Two examples of multi well-balanced schemes for shallow water type systems

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Outline

1. Shallow water systems and steady states
2. The shallow water MHD system
3. Shallow water system with volume fraction
4. Conclusion

▷ The shallow water system

(1)
$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x z = 0, \end{cases}$$

with $z(x)$ given, has the well-known steady states at rest

(2)
$$u = 0, \qquad h + z = cst.$$

There is only one degree of freedom in these steady states : if z is given then we deduce h (and u). We have three unknown h, u, z, and two relations in (2).

▷ Many works have been devoted to the construction of well-balanced schemes that preserve these steady states at rest.

▷ This remains an issue for related systems, in particular when several source terms are present. Examples are the Ripa model that has been studied in [Desveaux, Zenk, Berthon, Klingenberg], the Euler system with gravity [], the shallow water MHD system [Bouchut, Lhébrard], the Saint-Venant system with variable pressure, the Euler system with variable cross section...

▷ Here we consider systems with several families of steady states. We are going to show that in some cases it is possible to preserve several families of steady states at rest. And moreover to do it while satisfying a semi-discrete entropy ineqality.

The shallow water MHD system in 1d can be written as

$$\partial_t h + \partial_x(hu) = 0, \tag{3}$$

$$\partial_t(hu) + \partial_x(hu^2 + P) + gh\partial_x z - fhv = 0, \tag{4}$$

$$\partial_t(hv) + \partial_x(huv + P_\perp) + fhu = 0, \tag{5}$$

$$\partial_t(ha) + u\partial_x(ha) = 0, \tag{6}$$

$$\partial_t(hb) + \partial_x(hbu - hav) + v\partial_x(ha) = 0, \tag{7}$$

with

$$P = g\frac{h^2}{2} - ha^2, \quad P_\perp = -hab. \tag{8}$$

It has an energy inequality

$$\begin{aligned} &\partial_t\Big(\frac{1}{2}h(u^2+v^2) + \frac{1}{2}gh^2 + \frac{1}{2}h(a^2+b^2) + ghz\Big) \\ &+\partial_x\Big(\big(\tfrac{1}{2}h(u^2+v^2) + gh^2 + \tfrac{1}{2}h(a^2+b^2) + ghz\big)u - ha(au+bv)\Big) \le 0. \end{aligned} \tag{9}$$

The unknowns are h, u, v, a, b, and the topography z is given.

▷ The eigenvalues of the system are

$$u, u \pm |a|, u \pm \sqrt{a^2 + gh}. \quad (10)$$

The associated waves are called respectively material (or divergence) waves, Alfven waves and magnetogravity waves. There is an additional eigenvalue which is 0, and we shall call the associated wave the topography wave. The presence of the zero-order Coriolis terms proportional to f induces complex nonlinear waves. Here we shall assume that $f \equiv 0$.

▷ The system is nonconservative in the variables ha, hb. However ha jumps only through the material contacts, where u and v are continuous. Therefore, there is indeed no ambiguity in the non conservative products $u\partial_x(ha)$ and $v\partial_x(ha)$, that are well-defined. Concerning the nonconservative term $h\partial_x z$ in (4), it is well-defined for continuous topography z. Piecewise constant discontinuous z is considered however for discrete approximations.

▷ A striking property of the system is that four out of six of the waves are contact discontinuities, corresponding to linearly degenerate eigenvalues : the material contacts associated to the eigenvalue u, the left Alfven contacts associated to $u - |a|$, the right Alfven contacts associated to $u + |a|$, and the topography contacts associated to the eigenvalue 0. Resonance can occur, which means that these waves can collapse. It happens in particular when $u = 0$ or $u \pm |a| = 0$.

▷ Resonant steady states at rest for $a \neq 0$ satisfy

$$\begin{aligned} & u = 0, \quad v = cst, \quad hab = cst, \\ & \partial_x\left(g\frac{h^2}{2} - ha^2\right) + gh\partial_x z = 0. \end{aligned} \tag{11}$$

We have 6 unknowns h, u, v, a, b, z, and 4 relations in (11). Thus 2 degrees of freedom. The second line is a non integrable differential form : it cannot be characterized by a function of the unknown being constant. Thus there are several possible interpretations of nonconservative products.

