpha(x) \ge 0$ and $A(x) \le 0$, we have from (3.0.3) that

$$\begin{aligned}
L_1 &\le A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\
&= (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)|u|)\tfrac12(w-z) \\
&= \tfrac12(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)|u|)(v_1 - v + C(x) - B(x)) \\
&\le \tfrac12(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)|u|)(v_1 - v).
\end{aligned} \tag{3.2.7}$$

So, we have the following inequality from (2.0.9), (2.0.11), (3.0.3) and (3.2.7)

$$v_{1t} + a_1(x,t)v_{1x} + l_{13}(x,t)v + l_{14}(x,t)v_1 \le \varepsilon v_{1xx}, \tag{3.2.8}$$

where $l_{13}(x,t) \le 0, l_{14}(x,t)$ are suitable functions.

3.3. The case of $\alpha(x) \le 0$; $A(x) \ge 0$

Let $\alpha(x) \le 0$, $A(x) \ge 0$. First, at the points (x,t), where $B(x) \le \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s}ds$, we have from (3.0.2) that

$$\begin{aligned}
&L \le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(w-z) \\
&-\alpha(x)(z - \textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|z - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds| \\
&\le A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v_1-v) + (1-\varepsilon_1)B(x)B'(x) \\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(C(x)-B(x)) \\
&-\alpha(x)|B(x)+v-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&\le -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(w+z) - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= -A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}(v-v_1) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds(v_1+v+C(x)+B(x)) - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= (\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v_1 \\
&+(\tfrac{1}{2}\theta A(x)\textstyle\int_{-\infty}^{x}\beta(s)ds - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho} - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|)v \\
&-((1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) - \theta M A(x)\int_{-\infty}^{x}\beta(s)ds),
\end{aligned} \tag{3.3.1}$$

where

$$\begin{aligned}&(1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds)-\theta MA(x)\int_{-\infty}^x\beta(s)ds\\&>\tfrac{M}{2}\beta(x)-\theta MA(x)|\beta(x)|_{L^1(R)}\geq 0\end{aligned}\tag{3.3.2}$$

due to the conditions $|\beta(x)|_{L^1(R)}<\frac{M}{2}$ and $\theta MA(x)<\beta(x)$ in Theorem 1 and

$$\tfrac{1}{2}\theta A(x)\int_{-\infty}^x\beta(s)ds+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho}\geq 0.\tag{3.3.3}$$

Thus, we have from (3.3.1)-(3.3.3) that

$$v_t+a(x,t)v_x+l_{15}(x,t)v+l_{16}(x,t)v_1\leq\varepsilon v_{xx},\tag{3.3.4}$$

where $l_{15}(x,t),l_{16}(x,t)\leq 0$ are suitable functions.

Second, at the points (x,t), where $B(x)\geq\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds$, repeating the calculations of the part $A(x)$ in (3.3.1), we have

$$\begin{aligned}&L\leq(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}\\&-\alpha(x)(B(x)+v-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds)|B(x)+v-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds|\\&=(\tfrac{1}{2}\theta A(x)\int_{-\infty}^x\beta(s)ds+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v_1\\&+(\tfrac{1}{2}\theta A(x)\int_{-\infty}^x\beta(s)ds-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v\\&-((1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds)-\theta MA(x)\int_{-\infty}^x\beta(s)ds)\\&-\alpha(x)|B(x)+v-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds|v\\&-\alpha(x)(B(x)-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds)|B(x)+v-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds|\\&\leq(\tfrac{1}{2}\theta A(x)\int_{-\infty}^x\beta(s)ds+A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v_1\\&+(\tfrac{1}{2}\theta A(x)\int_{-\infty}^x\beta(s)ds-A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{2\rho})v\\&-((1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds)-\theta MA(x)\int_{-\infty}^x\beta(s)ds)\\&-\alpha(x)|B(x)+v-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds|v-\alpha(x)(B(x)-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds)|v|\\&-\alpha(x)(B(x)-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds)^2,\end{aligned}\tag{3.3.5}$$

where

$$\begin{aligned}&-((1-\varepsilon_1)\beta(x)(M-\int_{-\infty}^x\beta(s)ds)-\theta MA(x)\int_{-\infty}^x\beta(s)ds)\\&-\alpha(x)(B(x)-\int_{2\delta}^\rho\frac{\sqrt{P'(s)}}{s}ds)^2\\&\leq[-(1-\varepsilon_1)\beta(x)+\theta MA(x)-\alpha(x)B(x)]B(x)\\&\leq[-(1-\varepsilon_1)\beta(x)+\theta MA(x)-\tfrac{1}{2}\alpha(x)M]B(x)\leq 0\end{aligned}\tag{3.3.6}$$

because

$$2\int_{-\infty}^{\infty}\beta(s)ds < M \tag{3.3.7}$$

or

$$\int_{-\infty}^{x}\beta(s)ds < M - \int_{-\infty}^{x}\beta(s)ds \tag{3.3.8}$$

and

$$\theta MA(x) - \frac{1}{2}\alpha(x)M < \beta(s)ds \tag{3.3.9}$$

given in Theorem 1. Thus we have from (3.3.5) and (3.3.6) that

$$v_t + a(x,t)v_x + l_{17}(x,t)v + l_{18}(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.3.10}$$

where $l_{17}(x,t), l_{18}(x,t) \le 0$ are suitable functions.

Similarly, at the points (x,t), where $\alpha(x) \le 0$, $A(x) \ge 0$ and $C(x) \le \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds$, we have

$$\begin{aligned}
&L_1 \le -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\
&= -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(w - \int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho) \\
&-\alpha(x)(w - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|w - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds| \\
&\le -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1 + C(x)) \\
&-\alpha(x)(C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds| \\
&\le -(1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 + \theta A(x)C(x)\int_{2\delta}^{\rho}\frac{\sqrt{P'(\rho)}}{\rho}d\rho \\
&-\alpha(x)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v_1 \\
&= -(1-\varepsilon_1)C(x)\beta(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 + \frac{1}{2}\theta A(x)C(x)(w+z) \\
&-\alpha(x)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v_1 \\
&= -(1-\varepsilon_1)C(x)\beta(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v_1 \\
&+\frac{1}{2}\theta A(x)C(x)(v + v_1 + C(x) + B(x)) \\
&-\alpha(x)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v_1 \\
&= (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \frac{1}{2}\theta A(x)C(x) - \alpha(x)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|)v_1 \\
&+\frac{1}{2}\theta A(x)C(x)v - ((1-\varepsilon_1)\beta(x) - \theta MA(x))C(x) \\
&\le (A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \frac{1}{2}\theta A(x)C(x) - \alpha(x)|C(x) + v_1 - \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|)v_1 \\
&+\frac{1}{2}\theta A(x)C(x)v
\end{aligned} \tag{3.3.11}$$

because $\theta MA(x) < (1-\varepsilon_1)\beta(x)$ for a suitable small ε_1. Thus we have from (3.3.11) that

$$v_{1t} + a_1(x,t)v_{1x} + l_{19}(x,t)v + l_{20}(x,t)v_1 \le \varepsilon v_{1xx}, \tag{3.3.12}$$

where $l_{19}(x,t) \le 0, l_{20}(x,t)$ are suitable functions.

Finally, at the points (x,t), where $\alpha(x) \le 0$, $A(x) \ge 0$ and $C(x) \ge \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds$, we have from the calculations on the part of $A(x)$ in (3.3.11) that

$$\begin{aligned}
&L_1 \le (A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac{1}{2}\theta A(x)C(x))v_1 \\
&+\tfrac{1}{2}\theta A(x)C(x)v - ((1-\varepsilon_1)\beta(x) - \theta MA(x))C(x) \\
&-\alpha(x)(C(x) + v_1 - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds)|C(x) + v_1 - \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds| \\
&\le (A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \tfrac{1}{2}\theta A(x)C(x))v_1 \\
&+\tfrac{1}{2}\theta A(x)C(x)v - ((1-\varepsilon_1)\beta(x) - \theta MA(x))C(x) \\
&-\alpha(x)|C(x) + v_1 - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds|v_1 - \alpha(x)(C(x) - \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds)|v_1| \\
&-\alpha(x)(C(x) - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds)^2,
\end{aligned} \tag{3.3.13}$$

where

$$\begin{aligned}
&-((1-\varepsilon_1)\beta(x) - \theta MA(x))C(x) - \alpha(x)(C(x) - \textstyle\int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds)^2 \\
&\le -((1-\varepsilon_1)\beta(x) - \theta MA(x))C(x) - \alpha(x)C(x)^2 \\
&= [-(1-\varepsilon_1)\beta(x) + \theta MA(x) - \alpha(x)C(x)]C(x) \\
&\le [-(1-\varepsilon_1)\beta(x) + \theta MA(x) - \tfrac{3}{2}M\alpha(x)]C(x) \le 0
\end{aligned} \tag{3.3.14}$$

because

$$|\beta(x)|_{L^1(R)} < \frac{M}{2}, \quad \theta MA(x) - \frac{3}{2}M\alpha(x) < \beta(x). \tag{3.3.15}$$

Thus we have also an inequality

$$v_{1t} + a_1(x,t)v_{1x} + l_{21}(x,t)v + l_{22}(x,t)v_1 \le \varepsilon v_{1xx}, \tag{3.3.16}$$

where $l_{21}(x,t) \le 0, l_{22}(x,t)$ are suitable functions.

