The reality of the compound solution in magnetohydrodynamics

Axel Brandenburg, *brandenb@nordita.dk*
Christian Klingenberg, *klingen@mathematik.uni-wuerzburg.de*
Knut Waagan, *knutwa@math.uio.no*
Hvard Huru Bergene, *huru@phys.ntnu.no*
Yuan Lin, *yuan@mech.kth.se*
Mohammad Motamed, *mohamad@nada.kth.se*
Thomas Ramstad, *thomaram@tf.phys.ntnu.no*
Joni Virtanen, *jonvir@utu.fi*

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1 Introduction

The equations of magnetohydrodynamics (MHD) are obtained by coupling the Euler equations of gas dynamics and the Maxwell equations of electromagnetism. They describe the flow of a fluid in the presence of magnetic field. For ideal gases, the inviscid MHD equation in one dimension reads

$$\begin{aligned}
\partial_t \rho + \partial_x(\rho v_n) &= 0,\\
\partial_t \rho v_n + \partial_x(\rho v_n^2 + p + \tfrac{1}{2}\mathbf{B}_t^2) &= 0,\\
\partial_t \rho \mathbf{v_t} + \partial_x(\rho \mathbf{v}_t - B_n \mathbf{B}_t) &= 0,\\
\partial_t \rho \mathbf{B_t} + \partial_x(v_n \mathbf{B}_t - B_n \mathbf{v}_t) &= 0,\\
\partial_t E + \partial_x((E + p + \tfrac{1}{2}\mathbf{B}_t^2) v_n - B_n \mathbf{B}_t \cdot \mathbf{v}_t) &= 0,
\end{aligned} \tag{1}$$

where the variables are density ρ, flow velocity $\mathbf{v} = (v_x, v_y, v_z) = (v_n, \mathbf{v_t})$, magnetic field $\mathbf{B} = (B_n, \mathbf{B_t})$, and total energy E. The total energy is substituted by the pressure

$$E = \frac{1}{\gamma - 1} p + \frac{1}{2}\rho v_n^2 + \frac{1}{2}\rho \mathbf{v_t}^2 + \frac{1}{2}\mathbf{B_t}^2, \tag{2}$$

where γ is the adiabatic constant. Note that because of the divergence condition $\nabla \cdot \mathbf{B} = 0$ in 1D, B_n has to be constant, and therefore we consider it as a parameter.

By writing the system (1) in the quasi-linear form $U_t + A\,U_x = 0$, one can see that it has the characteristic velocities

$$\begin{aligned} &\lambda_1 = v - c_f, \lambda_2 = v - c_A, \lambda_3 = v - c_s, \\ &\lambda_4 = v, \\ &\lambda_5 = v + c_s, \lambda_6 = v + c_A, \lambda_7 = v + c_f, \end{aligned} \tag{3}$$

which are the eigenvalues of the Jacobian A. The fast, slow and Alfven velocities are computed as

$$\begin{aligned} c_{f,s} &= \sqrt{\frac{1}{2}\left(\frac{B_n^2 + \mathbf{B_t}^2}{\rho} + a^2\right) \pm \sqrt{\frac{1}{4}\left(\frac{B_n^2 + \mathbf{B_t}^2}{\rho} + a^2\right)^2 - a^2\frac{B_n^2}{\rho}}}, \\ c_A &= \sqrt{\frac{B_n^2}{\rho}}, \end{aligned} \tag{4}$$

where $a = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed.

Since $c_s \le c_A \le c_f$, the real eigenvalues (3) may coincide at certain points, and therefore the MHD equation is a non-strictly hyperbolic system. For example if $B_n = 0$, we get $c_s = c_a = 0$, and v is an eigenvalue of multiplicity 5.

2 Stability of the waves

In a Riemann problem for a hyperbolic system, the initial conditions are discontinuous. the solution of such problems is built from either discontinuities, like shock waves, or rarefaction waves. Due to the non-strict hyperbolicity of MHD equations, the wave structure of the Riemann problem for ideal MHD, governed by 7 waves, is more complicated than for a strictly hyperbolic system; First, the uniqueness of solution is not guaranteed. Second, the system may admit non-regular waves, including compound waves and over-compressive shocks.

