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BGK MODEL FOR TWO-COMPONENT GASES NEAR A GLOBAL MAXWELLIAN*

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Abstract. In this paper, we establish the existence of the unique global-in-time classical solutions to the two-component Bhatnagar–Gross–Krook (BGK) model suggested in [C. Klingenberg, M. Pirner, and G. Puppo, *Kinet. Relat. Models*, 10 (2017), pp. 445–465] when the initial data is a small perturbation of global equilibrium. For this, we carefully analyze the dissipative nature of the linearized two-component relaxation operator and observe that the partial dissipation from the intraspecies and the interspecies linearized relaxation operators are combined in a complementary manner to give rise to the desired dissipation estimate of the model. We also observe that the convergence rate of the distribution function increases as the momentum-energy interchange rate between the different components of the gas increases.

Key words. multicomponent gases, BGK model for multicomponent gas mixtures, Boltzmann equation for multicomponent gas mixtures, nonlinear energy method, classical solutions, asymptotic behavior

MSC codes. 82C40, 76P05, 35Q20, 35F16

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1. Introduction.

In this paper, we study the existence and the asymptotic behavior of the Bhatnagar–Gross–Krook (BGK) model for two-component gases suggested in [47]:

$$
\begin{aligned}
&\partial_t F_1 + v\cdot\nabla_x F_1 = n_1(\mathcal{M}_{11}-F_1) + n_2(\mathcal{M}_{12}-F_1),\\
&\partial_t F_2 + v\cdot\nabla_x F_2 = n_2(\mathcal{M}_{22}-F_2) + n_1(\mathcal{M}_{21}-F_2),\\
&F_1(x,v,0)=F_{10}(x,v), \qquad F_2(x,v,0)=F_{20}(x,v).
\end{aligned}
\tag{1.1}
$$

The distribution function $F_i(x,v,t)$ denotes the number density of the ith species particle at the phase point $(x,v)\in\mathbb{T}^3\times\mathbb{R}^3$ at time $t\in\mathbb{R}^+$ for $i=1,2$. The intraspecies

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Maxwell distributions in the BGK operator $\mathcal{M}_{ii}$ are defined as

$$\mathcal{M}_{ii} = \frac{n_i}{\sqrt{2\pi \frac{T_i}{m_i}}^3} \exp\left(-\frac{|v-U_i|^2}{2\frac{T_i}{m_i}}\right) \quad (i=1,2).$$

Here m_i $(i=1,2)$ denotes the mass of a molecule in the ith component, and we assume that $m_1 \geq m_2$ throughout the paper without loss of generality. The number density n_i, the bulk velocity U_i, and the temperature T_i of the ith particle are defined by

$$\begin{aligned} n_i(x,t) &= \int_{\mathbb{R}^3} F_i(x,v,t)dv, \\ U_i(x,t) &= \frac{1}{n_i}\int_{\mathbb{R}^3} F_i(x,v,t)vdv, \\ T_i(x,t) &= \frac{1}{3n_i}\int_{\mathbb{R}^3} F_i(x,v,t)m_i|v-U_i|^2dv. \end{aligned}$$

The interspecies Maxwellian distributions are defined by

$$\mathcal{M}_{12} = \frac{n_1}{\sqrt{2\pi \frac{T_{12}}{m_1}}^3} \exp\left(-\frac{|v-U_{12}|^2}{2\frac{T_{12}}{m_1}}\right), \qquad \mathcal{M}_{21} = \frac{n_2}{\sqrt{2\pi \frac{T_{21}}{m_2}}^3} \exp\left(-\frac{|v-U_{21}|^2}{2\frac{T_{21}}{m_2}}\right),$$

where the interspecies bulk velocities U_{12}, U_{21} and the interspecies temperatures T_{12}, T_{21} are defined by

$$\begin{aligned} U_{12} &= \delta U_1 + (1-\delta)U_2, \\ U_{21} &= \frac{m_1}{m_2}(1-\delta)U_1 + \left(1-\frac{m_1}{m_2}(1-\delta)\right)U_2, \end{aligned}$$

and

$$\begin{aligned} T_{12} &= \omega T_1 + (1-\omega)T_2 + \gamma|U_2-U_1|^2, \\ T_{21} &= (1-\omega)T_1 + \omega T_2 + \left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right) - \gamma\right)|U_2-U_1|^2. \end{aligned}$$

Here, the free parameters δ and ω denote the momentum interchange rate and the temperature interchange rate, respectively. In (1.1), $n_i(\mathcal{M}_{ii}-F_i)$ $(i=1,2)$ are the intraspecies relaxation operators for the ith gas component, while $n_j(\mathcal{M}_{ij}-F_i)$ $(i \neq j)$ are the interspecies relaxation operators between different components of the gas. We note that the interspecies relaxation operators describe the interchange of the macroscopic momentum and the temperature between two different species of gas. These relaxation operators satisfy the cancellation properties

$$\begin{aligned} &\int_{\mathbb{R}^3} (\mathcal{M}_{ii}-F_i)\left(1, m_i v, m_i|v|^2\right)dv = 0, \quad i=1,2 \\ &\int_{\mathbb{R}^3} (\mathcal{M}_{12}-F_1)dv = 0, \quad \int_{\mathbb{R}^3} (\mathcal{M}_{21}-F_2)dv = 0, \\ &\int_{\mathbb{R}^3} n_1(\mathcal{M}_{12}-F_1)m_1 v dv + \int_{\mathbb{R}^3} n_2(\mathcal{M}_{21}-F_2)m_2 v dv = 0, \\ &\int_{\mathbb{R}^3} n_1(\mathcal{M}_{12}-F_1)m_1|v|^2 dv + \int_{\mathbb{R}^3} n_2(\mathcal{M}_{21}-F_2)m_2|v|^2 dv = 0, \end{aligned}$$

leading to the following conservation laws of the density, total momentum, and total energy:

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F_1(x,v,t)dvdx = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F_2(x,v,t)dvdx = 0,\\
&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} \left(F_1(x,v,t)m_1v + F_2(x,v,t)m_2v\right)dvdx = 0,\\
&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} \left(F_1(x,v,t)m_1|v|^2 + F_2(x,v,t)m_2|v|^2\right)dvdx = 0.
\end{aligned}\tag{1.2}
$$

To ensure the positivity of all temperatures, the free parameters ω, δ, and γ are restricted to

$$
\frac{\frac{m_1}{m_2}-1}{1+\frac{m_1}{m_2}} \le \delta < 1, \qquad 0 \le \omega < 1,
$$

and

$$
0 \le \gamma \le \frac{m_1}{3}(1-\delta)\left[\left(1+\frac{m_1}{m_2}\right)\delta + 1 - \frac{m_1}{m_2}\right].
$$

For more details, see [47].

The main goal of this paper is to establish the global-in-time classical solution of the mixture BGK model when the initial data is close to global equilibrium. For this, we consider the following global equilibrium for each particle distribution function:

$$
\mu_1(v) = n_{10}\frac{\sqrt{m_1}^3}{\sqrt{2\pi}^3}e^{-\frac{m_1|v|^2}{2}}, \qquad \mu_2(v) = n_{20}\frac{\sqrt{m_2}^3}{\sqrt{2\pi}^3}e^{-\frac{m_2|v|^2}{2}},
$$

for some fixed parameters m_k and n_{k0}, $k=1,2$. We note that n_{k0} is the total mass of the kth component.

We then define the perturbations f_k $(k=1,2)$ by $F_k = \mu_k + \sqrt{\mu_k}f_k$ and rewrite the mixture BGK model (1.1) in terms of f_k as

$$
\begin{aligned}
&\partial_t f_1 + v\cdot\nabla_x f_1 = L_{11}(f_1) + L_{12}(f_1,f_2) + \Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2),\\
&\partial_t f_2 + v\cdot\nabla_x f_2 = L_{22}(f_2) + L_{21}(f_1,f_2) + \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2).
\end{aligned}\tag{1.3}
$$

On the right-hand side (R.H.S), L_{11} and L_{22} denote the linearized part of the intraspecies relaxation operators and L_{12} and L_{21} are the linearized operators for interspecies relaxation operators that include terms describing the interchange of momentum and energy. Finally, Γ_{11}, Γ_{22}, Γ_{12}, and Γ_{21} are nonlinear perturbations. For detailed derivation of (1.3), see section 2.

We introduce

$$
L(f_1,f_2) = (L_{11}(f_1) + L_{12}(f_1,f_2), L_{22}(f_2) + L_{21}(f_1,f_2)),
$$

and

$$
\Gamma(f_1,f_2) = (\Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2), \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2)),
$$

to rewrite (1.3) in the following succinct form:

$$
(\partial_t + v\cdot\nabla_x)(f_1,f_2) = L(f_1,f_2) + \Gamma(f_1,f_2).
$$

To state our main result, we need to set up several notations.

- The constant C in the estimates will be defined generically.
- $\langle\cdot,\cdot\rangle_{L^2_v}$ and $\langle\cdot,\cdot\rangle_{L^2_{x,v}}$ denote the standard L^2 inner product on $\mathbb{R}^3_v$ and $\mathbb{T}^3_x\times\mathbb{R}^3_v$, respectively:

$$\langle f,g\rangle_{L^2_v}=\int_{\mathbb{R}^3} f(v)g(v)dv, \quad \langle f,g\rangle_{L^2_{x,v}}=\int_{\mathbb{T}^3\times\mathbb{R}^3} f(x,v)g(x,v)dvdx.$$

- $\|\cdot\|_{L^2_v}$ and $\|\cdot\|_{L^2_{x,v}}$ denote the standard L^2 norms in $\mathbb{R}^3_v$ and $\mathbb{T}^3_x\times\mathbb{R}^3_v$, respectively:

$$\|f\|_{L^2_v}\equiv\left(\int_{\mathbb{R}^3}|f(v)|^2dv\right)^{\frac{1}{2}}, \quad \|f\|_{L^2_{x,v}}\equiv\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}|f(x,v)|^2dvdx\right)^{\frac{1}{2}}.$$

- We define an L^2 inner product between two vectors (f_1,f_2) and (g_1,g_2) as

$$\langle (f_1,f_2),(g_1,g_2)\rangle_{L^2_v}=\int_{\mathbb{R}^3}\left(f_1(v)g_1(v)+f_2(v)g_2(v)\right)dv,$$
$$\langle (f_1,f_2),(g_1,g_2)\rangle_{L^2_{x,v}}=\int_{\mathbb{T}^3\times\mathbb{R}^3}\left(f_1(x,v)g_1(x,v)+f_2(x,v)g_2(x,v)\right)dvdx.$$

- The standard L^2 norm of a vector denotes

$$\|\left(f(x,v),g(x,v)\right)\|_{L^2_v}=\left(\int_{\mathbb{R}^3}\left(|f(v)|^2+|g(v)|^2\right)dv\right)^{\frac{1}{2}},$$
$$\|\left(f(x,v),g(x,v)\right)\|_{L^2_{x,v}}=\left(\int_{\mathbb{T}^3\times\mathbb{R}^3}\left(|f(x,v)|^2+|g(x,v)|^2\right)dvdx\right)^{\frac{1}{2}}.$$

- We use the following notations for multi-indices differential operators:

$$\alpha=[\alpha_0,\alpha_1,\alpha_2,\alpha_3], \quad \beta=[\beta_1,\beta_2,\beta_3],$$

  and

$$\partial^\alpha_\beta=\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{v_1}^{\beta_1}\partial_{v_2}^{\beta_2}\partial_{v_3}^{\beta_3}.$$

- We employ the following convention for simplicity:

$$\partial^\alpha_\beta(f_1,f_2)=\left(\partial^\alpha_\beta f_1,\partial^\alpha_\beta f_2\right).$$

- We define the high-order energy norm $\mathcal{E}_{N_1,N_2}(f_1(t),f_2(t))$:

$$\mathcal{E}_{N_1,N_2}(f_1(t),f_2(t))=\sum_{\substack{|\alpha|\le N_1,\ |\beta|\le N_2\\ N_1+N_2=N}}\|\partial^\alpha_\beta\left(f_1(t),f_2(t)\right)\|^2_{L^2_{x,v}}.$$

  For notational simplicity, we use $\mathcal{E}(t)$ to denote $\mathcal{E}_{N_1,N_2}(f_1(t),f_2(t))$ when the dependency on (N_1,N_2) is not relevant.

We are now ready to state our main result.