Several subfamilies of steady states at rest are characterized by giving an additional relation between the unknowns :

▷ for $\sqrt{h}a = cst$ it gives

$$u = 0, \quad v = cst, \quad h + z = cst, \quad \sqrt{h}\,a = cst\ (\neq 0), \quad \sqrt{h}\,b = cst. \tag{12}$$

▷ for $ha = cst$ it gives

$$u = 0, \quad v = cst, \quad ha = cst\ (\neq 0), \quad b = cst, \quad h - \frac{a^2}{2g} + z = cst. \tag{13}$$

▷ Another family is for $a = 0$ (3 degrees of freedom, but integrable)

$$u = 0, \quad a = 0, \quad h + z = cst. \tag{14}$$

▷ We consider a finite volume scheme

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x_i}\Big(F_l(U_i^n, U_{i+1}^n, \Delta z_{i+1/2}) - F_r(U_{i-1}^n, U_i^n, \Delta z_{i-1/2})\Big), \tag{15}$$

where $U = (h, hu, hv, ha, hb)$ and $\Delta z_{i+1/2} = z_{i+1} - z_i$.

▷ We consider a cutoff parameter $\gamma \geq 1$ and we set $h_l^{\#} = 0$ if $h_l = 0$, otherwise for $h_l > 0$

$$\begin{cases} h_l^{\#} - \dfrac{a_l^2}{2g}\min\Big(\dfrac{h_l}{h_l^{\#}}, \gamma\Big)^2 = h_l - \dfrac{a_l^2}{2g} + z_l - z^{\#} & \text{if } h_l + (\gamma^2 - 1)\dfrac{a_l^2}{2g} \geq z^{\#} - z_l, \\ h_l^{\#} = 0 \text{ otherwise,} \end{cases} \tag{16}$$

with

$$z^{\#} = \max\left(z_l,\ z_r\right). \tag{17}$$

Indeed, the function $h \mapsto h - (a_l^2/2g)\min(h_l/h, \gamma)^2$ is increasing on $[0, \infty)$, and the condition on the data in (16) is for having a solution $h_l^{\#} \geq 0$ to the equation in the first line. In the case there is no nonnegative solution, we set $h_l^{\#} = 0$. Similarly we set on the right for $h_r > 0$

$$\begin{cases} h_r^{\#} - \dfrac{a_r^2}{2g}\min\Big(\dfrac{h_r}{h_r^{\#}}, \gamma\Big)^2 = h_r - \dfrac{a_r^2}{2g} + z_r - z^{\#} & \text{if } h_r + (\gamma^2 - 1)\dfrac{a_r^2}{2g} \geq z^{\#} - z_r, \\ h_r^{\#} = 0 \text{ otherwise.} \end{cases} \tag{18}$$

Then we have in any case

(19) $$0 \leq h_l^\# \leq h_l, \qquad 0 \leq h_r^\# \leq h_r.$$

We define then

(20) $$a_l^\# = \kappa_l a_l, \quad a_r^\# = \kappa_r a_r,$$

with

(21) $$\kappa_l = \min\Big(\frac{h_l}{h_l^\#}, \gamma\Big), \quad \kappa_r = \min\Big(\frac{h_r}{h_r^\#}, \gamma\Big),$$

(we set $\kappa_l = 1$ if $h_l = 0$, $\kappa_r = 1$ if $h_r = 0$), and

(22) $$U_l^\# = \big(h_l^\#, h_l^\# u_l, h_l^\# v_l, h_l^\# a_l^\#, h_l^\# b_l\big), \quad U_r^\# = \big(h_r^\#, h_r^\# u_r, h_r^\# v_r, h_r^\# a_r^\#, h_r^\# b_r\big).$$