3.4. The case of $\alpha(x) \le 0$; $A(x) \le 0$

Let $\alpha(x) \le 0$, $A(x) \le 0$. First, at the points (x,t), where $B(x) \le \int_{2\delta}^{\rho} \frac{\sqrt{P'(s)}}{s} ds$, we have from (3.0.2) that

$$
\begin{aligned}
L &\le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds - z) \\
&-\alpha(x)(B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds| \\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v+B(x)) \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&\le (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \theta A(x)B(x)\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \frac{1}{2}\theta A(x)B(x)(w+z) \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= (1-\varepsilon_1)B(x)B'(x) - A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v - \frac{1}{2}\theta A(x)B(x)(v+v_1+2M) \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&= -(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \frac{1}{2}\theta A(x)B(x))v \\
&-\frac{1}{2}\theta A(x)B(x)v_1 - ((1-\varepsilon_1)\beta(x)+\theta M A(x))B(x) \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v \\
&\le -(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \frac{1}{2}\theta A(x)B(x))v \\
&-\frac{1}{2}\theta A(x)B(x)v_1 - \alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v,
\end{aligned} \tag{3.4.1}
$$

where $-\frac{1}{2}\theta A(x)B(x) \ge 0$. Thus we obtain

$$
v_t + a(x,t)v_x + l_{23}(x,t)v + l_{24}(x,t)v_1 \le \varepsilon v_{xx}, \tag{3.4.2}
$$

where $l_{23}(x,t), l_{24}(x,t) \le 0$ are suitable functions.

Second, at the points (x,t), where $B(x) \ge \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds$, we have

$$
\begin{aligned}
L &\le (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho} \\
&-\alpha(x)(B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds| \\
&\le -(A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} + \frac{1}{2}\theta A(x)B(x))v \\
&-\frac{1}{2}\theta A(x)B(x)v_1 - ((1-\varepsilon_1)\beta(x)+\theta M A(x))B(x) \\
&-\alpha(x)|B(x)+v-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v - \alpha(x)(B(x)-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|v| \\
&-\alpha(x)(B(x)-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)^2,
\end{aligned} \tag{3.4.3}
$$

where

$$\begin{aligned}
&-((1-\varepsilon_1)\beta(x)+\theta MA(x))B(x)-\alpha(x)(B(x)-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)^2\\
&\le -((1-\varepsilon_1)\beta(x)+\theta MA(x))B(x)-\alpha(x)B(x)^2\\
&\le [-((1-\varepsilon_1)\beta(x)+\theta MA(x)-\tfrac{1}{2}M\alpha(x)]B(x)\le 0.
\end{aligned}\tag{3.4.4}$$

Thus we obtain the following inequality

$$v_t+a(x,t)v_x+l_{25}(x,t)v+l_{26}(x,t)v_1\le \varepsilon v_{xx},\tag{3.4.5}$$

where $l_{25}(x,t), l_{26}(x,t)\le 0$ are suitable functions.

Similarly, at the points (x,t), where $\alpha(x)\le 0$, $A(x)\le 0$ and $C(x)\le \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds$, we have

$$\begin{aligned}
&L_1\le -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}-\alpha(x)u|u|\\
&=-(1-\varepsilon_1)C(x)C'(x)+\tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(w-z)\\
&-\alpha(x)(w-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|w-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|\\
&=-(1-\varepsilon_1)C(x)C'(x)+\tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1-v+C(x)-B(x))\\
&-\alpha(x)(C(x)+v_1-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|C(x)+v_1-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|\\
&\le \tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1-v)-\alpha(x)|C(x)+v_1-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|v_1.
\end{aligned}\tag{3.4.6}$$

Thus we have

$$v_{1t}+a_1(x,t)v_{1x}+l_{27}(x,t)v+l_{28}(x,t)v_1\le \varepsilon v_{1xx},\tag{3.4.7}$$

where $l_{27}(x,t)\le 0, l_{28}(x,t)$ are suitable functions.

Finally, at the points (x,t), where $\alpha(x)\le 0$, $A(x)\le 0$ and $C(x)\ge \int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds$, we have from (3.4.6) that

$$\begin{aligned}
&L_1\le -(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}-\alpha(x)u|u|\\
&=-(1-\varepsilon_1)C(x)C'(x)+\tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(w-z)\\
&-\alpha(x)(w-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|w-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|\\
&=-(1-\varepsilon_1)C(x)C'(x)+\tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}(v_1-v+C(x)-B(x))\\
&-\alpha(x)(C(x)+v_1-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)|C(x)+v_1-\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds|\\
&\le -(1-\varepsilon_1)C(x)C'(x)-\tfrac{1}{2}A(x)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}v-l_{29}(x,t)v_1\\
&-\alpha(x)(C(x)-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)^2,
\end{aligned}\tag{3.4.8}$$

where

$$\begin{aligned}
&-(1-\varepsilon_1)C(x)C'(x)-\alpha(x)(C(x)-\textstyle\int_{2\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds)^2\\
&\leq -(1-\varepsilon_1)\beta(x)C(x)-\alpha(x)C(x)^2\\
&=(-(1-\varepsilon_1)\beta(x)-\alpha(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds))C(x)\\
&\leq(-(1-\varepsilon_1)\beta(x)-\tfrac{3}{2}M\alpha(x))C(x)\leq 0.
\end{aligned}\tag{3.4.9}$$

Thus we have

$$v_{1t}+a_1(x,t)v_{1x}+l_{29}(x,t)v_1+l_{30}(x,t)v\leq\varepsilon v_{1xx},\tag{3.4.10}$$

where $l_{29}(x,t), l_{30}(x,t)\leq 0$ are suitable functions, which completes the Proof of Theorem 1.

4. Proof of Theorem 2: the case of $1<\gamma<3$

In this section, we shall prove Theorem 2. As introduced in Section 2, we will only give the details how to obtain the coupled inequalities in (2.0.12).

When $P(\rho)=\frac{1}{\gamma}\rho^\gamma$, $1<\gamma<3$, we choose $l=0$, then the Riemann invariants of (1.0.3) are

$$z(\rho,u)=\frac{1}{\theta}\rho^\theta-u,\quad w(\rho,u)=\frac{1}{\theta}\rho^\theta+u.\tag{4.0.1}$$

We may rewrite (2.0.6) and (2.0.9) as follows

$$\begin{aligned}
&v_t+a(x,t)v_x+b(x,t)v\\
&+[-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2-\varepsilon B''(x)-\varepsilon_1B(x)B'(x)+2\delta\tfrac{\sqrt{P'(\rho)}}{\rho}B'(x)]\\
&\leq\varepsilon v_{xx}-\textstyle\int_0^\rho\frac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x)+(1-\varepsilon_1)B(x)B'(x)+B'(x)\sqrt{P'(\rho)}\\
&+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}+\alpha(x)u|u|\\
&=\varepsilon v_{xx}+\tfrac{\gamma-3}{\gamma-1}\rho^\theta B'(x)+(1-\varepsilon_1)B(x)B'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}+\alpha(x)u|u|
\end{aligned}\tag{4.0.2}$$

and

$$\begin{aligned}
&v_{1t}+a_1(x,t)v_{1x}+b_1(x,t)v_1\\
&+[-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2-\varepsilon C''(x)+\varepsilon_1C(x)C'(x)-2\delta\tfrac{\sqrt{P'(\rho)}}{\rho}C'(x)]\\
&\leq\varepsilon v_{1xx}+\textstyle\int_0^\rho\frac{\sqrt{P'(\rho)}}{\rho}d\rho C'(x)-(1-\varepsilon_1)C(x)C'(x)\\
&-C'(x)\sqrt{P'(\rho)}+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}-\alpha(x)u|u|\\
&=\varepsilon v_{1xx}-\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x)-(1-\varepsilon_1)C(x)C'(x)+A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}-\alpha(x)u|u|.
\end{aligned}\tag{4.0.3}$$

Since

$$2\delta\frac{\sqrt{P'(\rho)}}{\rho} = 2\delta\rho^{\frac{\gamma-3}{2}} \le (2\delta)^{\frac{\gamma-1}{2}}, \tag{4.0.4}$$

and $B(x)$ is strictly positive, we may choose $\beta(x)$ to be sufficiently smooth, $\varepsilon = o(\delta)$ and suitable relation between ε and ε_1 such that the following terms on the left-hand side of (4.0.2) and (4.0.3)

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_1 B(x)B'(x) + 2\delta\frac{\sqrt{P'(\rho)}}{\rho}B'(x) \ge 0, \tag{4.0.5}$$

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}C'(x)^2 - \varepsilon C''(x) + \varepsilon_1 C(x)C'(x) - 2\delta\frac{\sqrt{P'(\rho)}}{\rho}C'(x) \ge 0. \tag{4.0.6}$$

Let the terms on the right-hand side of (4.0.2) and (4.0.3) be

$$K = \tfrac{\gamma-3}{\gamma-1}\rho^\theta B'(x) + (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \tag{4.0.7}$$

and

$$K_1 = -\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x) - (1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u|. \tag{4.0.8}$$

By simple calculations,

$$\begin{aligned}
K &= \tfrac{\gamma-3}{\gamma-1}\rho^\theta B'(x) + (1-\varepsilon_1)B(x)B'(x) + A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} + \alpha(x)u|u| \\
&= \tfrac{\gamma-3}{4}(w+z)B'(x) + (1-\varepsilon_1)B(x)B'(x) \\
&+ A(x)\tfrac{\rho-2\delta}{\rho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2} + \tfrac{1}{4}\alpha(x)(w-z)|w-z| \\
&= \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^x\beta(s)ds) \\
&+ \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds)(v+v_1+2M) \\
&+ \tfrac{1}{4}\alpha(x)(v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds)|v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds|
\end{aligned} \tag{4.0.9}$$

and

$$\begin{aligned}
K_1 &= -\tfrac{\gamma-3}{\gamma-1}\rho^\theta C'(x) - (1-\varepsilon_1)C(x)C'(x) + A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho} - \alpha(x)u|u| \\
&= -\tfrac{\gamma-3}{4}(w+z)C'(x) - (1-\varepsilon_1)C(x)C'(x) \\
&+ A(x)\tfrac{\rho-2\delta}{\rho}\tfrac{w-z}{2}\theta\tfrac{w+z}{2} - \tfrac{1}{4}\alpha(x)(w-z)|w-z| \\
&= \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M + \textstyle\int_{-\infty}^x\beta(s)ds) \\
&+ \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds)(v+v_1+2M) \\
&- \tfrac{1}{4}\alpha(x)(v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds)|v_1 - v + 2\textstyle\int_{-\infty}^x\beta(s)ds|.
\end{aligned} \tag{4.0.10}$$

We shall analyze the functions K and K_1 point by point in the following several subsections.