THEOREM 1.1. *Let $N\ge 3$. We set the macroscopic quantities of the initial data to the same as that of the global equilibria:*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3}F_{k0}(x,v)\begin{pmatrix}1\\ m_kv\\ m_k|v|^2\end{pmatrix}dvdx=\int_{\mathbb{T}^3\times\mathbb{R}^3}\mu_k(v)\begin{pmatrix}1\\ m_kv\\ m_k|v|^2\end{pmatrix}dvdx,$$

for $k=1,2$. We define f_{k0} as $F_{k0}=\mu_k+\sqrt{\mu_k}f_{k0}$. Then there exists $\epsilon>0$ such that if $\mathcal{E}_{N_1,N_2}(f_{10},f_{20})<\epsilon$, then there exists a unique global-in-time classical solution of (1.1) *satisfying the following:*

- *The two distribution functions are nonnegative:*

$$F_k(x,v,t) = \mu_k + \sqrt{\mu_k} f_k \geq 0.$$

- *The conservation laws hold* (1.2).
- *The distribution functions converge exponentially to the global equilibrium:*

$$\mathcal{E}_{N_1,N_2}(f_1,f_2)(t) \leq Ce^{-\eta t}\mathcal{E}_{N_1,N_2}(f_{10},f_{20}).$$

In the case of $N_2 = 0$, that is, if $\mathcal{E}_{N_1,0}(f_{10},f_{20}) < \epsilon$, we have the following more detailed convergence estimate:

$$\mathcal{E}_{N_1,0}(f_1,f_2)(t) \leq Ce^{-\eta \min\{(1-\delta),(1-\omega)\}t}\mathcal{E}_{N_1,0}(f_{10},f_{20}).$$

- *Let (f_1,f_2) and $(\bar{f}_1,\bar{f}_2)$ be solutions corresponding to the initial data (f_{10},f_{20}) and $(\bar{f}_{10},\bar{f}_{20})$, respectively, then the system satisfies the following L^2 stability:*

$$\begin{aligned}\sum_{|\alpha|\leq N} &\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|_{L^2_{x,v}} \\ &\leq C\sum_{|\alpha|\leq N}\|\partial^\alpha(f_{10}-\bar{f}_{10}),\partial^\alpha(f_{20}-\bar{f}_{20})\|_{L^2_{x,v}}\end{aligned}$$

Remark 1.2. (1) The convergence rate in the case of $N_2 = 0$ shows that the higher interchange rate (δ and ω close to 0) gives the faster convergence rates. (2) In the mathematical perspective L^2 energy analysis is relevant in that the dissipative property of the linearized operator can be captured only in L^2 space. High-order regularity is required to close the nonlinear terms using Sobolev embeddings. From the physical point of view, the assumption on the small high-order energy norm on the initial data means that initially the gas velocity distribution is a small perturbation of the global equilibrium.

The main result is obtained by an extended application of the high-order energy methods for BGK models in [71, 74], which in turn is based on the energy methods developed to study the existence and asymptotic behavior of classical solutions for Boltzmann type equations near equilibriums [38, 39, 40]. The most important step in such energy methods is the derivation of the full dissipative mechanism of the linearized operator. Unlike the results mentioned above, we observe that the partial dissipation from the intraspecies and the interspecies linearized relaxation operators has to be combined in a complementary manner to obtain the desired dissipation estimate of the model. We also observe that the energy-momentum interchange rate is directly related to the asymptotic convergence rate. More precisely, to investigate the dissipative property of L, we decompose the linearized interspecies relaxation operator L_{ij} $(i \neq j)$ further into the mass interaction part L^1_{ij} and the momentum-energy interaction part L^2_{ij}.

so that $L_{12} = L^1_{12} + L^2_{12}$ and $L_{21} = L^1_{21} + L^2_{21}$. (Precise definitions of L^k_{12} and L^k_{21} will be given in section 2 for $(k = 1,2)$.) We first derive from an explicit computation that the intraspecies operator L_{ii} and the mass interaction part of the interspecies operator L^1_{12} and L^1_{21} give rise to the following partial dissipative estimate:

$$\begin{aligned}&\langle (L_{11}+L^1_{12})f_1,f_1\rangle_{L^2_{x,v}} + \langle (L_{22}+L^1_{21})f_2,f_2\rangle_{L^2_{x,v}} \\ &\quad = -(n_{10}+n_{20})\|(I-P_1,I-P_2)(f_1,f_2)\|^2_{L^2_{x,v}},\end{aligned} \tag{1.4}$$

where I is the identity operator, and P_k $(k=1,2)$ is the L^2 projection onto the linear space spanned by

$$\left\{\sqrt{\mu_k}, v\sqrt{\mu_k}, |v|^2\sqrt{\mu_k}\right\}.$$

We also used the notation $(I-P_1, I-P_2)(f_1,f_2) = (f_1 - P_1f_1, f_2 - P_2f_2)$ for simplicity. We note that the dissipation estimate above is too weak in that it involves 10-dimensional degeneracy, which is 4-dimensional bigger than the 6-dimensional conservation laws in (1.2). It is the additional dissipation from the momentum-energy interaction parts L^2_{12}, L^2_{21} of the interspecies operators L_{12} and L_{21} that make up for the deficiency,

$$\begin{aligned}\langle L^2_{12}, f_1\rangle_{L^2_{x,v}} + \langle L^2_{21}, f_2\rangle_{L^2_{x,v}} \leq &-\min\left\{(1-\delta),(1-\omega)\right\}(n_{10}+n_{20}) \\ &\times \Big(\|(P_1,P_2)(f_1,f_2)\|^2_{L^2_{x,v}} - \|P(f_1,f_2)\|^2_{L^2_{x,v}}\Big),\end{aligned} \tag{1.5}$$

where P is an orthonormal $L^2\times L^2$ projection on the space spanned by the following 6-dimensional basis:

$$\{(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),(m_1v\sqrt{\mu_1}, m_2v\sqrt{\mu_2}),\big((m_1|v|^2-3)\sqrt{\mu_1},(m_2|v|^2-3)\sqrt{\mu_2}\big)\}.$$

Then partial dissipation estimates (1.4) and (1.5) complement each other to give rise to the following two-component dissipation estimate for L:

$$\begin{aligned}\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} \leq -(n_{10}+n_{20})\Big(&\max\{\delta,\omega\}\|(I-P_1,I-P_2)(f_1,f_2)\|^2_{L^2_{x,v}} \\ &+\min\left\{(1-\delta),(1-\omega)\right\}\|(I-P)(f_1,f_2)\|^2_{L^2_{x,v}}\Big).\end{aligned} \tag{1.6}$$

The dissipation estimate (1.6), together with further analysis on the degeneracy part through the standard micro-macro decomposition, provides the following full coercivity depending on the interchange rates:

$$\langle L(\partial^\alpha(f_1,f_2)),\partial^\alpha(f_1,f_2)\rangle_{L^2_{x,v}} \leq -\eta\min\left\{(1-\delta),(1-\omega)\right\}\sum_{|\alpha|\leq N}\|(\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}.$$

Due to the presence of the momentum interchange rate δ and the energy interchange rate ω between different components in the dissipation estimate, we see that the larger interchange rate (when δ and ω are close to zero) leads to the stronger dissipation and, therefore, the faster convergence to the global equilibrium:

$$\sum_{|\alpha|\leq N}\|\partial^\alpha(f_1(t),f_2(t))\|^2_{L^2_{x,v}} \leq e^{-\eta\min\{(1-\delta),(1-\omega)\}t}\sum_{|\alpha|\leq N}\|\partial^\alpha(f_1(0),f_2(0))\|^2_{L^2_{x,v}}.$$

1.1. Literature review. We start with a review of the mathematical results of the mono-species BGK model. Perthame established the first result on global weak solutions for general initial data in [52]. In [53], the authors considered weighted-L^∞ bounds to obtain the uniqueness. Desvillettes considered the convergence to equilibrium in a weak sense [26]. Ukai proved the existence of the stationary solution on a finite interval with inflow boundary condition in [67]. In [75], the L^∞ work in [52] is generalized to an weighted L^p space. Classical solutions near-global equilibrium is constructed in [7] using the spectral analysis of Ukai [66] and by using the nonlinear energy method of Yan Guo [38, 39, 40] in [71]. The nonlinear energy method is then employed further to study several types of BGK models [5, 6, 44, 71, 73, 74]. Saint-Raymond considered the hydrodynamic limits of the BGK model in [58, 59]. For the numerical study of the BGK model, we refer to [8, 14, 22, 23, 24, 25, 48, 56, 57].

Various BGK models to describe the dynamics of multicomponent gases are proposed in the literature. The first mixture BGK model was suggested by Gross and Krook in [37] where conditions required for the two-species conservation laws are presented without precise definitions of equilibrium coefficients. Several physicists have proposed possible recipes for such equilibrium coefficients for the two-species BGK model in [35, 43, 60]. Extension to the N-species model was made in [33]. The first consistent multispecies BGK model was derived in [1], in which the authors used only a single relaxation operator instead of the combination of inter- and intraspecies relation operators. Fick's law and Newton's law are considered in [17] under the restriction $T_{ij} = T_{ji}$ for $i \neq j$ $(i, j = 1, \ldots, N)$. The mixture BGK model imitating the velocity and temperature relaxation of the mixture Boltzmann equation is suggested in [42] with the restriction $U_{ij} = U_{ji}$ and $T_{ij} = T_{ji}$. The authors in [47] suggested the mixture BGK model that can control the interchange rate of momentum and temperature through free parameters in a binary mixture. The arbitrary N-species mixture BGK model imitating the interchange of the momentum and temperature of the Boltzmann equation was suggested in [13].

The BGK model for gas mixtures has also been extended to the ES-BGK model, polyatomic molecules, chemical reactions, or the quantum case; see, for example, [4, 11, 12, 36, 46, 48, 55, 62, 72]. For the applications of the mixture BGK models, we refer to [8, 9, 14, 27, 28, 31, 54, 56]. The literature on mathematical analysis of the mixture BGK models is limited. The existence of mild solutions for [47] can be found in [45]. In [49], the authors prove exponential relaxation to equilibrium with explicit rates by constructing an entropy functional. The hypocoercivity for the model [47] is investigated in [49].

A review of the multispecies Boltzmann equation is in order. In [39], the author established the global existence for the mixture of a charged particle described by the Vlasov–Maxwell–Boltzmann equation. The mild solution and uniform L^1 stability are obtained in [41]. A mass diffusion problem of the mixture and the cross-species resonance is studied for a one-dimensional case in [61] based on the work in [50]. In [16], the author constructed the global-in-time mild solution near-global equilibrium for the mixture Boltzmann equation. The Vlasov–Poisson–Boltzmann equation was considered in [30] about large time asymptotic profiles when the different-species gases tend to two distinct global Maxwellians. In [32], the existence and uniqueness are constructed in spatially homogeneous settings when an initial data has upper and lower bounds for some polynomial moments. The authors in [15] obtained some energy estimates.

For physical or engineering references on the studies on multicomponent gases at the kinetic level, we refer to [2, 3, 51, 60, 61, 63, 64, 65, 67, 68, 70]. Some general reviews of the Boltzmann and the BGK model can be found in [10, 18, 19, 20, 21, 29, 34, 69].

This paper is organized as follows. In section 2, we linearized the system (1.1) to obtain (1.3). In section 3, we derive the dissipation estimate of the linearized relaxation operator. The local-in-time classical solution is constructed in section 4. In section 5, the full coercivity of L is recovered when the energy norm is sufficiently small. Last, we establish the global-in-time classical solution in section 6.

2. Linearization of the mixture BGK model.

2.1. Linearization of the mixture Maxwellian. In this part, we linearize the interspecies Maxwellian $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$. We first define the macroscopic projection on L^2_v and state the linearization result of the monospecies local Maxwellian $\mathcal{M}_{kk}$.

DEFINITION 2.1. *We define the macroscopic projection operator P_k in L^2_v for $k=1,2$:*

$$P_k f = \frac{1}{n_{k0}} \int_{\mathbb{R}^3} f\sqrt{\mu_k}dv\sqrt{\mu_k} + \frac{m_k}{n_{k0}} \int_{\mathbb{R}^3} fv\sqrt{\mu_k}dv \cdot v\sqrt{\mu_k} + \frac{1}{6n_{k0}} \int_{\mathbb{R}^3} f(m_k|v|^2-3)\sqrt{\mu_k}dv(m_k|v|^2-3)\sqrt{\mu_k}.$$

We denote the 5-dimensional basis as ($i=2,3,4$)

$$e_{k1} = \frac{1}{\sqrt{n_{k0}}}\sqrt{\mu_k}, \qquad e_{ki} = \sqrt{\frac{m_k}{n_{k0}}}v_{i-1}\sqrt{\mu_k}, \qquad e_{k5} = \frac{m_k|v|^2-3}{\sqrt{6n_{k0}}}\sqrt{\mu_k}. \tag{2.1}$$

The 5-dimensional basis set $\{e_{1i}\}_{i=1,\dots,5}$ and $\{e_{2i}\}_{i=1,\dots,5}$ construct an orthonormal basis in L^2_v, respectively. So, we can write

$$P_1 f = \sum_{1\le i\le 5} \langle f, e_{1i}\rangle_{L^2_v} e_{1i} \quad \textit{and} \quad P_2 f = \sum_{1\le i\le 5} \langle f, e_{2i}\rangle_{L^2_v} e_{2i}.$$

LEMMA 2.2 (see [71]). *The monospecies BGK Maxwellian $\mathcal{M}_{kk}$ is linearized as follows:*

$$\mathcal{M}_{kk}(F_k) = \mu_k + \sqrt{\mu_k}P_k f_k + \sqrt{\mu_k}\,\Gamma_{kk}(f_k,f_k),$$

where the nonlinear term $\Gamma_{kk}(f_k,f_k)$ is given by

$$\Gamma_{kk}(f_k,f_k) = \sum_{1\le i,j\le 5} \frac{1}{\sqrt{\mu_k}} \int_0^1 \frac{P_{ij}(n_{k\theta},U_{k\theta},T_{k\theta},v-U_{k\theta},U_{k\theta})}{R_{ij}(n_{k\theta},T_{k\theta})}\mathcal{M}''_{kk}(\theta)(1-\theta)d\theta \times \langle f_k, e_{ki}\rangle_{L^2_v}\langle f_k, e_{kj}\rangle_{L^2_v}$$

for $k=1,2$. The function $P_{ij}(x_1,\dots,x_5)$ denotes a generic polynomial depending on $(x_1,\dots,x_5)$ and $R_{ij}(x,y)$ denotes a generic monomial $R_{ij}(x,y)=x^ny^m$, where $n,m\in\mathbb{N}\cup\{0\}$. The precise definition of the macroscopic fields $(n_{k\theta},U_{k\theta},T_{k\theta})$ will be given in Proposition 2.3.