The left and right numerical fluxes are finally defined by

(23) $$\begin{aligned} F_l(U_l, U_r, \Delta z) &= \mathcal{F}_l(U_l^\#, U_r^\#) \\ &+ \Big(0, g\frac{h_l^2}{2} - h_l a_l^2 - g\frac{h_l^{\#2}}{2} + \kappa_l h_l a_l^2, 0, \\ &\qquad \kappa_l\big((ha)_l^\# - (ha)_l\big) u_l, \big((ha)_l^\# - (ha)_l\big) v_l\Big) \\ &+ (\kappa_l - 1)\Big(0, 0, 0, \mathcal{F}_l^{ha}(U_l^\#, U_r^\#), 0\Big), \end{aligned}$$

and a similar definition for F_r, where $\mathcal{F}_l$ and $\mathcal{F}_r$ are numerical fluxes associated to the problem without topography.

Theorem 1 The scheme with the numerical fluxes F_l, F_r with the above reconstruction satisfies the following properties.

(i) It is conservative in the variables h and hv,

(ii) It is consistent with (3)-(7) for smooth solutions,

(iii) It keeps the positivity of h under the CFL condition $\Delta t A(U_l^{\#}, U_r^{\#}) \leq \frac{1}{2}\min(\Delta x_l, \Delta x_r)$ with $A(.,.)$ the maximum speed of the homogeneous solver,

(iv) It satisfies a semi-discrete energy inequality associated to (9),

(v) It is well-balanced with respect to steady material and Alfven contact discontinuities without jump in topography,

(vi) It is well-balanced with respect to the steady states (14) corresponding to material and Alfven resonance.

(vii) It is well-balanced with respect to the steady states (13) that satisfy

$$\max\left(\frac{h_l}{h_r}, \frac{h_r}{h_l}\right) \leq \gamma. \tag{24}$$

(viii) The relation $ha = cst$ is preserved by the scheme provided that at each interface the data satisfy

$$\max\left(\frac{h_l}{h_l^{\#}}, \frac{h_r}{h_r^{\#}}\right) \leq \gamma \quad \text{whenever } h_l > 0 \text{ and } h_r > 0. \tag{25}$$

The formulas defining the scheme are deduced from the discrete entropy inequality, as follows. At the continuous level, the energy inequality (9) can be written

(26) $$\partial_t \tilde{E} + \partial_x \tilde{G} \leq 0,$$

with

(27) $$\tilde{E}(U,z) = E(U) + ghz, \quad \tilde{G}(U,z) = G(U) + ghzu,$$

and

(28) $$\begin{aligned} E(U) &= \frac{1}{2}h(u^2+v^2) + \frac{1}{2}gh^2 + \frac{1}{2}h(a^2+b^2), \\ G(U) &= E(U)u + P(U)u + P_\perp(U)v. \end{aligned}$$

As before, $U = (h, hu, hv, ha, hb)$ and P, $P_\perp$ are defined by (8). The scheme without topography satisfies a fully discrete energy inequality. It implies that it satisfies also a semi-discrete energy inequality, under the form

(29) $$\begin{aligned} &G(U_r) + E'(U_r)\left(\mathcal{F}_r(U_l,U_r) - F(U_r)\right) \leq \mathcal{G}(U_l,U_r), \\ &\mathcal{G}(U_l,U_r) \leq G(U_l) + E'(U_l)\left(\mathcal{F}_l(U_l,U_r) - F(U_l)\right), \end{aligned}$$

for all values of U_l, U_r, where E' is the derivative of E with respect to U, and $\mathcal{G}(U_l,U_r)$ is a consistent energy flux.

Then, for the scheme with topography, the characterization of the semi-discrete energy inequality writes

$$\text{(30)} \qquad \begin{aligned} \tilde{G}(U_r, z_r) + \tilde{E}'(U_r, z_r)\,(F_r - F(U_r)) \leq \tilde{\mathcal{G}}(U_l, U_r, z_l, z_r), \\ \tilde{\mathcal{G}}(U_l, U_r, z_l, z_r) \leq \tilde{G}(U_l, z_l) + \tilde{E}'(U_l, z_l)\,(F_l - F(U_l))\,, \end{aligned}$$

where $\tilde{E}$ and $\tilde{G}$ are defined by (27), $\tilde{E}'$ is the derivative of $\tilde{E}$ with respect to U, and $\tilde{\mathcal{G}}$ is an unknown consistent numerical energy flux. Let us choose