4.1. The case of $\alpha(x) \geq 0$; $A(x) \geq 0$

Let $\alpha(x) \geq 0$, $A(x) \geq 0$. First, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x} \beta(s)ds \leq 0$, we have from (4.0.9) that

$$\begin{aligned}
&K \leq \tfrac{3-\gamma}{4}(v + v_1 + 2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v + v_1 + 2M) \\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v \\
&= (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - v - 2M))v \\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v \\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))v_1 + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2 \\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds.
\end{aligned} \tag{4.1.1}$$

Since

$$\begin{aligned}
v_1 &= w - C(x) = \tfrac{1}{\theta}\rho^\theta + u - C(x) = \tfrac{2}{\theta}\rho^\theta - z - C(x) \\
&= \tfrac{2}{\theta}\rho^\theta - v - B(x) - C(x) = \tfrac{2}{\theta}\rho^\theta - v - 2M,
\end{aligned} \tag{4.1.2}$$

we have from (4.1.1) that

$$\begin{aligned}
&K \leq (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - v - 2M))v \\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1 v \\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds)v_1 + \tfrac{\gamma-1}{4\theta}A(x)\tfrac{\rho-2\delta}{\rho}\rho^\theta v_1 \\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds
\end{aligned} \tag{4.1.3}$$

where the coefficient before v_1

$$\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds + \tfrac{\gamma-1}{4\theta}A(x)\tfrac{\rho-2\delta}{\rho}\rho^\theta \geq 0 \tag{4.1.4}$$

and

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds \\
&\leq (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + (1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds + \textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) \\
&= \beta(x)((\tfrac{1-\gamma}{2} + \varepsilon_1)M + (2-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds) \leq 0
\end{aligned} \tag{4.1.5}$$

because $|\theta MA(x)| < \frac{(3-\gamma)}{\gamma+3}\beta(x) < \beta(x)$ and $2\int_{-\infty}^{\infty}\beta(s)ds < \frac{\gamma-1}{2}M$ as the conditions given in Theorem 2. Thus we have from (4.1.1)-(4.1.5) that

$$K \leq n_1(x,t)v + n_2(x,t)v_1, \tag{4.1.6}$$

where $n_1(x,t), n_2(x,t) \geq 0$ are two suitable functions.

Second, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \geq 0$, we have from (4.0.9) and the calculations on the part of $A(x)$ in (4.1.1)-(4.1.3) that

$$\begin{aligned}
&K \leq \tfrac{3-\gamma}{4}(v + v_1 + 2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v + v_1 + 2M)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&= n_3(x,t)v + \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))v_1 + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds\\
&= (n_3(x,t) - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1)v + \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds)v_1 + \tfrac{\gamma-1}{4\theta}A(x)\tfrac{\rho-2\delta}{\rho}\rho^{\theta}v_1\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds
\end{aligned} \tag{4.1.7}$$

for a suitable function $n_3(x,t)$. Furthermore,

$$\begin{aligned}
&\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&= \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)v_1 + \tfrac{1}{2}\alpha(x)\int_{-\infty}^{x}\beta(s)ds v_1 + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2,
\end{aligned} \tag{4.1.8}$$

where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \geq 0$, and

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&\leq (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + (1-\varepsilon_1)\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x)\\
&+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds\tfrac{\gamma-1}{4}M + \tfrac{1}{2}\int_{-\infty}^{x}\beta(s)ds\beta(x)\\
&\leq (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + 2\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) \leq 0
\end{aligned} \tag{4.1.9}$$

due to

$$\begin{aligned}&\tfrac{\gamma-1}{4}\alpha(x)M < \tfrac{(3-\gamma)(\gamma-1)}{8(\gamma+1)}\beta(x) < \tfrac{1}{2}\beta(x),\\ &\theta M|A(x)| < \tfrac{3-\gamma}{\gamma+3}\beta(x) < \tfrac{1}{2}\beta(x), \quad |\beta(x)|_{L^1(R)} < \tfrac{\gamma-1}{4}M.\end{aligned} \tag{4.1.10}$$

Thus we have also from (4.1.7)-(4.1.9) that

$$K \le n_4(x,t)v + n_5(x,t)v_1, \tag{4.1.11}$$

where $n_4(x,t), n_5(x,t) \ge 0$ are two suitable functions.

Similarly, at the points (x,t), where $\alpha(x) \ge 0$, $A(x) \ge 0$, we have from (4.0.10) that

$$\begin{aligned}&K_1 \le \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\ &+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v+v_1+2M)\\ &-\tfrac{1}{2}\alpha(x)|u|(v_1 - v + C(x) - B(x))\\ &\le (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + v_1 + 2M))v_1\\ &-\tfrac{1}{2}\alpha(x)|u|(v_1 - v)\\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - 2M))v - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v^2\\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds,\end{aligned} \tag{4.1.12}$$

where the coefficient before v

$$\begin{aligned}&\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - 2M) + \tfrac{1}{2}\alpha(x)|u|\\ &\ge \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma-1}{4}\theta M|A(x)| \ge 0\end{aligned} \tag{4.1.13}$$

because $|\theta MA(x)| < \frac{(3-\gamma)}{\gamma+3}\beta(x)$;

$$-\frac{\gamma-1}{8}A(x)\frac{\rho-2\delta}{\rho}v^2 \le 0 \tag{4.1.14}$$

and

$$\begin{aligned}&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\ &+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds\\ &\le (\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x) - (1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds + \tfrac{1}{2}\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) \le 0\end{aligned} \tag{4.1.15}$$

because

$$\theta MA(x) < \frac{1}{2}\beta(x), \quad |\beta(x)|_{L^1(R)} < \frac{\gamma-1}{2}M. \tag{4.1.16}$$

Thus we have from (4.1.12)-(4.1.15) that

$$K_1 \le n_6(x,t)v + n_7(x,t)v_1, \tag{4.1.17}$$

where $n_6(x,t) \ge 0, n_7(x,t)$ are two suitable functions.

Summing up the analysis above, at any point (x,t), where $\alpha(x) \ge 0$, $A(x) \ge 0$, we obtain the inequalities in (2.0.12).

4.2. The case of $\alpha(x) \ge 0$; $A(x) \le 0$

Let $\alpha(x) \ge 0$, $A(x) \le 0$. First, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \le 0$, we have from (4.0.9) that

$$\begin{aligned}
&K \le \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v+v_1+2M)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&= (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - v - 2M))v \qquad (4.2.1)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))v_1 + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds,
\end{aligned}$$

where the coefficient before v_1

$$\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M)\\
&\ge \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma-1}{8}|A(x)|\tfrac{\rho-2\delta}{\rho}(\tfrac{\gamma-1}{2}M + 2M) \qquad (4.2.2)\\
&= \tfrac{3-\gamma}{4}\beta(x) - \tfrac{\gamma+3}{8}\theta M|A(x)| \ge 0
\end{aligned}$$

because $|\theta MA(x)| < \frac{(3-\gamma)}{\gamma+3}\beta(x)$;

$$\frac{\gamma-1}{8}A(x)\frac{\rho-2\delta}{\rho}v_1^2 \le 0 \tag{4.2.3}$$

and

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds \qquad (4.2.4)\\
&\le (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + (1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds \le 0
\end{aligned}$$

due to $\int_{-\infty}^{\infty}\beta(s)ds < \frac{\gamma-1}{4}M$ as given in Theorem 2. Thus we have from (4.2.1)-(4.2.4) that

$$K \le n_8(x,t)v + n_9(x,t)v_1, \tag{4.2.5}$$

where $n_8(x,t), n_9(x,t) \ge 0$ are two suitable functions.