We will specifically study the Riemann problem with left state

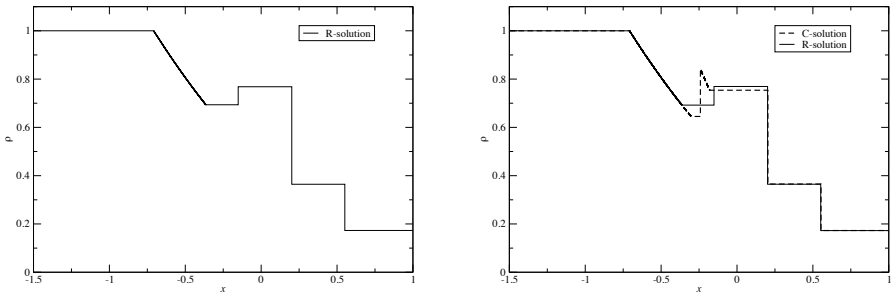
$$(\rho, v_n, \mathbf{v_t}, B_n, \mathbf{B_t}, p) = (1, 0, [0, 0], 1, [1, 0], 1)\,, \tag{5}$$

and right state

$$(\rho, v_n, \mathbf{v_t}, B_n, \mathbf{B_t}, p) = (0.2, 0, [0, 0], 1, [\cos\alpha, \sin\alpha], 0.2)\,, \tag{6}$$

as initial conditions, which was investigated by Manuel Torrilhon in [4]. Angle α is the initial twist angle of the planes of the magnetic fields. The exact solution for $\alpha = 3.0$ and $\alpha = \pi$ is shown in figure 1.

For certain initial data the Riemann problem for ideal MHD does not have a unique solution. In [5], the uniqueness condition has been derived as the following:



(a) $\alpha = 3.0$ (b) $\alpha = \pi$

Figure 1: Exact solutions of (1,5,6), calculated by Torrilhon [3].

Theorem. If for the initial conditions in the non-planar case ($\alpha \neq k\pi, k = 0, 1, 2$), the condition $[[\mathbf{v_t}]] = 0$, or in the coplanar case ($\alpha = \pi$), the condition $0 \neq [[\mathbf{v_t}]] \not\parallel [[\mathbf{B_t}]]$ holds, then the MHD Riemann problem has a unique regular solution.

Here, the quantity

$$[[\phi]] := \phi^{(1)} - \phi^{(0)}$$

describes the jump of the fields to right and left of the discontinuity.

From the theorem it follows that the solution of (5) and (6), when $\alpha = \pi$, is not unique since $[[\mathbf{v_t}]] = 0$. In this case the left and right states have magnetic fields in the same plane, and there are two solutions to the Riemann problem, which we will denote the *r-solution* and the *c-solution*, following [4]. The 'r' in *r-solution* stands for *regular* since it is composed of only Lax shocks, rarefactions, rotational discontinuities and a contact discontinuity, all having nice stability properties. The *c-solution*, however, has a *compound* wave (hence the 'c') travelling to the left, where a marginally over-compressive shock is directly followed by a slow rarefaction. For $\alpha \neq \pi$ there is only one solution, U_α to the Riemann problem (5), (6), and as $\alpha \to \pi$, U_α approaches the r-solution. Hence the *c-solution* is unstable to arbitrarily small perturbations of B out of the plane.

Therefore one might suggest that the *c-solution* does not have physical significance. However, we have assumed that there is no viscosity, which is an approximation. In the viscous equation the momentum equations are modified as follows:

$$\begin{aligned} \partial_t \rho v_n + \partial_x(\rho v_n^2 + p + \tfrac{1}{2}\mathbf{B}_t^2) &= \frac{2}{3}\nu\partial_x(\rho\,\partial_x v_n), \\ \partial_t \rho \mathbf{v_t} + \partial_x(\rho \mathbf{v}_t - B_n \mathbf{B}_t) &= \nu\partial_x(\rho\,\partial_x \mathbf{v_t}), \end{aligned} \tag{7}$$

If $\nu > 0$ the *c-solution* will be stable under small perturbations in phase space, including perturbations of B out of the plane. However this stability

region shrinks with ν, until it disappears at $\nu = 0$, see [2].

When doing numerical simulations of the inviscid equations, one also gets more complex behaviour than the inviscid analysis suggests. For $\alpha = \pi$, all the schemes we are aware of pick the *c-solution*. However, if α is slightly perturbed, at coarse grids the schemes will still tend to the *c-solution*, and begins to converge to the unique regular solution only when the grid is refined sufficiently. Since all the schemes add viscosity in a certain sense, it is plausible that the analysis of the viscous equation might provide some explanation for this. For coarser resolution the numerical viscosity is higher, so it is as though the data are within the stability region of the *c-solution* until we refine, and thereby reduce viscosity, sufficiently.

3 Solving the inviscid MHD equations

To solve the inviscid one dimensional MHD equations (1) numerically we used the Staggered Central Scheme by Tadmor and Nessyahu [1]. The scheme is 2. order i01 Devilś Haircut.mp3n time and space. Already on the second day of the workshop we were able to run a few test runs with an existing two-dimensional code written by Knut Waagan for $\alpha \neq \pi$.

The only modifications needed was specifying the initial conditions, and setting the dimensions to $n \times 1$. We were able to reproduce the convergence behavior discussed by Torrilhon in [4]. For coarse grids the solution is "c-like", and for finer grids the solution approaches the correct r-solution.