$$\text{(31)} \qquad \tilde{\mathcal{G}}(U_l, U_r, z_l, z_r) = \mathcal{G}(U_l^\#, U_r^\#) + \mathcal{F}^h(U_l^\#, U_r^\#) g z^\#,$$

where $\mathcal{F}^h$ is the common h-component of $\mathcal{F}_l$ and $\mathcal{F}_r$, and for some $z^\#$ that is defined below. Then, noticing that $\tilde{E}'(U, z) = E'(U) + gz(1, 0, 0, 0, 0)$, we can write the desired inequalities (30) as

$$\text{(32)} \qquad \begin{aligned} & G(U_r) + E'(U_r)\,(F_r - F(U_r)) + \mathcal{F}^h(U_l^\#, U_r^\#) g z_r \\ & \leq \mathcal{G}(U_l^\#, U_r^\#) + \mathcal{F}^h(U_l^\#, U_r^\#) g z^\#, \end{aligned}$$

$$\text{(33)} \qquad \begin{aligned} & \mathcal{G}(U_l^\#, U_r^\#) + \mathcal{F}^h(U_l^\#, U_r^\#) g z^\# \\ & \leq G(U_l) + E'(U_l)\,(F_l - F(U_l)) + \mathcal{F}^h(U_l^\#, U_r^\#) g z_l. \end{aligned}$$

▷ By using (29) evaluated at $U_l^\#$, $U_r^\#$ and comparing the result with (32) and (33), we get the sufficient conditions

$$(34)\qquad \begin{aligned} & G(U_r) + E'(U_r)\,(F_r - F(U_r)) + \mathcal{F}^h(U_l^\#, U_r^\#) g z_r \\ \le\ & G(U_r^\#) + E'(U_r^\#)\left(\mathcal{F}_r(U_l^\#, U_r^\#) - F(U_r^\#)\right) + \mathcal{F}^h(U_l^\#, U_r^\#) g z^\#, \end{aligned}$$

$$(35)\qquad \begin{aligned} & G(U_l^\#) + E'(U_l^\#)\left(\mathcal{F}_l(U_l^\#, U_r^\#) - F(U_l^\#)\right) + \mathcal{F}^h(U_l^\#, U_r^\#) g z^\# \\ \le\ & G(U_l) + E'(U_l)\,(F_l - F(U_l)) + \mathcal{F}^h(U_l^\#, U_r^\#) g z_l. \end{aligned}$$

This leads to the sufficient condition for the left side of the interface

$$(36)\qquad \left(h_r - \frac{a_r^2}{2g} - h_r^\# + \kappa_r^2 \frac{a_r^2}{2g} + z_r - z^\#\right) \mathcal{F}^h(U_l^\#, U_r^\#) \le 0.$$

▷ These inequalities lead to the appropriate definition of the reconstructed states $U_l^\#$, $U_r^\#$.

▷ The principle of the construction implies the consistency.