Second, the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \ge 0$, we have from (4.0.9) that

$$\begin{aligned}
&K \le \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M-\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v+v_1+2M)\\
&-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v\\
&+\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds + |v|)\\
&= n_{10}(x,t)v + \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)v_1\\
&+\tfrac{1}{2}\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds v_1 + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))v_1 + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\int_{-\infty}^{x}\beta(s)ds
\end{aligned} \tag{4.2.6}$$

for a suitable function $n_{10}(x,t)$, where $\frac{1}{4}\alpha(x)(v_1 + 2\int_{-\infty}^{x}\beta(s)ds) \ge 0$,

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&\le ((\tfrac{1-\gamma}{2}+\varepsilon_1)M + (1-\varepsilon_1)\textstyle\int_{-\infty}^{x}\beta(s)ds)\beta(x) + \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&\le ((\tfrac{1-\gamma}{2}+\varepsilon_1)M + (1-\varepsilon_1)\textstyle\int_{-\infty}^{x}\beta(s)ds)\beta(x) + \alpha(x)\int_{-\infty}^{x}\beta(s)ds\tfrac{\gamma-1}{4}M\\
&\le ((\tfrac{1-\gamma}{2}+\varepsilon_1)M\beta(x) + 2\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) \le 0
\end{aligned} \tag{4.2.7}$$

because

$$\frac{\gamma-1}{4}\alpha(x)M < \frac{3-\gamma}{\gamma+3}\beta(x) < \beta(x), \quad |\beta(x)|_{L^1(R)} < \frac{\gamma-1}{4}M \tag{4.2.8}$$

and

$$\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M)\\
&\ge \tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)(\tfrac{\gamma-1}{2}M + 2M)\\
&\ge \tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma+3}{8}\theta A(x)M \ge 0
\end{aligned} \tag{4.2.9}$$

because

$$|\beta(x)|_{L^1(R)} < \frac{\gamma-1}{4}M, \quad \theta|A(x)|M < \frac{3-\gamma}{\gamma+3}\beta(x). \tag{4.2.10}$$

Thus we have from (4.2.6), (4.2.7) and (4.2.9) that

$$K \le n_{11}(x,t)v + n_{12}(x,t)v_1, \tag{4.2.11}$$

where $n_{11}(x,t), n_{12}(x,t) \ge 0$ are two suitable functions.

Similarly, at the points (x,t), where $\alpha(x) \ge 0$, $A(x) \le 0$, we have from (4.0.10) that

$$\begin{aligned}
&K_1 \le \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M + \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&+(A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - \tfrac{1}{2}\alpha(x)|u|)(v_1 - v + C(x) - B(x)) \\
&\le (\tfrac{3-\gamma}{4}\beta(x) - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} + \tfrac{1}{2}\alpha(x)|u|)v \\
&+(\tfrac{3-\gamma}{4}\beta(x) + A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - \tfrac{1}{2}\alpha(x)|u|)v_1 \\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M + \textstyle\int_{-\infty}^{x}\beta(s)ds),
\end{aligned} \tag{4.2.12}$$

where

$$\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M + \textstyle\int_{-\infty}^{x}\beta(s)ds) \\
&\le (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) - (1-\varepsilon_1)\beta(x)\textstyle\int_{-\infty}^{x}\beta(s)ds \le 0.
\end{aligned} \tag{4.2.13}$$

Thus we have from (4.2.12) and (4.2.13) that

$$K_1 \le n_{14}(x,t)v + n_{15}(x,t)v_1, \tag{4.2.14}$$

where $n_{14}(x,t) \ge 0, n_{15}(x,t)$ are two suitable functions.

Summing up the analysis above, at any point (x,t), where $\alpha(x) \ge 0$, $A(x) \le 0$, we obtain the inequalities in (2.0.12).

4.3. *The case of $\alpha(x) \le 0$; $A(x) \ge 0$*

Let $\alpha(x) \le 0, A(x) \ge 0$. Repeating the proof of (4.1.6), at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \ge 0$, we have from (4.1.1) that

$$K \le n_{16}(x,t)v + n_{17}(x,t)v_1, \tag{4.3.1}$$

where $n_{16}(x,t), n_{17}(x,t) \ge 0$ are two suitable functions.

Similarly, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \le 0$, we have from (4.1.7) that

$$
\begin{aligned}
&K \le (n_3(x,t) - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1)v - \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds)v_1 + \tfrac{\gamma-1}{4\theta}A(x)\tfrac{\rho-2\delta}{\rho}\rho^{\theta}v_1\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds.
\end{aligned}
\tag{4.3.2}
$$

By using (4.1.2), we have

$$
\begin{aligned}
&-\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&= -\tfrac{1}{4}\alpha(x)v_1^2 - \alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds v_1 - \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&= -\tfrac{1}{4}\alpha(x)v_1(\tfrac{2}{\theta}\rho^{\theta} - v - 2M) - \alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds v_1 - \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2.
\end{aligned}
\tag{4.3.3}
$$

Thus we have from (4.3.2) that

$$
\begin{aligned}
&K \le (n_3(x,t) - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1 + \tfrac{1}{4}\alpha(x)v_1)v - \tfrac{1}{2\theta}\alpha(x)\rho^{\theta}v_1\\
&+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds + \tfrac{1}{2}\alpha(x)M)v_1 + \tfrac{\gamma-1}{4\theta}A(x)\tfrac{\rho-2\delta}{\rho}\rho^{\theta}v_1\\
&+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds - \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2,
\end{aligned}
\tag{4.3.4}
$$

where

$$
\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{4}A(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds + \tfrac{1}{2}\alpha(x)M\\
&\ge \tfrac{3-\gamma}{4}\beta(x) + \tfrac{1}{2}\alpha(x)M \ge 0
\end{aligned}
\tag{4.3.5}
$$

and

$$
\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds - \alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&\le (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + (1-\varepsilon_1)\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x)\\
&+\tfrac{1}{2}\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) - \alpha(x)\int_{-\infty}^{x}\beta(s)ds\tfrac{\gamma-1}{4}M\\
&\le (\tfrac{1-\gamma}{2} + \varepsilon_1)M\beta(x) + 2\textstyle\int_{-\infty}^{x}\beta(s)ds\beta(x) \le 0
\end{aligned}
\tag{4.3.6}
$$

because

$$\begin{aligned}&-\tfrac{\gamma-1}{4}\alpha(x)M < \tfrac{\gamma-1}{4}\tfrac{3-\gamma}{2(\gamma+1)}\beta(x) \le \tfrac{1}{2}\beta(x),\\ &\theta M|A(x)| < \tfrac{3-\gamma}{\gamma+3}\beta(x) < \tfrac{1}{2}\beta(x), \quad |\beta(x)|_{L^1(R)} < \tfrac{\gamma-1}{4}M.\end{aligned} \tag{4.3.7}$$

Thus we have also from (4.3.4)-(4.3.6) that

$$K \le n_{18}(x,t)v + n_{19}(x,t)v_1, \tag{4.3.8}$$

where $n_{18}(x,t), n_{19}(x,t) \ge 0$ are two suitable functions.

To obtain a similar estimate on K_1, first, repeating the proof of (4.1.17), at the points (x,t), where $-v + 2\int_{-\infty}^{x}\beta(s)ds \le 0$, we have from (4.1.12) that

$$\begin{aligned}&K_1 = \tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\ &+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)(v+v_1+2M)\\ &-\tfrac{1}{4}\alpha(x)(v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|\\ &\le (\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + v_1 + 2M))v_1\\ &-\tfrac{1}{4}\alpha(x)|v_1 - v + 2\textstyle\int_{-\infty}^{x}\beta(s)ds|v_1\\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - 2M))v - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v^2\\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds,\end{aligned} \tag{4.3.9}$$

and so

$$K_1 \le n_{20}(x,t)v + n_{21}(x,t)v_1, \tag{4.3.10}$$

where $n_{20}(x,t) \ge 0, n_{21}(x,t)$ are two suitable functions.

Second, at the points (x,t), where $-v + 2\int_{-\infty}^{x}\beta(s)ds \ge 0$, we have from (4.3.9) that

$$\begin{aligned}&K_1 \le n_{22}(x,t)v_1 - \tfrac{1}{4}\alpha(x)(v - 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds - 2M))v - \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v^2\\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds\end{aligned} \tag{4.3.11}$$

for a suitable function $n_{22}(x,t)$. By using

$$\begin{aligned}&v = z - B(x) = \tfrac{1}{\theta}\rho^{\theta} - u - B(x) = \tfrac{2}{\theta}\rho^{\theta} - w - B(x)\\ &= \tfrac{2}{\theta}\rho^{\theta} - v_1 - B(x) - C(x) = \tfrac{2}{\theta}\rho^{\theta} - v_1 - 2M,\end{aligned} \tag{4.3.12}$$

we have

$$
\begin{aligned}
&-\tfrac{1}{4}\alpha(x)(v-2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\
&=-\tfrac{1}{4}\alpha(x)v^2+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)dsv-\alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\\
&=-\tfrac{1}{4}\alpha(x)v(\tfrac{2}{\theta}\rho^\theta-v_1-2M)+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)dsv-\alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2.
\end{aligned}
\tag{4.3.13}
$$

Thus we have from (4.3.11) that

$$
\begin{aligned}
&K_1\le(n_{22}(x,t)+\tfrac{1}{4}\alpha(x)v)v_1-\tfrac{1}{2\theta}\alpha(x)\rho^\theta v\\
&+(\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds-2M)+\tfrac{1}{2}\alpha(x)M+\alpha(x)\int_{-\infty}^{x}\beta(s)ds)v\\
&+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds-\alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2,
\end{aligned}
\tag{4.3.14}
$$

where

$$
\begin{aligned}
&\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\
&+\tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds-\alpha(x)(\int_{-\infty}^{x}\beta(s)ds)^2\le 0
\end{aligned}
\tag{4.3.15}
$$

due to the proof of (4.3.6), and

$$
\begin{aligned}
&\tfrac{3-\gamma}{4}\beta(x)+\tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds-2M)+\tfrac{1}{2}\alpha(x)M+\alpha(x)\int_{-\infty}^{x}\beta(s)ds\\
&\ge\tfrac{3-\gamma}{4}\beta(x)-\tfrac{\gamma-1}{4}MA(x)+\tfrac{1}{2}\alpha(x)M+\tfrac{\gamma-1}{4}M\alpha(x)\\
&\ge\tfrac{3-\gamma}{4}\beta(x)-\tfrac{1}{2}\tfrac{3-\gamma}{\gamma+3}\beta(x)-\tfrac{\gamma+1}{4}\tfrac{3-\gamma}{2(\gamma+1)}\beta(x)\ge 0
\end{aligned}
\tag{4.3.16}
$$

because the conditions in Theorem 2. Thus we have from (4.3.14)-(4.3.16) that

$$
K_1\le n_{23}(x,t)v+n_{24}(x,t)v_1,\tag{4.3.17}
$$

where $n_{23}(x,t)\ge 0, n_{24}(x,t)$ are two suitable functions.