Proof. The linearization of the monospecies BGK Maxwellian $\mathcal{M}_{kk}$ is in [71] for the case $n_{k0}=1$ and $m_k=1$. For a general n_{k0} and m_k, the linearization of $\mathcal{M}_{kk}$ is a special case of the linearization of $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$ with the choice $\delta=\omega=1$ (see (2.10) and (2.11), respectively). ∎

PROPOSITION 2.3. *The two-species BGK Maxwellians $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$ are linearized as follows:*

$$\mathcal{M}_{12}(F) = \mu_1 + P_1 f_1\sqrt{\mu_1} + (1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v} - \langle f_1,e_{1i}\rangle_{L^2_v}\right)e_{1i}\sqrt{\mu_1} + (1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v} - \langle f_1,e_{15}\rangle_{L^2_v}\right)e_{15}\sqrt{\mu_1} + \sqrt{\mu_1}\Gamma_{12}(f_1,f_2),$$

and

$$\mathcal{M}_{21}(F)=\mu_2+P_2f_2\sqrt{\mu_2}+\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1,e_{1i}\rangle_{L^2_v}-\langle f_2,e_{2i}\rangle_{L^2_v}\right)e_{2i}\sqrt{\mu_2}$$
$$+(1-\omega)\left(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1,e_{15}\rangle_{L^2_v}-\langle f_2,e_{25}\rangle_{L^2_v}\right)e_{25}\sqrt{\mu_2}+\sqrt{\mu_2}\Gamma_{21}(f_1,f_2).$$

We give the precise definition of the nonlinear terms Γ_{12} *and* Γ_{21} *in section* 2.2.

Proof. We first define the following transition of the macroscopic fields, which shows the transition from the macroscopic fields of global Maxwellian ($\theta=0$) to the macroscopic fiels of the local Maxwellian ($\theta=1$) (we note that $(n_{k\theta},U_{k\theta},T_{k\theta})|_{\theta=1}=(n_k,U_k,T_k)$ and $(n_{k\theta},U_{k\theta},T_{k\theta})|_{\theta=0}=(n_{k0},0,1)$):

$$n_{k\theta}=\theta n_k+(1-\theta)n_{k0},\quad n_{k\theta}U_{k\theta}=\theta n_kU_k,\quad G_{k\theta}=\theta G_k, \tag{2.2}$$

where

$$G_k=\frac{3n_kT_k+m_kn_k|U_k|^2-3n_k}{\sqrt{6}},$$

for $k=1,2$. We also denote two-species macroscopic fields as

(2.3)
$$\begin{aligned}
&U_{12\theta}=\delta U_{1\theta}+(1-\delta)U_{2\theta},\\
&U_{21\theta}=\frac{m_1}{m_2}(1-\delta)U_{1\theta}+\left(1-\frac{m_1}{m_2}(1-\delta)\right)U_{2\theta},\\
&T_{12\theta}=\omega T_{1\theta}+(1-\omega)T_{2\theta}+\gamma|U_{2\theta}-U_{1\theta}|^2,\\
&T_{21\theta}=(1-\omega)T_{1\theta}+\omega T_{2\theta}+\left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right)-\gamma\right)|U_{2\theta}-U_{1\theta}|^2.
\end{aligned}$$

Then we consider the two-species BGK Maxwellians $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$, which depend on θ:

$$\mathcal{M}_{12}(\theta)=\frac{n_{1\theta}}{\sqrt{2\pi\frac{T_{12\theta}}{m_1}}^3}\exp\left(-\frac{|v-U_{12\theta}|^2}{2\frac{T_{12\theta}}{m_1}}\right),$$
$$\mathcal{M}_{21}(\theta)=\frac{n_{2\theta}}{\sqrt{2\pi\frac{T_{21\theta}}{m_2}}^3}\exp\left(-\frac{|v-U_{21\theta}|^2}{2\frac{T_{21\theta}}{m_2}}\right).$$

The definition of $n_{k\theta},U_{k\theta},T_{k\theta}$ gives

$$(n_{k\theta},U_{k\theta},T_{k\theta})\,|_{\theta=1}=(n_k,U_k,T_k)\quad \textit{and}\quad (n_{k\theta},U_{k\theta},T_{k\theta})\,|_{\theta=0}=(n_{k0},0,1),$$

so we have

$$\mathcal{M}_{12}(1)=\mathcal{M}_{12},\quad \mathcal{M}_{12}(0)=\mu_1,$$

and

$$\mathcal{M}_{21}(1)=\mathcal{M}_{21},\quad \mathcal{M}_{21}(0)=\mu_2,$$

where we used $U_{120} = U_{210} = 0$ and $T_{120} = T_{210} = 1$. We apply the Taylor expansion to $\mathcal{M}_{12}(\theta)$ and $\mathcal{M}_{21}(\theta)$:

$$\mathcal{M}_{12}(1) = \mu_1 + \mathcal{M}'_{12}(0) + \int_0^1 \mathcal{M}''_{12}(\theta)(1-\theta)d\theta,$$

and

$$\mathcal{M}_{21}(1) = \mu_2 + \mathcal{M}'_{21}(0) + \int_0^1 \mathcal{M}''_{21}(\theta)(1-\theta)d\theta.$$

By the chain rule, we compute the linear term $\mathcal{M}'_{ij}(0)$:

(2.4)

$$\mathcal{M}'_{ij}(0) = \left(\frac{d(n_{1\theta}, n_{1\theta}U_{1\theta}, G_{1\theta}, n_{2\theta}, n_{2\theta}U_{2\theta}, G_{2\theta})}{d\theta}\right)^T \times \left(\frac{\partial(n_{1\theta}, n_{1\theta}U_{1\theta}, G_{1\theta}, n_{2\theta}, n_{2\theta}U_{2\theta}, G_{2\theta})}{\partial(n_{1\theta}, U_{1\theta}, T_{1\theta}, n_{2\theta}, U_{2\theta}, T_{2\theta})}\right)^{-1} \left(\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta},n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{ij}(\theta)\right)\Big|_{\theta=0},$$

for $(i,j) = (1,2)$ or $(2,1)$. Although $\mathcal{M}_{12}$ does not depend on n_2, we use the above form for the convenience of the calculation. In this proposition, we focus on the linear terms $\mathcal{M}'_{12}(0)$ and $\mathcal{M}'_{21}(0)$. The exact form of the nonlinear terms will be presented in section 2.2. The remaining proof proceeds by stating some auxiliary lemmas below. ∎

LEMMA 2.4 (see [71]). *Let us define*

$$G = \frac{3nT + mn|U|^2 - 3n}{\sqrt{6}}.$$

Then we have

$$J = \frac{\partial(n, nU, G)}{\partial(n, U, T)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ U_1 & n & 0 & 0 & 0 \\ U_2 & 0 & n & 0 & 0 \\ U_3 & 0 & 0 & n & 0 \\ \frac{3T+m|U|^2-3}{\sqrt{6}} & \frac{2nU_1m}{\sqrt{6}} & \frac{2nU_2m}{\sqrt{6}} & \frac{2nU_3m}{\sqrt{6}} & \frac{3n}{\sqrt{6}} \end{bmatrix},$$

and

$$J^{-1} = \left(\frac{\partial(n, nU, G)}{\partial(n, U, T)}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{U_1}{n} & \frac{1}{n} & 0 & 0 & 0 \\ -\frac{U_2}{n} & 0 & \frac{1}{n} & 0 & 0 \\ -\frac{U_3}{n} & 0 & 0 & \frac{1}{n} & 0 \\ \frac{m|U|^2-3T+3}{3n} & -\frac{2m}{3}\frac{U_1}{n} & -\frac{2m}{3}\frac{U_2}{n} & -\frac{2m}{3}\frac{U_3}{n} & \sqrt{\frac{2}{3}}\frac{1}{n} \end{bmatrix}.$$

Proof. In the case of $m_i = 1$, it is proved in [71], and by the same explicit calculation, we can extend the result for general m_i. We omit it. ∎

2.1.1. Linearization of $\mathcal{M}_{12}$. We first consider the calculation of $\mathcal{M}'_{12}(0)$ in (2.4).

LEMMA 2.5. *We have*

(1) $\left.\frac{\partial\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}}\right|_{\theta=0}=\frac{1}{n_{10}}\mu_1,$ (2) $\left.\frac{\partial\mathcal{M}_{12}(\theta)}{\partial U_{1\theta}}\right|_{\theta=0}=\delta m_1 v\mu_1,$

(3) $\left.\frac{\partial\mathcal{M}_{12}(\theta)}{\partial T_{1\theta}}\right|_{\theta=0}=\omega\frac{m_1|v|^2-3}{2}\mu_1,$ (4) $\left.\frac{\partial\mathcal{M}_{12}(\theta)}{\partial U_{2\theta}}\right|_{\theta=0}=(1-\delta)m_1 v\mu_1,$

(5) $\left.\frac{\partial\mathcal{M}_{12}(\theta)}{\partial T_{2\theta}}\right|_{\theta=0}=(1-\omega)\frac{m_1|v|^2-3}{2}\mu_1.$

Proof. For readability, we ignore the dependence on θ.
(1) By an explicit computation, we have

$$\frac{\partial\mathcal{M}_{12}}{\partial n_1}=\frac{1}{n_1}\mathcal{M}_{12}.$$

(2) Note that both U_{12} and T_{12} depend on U_1. So, the chain rule gives

$$\begin{aligned}\frac{\partial\mathcal{M}_{12}}{\partial U_1}&=\frac{\partial U_{12}}{\partial U_1}\frac{\partial\mathcal{M}_{12}}{\partial U_{12}}+\frac{\partial T_{12}}{\partial U_1}\frac{\partial\mathcal{M}_{12}}{\partial T_{12}}\\&=\delta m_1\frac{v-U_{12}}{T_{12}}\mathcal{M}_{12}-2\gamma(U_2-U_1)\left(-\frac{3}{2}\frac{1}{T_{12}}+\frac{m_1|v-U_{12}|^2}{2T_{12}^2}\right)\mathcal{M}_{12}.\end{aligned}$$

(3) An explicit calculation gives

$$\frac{\partial\mathcal{M}_{12}}{\partial T_1}=\frac{\partial T_{12}}{\partial T_1}\frac{\partial\mathcal{M}_{12}}{\partial T_{12}}=\omega\left(-\frac{3}{2}\frac{1}{T_{12}}+\frac{m_1|v-U_{12}|^2}{2T_{12}^2}\right)\mathcal{M}_{12}.$$

(4) Similar to case (2), both U_{12} and T_{12} depend on U_2.

$$\begin{aligned}\frac{\partial\mathcal{M}_{12}}{\partial U_2}&=\frac{\partial U_{12}}{\partial U_2}\frac{\partial\mathcal{M}_{12}}{\partial U_{12}}+\frac{\partial T_{12}}{\partial U_2}\frac{\partial\mathcal{M}_{12}}{\partial T_{12}}\\&=(1-\delta)m_1\frac{v-U_{12}}{T_{12}}\mathcal{M}_{12}+2\gamma(U_2-U_1)\left(-\frac{3}{2}\frac{1}{T_{12}}+\frac{m_1|v-U_{12}|^2}{2T_{12}^2}\right)\mathcal{M}_{12}.\end{aligned}$$

(5) By an explicit computation, we have

$$\frac{\partial\mathcal{M}_{12}}{\partial T_2}=\frac{\partial T_{12}}{\partial T_2}\frac{\partial\mathcal{M}_{12}}{\partial T_{12}}=(1-\omega)\left(-\frac{3}{2}\frac{1}{T_{12}}+\frac{m_1|v-U_{12}|^2}{2T_{12}^2}\right)\mathcal{M}_{12}.$$

Substituting

$$(n_{1\theta},U_{1\theta},T_{1\theta},U_{2\theta},T_{2\theta})\big|_{\theta=0}=(n_{10},U_{10},T_{10},U_{20},T_{20})=(n_{10},0,1,0,1) \tag{2.5}$$

and

$$U_{12\theta}|_{\theta=0}=U_{21\theta}|_{\theta=0}=0,\qquad T_{12\theta}|_{\theta=0}=T_{21\theta}|_{\theta=0}=1, \tag{2.6}$$

in the above computations, we get the desired result. ∎

Now we proceed with the proof of Proposition 2.3 for $\mathcal{M}_{12}(F)$. By the definition of the transition of the macroscopic fields (2.2) and the definition of the basis (2.1), we have

(2.7)
$$\begin{aligned}\frac{d(n_{k\theta},n_{k\theta}U_{k\theta},G_{k\theta})}{d\theta}&=\left(\int_{\mathbb{R}^3}f_k\sqrt{\mu_k}dv,\int_{\mathbb{R}^3}f_k v\sqrt{\mu_k}dv,\int_{\mathbb{R}^3}f_k\frac{m_k|v|^2-3}{\sqrt{6}}\sqrt{\mu_k}dv\right)\\&=\left(\langle f_k,e_{k1}\rangle_{L^2_v},\langle f_k,e_{k2}\rangle_{L^2_v},\langle f_k,e_{k3}\rangle_{L^2_v},\langle f_k,e_{k4}\rangle_{L^2_v},\langle f_k,e_{k5}\rangle_{L^2_v}\right),\end{aligned}$$

for $k=1,2$. For notational brevity, we define

$$J_{k\theta}=\frac{\partial(n_{k\theta},n_{k\theta}U_{k\theta},G_{k\theta})}{\partial(n_{k\theta},U_{k\theta},T_{k\theta})}.$$