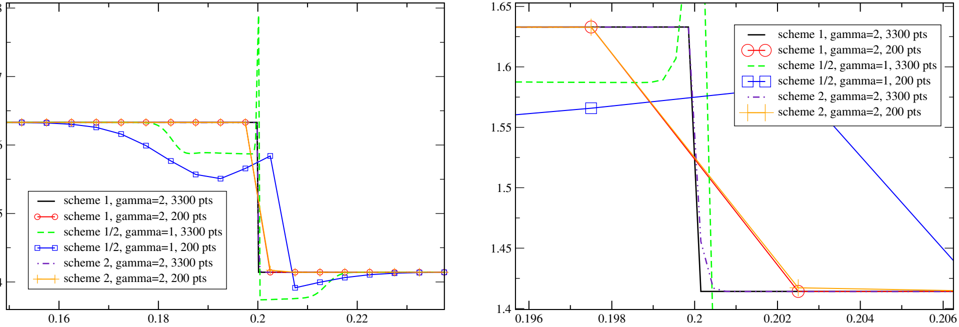

FIGURE – Zoom of component b for Test 1 at the material resonance at time $t = 0.02$ computed at first order, with either a high resolution of 3300 points or a low resolution of 200 points, with different values of γ, and either the first scheme [] or the second scheme of Theorem 1 (they are identical when $\gamma = 1$ and are denoted by scheme 1/2). The value $\gamma = 1$ leads to a slight overshoot while the value $\gamma = 2$ does not. The right picture is a further zoom of the left one. We observe that even at the material resonance, the schemes 1 and 2 give almost the same results, the difference can only be seen on the right picture for 200 points and $\gamma = 2$. The scheme 1 gives the exact solution, in accordance with Theorem 1 since here the contact discontinuity is of the type (12). The scheme 2 does not give the exact solution but is nevertheless extremely accurate.

▷ We consider the system

$$\partial_t(\varphi h) + \partial_x(\varphi h v) = 0, \tag{37}$$

$$\partial_t(\varphi h v) + \partial_x\left(\varphi h v^2 + \frac{1}{2} g_c \varphi h^2\right) + g_c \varphi h \partial_x z = 0, \tag{38}$$

$$\partial_t \varphi + v \partial_x \varphi = 0, \tag{39}$$

$h \geq 0$: thickness, $\varphi > 0$: volume fraction, v : velocity, $g_c > 0$: gravity, z : topography.

▷The system can be interpreted as the classical full gas dynamics system with $\gamma = 2$ (by setting the density to $\rho = \varphi h$) with force $-g_c \partial_x z$, where the roles of energy and entropy have been reversed.

▷ This system is relevant in the modeling of two-phase granular flows : φ is the volume fraction of solid in a solid/fluid mixture.

▷ A conservative form for (39) is

$$\partial_t(h\varphi^\alpha) + \partial_x(h\varphi^\alpha v) = 0, \tag{40}$$

for some α with $\alpha \neq 1$ in order to be independent of (37).

▷ Even for weak solutions with discontinuities, the conservation laws (40) (when α varies) are all equivalent, because of the combination with (37).

▷ The system is completed with an entropy (energy) inequality

$$\partial_t\Big(\varphi h\frac{v^2}{2} + g_c\varphi hz + g_c\varphi\frac{h^2}{2}\Big) + \partial_x\Big(\varphi h\frac{v^2}{2}v + g_c\varphi h(h+z)v\Big) \leq 0. \tag{41}$$

This energy is convex with respect to the conservative variables $(\varphi h, \varphi hv, h\varphi^\alpha)$ if and only if $\alpha \geq 1/2$ (and still $\alpha \neq 1$).

▷ Other entropy inequalities are

$$\partial_t\big(\varphi h\psi(\varphi^{\alpha-1})\big) + \partial_x\big(\varphi h\psi(\varphi^{\alpha-1})v\big) \leq 0, \quad \text{for } \psi \text{ convex}, \tag{42}$$

This provides the minimum and maximum principle on φ, by taking $\psi(\xi) = (k-\xi)_+$ and $\psi(\xi) = (\xi - k)_+$ for an arbitrary constant $k \geq 0$.

▷ The steady states at rest are characterized by

$$v = 0, \qquad \varphi \partial_x (h+z) + \frac{1}{2} h \partial_x \varphi = 0. \tag{43}$$

▷ 4 unknowns h, φ, v, z and 2 relations, thus 2 degrees of freedom.

▷ Several discontinuous solutions can be obtained depending on the way we understand the nonconservative products.

▷ If we impose an additional relation between h and φ, this defines a subfamily of steady states. We can choose in particular $\varphi = cst$ (first family), or $h\varphi^{1-\alpha} = cst$ (second family).

▷ We consider the conservative variable

$$U = \big(\varphi h, \varphi h v, h\varphi^{\alpha}\big) \tag{44}$$

for some fixed $\alpha > 1$.