Summing up the analysis above, at any point (x,t), where $\alpha(x)\le 0$, $A(x)\ge 0$, we have (2.0.12).

4.4. *The case of* $\alpha(x)\le 0$; $A(x)\le 0$

Let $\alpha(x)\le 0$, $A(x)\le 0$. First, at the points (x,t), where $v_1+2\int_{-\infty}^{x}\beta(s)ds\ge 0$, repeating the proof of (4.2.1) we have a similar inequality like (4.2.5)

$$
K\le n_{25}(x,t)v+n_{26}(x,t)v_1,\tag{4.4.1}
$$

where $n_{25}(x,t), n_{26}(x,t)\ge 0$ are two suitable functions.

Second, at the points (x,t), where $v_1 + 2\int_{-\infty}^{x}\beta(s)ds \le 0$, repeating the proof of (4.2.6), we have

$$\begin{aligned} &K \le n_{27}(x,t)v - \tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2 \\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M))v_1 + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}v_1^2 \\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) + \tfrac{\gamma-1}{2}MA(x)\tfrac{\rho-2\delta}{\rho}\textstyle\int_{-\infty}^{x}\beta(s)ds \end{aligned} \tag{4.4.2}$$

for a suitable function $n_{27}(x,t)$. Using (4.1.2), we have

$$\begin{aligned} &-\tfrac{1}{4}\alpha(x)(v_1 + 2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2 \\ &= -\tfrac{1}{4}\alpha(x)v_1^2 - \alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds v_1 - \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2 \\ &= -\tfrac{1}{4}\alpha(x)v_1(\tfrac{2}{\theta}\rho^{\theta} - v - 2M) - \alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds v_1 - \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2. \end{aligned} \tag{4.4.3}$$

Thus we have from (4.4.2) that

$$\begin{aligned} &K \le (n_{27}(x,t) + \tfrac{1}{4}\alpha(x)v_1)v - (\tfrac{1}{2\theta}\rho^{\theta} + \textstyle\int_{-\infty}^{x}\beta(s)ds)\alpha(x)v_1 \\ &+(\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M) + \tfrac{1}{2}\alpha(x)M)v_1 \\ &+\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) - \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2, \end{aligned} \tag{4.4.4}$$

where

$$-(\tfrac{1}{2\theta}\rho^{\theta} + \textstyle\int_{-\infty}^{x}\beta(s)ds)\alpha(x) \ge 0, \tag{4.4.5}$$

$$\begin{aligned} &\tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)\tfrac{\rho-2\delta}{\rho}(2\textstyle\int_{-\infty}^{x}\beta(s)ds + 2M) + \tfrac{1}{2}\alpha(x)M \\ &\ge \tfrac{3-\gamma}{4}\beta(x) + \tfrac{\gamma-1}{8}A(x)(\tfrac{\gamma-1}{2}M + 2M) - \tfrac{3-\gamma}{4(\gamma+1)}\beta(x) \\ &\ge \tfrac{3-\gamma}{8}\beta(x) + \tfrac{\gamma+3}{8}\theta MA(x) \ge 0 \end{aligned} \tag{4.4.6}$$

and

$$\tfrac{3-\gamma}{2}M\beta(x) - (1-\varepsilon_1)\beta(x)(M - \textstyle\int_{-\infty}^{x}\beta(s)ds) - \alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2 \le 0 \tag{4.4.7}$$

due to the proof of (4.3.6). Thus we have

$$K \le n_{28}(x,t)v + n_{29}(x,t)v_1, \tag{4.4.8}$$

where $n_{28}(x,t), n_{29}(x,t) \ge 0$ are two suitable functions.

Similarly, at the points (x,t), where $-v+2\int_{-\infty}^{x}\beta(s)ds\le 0$, we have from (4.0.10) that

$$\begin{aligned}&K_1=\tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&+\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}(v_1-v+C(x)-B(x))\\&-\tfrac{1}{4}\alpha(x)(v_1-v+2\textstyle\int_{-\infty}^{x}\beta(s)ds)|v_1-v+2\textstyle\int_{-\infty}^{x}\beta(s)ds|\\&\le n_{30}(x,t)v_1+(\tfrac{3-\gamma}{4}\beta(x)-A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho})v\\&+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\le n_{30}(x,t)v_1+n_{31}(x,t)v,\end{aligned}\tag{4.4.9}$$

where $n_{30}(x,t)$ is a suitable function, and

$$n_{31}(x,t)=\tfrac{3-\gamma}{4}\beta(x)-\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\ge 0.\tag{4.4.10}$$

Finally, at the points (x,t), where $-v+2\int_{-\infty}^{x}\beta(s)ds\ge 0$, we have from (4.0.10) and (4.3.13) that

$$\begin{aligned}&K_1=\tfrac{3-\gamma}{4}(v+v_1+2M)\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&+\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}(v_1-v+C(x)-B(x))\\&-\tfrac{1}{4}\alpha(x)(v_1-v+2\textstyle\int_{-\infty}^{x}\beta(s)ds)|v_1-v+2\textstyle\int_{-\infty}^{x}\beta(s)ds|\\&\le n_{32}(x,t)v_1+(\tfrac{3-\gamma}{4}\beta(x)-\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho})v-\tfrac{1}{4}\alpha(x)(v-2\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\&+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&=(n_{32}(x,t)+\tfrac{1}{4}\alpha(x)v)v_1+\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)\\&-\alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\\&+(\tfrac{3-\gamma}{4}\beta(x)-\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds+\tfrac{1}{2}\alpha(x)M)v\end{aligned}\tag{4.4.11}$$

for a suitable function $n_{32}(x,t)$, where

$$\tfrac{3-\gamma}{2}M\beta(x)-(1-\varepsilon_1)\beta(x)(M+\textstyle\int_{-\infty}^{x}\beta(s)ds)-\alpha(x)(\textstyle\int_{-\infty}^{x}\beta(s)ds)^2\le 0\tag{4.4.12}$$

due to the proof of (4.3.6), and

$$\begin{aligned}&\tfrac{3-\gamma}{4}\beta(x)-\tfrac{1}{2}A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds+\tfrac{1}{2}\alpha(x)M\\&\ge\tfrac{3-\gamma}{4}\beta(x)+\alpha(x)\textstyle\int_{-\infty}^{x}\beta(s)ds+\tfrac{1}{2}\alpha(x)M\ge 0\end{aligned}\tag{4.4.13}$$

due to the proof of (4.3.16). Thus for all points (x,t), where $\alpha(x) \le 0$, $A(x) \le 0$, we have the inequality

$$K_1 \le n_{33}(x,t)v_1 + n_{34}(x,t)v, \quad n_{34}(x,t) \ge 0 \tag{4.4.14}$$

and so the inequalities in (2.0.12), which complete the Proof of Theorem 2.

5. Proof of Theorem 3: existence of global solutions

In this section, we shall prove that there exists a subsequence of the viscosity-flux approximate solutions $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ of the Cauchy problem (1.0.13) and (1.0.8), which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as δ, ε tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.0.1)-(1.0.2).

First of all, from the upper estimates (1.0.17) and (1.0.18) given in Theorems 1-2, we can use the Riemann invariants (1.0.10) to obtain the estimate on $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$ directly

$$2\delta \le \rho^{\delta,\varepsilon}(x,t) \le M(x), \quad |u^{\delta,\varepsilon}(x,t)| \le M(x), \tag{5.0.1}$$

where $M(x)$ is a nonnegative, bounded function, which depends on the bound of the initial data, but independent of ε, δ.

By simple calculations, for smooth solutions, the following two systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = 0, \end{cases} \tag{5.0.2}$$

where $P_1(\rho,\delta)$ being given in (1.0.7), and

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ u_t + (\frac{1}{2}u^2 + \int_{2\delta}^{\rho} \frac{(t-2\delta)P'(t)}{t^2} dt)_x = 0 \end{cases} \tag{5.0.3}$$

are equivalent, and particularly, both systems have the same entropy-entropy flux pairs. Thus any entropy-entropy flux pair $(\eta(\rho,m), q(\rho,m))$ of system (5.0.2) satisfies the additional system

$$q_\rho = u\eta_\rho + \frac{(\rho-2\delta)P'(\rho)}{\rho^2}\eta_u, \quad q_u = (\rho-2\delta)\eta_\rho + u\eta_u. \tag{5.0.4}$$

Eliminating the q from (5.0.4), we have

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{5.0.5}$$

Therefore, system (5.0.2) and system (1.0.3) have the same entropies [23].

We recall that for the case of polytropic gas, any weak entropy [5] can be represented by the following explicit formula:

$$\eta_0(\rho,u)=\rho\int_0^1[\tau(1-\tau)]^\lambda g(u+\rho^\theta-2\rho^\theta\tau)d\tau, \tag{5.0.6}$$

where $\theta=\frac{\gamma-1}{2}$, $\lambda=\frac{3-\gamma}{2(\gamma-1)}$ and g is a smooth function.