Then applying Lemma 2.4 gives

$$J_{k\theta}^{-1}\big|_{\theta=0}=\mathrm{diag}\left(1,\frac{1}{n_{k0}},\frac{1}{n_{k0}},\frac{1}{n_{k0}},\sqrt{\frac{2}{3}}\frac{1}{n_{k0}}\right),$$

and

$$\left(\frac{\partial(n_{1\theta},n_{1\theta}U_{1\theta},G_{1\theta},n_{2\theta},n_{2\theta}U_{2\theta},G_{2\theta})}{\partial(n_{1\theta},U_{1\theta},T_{1\theta},n_{2\theta},U_{2\theta},T_{2\theta})}\right)^{-1}\Bigg|_{\theta=0}=\begin{bmatrix} J_{1\theta}^{-1}\big|_{\theta=0} & 0\\ 0 & J_{2\theta}^{-1}\big|_{\theta=0}\end{bmatrix}, \tag{2.8}$$

where we used

$$\begin{bmatrix} J_1 & 0\\ 0 & J_2\end{bmatrix}^{-1}=\begin{bmatrix} J_1^{-1} & 0\\ 0 & J_2^{-1}\end{bmatrix}. \tag{2.9}$$

We substitute (2.7), (2.8), and Lemma 2.5 into (2.4) to obtain

$$\begin{aligned}\mathcal{M}_{12}'(0)=&\frac{\mu_1}{n_{10}}\int_{\mathbb{R}^3}f_1\sqrt{\mu_1}dv+\frac{\delta m_1 v\mu_1}{n_{10}}\int_{\mathbb{R}^3}f_1 v\sqrt{\mu_1}dv\\ &+\omega\frac{m_1|v|^2-3}{2}\mu_1\sqrt{\frac{2}{3}}\frac{1}{n_{10}}\int_{\mathbb{R}^3}f_1\frac{m_1|v|^2-3}{\sqrt{6}}\sqrt{\mu_1}dv\\ &+\frac{(1-\delta)m_1 v\mu_1}{n_{20}}\int_{\mathbb{R}^3}f_2 v\sqrt{\mu_2}dv\\ &+(1-\omega)\frac{m_1|v|^2-3}{2}\mu_1\sqrt{\frac{2}{3}}\frac{1}{n_{20}}\int_{\mathbb{R}^3}f_2\frac{m_2|v|^2-3}{\sqrt{6}}\sqrt{\mu_2}dv.\end{aligned}$$

Using the definition of the basis in (2.1), we simplify it as follows:

(2.10)

$$\begin{aligned}\mathcal{M}_{12}'(0)=&\langle f_1,e_{11}\rangle_{L^2_v}e_{11}\sqrt{\mu_1}+\delta\sum_{2\le i\le 4}\langle f_1,e_{1i}\rangle_{L^2_v}e_{1i}\sqrt{\mu_1}+\omega\langle f_1,e_{15}\rangle_{L^2_v}e_{15}\sqrt{\mu_1}\\ &+(1-\delta)\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\sum_{2\le i\le 4}\langle f_2,e_{2i}\rangle_{L^2_v}e_{1i}\sqrt{\mu_1}+(1-\omega)\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}e_{15}\sqrt{\mu_1}.\end{aligned}$$

Adding and subtracting the term

$$(1-\delta)\sum_{2\le i\le 4}\langle f_1,e_{1i}\rangle_{L^2_v}e_{1i}\sqrt{\mu_1}+(1-\omega)\langle f_1,e_{15}\rangle_{L^2_v}e_{15}\sqrt{\mu_1}$$

gives

$$\begin{aligned}\mathcal{M}_{12}'(0)=&P_1 f_1\sqrt{\mu_1}+(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v}-\langle f_1,e_{1i}\rangle_{L^2_v}\right)e_{1i}\sqrt{\mu_1}\\ &+(1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}-\langle f_1,e_{15}\rangle_{L^2_v}\right)e_{15}\sqrt{\mu_1}.\end{aligned}$$

This completes the proof for the linearization of $\mathcal{M}_{12}$.

2.1.2. Linearization of $\mathcal{M}_{21}$. Now we consider the calculation of $\mathcal{M}_{21}$ in (2.4).

LEMMA 2.6. *We have*

(1) $\displaystyle \frac{\partial \mathcal{M}_{21\theta}}{\partial n_{2\theta}}\bigg|_{\theta=0} = \frac{1}{n_{20}}\mu_2,$ (2) $\displaystyle \frac{\partial \mathcal{M}_{21\theta}}{\partial U_{2\theta}}\bigg|_{\theta=0} = \left(1-\frac{m_1}{m_2}(1-\delta)\right) m_2 v\mu_2,$

(3) $\displaystyle \frac{\partial \mathcal{M}_{21\theta}}{\partial T_{2\theta}}\bigg|_{\theta=0} = \omega\frac{m_2|v|^2-3}{2}\mu_2,$ (4) $\displaystyle \frac{\partial \mathcal{M}_{21\theta}}{\partial U_{1\theta}}\bigg|_{\theta=0} = \frac{m_1}{m_2}(1-\delta)m_2 v\mu_2,$

(5) $\displaystyle \frac{\partial \mathcal{M}_{21\theta}}{\partial T_{1\theta}}\bigg|_{\theta=0} = (1-\omega)\frac{m_2|v|^2-3}{2}\mu_2.$

Proof. (1) By an explicit computation, we have

$$\frac{\partial \mathcal{M}_{21}}{\partial n_2} = \frac{1}{n_2}\mathcal{M}_{21}.$$

(2) Note that U_{21} and T_{21} depend on U_2. The chain rule gives

$$\frac{\partial \mathcal{M}_{21}}{\partial U_2} = \frac{\partial U_{21}}{\partial U_2}\frac{\partial \mathcal{M}_{21}}{\partial U_{21}} + \frac{\partial T_{21}}{\partial U_2}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}}.$$

So we differentiate

$$\frac{\partial U_{21}}{\partial U_2}\frac{\partial \mathcal{M}_{21}}{\partial U_{21}} = \left(1-\frac{m_1}{m_2}(1-\delta)\right) m_2\frac{v-U_{21}}{T_{21}}\mathcal{M}_{21},$$

and

$$\begin{aligned}\frac{\partial T_{21}}{\partial U_2}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}} &= 2\left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right)-\gamma\right)\\ &\quad\times (U_2-U_1)\left(-\frac{3}{2}\frac{1}{T_{21}}+\frac{m_2|v-U_{21}|^2}{2T_{21}^2}\right)\mathcal{M}_{21}.\end{aligned}$$

(3) We have

$$\frac{\partial \mathcal{M}_{21}}{\partial T_2} = \frac{\partial T_{21}}{\partial T_2}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}} = \omega\left(-\frac{3}{2}\frac{1}{T_{21}}+\frac{m_2|v-U_{21}|^2}{2T_{21}^2}\right)\mathcal{M}_{21}.$$

(4) Since both U_{21} and T_{21} depend on U_1,

$$\frac{\partial \mathcal{M}_{21}}{\partial U_1} = \frac{\partial U_{21}}{\partial U_1}\frac{\partial \mathcal{M}_{21}}{\partial U_{21}} + \frac{\partial T_{21}}{\partial U_1}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}},$$

we compute

$$\frac{\partial U_{21}}{\partial U_1}\frac{\partial \mathcal{M}_{21}}{\partial U_{21}} = \frac{m_1}{m_2}(1-\delta)m_2\frac{v-U_{21}}{T_{21}}\mathcal{M}_{21},$$

and

$$\begin{aligned}\frac{\partial T_{21}}{\partial U_1}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}} &= -2\left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right)-\gamma\right)\\ &\quad\times (U_2-U_1)\left(-\frac{3}{2}\frac{1}{T_{21}}+\frac{m_2|v-U_{21}|^2}{2T_{21}^2}\right)\mathcal{M}_{21}.\end{aligned}$$

(5) We have

$$\frac{\partial \mathcal{M}_{21}}{\partial T_1} = \frac{\partial T_{21}}{\partial T_1}\frac{\partial \mathcal{M}_{21}}{\partial T_{21}} = (1-\omega)\left(-\frac{3}{2}\frac{1}{T_{21}} + \frac{m_2|v-U_{21}|^2}{2T_{21}^2}\right)\mathcal{M}_{21}.$$

Similar to Lemma 2.5, substituting (2.5) and (2.6) in the above calculations gives desired results. ∎

Substituting (2.7), (2.8), and Lemma 2.6 into (2.4) yields

$$\begin{aligned}\mathcal{M}'_{21}(0) = &\frac{\mu_2}{n_{20}}\int_{\mathbb{R}^3} f_2\sqrt{\mu_2}dv + \frac{\left(1-\frac{m_1}{m_2}(1-\delta)\right)m_2 v\mu_2}{n_{20}}\int_{\mathbb{R}^3} f_2 v\sqrt{\mu_2}dv \\ &+\omega\frac{m_2|v|^2-3}{2}\mu_2\sqrt{\frac{2}{3}}\frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2\frac{m_2|v|^2-3}{\sqrt{6}}\sqrt{\mu_2}dv \\ &+\frac{\frac{m_1}{m_2}(1-\delta)m_2 v\mu_2}{n_{10}}\int_{\mathbb{R}^3} f_1 v\sqrt{\mu_1}dv \\ &+(1-\omega)\frac{m_2|v|^2-3}{2}\mu_2\sqrt{\frac{2}{3}}\frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1\frac{m_1|v|^2-3}{\sqrt{6}}\sqrt{\mu_1}dv.\end{aligned}$$

Using the notation of the basis in (2.1), it is equal to

$$\begin{aligned}\mathcal{M}'_{21}(0) = &\langle f_2, e_{21}\rangle_{L^2_v} e_{21}\sqrt{\mu_2} \\ &+\left(1-\frac{m_1}{m_2}(1-\delta)\right)\sum_{2\le i\le 4}\langle f_2, e_{2i}\rangle_{L^2_v} e_{2i}\sqrt{\mu_2} + \omega\langle f_2, e_{25}\rangle_{L^2_v} e_{25}\sqrt{\mu_2} \\ &+\frac{m_1}{m_2}(1-\delta)\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\sum_{2\le i\le 4}\langle f_1, e_{1i}\rangle_{L^2_v} e_{2i}\sqrt{\mu_2} \\ &+(1-\omega)\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1, e_{15}\rangle_{L^2_v} e_{25}\sqrt{\mu_2}.\end{aligned} \tag{2.11}$$

Adding and subtracting the term

$$\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\langle f_2, e_{2i}\rangle_{L^2_v} e_{2i}\sqrt{\mu_2} + (1-\omega)\langle f_2, e_{25}\rangle_{L^2_v} e_{25}\sqrt{\mu_2}$$

gives

$$\begin{aligned}\mathcal{M}'_{21}(0) = &P_2 f_2\sqrt{\mu_2} \\ &+\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1, e_{1i}\rangle_{L^2_v} - \langle f_2, e_{2i}\rangle_{L^2_v}\right) e_{2i}\sqrt{\mu_2} \\ &+(1-\omega)\left(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1, e_{15}\rangle_{L^2_v} - \langle f_2, e_{25}\rangle_{L^2_v}\right) e_{25}\sqrt{\mu_2}.\end{aligned}$$

This completes the proof for the linearization of $\mathcal{M}_{21}$.