Since the code was two-dimensional it was quite time consuming to run on finer grids ($>$ 10000 grid points). The goal was to run on very fine grids ($>$ 100000 grid points), so we stripped the 2nd dimension from the code, giving a purely one-dimensional code. Then the code ran about 10 times faster. The code running on a 1.6 GHz portable computer ran 10000 grid points to time t=0.4 in a couple of hours, while 50000 grid points required about 24 hours. In order to run on even finer grids we started parallelizing the code.

3.1 Parallel version

The parallelization was done using domain decomposition. Assuming that we have N processors (and domains) numbered by $i \in [0, N-1]$ and n total grid points. The the domain of processor i have $n_i = n/N$ grid points. The division is integer, and in case the modulo is nonzero, each processor with rank less than the modulo get one more grid point. In addition the grid has two[1]ghost cells on both sides of the domain. The ghost cells are set by communicating the corresponding cells from the neighbouring domains. Figure 2 shows an example for $n = 64$ and $N = 6$. The arrows shows the direction of communication. If the ghost cells lie outside the boundaries of the total grid they are set using Neumann boundary conditions. The communications are done using MPI which is widely

[1]Actually we use three ghost cells to avoid communicating twice each time step.

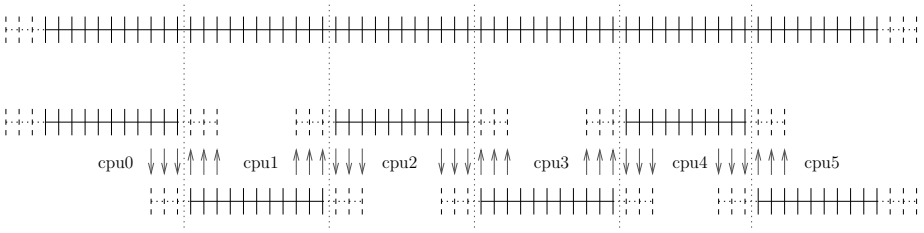


Figure 2: An example of domain decomposition of a system with $n = 64$ grid points distributed to $N = 6$ processors. Arrows show the direction of communication.

available and may run on both massively parallel computers and workstation clusters.

The parallelization was done just in time to run on 100000 grid points before last preparations for the next days final presentations. Using 32 2,6GHz processors on the Horseshoe cluster in Odense (Denmark) the running time was less than 1 hour.

3.2 Results

In figure 3 we have plotted the density, ρ and the y-component of the magnetic field B_y for different number of grid points using (5,6) and $\alpha = 3.0$. Note that for coarse grids we get a solution that is similar to a *c-solution*, and that for the

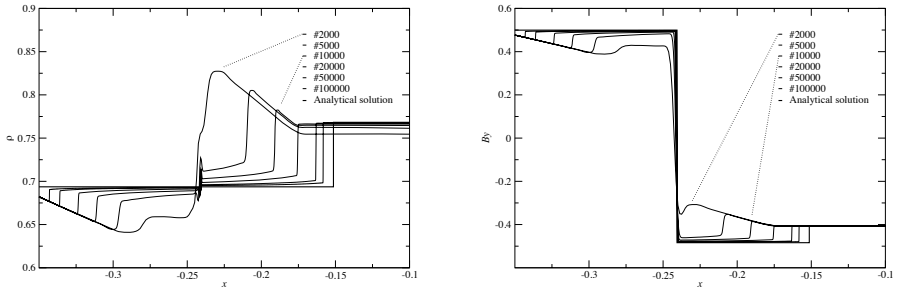


Figure 3: The transition from a "c-like" solution towards the exact r-solution for $\alpha = 3.0$.

3.3 Numerical stability of the r-solution

For $\alpha = \pi$ we were not able to get the r-solution at all numerically. But as a step towards achieving that, we decided to sample the r-solution at some time,

and use it as the initial condition for the Nessuahu-Tadmor scheme. In order to do so we obtained from M. Torrilhon his calculation of the exact solution to the Riemann problem (5)-(6). The simulation did not simply evolve the r-solution as one would expect. Instead the left-going rotational discontinuity broke up into several new structures that eventually interacted with the slow shock - Hence the self-similarity of the r-solution was destroyed. We believe that this failure is due to a general difficulty in resolving rotational discontinuities numerically, since similar phenomena could be observed for other Riemann problems consisting only of a rotational discontinuity. There seemed to be some analogy with the issue of resolving shear waves in hydrodynamics. However this observation does not contradict the presence of rotational discontinuities in numerical solutions of other Riemann problems. In that case they emerge in a stable fashion out of other structures. It is possible that some kind of smoothing of the initial rotational discontinuity would solve this issue.