Secondly, for general pressure $P(\rho)$, we have the following lemma (See also Theorem 2 in [23])

Lemma 5. *Suppose the viscosity-flux approximate solutions $(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))$ of the Cauchy problem (1.0.13) and (1.0.8) are uniformly bounded in L^∞ space, and the limit*

$$\lim_{\rho\to 0}\frac{(P'(\rho))^{\frac{3}{2}}}{\rho P''(\rho)}=e, \tag{5.0.7}$$

where $e\geq 0$ is a constant. If the weak entropy-entropy flux pair $(\eta(\rho,u),q(\rho,u))$ of system (1.0.3) is in the form $\eta(\rho,u)=\rho H(\rho,u)$ and $H_u(\rho,u)$, $H_{uu}(\rho,u)$, $H_{uuu}(\rho,u)$ are continuous on $0\leq\rho\leq M_1$, $|u|\leq M_1$, where M_1 is a positive constant, then

$$\eta_t(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+q_x(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t)) \tag{5.0.8}$$

is compact in $H^{-1}_{loc}(R\times R^+)$ as $\varepsilon=o\big(\frac{P'(2\delta)}{2\delta}\big)$ and δ tends to zero, with respect to the viscosity solutions $(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t)$ of the Cauchy problem (1.0.13) and (1.0.8).

Proof of Lemma 5. One can easily check that system (5.0.2) has a convex entropy

$$\eta^\star=\frac{\rho u^2}{2}+\int_{2\delta}^{\rho}\frac{(\rho-t)P'(t)}{t}dt, \tag{5.0.9}$$

with corresponding entropy flux

$$q^\star=\frac{\rho u^3}{2}-\frac{\delta u^3}{3}+u(\rho-2\delta)\int_{2\delta}^{\rho}\frac{P'(t)}{t}dt. \tag{5.0.10}$$

To show that

$$\varepsilon(\rho_x,m_x)\cdot\nabla^2\eta^\star(\rho,m)\cdot(\rho_x,m_x)^T \tag{5.0.11}$$

is bounded in $L^1_{loc}(R\times R^+)$, we multiply (1.0.13) by $(\eta^\star_\rho,\eta^\star_m)$. It follows that

$$\varepsilon\frac{P'(\rho)}{\rho}\rho_x^2+\varepsilon\frac{1}{\rho}[\frac{m}{\rho}\rho_x-m_x]^2=\varepsilon\frac{P'(\rho)}{\rho}\rho_x^2+\varepsilon\rho u_x^2 \tag{5.0.12}$$

is bounded in $L^1_{loc}(R\times R^+)$.

We rewrite system (1.0.13) by the following equivalent system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = A(x)(\rho - 2\delta)u + \varepsilon\rho_{xx} \\ u_t + (\frac{1}{2}u^2 + \int_{2\delta}^{\rho} \frac{(t-2\delta)P'(t)}{t^2} dt)_x = \varepsilon u_{xx} + \frac{2\varepsilon}{\rho}\rho_x u_x - \alpha(x)|u|u. \end{cases} \tag{5.0.13}$$

Let $(\eta(\rho,u), q(\rho,u))$, $(\eta(\rho,u), q_1(\rho,u,\delta))$ be the entropy-entropy flux pairs of systems (1.0.3), (5.0.2) respectively since they have the same entropy equation (5.0.5), but different entropy fluxes.

Multiplying system (5.0.13) by (η_ρ, η_u), we obtain the relation

$$\begin{aligned} & \eta(\rho,m)_t + q(\rho,m)_x \\ = \;& \varepsilon\eta(\rho,m)_{xx} - (q_1(\rho,m,\delta) - q(\rho,m))_x + \tfrac{2\varepsilon}{\rho}\eta_u\rho_x u_x \\ - \;& \varepsilon(\eta_{\rho\rho}\rho_x^2 + 2\eta_{\rho u}\rho_x u_x + \eta_{uu}u_x^2) + A(x)(\rho-2\delta)u\eta_\rho - \alpha(x)|u|u\eta_u. \end{aligned} \tag{5.0.14}$$

By using the entropy equation (5.0.5), we obtain

$$\begin{aligned} \eta_\rho &= \int_0^\rho \tfrac{P'(\tau)}{\tau^2}\eta_{uu}(\tau,u)d\tau + g(u) \\ &= \int_0^\rho \tfrac{P'(\tau)}{\tau} H_{uu}(\tau,u)d\tau + g(u) \end{aligned} \tag{5.0.15}$$

since $\eta(\rho,u) = \rho H(\rho,u)$, where $g(u)$ is an arbitrary smooth function. Furthermore, by integrating (5.0.15), we get

$$\eta = \int_0^\rho \int_0^t \frac{P'(\tau)}{\tau} H_{uu}(\tau,u)d\tau dt + g(u)\rho \tag{5.0.16}$$

since $\eta(0,u) = 0$. Then

$$\eta_u = \int_0^\rho \int_0^t \frac{P'(\tau)}{\tau} H_{uuu}(\tau,u)d\tau dt + g'(u)\rho \tag{5.0.17}$$

and

$$\eta_{\rho u} = \int_0^\rho \frac{P'(\tau)}{\tau} H_{uuu}(\tau,u)d\tau + g'(u). \tag{5.0.18}$$

By substituting (5.0.17), (5.0.18) into (5.0.14) and by using entropy equation (5.0.5), we get the following equality

$$\eta(\rho,m)_t + q(\rho,m)_x = I_1 + I_2 + I_3, \tag{5.0.19}$$

where

$$I_1 = \varepsilon\eta(\rho,m)_{xx} - (q_1(\rho,m,\delta) - q(\rho,m))_x, \tag{5.0.20}$$

$$I_2 = -\varepsilon\big(\frac{P'(\rho)}{\rho}H_{uu}(\rho,u)\rho_x^2 + \rho H_{uu}u_x^2\big) + A(x)(\rho-2\delta)u\eta_\rho - \alpha(x)|u|u\eta_u, \tag{5.0.21}$$

$$I_3 = -2\varepsilon\big(\int_0^\rho \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau - \frac{1}{\rho}\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt\big)\rho_x u_x. \tag{5.0.22}$$

For any $\phi \in C_0^1(R\times R^+)$ with $S = \operatorname{supp}\phi$,

$$\begin{aligned}
&|\int_0^\infty\int_{-\infty}^\infty \varepsilon\eta(\rho,m)_{xx}\phi dxdt| = |\int_0^\infty\int_{-\infty}^\infty \varepsilon(\rho H(\rho,u))_x\phi_x dxdt| \\
&\le \varepsilon|\big(\eta_\rho\rho_x + H_u\rho u_x\big)\phi_x|dxdt \\
&\le M\big[\big(\textstyle\int\int_S \varepsilon\frac{P'(\rho)}{\rho}(\rho_x)^2\frac{\varepsilon\rho}{P'(\rho)}dxdt\big)^{\frac12} \\
&+\big(\textstyle\int\int_S \varepsilon^2\rho^2u_x^2dxdt\big)^{\frac12}\big]\big(\textstyle\int\int_S(\phi_x)^2dxdt\big)^{\frac12} \to 0
\end{aligned} \tag{5.0.23}$$

since $\varepsilon = o\big(\frac{P'(2\delta)}{2\delta}\big)$ or $\frac{\varepsilon\rho^\varepsilon}{P'(\rho^\varepsilon)} \to 0$ as $\varepsilon,\delta\to 0$. Since $q_1(\rho,m,\delta)-q(\rho,m)$ tends to zero as δ tends to zero, we get the compactness of I_1 in $H^{-1}_{loc}(R\times R^+)$. Using (5.0.12), we know that I_2 is bounded in $L^1_{loc}(R\times R^+)$, and hence compact in $W^{-1,\alpha}_{loc}(R\times R^+)$, for some $\alpha\in(1,2)$, by the Sobolev embedding theorems. Using the Vol'pert theorem and the limit given in (5.0.7), we have the following estimates

$$\begin{aligned}
&|\textstyle\int_0^\rho \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau| \le M|\int_0^\rho \frac{P'(\tau)}{\tau}d\tau| \le M_1\sqrt{P'(\rho)}, \\
&|\textstyle\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}H_{uuu}(\tau,u)d\tau dt| \le M|\int_0^\rho\int_0^t \frac{P'(\tau)}{\tau}d\tau dt| \le M_1\rho\sqrt{P'(\rho)}.
\end{aligned} \tag{5.0.24}$$

Using these estimates, we get the boundedness of I_3 in $L^1_{loc}(R\times R^+)$ and hence the compactness in $W^{-1,\alpha}_{loc}(R\times R^+)$, for some $\alpha\in(1,2)$, by the Sobolev embedding theorems.

Therefore the right-hand side of (5.0.19) is compact in $W^{-1,\alpha}_{loc}(R\times R^+)$ for some $\alpha\in(1,2)$, but the left-hand side is bounded in $W^{-1,\infty}(R\times R^+)$. This implies the compactness of $\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})_t + q(\rho^\varepsilon,m^\varepsilon)_x$ in $H^{-1}_{loc}(R\times R^+)$ and hence the proof of Lemma 5 by the Murat theorem [36].