2.2. Linearization of the mixture BGK model. In this part, we linearize the mixture BGK model (1.1). Applying the linearization of the BGK Maxwellian lemma, that is, Lemma 2.2, and Proposition 2.3, we substitute $F_1=\mu_1+\sqrt{\mu_1}f_1$ on $(1.1)_1$ and divide it by $\sqrt{\mu_1}$ to have

$$\begin{aligned}\partial_t f_1+v\cdot\nabla_x f_1=&\,n_1\Big(P_1f_1-f_1+\frac{1}{\sqrt{\mu_1}}\int_0^1\mathcal{M}_{11}''(\theta)(1-\theta)d\theta\Big)\\&+n_2\Big(P_1f_1-f_1+\frac{1}{\sqrt{\mu_1}}\int_0^1\mathcal{M}_{12}''(\theta)(1-\theta)d\theta\Big)\\&+n_2\bigg[(1-\delta)\sum_{2\le i\le 4}\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v}-\langle f_1,e_{1i}\rangle_{L^2_v}\bigg)e_{1i}\\&+(1-\omega)\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}-\langle f_1,e_{15}\rangle_{L^2_v}\bigg)e_{15}\bigg].\end{aligned}$$

Splitting n_k by $n_k=(n_k-n_{k0})+n_{k0}$,

$$n_k=n_k-n_{k0}+n_{k0}=\int_{\mathbb{R}^3}f_k\sqrt{\mu_k}dv+n_{k0}=\sqrt{n_{k0}}\langle f_k,e_{k1}\rangle_{L^2_v}+n_{k0},\tag{2.12}$$

we can have the following linearized equation:

$$\partial_t f_1+v\cdot\nabla_x f_1=L_{11}(f_1)+L_{12}(f_1,f_2)+\Gamma_{11}(f_1)+\Gamma_{12}(f_1,f_2),\tag{2.13}$$

where $L_{11}(f_1)=n_{10}(P_1f_1-f_1)$. The linear term L_{12} is decomposed as $L_{12}=L_{12}^1+L_{12}^2$ with $L_{12}^1=n_{20}(P_1f_1-f_1)$. And L_{12}^2 denotes the linear term describing the interchange of momentum and temperature of each species as follows:

$$\begin{aligned}L_{12}^2(f_1,f_2)=&\,n_{20}\bigg[(1-\delta)\sum_{2\le i\le 4}\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v}-\langle f_1,e_{1i}\rangle_{L^2_v}\bigg)e_{1i}\\&+(1-\omega)\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}-\langle f_1,e_{15}\rangle_{L^2_v}\bigg)e_{15}\bigg].\end{aligned}\tag{2.14}$$

The nonlinear terms Γ_{11} and Γ_{12} denote

$$\begin{aligned}\Gamma_{11}(f_1)=&\,(n_1-n_{10})(P_1f_1-f_1)+n_1\frac{1}{\sqrt{\mu_1}}\int_0^1\mathcal{M}_{11}''(\theta)(1-\theta)d\theta,\\\Gamma_{12}(f_1,f_2)=&\,(n_2-n_{20})(P_1f_1-f_1)+n_2\frac{1}{\sqrt{\mu_1}}\int_0^1\mathcal{M}_{12}''(\theta)(1-\theta)d\theta\\&+(n_2-n_{20})\bigg[(1-\delta)\sum_{2\le i\le 4}\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v}-\langle f_1,e_{1i}\rangle_{L^2_v}\bigg)e_{1i}\\&+(1-\omega)\bigg(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}-\langle f_1,e_{15}\rangle_{L^2_v}\bigg)e_{15}\bigg].\end{aligned}$$

Similarly, we substitute $F_2=\mu_2+\sqrt{\mu_2}f_2$ on $(1.1)_2$ and divide it by $\sqrt{\mu_2}$ to have

$$\begin{aligned}\partial_t f_2+v\cdot\nabla_x f_2=&\,n_2(P_2f_2-f_2+\frac{1}{\sqrt{\mu_2}}\int_0^1\mathcal{M}_{22}''(\theta)(1-\theta)d\theta)\\&+n_1(P_2f_2-f_2+\frac{1}{\sqrt{\mu_2}}\int_0^1\mathcal{M}_{21}''(\theta)(1-\theta)d\theta)\\&+n_1\bigg[\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\bigg(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1,e_{1i}\rangle_{L^2_v}-\langle f_2,e_{2i}\rangle_{L^2_v}\bigg)e_{2i}\\&+(1-\omega)\bigg(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1,e_{15}\rangle_{L^2_v}-\langle f_2,e_{25}\rangle_{L^2_v}\bigg)e_{25}\bigg],\end{aligned}$$

which yields

$$\partial_t f_2 + v\cdot\nabla_x f_2 = L_{22}(f_2) + L^2_{21}(f_1,f_2) + \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2), \tag{2.15}$$

where $L_{22}(f_2) = n_{20}(P_2f_2 - f_2)$. The linear term L_{21} also decomposed as $L_{21} = L^1_{21} + L^2_{21}$ with $L^1_{21} = n_{10}(P_2f_2 - f_2)$. And L^2_{21} denotes the interchange of the momentum and temperature between other species.

$$\begin{aligned} L^2_{21}(f_1,f_2) = n_{10}\Bigg[&\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1,e_{1i}\rangle_{L^2_v} - \langle f_2,e_{2i}\rangle_{L^2_v}\right)e_{2i}\\ &+(1-\omega)\left(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1,e_{15}\rangle_{L^2_v} - \langle f_2,e_{25}\rangle_{L^2_v}\right)e_{25}\Bigg].\end{aligned}$$

The nonlinear terms Γ_{22} and Γ_{21} denote

$$\begin{aligned} \Gamma_{22}(f_2) &= (n_2-n_{20})(P_2f_2-f_2) + n_2\frac{1}{\sqrt{\mu_2}}\int_0^1 \mathcal{M}''_{22}(\theta)(1-\theta)d\theta,\\ \Gamma_{21}(f_1,f_2) &= (n_1-n_{10})(P_2f_2-f_2) + n_1\frac{1}{\sqrt{\mu_2}}\int_0^1 \mathcal{M}''_{21}(\theta)(1-\theta)d\theta\\ &\quad + (n_1-n_{10})\Bigg[\frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1,e_{1i}\rangle_{L^2_v} - \langle f_2,e_{2i}\rangle_{L^2_v}\right)e_{2i}\\ &\quad +(1-\omega)\left(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1,e_{15}\rangle_{L^2_v} - \langle f_2,e_{25}\rangle_{L^2_v}\right)e_{25}\Bigg].\end{aligned}$$

Overall, we can write the linearized mixture BGK model (1.1) as

$$\begin{aligned} &\partial_t f_1 + v\cdot\nabla_x f_1 = L_{11}(f_1) + L_{12}(f_1,f_2) + \Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2),\\ &\partial_t f_2 + v\cdot\nabla_x f_2 = L_{22}(f_2) + L_{21}(f_1,f_2) + \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2),\\ &f_1(x,v,0) = f_{10}(x,v), \qquad f_2(x,v,0) = f_{20}(x,v),\end{aligned} \tag{2.16}$$

where $f_{10} = (F_{10}-\mu_1)/\sqrt{\mu_1}$, and $f_{20} = (F_{20}-\mu_2)/\sqrt{\mu_2}$. The linearized mixture BGK model (2.16) satisfies the following conservation laws:

$$\begin{aligned} &\int_{\mathbb{T}^3\times\mathbb{R}^3}\sqrt{\mu_1}f_1(x,v,t)dvdx = \int_{\mathbb{T}^3\times\mathbb{R}^3}\sqrt{\mu_2}f_2(x,v,t)dvdx = 0,\\ &\int_{\mathbb{T}^3\times\mathbb{R}^3}\left(\sqrt{\mu_1}f_1(x,v,t)m_1v + \sqrt{\mu_2}f_2(x,v,t)m_2v\right)dvdx = 0,\\ &\int_{\mathbb{T}^3\times\mathbb{R}^3}\left(\sqrt{\mu_1}f_1(x,v,t)m_1|v|^2 + \sqrt{\mu_2}f_2(x,v,t)m_2|v|^2\right)dvdx = 0.\end{aligned} \tag{2.17}$$

3. Dissipative property of the linearized relaxation operator. In this part, we investigate the dissipative property of the linearized two-component relaxation operator. For simplicity of the notation, we denote the linear operator and the nonlinear perturbation as the vector forms,

$$\begin{aligned} L_1 &= L_{11}(f_1) + L_{12}(f_1,f_2),\\ L_2 &= L_{22}(f_2) + L_{21}(f_1,f_2),\end{aligned}$$

and

$$\begin{aligned} \Gamma_1 &= \Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2),\\ \Gamma_2 &= \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2),\end{aligned}$$

then we can write (2.13) and (2.15) as

$$(\partial_t + v\cdot\nabla_x)(f_1,f_2) = L(f_1,f_2) + \Gamma(f_1,f_2), \tag{3.1}$$

where $L(f_1,f_2) = (L_1,L_2)$ and $\Gamma(f_1,f_2) = (\Gamma_1,\Gamma_2)$. We also define the following 6-dimensional orthonormal basis:

$$\begin{aligned}
E_1 &= \frac{1}{\sqrt{n_{10}}}(\sqrt{\mu_1},0), \quad E_2 = \frac{1}{\sqrt{n_{20}}}(0,\sqrt{\mu_2}),\\
E_i &= \frac{1}{\sqrt{m_1 n_{10} + m_2 n_{20}}}\left(m_1 v_{i-2}\sqrt{\mu_1}, m_2 v_{i-2}\sqrt{\mu_2}\right) \quad (i=3,4,5),\\
E_6 &= \frac{1}{\sqrt{6n_{10}+6n_{10}}}\left((m_1|v|^2-3)\sqrt{\mu_1},(m_2|v|^2-3)\sqrt{\mu_2}\right).
\end{aligned}$$

We also denote $E_i = (E_i^1, E_i^2)$ for $i=1,\dots,6$. The macroscopic projection operator for mixture can be written as

$$P(f_1,f_2) = \sum_{1\le i\le 6}\langle (f_1,f_2),E_i\rangle_{L^2_v} E_i.$$

The following is the main result of this section.

PROPOSITION 3.1. *We have the following dissipation property for the linear operator L:*

$$\begin{aligned}
\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} \le -(n_{10}+n_{20})\Big(&\max\{\delta,\omega\}\|(I-P_1,I-P_2)(f_1,f_2)\|^2_{L^2_{x,v}}\\
&+\min\{(1-\delta),(1-\omega)\}\|(I-P)(f_1,f_2)\|^2_{L^2_{x,v}}\Big).
\end{aligned}$$

Proof. By an explicit computation, we have

$$\begin{aligned}
\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} &= \langle L_1 f_1, f_1\rangle_{L^2_{x,v}} + \langle L_2 f_2, f_2\rangle_{L^2_{x,v}}\\
&= -(n_{10}+n_{20})\|(I-P_1,I-P_2)(f_1,f_2)\|_{L^2_{x,v}} + \langle L^2_{12},f_1\rangle_{L^2_{x,v}} + \langle L^2_{21},f_2\rangle_{L^2_{x,v}}.
\end{aligned} \tag{3.2}$$

We decompose the proof into the following four steps.

Step 1. We consider the dissipation from the momentum and temperature interchange part of the interspecies linearized relaxation operator. We claim that

$$\langle L^2_{12},f_1\rangle_{L^2_v} + \langle L^2_{21},f_2\rangle_{L^2_v} \le 0,$$

and the equality holds if and only if

$$\frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1 v\sqrt{\mu_1}dv = \frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2 v\sqrt{\mu_2}dv,$$

and

$$\frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1(m_1|v|^2-3)\sqrt{\mu_1}dv = \frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2(m_2|v|^2-3)\sqrt{\mu_2}dv.$$

• *Proof of the claim.* By the definition of L^2_{12} in (2.14), we have

$$\begin{aligned}
\langle L^2_{12},f_1\rangle_{L^2_v} &= (1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v} - \langle f_1,e_{1i}\rangle_{L^2_v}\right)\langle f_1,e_{1i}\rangle_{L^2_v} n_{20}\\
&\quad + (1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v} - \langle f_1,e_{15}\rangle_{L^2_v}\right)\langle f_1,e_{15}\rangle_{L^2_v} n_{20}\\
&= I_1 + I_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}\langle L_{21}^2, f_2\rangle_{L_v^2} &= \frac{m_1}{m_2}(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{20}}{n_{10}}}\sqrt{\frac{m_2}{m_1}}\langle f_1, e_{1i}\rangle_{L_v^2} - \langle f_2, e_{2i}\rangle_{L_v^2}\right)\langle f_2, e_{2i}\rangle_{L_v^2} n_{10}\\ &\quad + \frac{m_1}{m_2}(1-\omega)\left(\sqrt{\frac{n_{20}}{n_{10}}}\langle f_1, e_{15}\rangle_{L_v^2} - \langle f_2, e_{25}\rangle_{L_v^2}\right)\langle f_2, e_{25}\rangle_{L_v^2} n_{10}\\ &= I_3 + I_4.\end{aligned}$$

By an explicit computation, we have

$$\begin{aligned}I_1 + I_3 &= -(1-\delta)n_{20}\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2, e_{2i}\rangle_{L_v^2} - \langle f_1, e_{1i}\rangle_{L_v^2}\right)^2\\ &= -(1-\delta)m_1 n_{10} n_{20}\left(\frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2 v\sqrt{\mu_2}dv - \frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1 v\sqrt{\mu_1}dv\right)^2 \le 0,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}I_2 + I_4 &= -(1-\omega)n_{20}\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2, e_{25}\rangle_{L_v^2} - \langle f_1, e_{15}\rangle_{L_v^2}\right)^2\\ &= -(1-\omega)\frac{n_{10}n_{20}}{6}\left(\frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right.\\ &\qquad \left. -\frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1(m_1|v|^2-3)\sqrt{\mu_1}dv\right)^2\\ &\le 0.\end{aligned} \tag{3.4}$$

which proves the claim of this step.