▷ We consider a given scheme for the problem without topography

$$\mathcal{F}(U_l, U_r) = \big(\mathcal{F}^0(U_l, U_r), \mathcal{F}^1(U_l, U_r), \mathcal{F}^2(U_l, U_r)\big), \tag{45}$$

corresponding to the conservative problem with flux

$$F(U) = \big(\varphi h v, \varphi h v^2 + \frac{1}{2} g_c \varphi h^2, h\varphi^{\alpha} v\big). \tag{46}$$

We shall assume that the volume fraction flux is given by the classical upwind passive transport flux

$$\mathcal{F}^2(U_l, U_r) = \begin{cases} \mathcal{F}^0(U_l, U_r)\varphi_l^{\alpha-1} & \text{if } \mathcal{F}^0(U_l, U_r) \geq 0, \\ \mathcal{F}^0(U_l, U_r)\varphi_r^{\alpha-1} & \text{if } \mathcal{F}^0(U_l, U_r) \leq 0. \end{cases} \tag{47}$$

Example : the Suliciu relaxation solver.

▷ The entropy and entropy flux of the system are

$$\eta(U) = \varphi h \frac{v^2}{2} + g_c \varphi \frac{h^2}{2}, \qquad G(U) = \big(\varphi h \frac{v^2}{2} + g_c \varphi h^2\big) v. \tag{48}$$

The entropy and entropy flux of the system with topography are

$$\widetilde{\eta}(U, z) = \eta(U) + g_c \varphi h z, \qquad \widetilde{G}(U, z) = G(U) + g_c \varphi h z v. \tag{49}$$

▷ We define now the reconstructed states

$$
(50) \qquad h_l^* = \begin{cases} h_l - \dfrac{\alpha-1}{\alpha-1/2}\Delta z_+^* & \text{if } \dfrac{\alpha-1}{\alpha-1/2}\Delta z_+^* \leq h_l\Big(1-\big(\dfrac{\varphi_r}{\varphi_l}\big)^{\alpha-1}\Big)_+, \\ \Big(h_l + \dfrac{h_l}{2(\alpha-1)}\Big(1-\big(\dfrac{\varphi_r}{\varphi_l}\big)^{\alpha-1}\Big)_+ - \Delta z_+^*\Big)_+ & \text{otherwise,} \end{cases}
$$

$$
(51) \qquad h_r^* = \begin{cases} h_r - \dfrac{\alpha-1}{\alpha-1/2}\Delta z_-^* & \text{if } \dfrac{\alpha-1}{\alpha-1/2}\Delta z_-^* \leq h_r\Big(1-\big(\dfrac{\varphi_l}{\varphi_r}\big)^{\alpha-1}\Big)_+, \\ \Big(h_r + \dfrac{h_r}{2(\alpha-1)}\Big(1-\big(\dfrac{\varphi_l}{\varphi_r}\big)^{\alpha-1}\Big)_+ - \Delta z_-^*\Big)_+ & \text{otherwise.} \end{cases}
$$

Then we have

$$
(52) \qquad 0 \leq h_l^* \leq h_l, \quad 0 \leq h_r^* \leq h_r.
$$

We also define the reconstructed volume fractions

$$(53)\qquad \varphi_l^* = \begin{cases} \varphi_l & \text{if} \quad \Delta z \leq 0 \text{ or } \varphi_r \geq \varphi_l, \\ \max\Big(\varphi_l\big(\frac{h_l^*}{h_l}\big)^{1/(\alpha-1)}, \varphi_r\Big) & \text{if} \quad \Delta z \geq 0 \text{ and } \varphi_r \leq \varphi_l, \end{cases}$$

$$(54)\qquad \varphi_r^* = \begin{cases} \varphi_r & \text{if} \quad \Delta z \geq 0 \text{ or } \varphi_l \geq \varphi_r, \\ \max\Big(\varphi_r\big(\frac{h_r^*}{h_r}\big)^{1/(\alpha-1)}, \varphi_l\Big) & \text{if} \quad \Delta z \leq 0 \text{ and } \varphi_l \leq \varphi_r, \end{cases}$$

that satisfy

$$(55)\qquad \varphi_l^*, \varphi_r^* \in [\varphi_l, \varphi_r], \qquad \varphi_l^* \leq \varphi_l, \quad \varphi_r^* \leq \varphi_r.$$