It is clear that the condition (5.0.7) is true for the polytropic gas, $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, and the weak entropy given in (5.0.6) satisfies the conditions in Lemma 5. Then there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t),u(x,t))$ as δ,ε tend to zero by using the compactness framework given in [3,5,16] for $1<\gamma<3$ and in [17] for $\gamma\ge 3$. It is easy to prove that the limit $(\rho(x,t),u(x,t))$ satisfies (1.0.20). Moreover, for any weak convex entropy-entropy flux pair $(\eta(\rho,u),q(\rho,u))$ of system (1.0.3), we multiply (1.0.13) by (η_ρ,η_m) to obtain that

$$
\begin{aligned}
&\eta_t(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+q_x(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))+\delta q_{1x}(\rho^{\delta,\varepsilon}(x,t),u^{\delta,\varepsilon}(x,t))\\
&=\varepsilon\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})_{xx}-\varepsilon(\rho_x^{\delta,\varepsilon},m_x^{\delta,\varepsilon})\cdot\nabla^2\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\cdot(\rho_x^{\delta,\varepsilon},m_x^{\delta,\varepsilon})^T\\
&+A(x)(\rho^{\delta,\varepsilon}-2\delta)u^{\delta,\varepsilon}\eta_\rho(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\\
&+(A(x)(\rho^{\delta,\varepsilon}-2\delta)(u^{\delta,\varepsilon})^2-\alpha(x)m^{\delta,\varepsilon}|u^{\delta,\varepsilon}|)\eta_m(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\\
&\le\varepsilon\eta(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})_{xx}+A(x)(\rho^{\delta,\varepsilon}-2\delta)u^{\delta,\varepsilon}\eta_\rho(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon})\\
&+(A(x)(\rho^{\delta,\varepsilon}-2\delta)(u^{\delta,\varepsilon})^2-\alpha(x)m^{\delta,\varepsilon}|u^{\delta,\varepsilon}|)\eta_m(\rho^{\delta,\varepsilon},m^{\delta,\varepsilon}),
\end{aligned}
\tag{5.0.25}
$$

where $q+\delta q_1$ is the entropy flux of system (5.0.2) corresponding to the entropy η. Thus the entropy inequality (1.0.21) is proved if we multiply a test function to (5.0.25) and let ε,δ go to zero. **Theorem 3 is proved.**

6. Proof of Theorem 4

In this section, we shall prove Theorem 4. By simple calculations, the two eigenvalues of (1.0.25) are

$$\lambda_1=u-1,\quad \lambda_2=u+1 \tag{6.0.1}$$

with corresponding Riemanna invariants

$$z(v,m)=\ln v-\frac{m}{v},\quad w(v,m)=\ln v+\frac{m}{v}, \tag{6.0.2}$$

where $m=vu$.

We consider the Cauchy problem of the parabolic system (1.0.26) with initial data

$$(v(x,0),v(x,0)u(x,0))=(v_0^\delta(x),v_0^\delta(x)u_0^\delta(x)), \tag{6.0.3}$$

where

$$(v_0^\delta(x),u_0^\delta(x))=(a(x)\rho_0(x)+2\delta,u_0(x))*G^\delta, \tag{6.0.4}$$

$$(A^\delta(x),\alpha^\delta(x))=(A(x),\alpha(x))*G^\delta, \tag{6.0.5}$$

and G^δ is a mollifier.

Then by the conditions given in Theorem 4, we have

$$(v_0^\delta(x),u_0^\delta(x))\in C^\infty(R)\times C^\infty(R), \tag{6.0.6}$$

$$v_0^\delta(x)\ge 2\delta,\quad v_0^\delta(x)+|u_0^\delta(x))|\le M \tag{6.0.7}$$

and

$$A^\delta(x)\in C^\infty(R)\cap L^1(R),\quad \alpha^\delta(x)\in C^\infty(R)\cap L^1(R). \tag{6.0.8}$$

Similarly, as did in Section 2, we multiply (1.0.26) by (w_v, w_m) and (z_v, z_m), respectively, to obtain

$$
\begin{aligned}
& z_t + \lambda_1^\delta z_x - A^\delta(x) - \alpha^\delta(x)u|u| \\
& = \varepsilon z_{xx} - \varepsilon(z_{vv}v_x^2 + 2z_{vm}v_x m_x + z_{mm}m_x^2) \\
& = \varepsilon z_{xx} + \tfrac{2\varepsilon}{v} v_x z_x - \tfrac{\varepsilon v_x^2}{v^2}
\end{aligned}
\tag{6.0.9}
$$

and

$$
\begin{aligned}
& w_t + \lambda_2^\delta w_x + A^\delta(x) + \alpha^\delta(x)u|u| \\
& = \varepsilon w_{xx} - \varepsilon(w_{vv}v_x^2 + 2w_{vm}v_x m_x + w_{mm}m_x^2) \\
& = \varepsilon w_{xx} + \tfrac{2\varepsilon}{v} v_x w_x - \tfrac{\varepsilon v_x^2}{v^2},
\end{aligned}
\tag{6.0.10}
$$

where $\lambda_1^\delta = u - \frac{v-2\delta}{v}, \lambda_2^\delta = u + \frac{v-2\delta}{v}$.

Let $X(x) = 3(|A^\delta(x)| + |\alpha^\delta(x)|)$, then $|X(x)|_{L^1(R)} \le \frac{1}{2}$ by the condition (1.0.22).

Making the transformations of $z = z_1 + B(x)$, $w = w_1 + C(x)$, where

$$
B(x) = M - \int_{-\infty}^{x} X(s)ds > \frac{1}{2}, C(x) = M + \int_{-\infty}^{x} X(s)ds > \frac{1}{2}, \tag{6.0.11}
$$

for a positive constant $M > 1$, we have from (6.0.9)-(6.0.10) that

$$
\begin{aligned}
& z_{1t} + \lambda_1^\delta z_{1x} - B'(x)z_1 - B'(x)B(x) \\
& + B'(x)\ln v - B'(x)\tfrac{v-2\delta}{v} - A^\delta(x) - \alpha^\delta(x)u|u| \\
& = \varepsilon z_{1xx} + \varepsilon B''(x) + \tfrac{2\varepsilon}{v} v_x z_{1x} + \tfrac{2\varepsilon}{v} v_x B'(x) - \tfrac{\varepsilon v_x^2}{v^2} \\
& = \varepsilon z_{1xx} + \varepsilon B''(x) + \tfrac{2\varepsilon}{v} v_x z_{1x} - \varepsilon(\tfrac{v_x}{v} - B'(x))^2 + \varepsilon B'^2(x) \\
& \le \varepsilon z_{1xx} + \varepsilon B''(x) + \tfrac{2\varepsilon}{v} v_x z_{1x} + \varepsilon B'^2(x)
\end{aligned}
\tag{6.0.12}
$$

and

$$
\begin{aligned}
& w_{1t} + \lambda_2^\delta w_{1x} + C'(x)w_1 + C'(x)C(x) \\
& - C'(x)\ln v + C'(x)\tfrac{v-2\delta}{v} + A^\delta(x) + \alpha^\delta(x)u|u| \\
& = \varepsilon w_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{v} v_x w_{1x} + \tfrac{2\varepsilon}{v} v_x C'(x) - \tfrac{\varepsilon v_x^2}{v^2} \\
& = \varepsilon w_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{v} v_x w_{1x} - \varepsilon(\tfrac{v_x}{v} - C'(x))^2 + \varepsilon C'^2(x) \\
& \le \varepsilon w_{1xx} + \varepsilon C''(x) + \tfrac{2\varepsilon}{v} v_x w_{1x} + \varepsilon C'^2(x).
\end{aligned}
\tag{6.0.13}
$$

Clearly, we can choose a suitable small positive constant ε_1 and $\varepsilon = o(\varepsilon_1)$, $\tau = o(\varepsilon_1)$ such that the following terms in (6.0.12) and (6.0.13) satisfy

$$\begin{cases} -\varepsilon_1 B'(x)B(x) - \varepsilon B''(x) - \varepsilon B'^2(x) = \varepsilon_1 X(x) + \varepsilon X'(x) - \varepsilon X^2(x) \geq 0, \\ \varepsilon_1 C'(x)C(x) - \varepsilon C''(x) - \varepsilon C'^2(x) = \varepsilon_1 X(x) - \varepsilon X'(x) - \varepsilon X^2(x) \geq 0. \end{cases} \tag{6.0.14}$$

Since the initial date $v_0^\delta(x) \geq 2\delta$, we may obtain the a priori estimate $v^{\delta,\varepsilon,\tau} \geq 2\delta$ by applying the maximum principle to the first equation in (1.0.26).

Now, under the conditions in Theorem 4, by using (6.0.12), (6.0.13) and (6.0.14), we prove the following inequalities

$$\begin{cases} z_{1t} + b_1(x,t)z_{1x} + b_2(x,t)z_1 + b_3(x,t)w_1 \leq \varepsilon z_{1xx}, \\ w_{1t} + c_1(x,t)w_{1x} + c_2(x,t)w_1 + c_3(x,t)z_1 \leq \varepsilon w_{1xx}, \end{cases} \tag{6.0.15}$$

where $b_i(x,t), c_i(x,t), i = 1,2,3$, are suitable functions satisfying the necessary conditions $b_3(x,t) \leq 0, c_3(x,t) \leq 0$.

Proof of (6.0.15). We prove (6.0.15) in several cases for two different groups of points (x,t), where $\alpha^\delta(x) \geq 0$ or $\alpha^\delta(x) \leq 0$.