Step 2. To estimate the gap of the macroscopic projection (P_1, P_2) with P, we compute the following term:

$$\|(P_1, P_2)(f_1, f_2) - P(f_1, f_2)\|_{L_{x,v}^2}^2.$$

We note that the element of $(P_1, P_2)(f_1, f_2)$ can be written as the linear combination of the 10-dimensional basis

$$\{(\sqrt{\mu_1}, 0), (0, \sqrt{\mu_2}), (v\sqrt{\mu_1}, 0), (0, v\sqrt{\mu_2}), \left(|v|^2\sqrt{\mu_1}, 0\right), \left(0, |v|^2\sqrt{\mu_2}\right)\}$$

so that $(P_1, P_2)P = P$. Therefore,

$$\|(P_1, P_2)(f_1, f_2) - P(f_1, f_2)\|_{L_{x,v}^2}^2 = \|(P_1, P_2)(f_1, f_2)\|_{L_{x,v}^2}^2 - \|P(f_1, f_2)\|_{L_{x,v}^2}^2.$$

Since we have

$$\begin{aligned}\int_{\mathbb{R}^3}|P_k f_k|^2 dv &= \frac{1}{n_{k0}}\left(\int_{\mathbb{R}^3} f_k\sqrt{\mu_k}dv\right)^2 + \frac{m_k}{n_{k0}}\left(\int_{\mathbb{R}^3} f_k v\sqrt{\mu_k}dv\right)^2\\ &\quad + \frac{1}{6n_{k0}}\left(\int_{\mathbb{R}^3} f_k(m_k|v|^2-3)\sqrt{\mu_k}dv\right)^2,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3}|P(f_1,f_2)|^2dv\\&=\frac{1}{n_{10}}\left(\int_{\mathbb{R}^3}f_1\sqrt{\mu_1}dv\right)^2+\frac{1}{n_{20}}\left(\int_{\mathbb{R}^3}f_2\sqrt{\mu_2}dv\right)^2\\&\quad+\frac{1}{m_1n_{10}+m_2n_{20}}\left(\int_{\mathbb{R}^3}f_1m_1v\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f_2m_2v\sqrt{\mu_2}dv\right)^2\\&\quad+\frac{1}{6n_{10}+6n_{20}}\left(\int_{\mathbb{R}^3}f(m_1|v|^2-3)\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f(m_2|v|^2-3)\sqrt{\mu_2}dv\right)^2,\end{aligned}$$

which follows directly from explicit computations, we can write

$$\|(P_1f_1,P_2f_2)\|^2_{L^2_{x,v}}-\|P(f_1,f_2)\|^2_{L^2_{x,v}}=II_1+II_2,$$

where

(3.5)

$$\begin{aligned}II_1&=\frac{1}{m_1n_{10}}\left(\int_{\mathbb{R}^3}f_1m_1v\sqrt{\mu_1}dv\right)^2+\frac{1}{m_2n_{20}}\left(\int_{\mathbb{R}^3}f_2m_2v\sqrt{\mu_2}dv\right)^2\\&\quad-\frac{1}{m_1n_{10}+m_2n_{20}}\left(\left(\int_{\mathbb{R}^3}f_1m_1v\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f_2m_2v\sqrt{\mu_2}dv\right)^2\right)\\&=\frac{1}{m_1n_{10}+m_2n_{20}}\left[\sqrt{\frac{m_2n_{20}}{m_1n_{10}}}\int_{\mathbb{R}^3}f_1m_1v\sqrt{\mu_1}dv-\sqrt{\frac{m_1n_{10}}{m_2n_{20}}}\int_{\mathbb{R}^3}f_2m_2v\sqrt{\mu_2}dv\right]^2\end{aligned}$$

and

(3.6)

$$\begin{aligned}II_2&=\frac{1}{6n_{10}}\left(\int_{\mathbb{R}^3}f_1(m_1|v|^2-3)\sqrt{\mu_1}dv\right)^2+\frac{1}{6n_{20}}\left(\int_{\mathbb{R}^3}f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right)^2\\&\quad-\frac{1}{6n_{10}+6n_{20}}\left(\left(\int_{\mathbb{R}^3}f(m_1|v|^2-3)\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f(m_2|v|^2-3)\sqrt{\mu_2}dv\right)^2\right)\\&=\frac{1}{6n_{10}+6n_{20}}\left[\sqrt{\frac{n_{20}}{n_{10}}}\int_{\mathbb{R}^3}f_1(m_1|v|^2-3)\sqrt{\mu_1}dv\right.\\&\quad\left.-\sqrt{\frac{n_{10}}{n_{20}}}\int_{\mathbb{R}^3}f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right]^2.\end{aligned}$$

Step 3. In this step, we compare $\langle L^2_{12},f_1\rangle_{L^2_{x,v}}+\langle L^2_{21},f_2\rangle_{L^2_{x,v}}$ with $\|(P_1,P_2)(f_1,f_2)-P(f_1,f_2)\|^2_{L^2_{x,v}}$ computed in Step 1 and Step 2, respectively. We claim that

$$\begin{aligned}\langle L^2_{12},f_1\rangle_{L^2_{x,v}}+\langle L^2_{21},f_2\rangle_{L^2_{x,v}}&\le-\min\{(1-\delta),(1-\omega)\}(n_{10}+n_{20})\\&\quad\times\left(\|(P_1,P_2)(f_1,f_2)\|^2_{L^2_{x,v}}-\|P(f_1,f_2)\|^2_{L^2_{x,v}}\right).\end{aligned}\tag{3.7}$$

which is equivalent to

$$(n_{10}+n_{20})(II_1+II_2)\le-\max\left\{\frac{1}{1-\delta},\frac{1}{1-\omega}\right\}[(I_1+I_3)+(I_2+I_4)],\tag{3.8}$$

where I_i $(i=1,2,3,4)$ are defined in Step 1, and II_i $(i=1,2)$ are defined in (3.5) and (3.6). We first compare II_2 with I_2+I_4. Multiplying $(n_{10}+n_{20})$ on (3.6) yields

$$(n_{10}+n_{20})II_2 = \frac{1}{6}\left[\sqrt{\frac{n_{20}}{n_{10}}}\int_{\mathbb{R}^3} f_1(m_1|v|^2-3)\sqrt{\mu_1}dv - \sqrt{\frac{n_{10}}{n_{20}}}\int_{\mathbb{R}^3} f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right]^2,$$

which is equal to $-\frac{1}{1-\omega}(I_2+I_4)$ by (3.4):

$$(n_{10}+n_{20})II_2 = -\frac{1}{1-\omega}(I_2+I_4). \tag{3.9}$$

Second, we compare II_1 with I_1+I_3. We multiply $(n_{10}+n_{20})$ on (3.5):

$$\begin{aligned}(n_{10}+n_{20})II_1 &= \frac{n_{10}+n_{20}}{m_1n_{10}+m_2n_{20}}\left[\sqrt{\frac{m_2n_{20}}{m_1n_{10}}}\int_{\mathbb{R}^3} f_1m_1v\sqrt{\mu_1}dv - \sqrt{\frac{m_1n_{10}}{m_2n_{20}}}\int_{\mathbb{R}^3} f_2m_2v\sqrt{\mu_2}dv\right]^2 \\ &\le \frac{1}{m_2}\left[\sqrt{\frac{m_2n_{20}}{m_1n_{10}}}\int_{\mathbb{R}^3} f_1m_1v\sqrt{\mu_1}dv - \sqrt{\frac{m_1n_{10}}{m_2n_{20}}}\int_{\mathbb{R}^3} f_2m_2v\sqrt{\mu_2}dv\right]^2\end{aligned}$$

where we used the assumption $m_1\ge m_2$. From (3.3), we compute

$$-m_2(I_1+I_3) = (1-\delta)m_1m_2n_{10}n_{20}\left(\frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2v\sqrt{\mu_2}dv - \frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1v\sqrt{\mu_1}dv\right)^2$$

which means that

$$(n_{10}+n_{20})II_1 \le -\frac{1}{1-\delta}(I_1+I_3). \tag{3.10}$$

Combining the estimates (3.9) and (3.10) yields the desired estimate (3.8).

Step 4. Finally, we go back to the estimate (3.2). Applying (3.7) on (3.2) yields

$$\begin{aligned}\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} &\le (n_{10}+n_{20})\left(\|(P_1,P_2)(f_1,f_2)\|^2_{L^2_{x,v}} - \|(f_1,f_2)\|^2_{L^2_{x,v}}\right) \\ &\quad - \min\{(1-\delta),(1-\omega)\}(n_{10}+n_{20})\left(\|(P_1,P_2)(f_1,f_2)\|^2_{L^2_{x,v}} - \|P(f_1,f_2)\|^2_{L^2_{x,v}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}}}{n_{10}+n_{20}} &\le -\|(f_1,f_2)\|^2_{L^2_{x,v}} + \max\{\delta,\omega\}\|(P_1,P_2)(f_1,f_2)\|^2_{L^2_{x,v}} \\ &\quad + \min\{(1-\delta),(1-\omega)\}\|P(f_1,f_2)\|^2_{L^2_{x,v}}.\end{aligned}$$

Finally, by splitting $1 = \max\{\delta,\omega\} + \min\{(1-\delta),(1-\omega)\}$ on the coefficient of $\|(f_1,f_2)\|^2_{L^2}$, we conclude that

$$\begin{aligned}\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} &\le -(n_{10}+n_{20})\Big(\max\{\delta,\omega\}\|(I-P_1,I-P_2)(f_1,f_2)\|^2_{L^2_{x,v}} \\ &\quad + \min\{(1-\delta),(1-\omega)\}\|(I-P)(f_1,f_2)\|^2_{L^2_{x,v}}\Big).\end{aligned}$$

∎

LEMMA 3.2. *The kernel of the linear operator L satisfies*

$$\mathrm{Ker}L = span\{(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),\\ (m_1 v\sqrt{\mu_1}, m_2 v\sqrt{\mu_2}),\big((m_1|v|^2-3)\sqrt{\mu_1},(m_2|v|^2-3)\sqrt{\mu_2}\big)\}.$$

Proof. We prove the following equivalence condition:

$$\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} = 0 \qquad \Leftrightarrow \qquad L(f_1,f_2)=0.$$

($\Leftarrow$) This is trivial.

($\Rightarrow$) By Proposition 3.1, $\langle L(f_1,f_2),(f_1,f_2)\rangle_{L^2_{x,v}} = 0$ implies $(f_1,f_2) = P(f_1,f_2)$. Now it is enough to show that $L(P(f_1,f_2))=0$. By direct computation,

$$L(P(f_1,f_2)) = (n_{10}+n_{20})((P_1,P_2)(P(f_1,f_2)) - P(f_1,f_2))\\ + (L^2_{12}(Pf)+L^2_{21}(Pf)).$$

The first term is equal to 0 since $(P_1,P_2)P=P$. From Step 1 of Proposition 3.1, we can observe that $A_1=A_2=0$ implies $L^2_{12}=L^2_{21}=0$, where

$$A_1 = \frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1 v\sqrt{\mu_1}dv - \frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2 v\sqrt{\mu_2}dv,$$
$$A_2 = \frac{1}{n_{10}}\int_{\mathbb{R}^3} f_1(m_1|v|^2-3)\sqrt{\mu_1}dv - \frac{1}{n_{20}}\int_{\mathbb{R}^3} f_2(m_2|v|^2-3)\sqrt{\mu_2}dv.$$

Thus we want to prove that $A_1=A_2=0$ when $(f_1,f_2)=P(f_1,f_2)=\sum_{1\le k\le 6}\langle(f_1,f_2), E_k\rangle_{L^2_v}E_k$. From the orthogonality of the basis E^1_k with $v_i\sqrt{\mu_1}$,

$$\begin{aligned}A_1 &= \frac{1}{n_{10}}\int_{\mathbb{R}^3}\sum_{1\le k\le 6}\left[\langle (f_1,f_2),E_k\rangle_{L^2_v}E^1_k\right] v_i\sqrt{\mu_1}dv\\ &\quad - \frac{1}{n_{20}}\int_{\mathbb{R}^3}\sum_{1\le k\le 6}\left[\langle (f_1,f_2),E_k\rangle_{L^2_v}E^2_k\right] v_i\sqrt{\mu_2}dv\\ &= \langle (f_1,f_2),E_{i+2}\rangle_{L^2_v}\left(\frac{1}{n_{10}}\int_{\mathbb{R}^3}E^1_{i+2}v_i\sqrt{\mu_1}dv - \frac{1}{n_{20}}\int_{\mathbb{R}^3}E^2_{i+2}v_i\sqrt{\mu_2}dv\right),\end{aligned}$$

for $i=1,2,3$. By definition of E_{i+2}, we have

$$A_1 = \frac{\langle (f_1,f_2),E_{i+2}\rangle_{L^2_v}}{\sqrt{m_1n_{10}+m_2n_{20}}}\left(\frac{1}{n_{10}}\int_{\mathbb{R}^3} m_1v_i^2\mu_1dv - \frac{1}{n_{20}}\int_{\mathbb{R}^3} m_2v_i^2\mu_2dv\right) = 0.$$

Similarly, we compute

$$\begin{aligned}A_2 &= \frac{1}{n_{10}}\int_{\mathbb{R}^3}\sum_{1\le k\le 6}\left[\langle (f_1,f_2),E_k\rangle_{L^2_v}E^1_k\right](m_1|v|^2-3)\sqrt{\mu_1}dv\\ &\quad - \frac{1}{n_{20}}\int_{\mathbb{R}^3}\sum_{1\le k\le 6}\left[\langle (f_1,f_2),E_k\rangle_{L^2_v}E^2_k\right](m_2|v|^2-3)\sqrt{\mu_2}dv\\ &= \frac{\langle (f_1,f_2),E_6\rangle_{L^2_v}}{\sqrt{6n_{10}+6n_{10}}}\left(\frac{1}{n_{10}}\int_{\mathbb{R}^3}(m_1|v|^2-3)^2\mu_1dv - \frac{1}{n_{20}}\int_{\mathbb{R}^3}(m_2|v|^2-3)^2\mu_2dv\right)\\ &= 0,\end{aligned}$$

where we used

$$\int_{\mathbb{R}^3}(m_i|v|^2-3)^2\mu_idv = \int_{\mathbb{R}^3}(m_i^2|v|^4-6m_i|v|^2+9)\mu_idv = 6n_{i0}.$$

Thus, $L^2_{12}(Pf)=L^2_{21}(Pf)=0$. Therefore, we conclude that $L(P(f_1,f_2))=0$ and the kernel of L is spanned by the basis of P. This completes the proof. ∎

Remark 3.3. Note that in the extreme cases $\delta=1$ or $\omega=1$, we have as follow:

- For $\delta=1$ and $0\le\omega<1$

$$\begin{aligned}\text{Ker}L=span\{&(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),(v\sqrt{\mu_1},0),(0,v\sqrt{\mu_2})\\&\big((m_1|v|^2-3)\sqrt{\mu_1},(m_2|v|^2-3)\sqrt{\mu_2}\big)\}.\end{aligned}$$

- For $0\le\delta<1$ and $\omega=1$

$$\begin{aligned}\text{Ker}L=span\{&(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),(m_1v\sqrt{\mu_1},m_2v\sqrt{\mu_2}),\\&\big(|v|^2\sqrt{\mu_1},0\big),\big(0,|v|^2\sqrt{\mu_2}\big)\}.\end{aligned}$$

- For $\delta=\omega=1$

$$\begin{aligned}\text{Ker}L=span\{&(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),(v\sqrt{\mu_1},0),(0,v\sqrt{\mu_2})\\&\big(|v|^2\sqrt{\mu_1},0\big),\big(0,|v|^2\sqrt{\mu_2}\big)\}.\end{aligned}$$

However, $\delta=1$ or $\omega=1$ corresponds respectively to the cases where no interchange of momentum or temperature occurs. We exclude the cases in the sequel.