3.4 1-D simulation using the Pencil code

Although the pre-existing Pencil code was able to provide the necessary tool for the numerical computing, it – as a fully three-dimensional – was a bit supernumerary for our purposes. Namely, as we needed only one-dimensional analysis and only certain 1-D related operations, the inclusion of un-needed extra dimensions and complex numerical algorithms were slowing down the simulations unnecessarily. Thus, we decided to write a new, lighter 1-D code basing on the Pencil code in order to reduce computing time in the forthcoming simulation runs.

The "dimension reduction" was performed so that one of the group leaders as well as one of the members started to work on the pre-existing code using CVS [2] for keeping track of changes and dealing with (and avoiding, as much as possible) conflicting modifications of the source codes. This approach turned out to be successful, and at the end of the week we had managed to do nearly all the modifications needed to have 1) modules for the physical interactions and properties, 2) mathematical formulas for needed derivatives and other operations, and 3) data handling and visualization procedures turned or rewritten to work in the 1-D scheme.

Taking into account the limited amount of time for working within the frames of the workshop, the (re)writing of the whole simulation code package was, obviously, too wide a project to be completed in order to have a working and tested code for the rest of the groups, so the contribution of the coding part of the work group to the overall research of the group remained minor. On the pedagogical side, on the other hand, this part of the project was very fruitful: getting in touch with larger scale coding projects and learning to use certain development and code writing tools effectively, together with seeing alternative uses for some familiar tools, was very instructing.

As for the future, there is still more work to do in order to get the code fully

[2] Concurrent Versioning System, see `https://www.cvshome.org/`

working in every aspects, including not only the dimensional conversion and rewriting of the code, but also testing and documentation as well; this project is very likely to continue beyond the workshop.

3.5 Regular and compound solutions in the 1D viscous case

We investigate numerically solutions to the MHD problem where the RHS of Eq. (1) do not vanish and the momentum equations will take the form of Eq. (7). Initially a regular shock is applied, and it is the aim of this section to look at the non-uniqueness in solutions for different values of α and ν, hence the angle between $\mathbf{B}_t$ before and after the shock and the viscosity.

In the case where we have the least viscosity the solutions have more numerical noise due to the discretization of the grid. We used grid sizes of hence 20000 and 50000 grid points in the x-direction due to the scheme described previously. The runs were carried out on a single CPU with the serial version of the Pencil Code and the results are shown in the figures below.

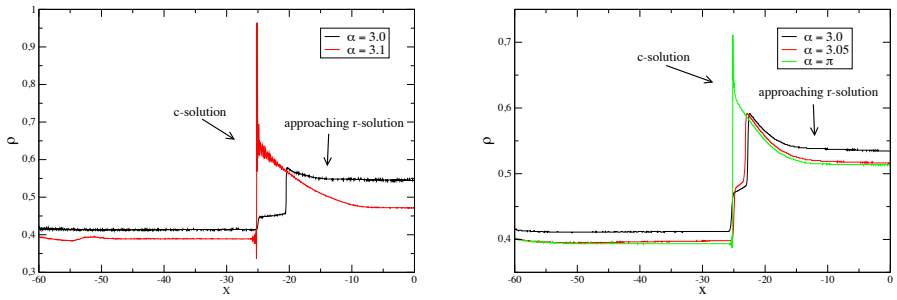


Figure 4: Numerical solutions with $\nu = 0.002$ and $\nu = 0.005$ with different α. The runs are done with 50000 grid points and took several hours on a single CPU.

In the results we see that for $\alpha = \pi$ and close to $\alpha = \pi$ the solution is compound like, while in the case of $alpha < 3.1$ the solutions tend towards a regular wave.

References

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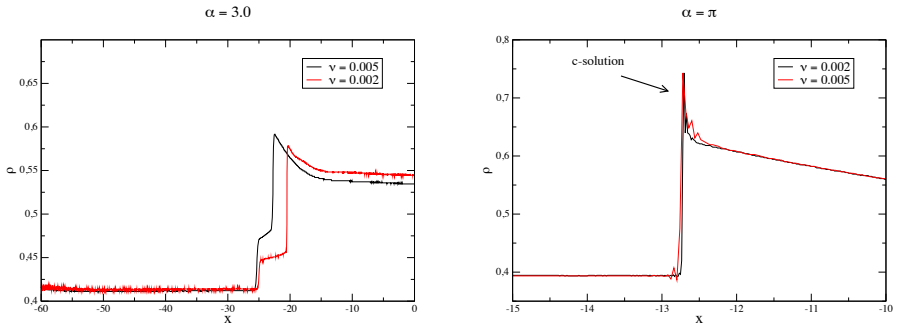

Figure 5: Numerical solutions with $\alpha = 3.0$ in the upper figure and $\alpha = \pi$ in the lower with different ν. The solutions for $\alpha = 3.0$ case is clearly tending towards a regular wave solutions.

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