Finally the reconstructed states are

$$(56)\qquad U_l^* = \big(\varphi_l^* h_l^*, \varphi_l^* h_l^* v_l, h_l^* (\varphi_l^*)^\alpha\big), \qquad U_r^* = \big(\varphi_r^* h_r^*, \varphi_r^* h_r^* v_r, h_r^* (\varphi_r^*)^\alpha\big).$$

The numerical fluxes are defined by

$$F_l(U_l, U_r, \Delta z) = \mathcal{F}(U_l^*, U_r^*) + \Big(0, g_c \varphi_l \frac{h_l^2}{2} - g_c \varphi_l^* \frac{(h_l^*)^2}{2}, 0\Big),$$

$$F_r(U_l, U_r, \Delta z) = \mathcal{F}(U_l^*, U_r^*) + \Big(0, g_c \varphi_r \frac{h_r^2}{2} - g_c \varphi_r^* \frac{(h_r^*)^2}{2}, 0\Big).$$

Theorem 2 Assume that the homogeneous solver $\mathcal{F}$ is upwind for the φ variable. Then the numerical scheme defined by the left/right numerical fluxes (19) satisfies
(i) It is conservative in the first and third components, and reduces to the homogeneous numerical flux $\mathcal{F}$ when $\Delta z = 0$.
(ii) It is well-balanced for the steady states at rest for which $\varphi = cst$.
(iii) It is well-balanced for the steady states at rest for which $h\varphi^{1-\alpha} = cst$.
(iv) The height h remains nonnegative if the homogeneous solver has this property.
(v) It satisfies a semi-discrete energy inequality if the homogeneous solver does.
(vi) It is consistent with the system.
(vii) If the initial volume fraction φ is constant, it remains constant.
(viii) The volume fraction φ satisfies the minimum principle (but not the maximum principle), which means that for any $k \geq 0$, the inequality $\varphi \geq k$ remains true if it holds initially.

▷ The principle of the reconstruction relies on writing the semi-discrete entropy inequality. It yields the sufficient condition for the left side of the interface condition

$$\mathcal{F}^0(U_l^*, U_r^*)\left(\frac{\alpha - 1/2}{\alpha - 1}(h_l^* - h_l) - \frac{1}{2(\alpha - 1)}\left(h_l^*(\varphi_l^*)^{1-\alpha} - h_l\varphi_l^{1-\alpha}\right)(\varphi_{l/r}^*)^{\alpha-1} + \Delta z_+^*\right) \leq 0.$$

▷ Then the definition of h_l^* and φ_l^* are such that this is identically zero when this is possible, taking into acount the necessary monotone variations.

▷ A generalized model has height h, velocity v, a transported variable r, and two topographies b_1, b_2.

$$\partial_t r + v\partial_x r = 0,$$

$$\partial_t h + \partial_x(hv) = 0,$$

$$h(\partial_t v + v\partial_x v) + ghM_1(r)\partial_x b_1 + ghM_2(r)\partial_x b_2 + g\partial_x(r\frac{h^2}{2}) = 0,$$

where $M_1(r)$ and $M_2(r)$ are two nonlinearities that play symmetric roles.

▷ The solutions at rest to the system are those for which $v = 0$ and

$$M_1(r)\partial_x b_1 + M_2(r)\partial_x b_2 + r\partial_x h + \frac{h}{2}\partial_x r = 0.$$

▷ This model includes alltogether the system with variable volume fraction with $M_1(r) = 1$, and the ripa model if $M_2(r) = r$. It is involved in the modeling of granular mixtures (work in progress).

▷ Defining robust well balanced schemes for shallow water type systems with several right-hand sides is difficult in general. There are multiple steady states, even if we consider only those at rest.

▷ If is not straightforward to define reconstructed states that manage correctly with several families of steady states, defining "multi well-balanced schemes".

▷ We have shown that using the formulation of the semi-discrete entropy inequality enables to find appropriate formulas in some cases : shallow water MHD system, Euler system with volume fraction.

▷ The method is quite systematic, in the sense that we can apply it without a priori idea of what could be the formula for the reconstructed states. It gives the consistency as a byproduct.