4. Local existence. In this section, we prove the local-in-time existence of the mixture BGK model. We start with estimates of the macroscopic fields.

4.1. Estimate of the macroscopic fields.

LEMMA 4.1. *For sufficiently small $\mathcal{E}(t)$, there exists a positive constant $C>0$, such that*

$$\begin{aligned}&(1)\ |n_{k\theta}(x,t)-n_{k0}|\le C\sqrt{\mathcal{E}(t)},\\&(2)\ |U_{ij\theta}(x,t)|\le C\sqrt{\mathcal{E}(t)},\\&(3)\ |T_{ij\theta}(x,t)-1|\le C\sqrt{\mathcal{E}(t)},\end{aligned}$$

for $k=1,2$ and $(i,j)=(1,2)$ or $(2,1)$.

Proof. We recall the estimates for the mono-species macroscopic fields in [71]:

$$|n_{k\theta}(x,t)-n_{k0}|,\quad |U_{k\theta}(x,t)|,\quad |T_{k\theta}(x,t)-1|\le C\sqrt{\mathcal{E}(t)}.$$

Therefore, from the definition of $U_{12\theta}$, $U_{21\theta}$, $T_{12\theta}$, and $T_{21\theta}$ in (2.3), we have

$$\begin{aligned}&|U_{12\theta}|\le\delta|U_{1\theta}|+(1-\delta)|U_{2\theta}|\le C\sqrt{\mathcal{E}(t)},\\&|U_{21\theta}|\le\frac{m_1}{m_2}(1-\delta)|U_{1\theta}|+\left(1-\frac{m_1}{m_2}(1-\delta)\right)|U_{2\theta}|\le C\sqrt{\mathcal{E}(t)},\\&|T_{12\theta}|=\omega|T_{1\theta}|+(1-\omega)|T_{2\theta}|+\gamma|U_{2\theta}-U_{1\theta}|^2\le C\sqrt{\mathcal{E}(t)}+C\mathcal{E}(t),\end{aligned}$$

and

$$\begin{aligned}|T_{21\theta}|&=(1-\omega)|T_{1\theta}|+\omega|T_{2\theta}|+\left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right)-\gamma\right)|U_{2\theta}-U_{1\theta}|^2\\&\le C\sqrt{\mathcal{E}(t)}+C\mathcal{E}(t),\end{aligned}$$

for sufficiently small $\mathcal{E}(t)$. ∎

LEMMA 4.2. *For $|\alpha| \geq 1$ and sufficiently small $\mathcal{E}(t)$, there exists a positive constant $C_\alpha > 0$ such that*

$$(1)\ |\partial^\alpha n_{k\theta}(x,t)| \leq C_\alpha \|\partial^\alpha f_k\|_{L^2_v},$$

$$(2)\ |\partial^\alpha U_{ij\theta}(x,t)| \leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1} f_k\|_{L^2_v},$$

$$(3)\ |\partial^\alpha T_{ij\theta}(x,t)| \leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1} f_k\|_{L^2_v} + C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1}(f_1,f_2)\|^2_{L^2_v},$$

for $k=1,2$ and $(i,j)=(1,2)$ or $(2,1)$.

Proof. We recall (2.3) and use the estimates from [71]:

$$|\partial^\alpha n_{k\theta}(x,t)|,\ |\partial^\alpha U_{k\theta}(x,t)|,\ |\partial^\alpha T_{k\theta}(x,t)| \leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1} f_k\|_{L^2_v} \quad (k=1,2), \tag{4.1}$$

to get

$$|\partial^\alpha U_{12\theta}| \leq \delta|\partial^\alpha U_{1\theta}| + (1-\delta)|\partial^\alpha U_{2\theta}| \leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1}(f_1,f_2)\|_{L^2_v},$$

$$|\partial^\alpha U_{21\theta}| \leq \frac{m_1}{m_2}(1-\delta)|\partial^\alpha U_{1\theta}| + \left(1-\frac{m_1}{m_2}(1-\delta)\right)|\partial^\alpha U_{2\theta}| \leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1}(f_1,f_2)\|_{L^2_v},$$

and

$$\begin{aligned}
|\partial^\alpha T_{12\theta}| &= \omega|\partial^\alpha T_{1\theta}| + (1-\omega)|\partial^\alpha T_{2\theta}| + \gamma\partial^\alpha|U_{2\theta}-U_{1\theta}|^2,\\
|\partial^\alpha T_{21\theta}| &= (1-\omega)|\partial^\alpha T_{1\theta}| + \omega|\partial^\alpha T_{2\theta}|\\
&\quad + \left(\frac{1}{3}m_1(1-\delta)\left(\frac{m_1}{m_2}(\delta-1)+1+\delta\right)-\gamma\right)\partial^\alpha|U_{2\theta}-U_{1\theta}|^2.
\end{aligned}$$

Then by Young's inequality and using $(4.1)_2$, we have

$$\begin{aligned}
\partial^\alpha|U_{2\theta}-U_{1\theta}|^2 &= \sum_{\alpha_1+\alpha_2=\alpha} 2\partial^{\alpha_1}(U_{2\theta}-U_{1\theta})\cdot\partial^{\alpha_2}(U_{2\theta}-U_{1\theta})\\
&\leq C_\alpha \sum_{|\alpha_1|\leq|\alpha|} \|\partial^{\alpha_1}(f_1,f_2)\|^2_{L^2_v},
\end{aligned}$$

which gives the desired result. □

4.2. Estimate of the nonlinear term. We now consider the estimates of nonlinear perturbation Γ.

LEMMA 4.3. *There exist nonnegative integers λ, ν, ξ and general polynomial $\mathcal{P}_{lm}$ satisfying*

$$\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{ij}(\theta)\}_{l,m} = \frac{\mathcal{P}_{lm}(n_{1\theta},n_{2\theta},U_{1\theta},U_{2\theta},T_{1\theta},T_{2\theta},v-U_{ij\theta})}{n^\lambda_{1\theta}n^\nu_{2\theta}T^\xi_{ij\theta}}\mathcal{M}_{ij}(\theta),$$

where $\mathcal{P}_{lm}(x_1,\dots,x_n) = \sum_k a_k x_1^{k_1}\cdots x_n^{k_n}$ and the indices $k_1,\dots,k_n$ are nonnegative integer, $ij=12$ or $ij=21$, and $1\leq l,m\leq 10$.

Proof. The estimates of $\mathcal{M}_{12}(\theta)$ and $\mathcal{M}_{21}(\theta)$ are similar. We only consider the former case. We compute

$$\nabla_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta) = \left(\frac{\partial(n_{1\theta}, n_{1\theta}U_{1\theta}, G_{1\theta}, n_{2\theta}, n_{2\theta}U_{2\theta}, G_{2\theta})}{\partial(n_{1\theta}, U_{1\theta}, T_{1\theta}, n_{2\theta}, U_{2\theta}, T_{2\theta})}\right)^{-1} \times \nabla_{(n_{1\theta},U_{1\theta},T_{1\theta},n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta).$$

Then, as in (2.9), we have

$$\begin{aligned}\nabla_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta) &= \begin{bmatrix} J_{1\theta}^{-1} & 0 \\ 0 & J_{2\theta}^{-1}\end{bmatrix} \times \nabla_{(n_{1\theta},U_{1\theta},T_{1\theta},n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta) \\ &= \begin{bmatrix} J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta) \\ J_{2\theta}^{-1}\nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta)\end{bmatrix}.\end{aligned}$$

Applying the same process one more time, we get

$$\begin{aligned}\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta) &= \begin{bmatrix} J_{1\theta}^{-1} & 0 \\ 0 & J_{2\theta}^{-1}\end{bmatrix} \\ &\quad \times \nabla_{(n_{1\theta},U_{1\theta},T_{1\theta},n_{2\theta},U_{2\theta},T_{2\theta})} \begin{bmatrix} J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta) \\ J_{2\theta}^{-1}\nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta)\end{bmatrix},\end{aligned}$$

where the second line on the R.H.S. is equal to

$$\begin{bmatrix}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\left(J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\right) & \nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\left(J_{2\theta}^{-1}\nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta)\right) \\ \nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\left(J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\right) & \nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\left(J_{2\theta}^{-1}\nabla_{(n_{2\theta},U_{2\theta},T_{2\theta})}\mathcal{M}_{12}(\theta)\right)\end{bmatrix}.$$

Thus we get

$$\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix},$$

where

$$T_{ij} = J_{i\theta}^{-1}\nabla_{(n_{i\theta},U_{i\theta},T_{i\theta})}\left(J_{j\theta}^{-1}\nabla_{(n_{j\theta},U_{j\theta},T_{j\theta})}\mathcal{M}_{12}(\theta)\right),$$

for $i,j=1,2$. Each T_{ij} is a 5×5 matrix. For simplicity, we only consider the $(1,1)$ and $(1,2)$ components of $\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)$. We can treat other components similarly. Recall that the first row of $J_{1\theta}^{-1}$ is $(1,0,0,0,0)$ so that

$$\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)\}_{11} = \frac{\partial}{\partial n_{1\theta}}\frac{\partial \mathcal{M}_{12}(\theta)}{\partial n_{1\theta}} = \frac{\partial}{\partial n_{1\theta}}\left(\frac{1}{n_{1\theta}}\mathcal{M}_{12}(\theta)\right) = 0.$$

Now we consider the $(1,2)$ component of $\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)$ which is an inner product of the first row of $J_{1\theta}^{-1}$ which is $(1,0,0,0,0)$, and the second column of $\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}$ $\{J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\}$. Thus, we only need the (1,2) component of $\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}$ $\{J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\}$:

$$\begin{aligned}\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)\}_{12} &= \left[\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\{J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\}\right]_{12} \\ &= \frac{\partial}{\partial n_{1\theta}}\left[J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\right]_2.\end{aligned} \tag{4.2}$$

The second component of $[J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)]$ is equal to the inner product of the second row of $J_{1\theta}^{-1}$ and $\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)$:

$$\begin{aligned}\left[J_{1\theta}^{-1}\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\right]_2 &= \left(-\frac{U_{1\theta}}{n_{1\theta}},\frac{1}{n_{1\theta}},0,0,0\right)\cdot\nabla_{(n_{1\theta},U_{1\theta},T_{1\theta})}\mathcal{M}_{12}(\theta)\\ &= -\frac{U_{1\theta}}{n_{1\theta}}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}}+\frac{1}{n_{1\theta}}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial U_{11\theta}}.\end{aligned}$$

Substituting this into (4.2) gives

$$\begin{aligned}\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)\}_{12} &= \frac{\partial}{\partial n_{1\theta}}\left(-\frac{U_{11\theta}}{n_{1\theta}}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}}+\frac{1}{n_{1\theta}}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial U_{11\theta}}\right)\\ &= \frac{U_{11\theta}}{n_{1\theta}^2}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}}-\frac{U_{11\theta}}{n_{1\theta}}\frac{\partial^2\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}^2}-\frac{1}{n_{1\theta}^2}\frac{\partial\mathcal{M}_{12}(\theta)}{\partial U_{11\theta}}\\ &\quad+\frac{1}{n_{1\theta}}\frac{\partial^2\mathcal{M}_{12}(\theta)}{\partial n_{1\theta}\partial U_{11\theta}}.\end{aligned}$$

Then, from Lemma 2.5(1) and (2), we have

$$\begin{aligned}&\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)\}_{12} = \frac{U_{11\theta}}{n_{1\theta}^3}\mathcal{M}_{12}(\theta)\\ &\quad+\frac{1}{n_{1\theta}}\left(\delta m_1\frac{v-U_{12\theta}}{T_{12\theta}}-2\gamma(U_{2\theta}-U_{1\theta})\left(-\frac{3}{2}\frac{1}{T_{12\theta}}+\frac{m_1|v-U_{12\theta}|^2}{2T_{12\theta}^2}\right)\right)\mathcal{M}_{12}(\theta).\end{aligned}$$

We observe that the $(1,2)$ component of $\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{12}(\theta)$ is expressed in the form presented in this lemma. ◻

We are now ready to estimate the nonlinear terms. The intraspecies part is established in [71].