Case (I). At the points (x,t), where $\alpha^\delta(x) \geq 0, v(x,t) \leq 1$ and $w_1 + 2\int_{-\infty}^x X(s)ds \leq 0$, the following terms in (6.0.12)

$$\begin{aligned} I_1 &= -(1-\varepsilon_1)B'(x)B(x) + B'(x)\ln v - B'(x)\tfrac{v-2\delta}{v} - A^\delta(x) - \alpha^\delta(x)u|u| \\ &\geq (1-\varepsilon_1)X(x)(M - \textstyle\int_{-\infty}^x X(s)ds) - \tfrac{1}{3}X(x) \\ &- \tfrac{1}{4}\alpha^\delta(x)(w_1 - z_1 + 2\textstyle\int_{-\infty}^x X(s)ds)|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds| \\ &\geq -\tfrac{1}{4}\alpha^\delta(x)(w_1 - z_1 + 2\textstyle\int_{-\infty}^x X(s)ds)|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds| \\ &\geq \tfrac{1}{4}\alpha^\delta(x)|w_1 - z_1 + 2\textstyle\int_{-\infty}^x X(s)ds|z_1. \end{aligned} \tag{6.0.16}$$

Case (II). At the points (x,t), where $\alpha^\delta(x) \geq 0, v(x,t) \leq 1$ and $w_1 + 2\int_{-\infty}^x X(s)ds \geq 0$,

$$\begin{aligned} I_1 &\geq (1-\varepsilon_1)X(x)(M - \textstyle\int_{-\infty}^x X(s)ds) - \tfrac{1}{3}X(x) \\ &+ \tfrac{1}{4}\alpha^\delta(x)|w_1 - z_1 + 2\textstyle\int_{-\infty}^x X(s)ds|z_1 \\ &- \tfrac{1}{4}\alpha^\delta(x)(w_1 + 2\textstyle\int_{-\infty}^x X(s)ds)|z_1| - \tfrac{1}{4}\alpha^\delta(x)(w_1 + 2\int_{-\infty}^x X(s)ds)^2 \\ &= (1-\varepsilon_1)X(x)(M - \textstyle\int_{-\infty}^x X(s)ds) - \tfrac{1}{3}X(x) - \alpha^\delta(x)(\int_{-\infty}^x X(s)ds)^2 \\ &+ d(x,t)z_1 + e(x,t)w_1 \geq d(x,t)z_1 + e(x,t)w_1, \end{aligned} \tag{6.0.17}$$

where $e(x,t) = -\frac{1}{4}\alpha^\delta(x)(w_1 + 4\int_{-\infty}^x X(s)ds) \leq 0$, because

$$
\begin{aligned}
&(1-\varepsilon_1)X(x)(M-\textstyle\int_{-\infty}^{x}X(s)ds)-\frac{1}{3}X(x)-\alpha^\delta(x)(\textstyle\int_{-\infty}^{x}X(s)ds)^2\\
&\geq \frac{1}{2}(1-\varepsilon_1)X(x)-\frac{1}{3}X(x)-\frac{1}{12}X(x)\geq 0.
\end{aligned}
\tag{6.0.18}
$$

Case (III). At the points (x,t), where $\alpha^\delta(x)\geq 0$, $v(x,t)>1$ and $w_1+2\int_{-\infty}^{x}X(s)ds\leq 0$, we have $\frac{v-2\delta}{v}\geq 1-\varepsilon_2>0$ for a small $\varepsilon_2>0$, and $B'(x)\ln v=-X(x)(\frac{1}{2}(w_1+z_1)+M)$. Then

$$
\begin{aligned}
&I_1\geq(1-\varepsilon_1)X(x)(M-\textstyle\int_{-\infty}^{x}X(s)ds)-\frac{1}{2}(w_1+z_1)X(x)-MX(x)\\
&+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)+\alpha^\delta(x)|w_1-z_1+2\textstyle\int_{-\infty}^{x}X(s)ds|z_1\\
&\geq-\frac{1}{2}(w_1+z_1)X(x)+\alpha^\delta(x)|w_1-z_1+2\textstyle\int_{-\infty}^{x}X(s)ds|z_1
\end{aligned}
\tag{6.0.19}
$$

because

$$
\begin{aligned}
&(1-\varepsilon_1)X(x)(M-\textstyle\int_{-\infty}^{x}X(s)ds)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)\\
&\geq X(x)(1-\varepsilon_2-\varepsilon_1M-\frac{1}{2}-\frac{1}{3})\geq 0
\end{aligned}
\tag{6.0.20}
$$

for small ε_1 and ε_2.

Case (IV). At the points (x,t), where $\alpha^\delta(x)\geq 0$, $v(x,t)>1$ and $w_1+2\int_{-\infty}^{x}X(s)ds\geq 0$,

$$
\begin{aligned}
&I_1\geq(1-\varepsilon_1)X(x)(M-\textstyle\int_{-\infty}^{x}X(s)ds)-\frac{1}{2}(w_1+z_1)X(x)-MX(x)\\
&+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)-\alpha^\delta(x)(\textstyle\int_{-\infty}^{x}X(s)ds)^2\\
&+d(x,t)z_1+e(x,t)w_1\geq-\frac{1}{2}(w_1+z_1)X(x)+d(x,t)z_1+e(x,t)w_1,
\end{aligned}
\tag{6.0.21}
$$

because

$$
\begin{aligned}
&(1-\varepsilon_1)X(x)(M-\textstyle\int_{-\infty}^{x}X(s)ds)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)\\
&-\alpha^\delta(x)(\textstyle\int_{-\infty}^{x}X(s)ds)^2\geq X(x)(1-\varepsilon_2-\varepsilon_1M-\frac{1}{2}-\frac{1}{3}-\frac{1}{12})\geq 0,
\end{aligned}
\tag{6.0.22}
$$

where $d(x,t),e(x,t)$ are given in (6.0.17). Thus we obtain the proof of the first inequality in (6.0.15) at the points (x,t), where $\alpha^\delta(x)\geq 0$.

Now we prove the second inequality in (6.0.15). At the points (x,t), where $\alpha^\delta(x)\geq 0$ and $v(x,t)\leq 1$, the following terms in (6.0.13)

$$
\begin{aligned}
&I_2=(1-\varepsilon_1)C'(x)C(x)-C'(x)\ln v+C'(x)\frac{v-2\delta}{v}+A^\delta(x)+\alpha^\delta(x)u|u|\\
&\geq(1-\varepsilon_1)X(x)(M+\textstyle\int_{-\infty}^{x}X(s)ds)-\frac{1}{3}X(x)\\
&+\frac{1}{4}\alpha^\delta(x)(w_1-z_1+2\textstyle\int_{-\infty}^{x}X(s)ds)|w_1-z_1+2\textstyle\int_{-\infty}^{x}X(s)ds|\\
&\geq\frac{1}{4}\alpha^\delta(x)(w_1-z_1)|w_1-z_1+2\textstyle\int_{-\infty}^{x}X(s)ds|;
\end{aligned}
\tag{6.0.23}
$$

and at the points (x,t), where $\alpha^\delta(x)\geq 0$ and $v(x,t)>1$,

$$
\begin{aligned}
I_2 &\geq (1-\varepsilon_1)X(x)(M+\textstyle\int_{-\infty}^x X(s)ds) - \frac{1}{2}(w_1+z_1)X(x) - MX(x) \\
&+(1-\varepsilon_2)X(x) - \frac{1}{3}X(x) + \frac{1}{4}\alpha^\delta(x)(w_1-z_1)|w_1-z_1+2\textstyle\int_{-\infty}^x X(s)ds| \\
&\geq -\frac{1}{2}(w_1+z_1)X(x) + \frac{1}{4}\alpha^\delta(x)(w_1-z_1)|w_1-z_1+2\textstyle\int_{-\infty}^x X(s)ds|.
\end{aligned} \tag{6.0.24}
$$

Thus we obtain the proof of (6.0.15) at the points (x,t), where $\alpha^\delta(x) \geq 0$. Similarly, we may prove (6.0.15) also at the points (x,t), where $\alpha^\delta(x) \leq 0$.

Under the conditions given in (1.0.24), it is clear that $z_1(x,0) \leq 0$, $w_1(x,0) \leq 0$, so, we may apply the maximum principle to (6.0.15) to obtain the estimates

$$
2\delta \leq v^{\delta,\varepsilon} \leq M_1, \quad \ln v^{\delta,\varepsilon} - M_2 \leq u^{\delta,\varepsilon} \leq M_2 - \ln v^{\delta,\varepsilon}, \quad |m^{\delta,\varepsilon}| \leq M_3, \tag{6.0.25}
$$

where $M_i, i = 1,2,3$ are suitable positive constants, independent of ε, δ and the time t.

By applying the general contracting mapping principle to an integral representation of (1.0.26), with the help of the lower, positive estimate and the L^∞ estimates given in (6.0.25), we can obtain the existence and uniqueness of smooth solution of the Cauchy problem (1.0.26)-(6.0.3). Applying the convergence frame given in [11] we have the pointwise convergence

$$
(v^{\delta,\varepsilon}(x,t), m^{\delta,\varepsilon}(x,t)) \to (v(x,t), m(x,t)) \text{ a.e., as } \delta, \varepsilon \to 0 \tag{6.0.26}
$$

or

$$
(\rho^{\delta,\varepsilon}(x,t), (\rho^{\delta,\varepsilon}u^{\delta,\varepsilon})(x,t)) \to (\rho(x,t), (\rho u)(x,t)) \text{ a.e., as } \delta, \varepsilon \to 0. \tag{6.0.27}
$$

Furthermore, in a similar way as given in [29], we may prove that the limit $(\rho(x,t), u(x,t))$ satisfies system (1.0.1) in the sense of distributions and the Lax entropy condition (1.0.21). So, we complete the proof of Theorem 4.

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Data availability

No data was used for the research described in the article.

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