LEMMA 4.4 (see [71]). *For sufficiently small $\mathcal{E}(t)$, we have the following inequality for $k=1,2$:*

$$\langle\partial^\alpha_\beta\Gamma_{kk}(f_k),g\rangle_{L^2_v}\le C\sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\|\partial^{\alpha_1}f_k\|_{L^2_v}\|\partial^{\alpha_2}f_k\|_{L^2_v}\|g\|_{L^2_v}.$$

Thus we focus on the interspecies part.

LEMMA 4.5. *Let $N\ge 3$ and $|\alpha|+|\beta|\le N$. For sufficiently small $\mathcal{E}(t)$, we have*

$$\langle\partial^\alpha_\beta\Gamma_{ij},g\rangle_{L^2_v}\le C\sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\|\partial^{\alpha_1}(f_1,f_2)\|_{L^2_v}\|\partial^{\alpha_2}_\beta(f_1,f_2)\|_{L^2_v}\|g\|_{L^2_v},$$

for $(i,j)=(1,2)$ or $(2,1)$.

Proof. We only consider the Γ_{12} since the estimate of Γ_{21} is similar. Therefore, we focus on the estimates of the nonlinear terms Γ_{12} and Γ_{21}. For convenience, we divide Γ_{12} into three parts:

$$\Gamma_{12}=\Gamma_{12A}+\Gamma_{12B}+\Gamma_{12C},$$

where

$$\begin{aligned}
\Gamma_{12A} &= (n_2-n_{20})(P_1f_1-f_1),\\
\Gamma_{12B} &= n_2\frac{1}{\sqrt{\mu_1}}\int_0^1 \mathcal{M}_{12}''(\theta)(1-\theta)d\theta,\\
\Gamma_{12C} &= (n_2-n_{20})\Bigg[(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle f_2,e_{2i}\rangle_{L^2_v}-\langle f_1,e_{1i}\rangle_{L^2_v}\right)e_{1i}\\
&\quad+(1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle f_2,e_{25}\rangle_{L^2_v}-\langle f_1,e_{15}\rangle_{L^2_v}\right)e_{15}\Bigg].
\end{aligned}$$

We first write Γ_{12B} in a concise form before we delve into the estimate. For this, compute applying the chain rule twice on $\mathcal{M}_{ij}$:

$$\begin{aligned}
&\mathcal{M}_{ij}''(\theta)\\
&\quad=\frac{d}{d\theta}\bigg(\frac{dn_{\theta 1}}{d\theta}\frac{d\mathcal{M}_{ij}}{dn_{\theta 1}}+\frac{d(n_{\theta 1}U_{\theta 1})}{d\theta}\frac{d\mathcal{M}_{ij}}{d(n_{\theta 1}U_{\theta 1})}+\frac{dG_{\theta 1}}{d\theta}\frac{d\mathcal{M}_{ij}}{dG_{\theta 1}}\\
&\qquad+\frac{dn_{\theta 2}}{d\theta}\frac{d\mathcal{M}_{ij}}{dn_{\theta 2}}+\frac{d(n_{\theta 2}U_{\theta 2})}{d\theta}\frac{d\mathcal{M}_{ij}}{d(n_{\theta 2}U_{\theta 2})}+\frac{dG_{\theta 2}}{d\theta}\frac{d\mathcal{M}_{ij}}{dG_{\theta 2}}\bigg)\\
&\quad=(n_1-n_{10},n_1U_1,G_1,n_2-n_{20},n_2U_2,G_2)^T\left\{\nabla^2_{(n_{1\theta},n_{1\theta}U_{1\theta},G_{1\theta},n_{2\theta},n_{2\theta}U_{2\theta},G_{2\theta})}\mathcal{M}_{ij}(\theta)\right\}\\
&\qquad\times(n_1-n_{10},n_1U_1,G_1,n_2-n_{20},n_2U_2,G_2).
\end{aligned}$$

Therefore, if we define

$$H_k=(n_k,n_kU_k,G_k),\quad \textit{and}\quad H_{k\theta}=(n_{k\theta},n_{k\theta}U_{k\theta},G_{k\theta}),\tag{4.3}$$

we can rewrite Γ_{12B} as

$$\begin{aligned}
\Gamma_{12B}&=\frac{n_2}{\sqrt{\mu_1}}(H_1-H_{10},H_2-H_{20})^T\\
&\quad\times\int_0^1\left\{\nabla^2_{(H_{1\theta},H_{2\theta})}\mathcal{M}_{ij}(\theta)\right\}(1-\theta)d\theta(H_1-H_{10},H_2-H_{20}).
\end{aligned}$$

Now we estimate each part of Γ_{12}.

- *Estimate of* Γ_{12A}: We take a derivative ∂^α_β on Γ_{12A}:

$$\partial^\alpha_\beta\Gamma_{12A}=\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1}\partial^{\alpha_1}(n_2-n_{20})\partial^{\alpha_2}_\beta(P_1f_1-f_1).$$

From (2.12), we have

$$\partial^\alpha(n_2-n_{20})\le C\|\partial^\alpha f_2\|_{L^2_v}.\tag{4.4}$$

For an estimate of the macroscopic projection P_1f_1, since $\partial_\beta e_{1i}$ has an exponential decay, we get

$$\|\partial^\alpha_\beta P_1f_1\|_{L^2_v}=\|\partial_\beta P_1\partial^\alpha f_1\|_{L^2_v}\le C_\beta\|\partial^\alpha f_1\|_{L^2_v}.$$

Thus we have

$$\langle\partial^\alpha_\beta(P_1f_1-f_1),g\rangle_{L^2_v}\le C\left(\|\partial^\alpha f_1\|_{L^2_v}+\|\partial^\alpha_\beta f_1\|_{L^2_v}\right)\|g\|_{L^2_v}.\tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\langle \partial_\beta^\alpha \Gamma_{12A}, g\rangle_{L_v^2} \le C \sum_{|\alpha_1|+|\alpha_2|+|\beta|\le N} \|\partial^{\alpha_1} f_2\|_{L_v^2} \left(\|\partial^{\alpha_2} f_1\|_{L_v^2} + \|\partial_\beta^{\alpha_2} f_1\|_{L_v^2}\right) \|g\|_{L_v^2}.$$

• *Estimate of* Γ_{12C}: We take a derivative ∂_β^α on Γ_{12C}:

$$\begin{aligned}\partial_\beta^\alpha \Gamma_{12C} = &\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1} \partial^{\alpha_1}(n_2 - n_{20})\\ &\times \Bigg[(1-\delta) \sum_{2\le i\le 4} \left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}} \langle \partial^{\alpha_2} f_2, e_{2i}\rangle_{L_v^2} - \langle \partial^{\alpha_2} f_1, e_{1i}\rangle_{L_v^2}\right) \partial_\beta e_{1i}\\ &+ (1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}} \langle \partial^{\alpha_2} f_2, e_{25}\rangle_{L_v^2} - \langle \partial^{\alpha_2} f_1, e_{15}\rangle_{L_v^2}\right) \partial_\beta e_{15}\Bigg].\end{aligned}$$

Since each e_{1i} and e_{2i} has exponential decay for $i = 1, \ldots, 5$, we can have

$$\langle \partial^\alpha f_1, e_{1i}\rangle_{L_v^2} \le C\|\partial^\alpha f_1\|_{L_v^2}, \qquad \langle \partial^\alpha f_2, e_{2i}\rangle_{L_v^2} \le C\|\partial^\alpha f_2\|_{L_v^2}, \tag{4.6}$$

and

$$\langle \partial_\beta e_{1i}, g\rangle_{L_v^2} \le C\|g\|_{L_v^2}, \qquad \langle \partial_\beta e_{2i}, g\rangle_{L_v^2} \le C\|g\|_{L_v^2}. \tag{4.7}$$

Thus by using (4.4), (4.6), and (4.7), we get

$$\langle \partial_\beta^\alpha \Gamma_{12C}, g\rangle_{L_v^2} \le C \sum_{|\alpha_1|+|\alpha_2|\le|\alpha|} \|\partial^{\alpha_1} f_2\|_{L_v^2} \|\partial^{\alpha_2}(f_1, f_2)\|_{L_v^2} \|g\|_{L_v^2}.$$

• *Estimate of* Γ_{12B}: Taking ∂_β^α on Γ_{12B} gives

(4.8)
$$\begin{aligned}\partial_\beta^\alpha \Gamma_{12B} = &\sum_{\sum \alpha_i=\alpha} C_{\alpha_i} \partial^{\alpha_0} n_2 \partial^{\alpha_1}(H_1 - H_{10}, H_2 - H_{20})^T\\ &\times \int_0^1 \partial_\beta^{\alpha_2} \left\{\frac{1}{\sqrt{\mu_1}} \nabla^2_{(H_{1\theta}, H_{2\theta})} \mathcal{M}_{12}(\theta)\right\} (1-\theta) d\theta \partial^{\alpha_3}(H_1 - H_{10}, H_2 - H_{20}).\end{aligned}$$

By the definition of H_k in (4.3), applying (2.7) yields

$$\partial^\alpha(H_k - H_{k0}) = \partial^\alpha(n_k - n_{k0}, n_k U_k, G_k) = \left(\langle \partial^\alpha f_k, e_{k1}\rangle_{L_v^2}, \ldots, \langle \partial^\alpha f_k, e_{k5}\rangle_{L_v^2}\right),$$

for $k = 1, 2$. Thus we have

$$|\partial^\alpha(H_k - H_{k0})| \le C\|\partial^\alpha f_k\|_{L_v^2}. \tag{4.9}$$

For notational simplicity, we set

$$A_{lm} = \int_0^1 \partial_\beta^{\alpha_2} \left\{\frac{1}{\sqrt{\mu_1}} \nabla^2_{(H_{1\theta}, H_{2\theta})} \mathcal{M}_{12}(\theta)\right\}_{l,m} (1-\theta) d\theta.$$

Then by Lemma 4.3, we can write it as

(4.10)
$$A_{lm} = \int_0^1 \partial_\beta^{\alpha_2} \left\{\frac{1}{\sqrt{\mu_1}} \frac{\mathcal{P}_{lm}(n_{1\theta}, n_{2\theta}, U_{1\theta}, U_{2\theta}, T_{1\theta}, T_{2\theta}, v - U_{12\theta})}{n_{1\theta}^\lambda n_{2\theta}^\nu T_{12\theta}^\xi} \mathcal{M}_{12}(\theta)\right\} (1-\theta) d\theta.$$

By the product rule, we have

$$\partial^\alpha_\beta\left\{\frac{\mathcal{P}_{lm}(n_{1\theta},n_{2\theta},U_{1\theta},U_{2\theta},T_{1\theta},T_{2\theta},v-U_{12\theta})}{n_{1\theta}^\lambda n_{2\theta}^\nu T_{12\theta}^\xi}\right\}$$
$$=C_\alpha\sum_{\sum\alpha_i=\alpha}\Bigg\{\mathcal{P}_{lm}(\partial^{\alpha_1}n_{1\theta},\partial^{\alpha_2}n_{2\theta},\partial^{\alpha_3}U_{1\theta},\partial^{\alpha_4}U_{2\theta},\partial^{\alpha_5}T_{1\theta},\partial^{\alpha_6}T_{2\theta},\partial^{\alpha_7}_\beta(v-U_{12\theta}))$$
$$\times\,\partial^{\alpha_8}\frac{1}{n_{1\theta}^\lambda n_{2\theta}^\nu T_{12\theta}^\xi}\Bigg\}$$

If $|\alpha_i|\le N-2$, then by Sobolev embedding $H^2\subset\subset L^\infty$ and Lemma 4.2, we have

$$|\partial^\alpha n_{k\theta}(x,t)|+|\partial^\alpha U_{k\theta}(x,t)|+|\partial^\alpha T_{k\theta}(x,t)|\le C\|\partial^\alpha f_k\|_{L^2_v}\le\sqrt{\mathcal{E}(t)}.$$

Since $N\ge 3$, there is at most one α_i that exceeds $N-2$. Thus, for sufficiently small $\mathcal{E}(t)$, we have

$$\partial^\alpha_\beta\left\{\frac{\mathcal{P}_{lm}(n_{1\theta},n_{2\theta},U_{1\theta},U_{2\theta},T_{1\theta},T_{2\theta},v-U_{12\theta})}{n_{1\theta}^\lambda n_{2\theta}^\nu T_{12\theta}^\xi}\right\}\le C\sqrt{\mathcal{E}(t)}\|\partial^\alpha f\|_{L^2_v}\mathcal{P}_{lm}(v).$$

Substituting it in (4.10) yields

$$A_{lm}\le C\sqrt{\mathcal{E}(t)}\|\partial^\alpha f\|_{L^2_v}\mathcal{P}_{lm}(v)\partial^\alpha_\beta\exp\left(-\frac{|v-U_{12\theta}|^2}{2\frac{T_{12\theta}}{m_1}}+\frac{m_1|v|^2}{4}\right).$$

Similarly, the derivative of the exponential part can be bounded as follows:

$$\partial^\alpha_\beta\exp\left(-\frac{|v-U_{12\theta}|^2}{2\frac{T_{12\theta}}{m_1}}+\frac{m_1|v|^2}{4}\right)$$
$$\le C\sqrt{\mathcal{E}(t)}\|\partial^\alpha f\|_{L^2_v}\mathcal{P}_{lm}(v)\exp\left(-\frac{|v-U_{12\theta}|^2}{2\frac{T_{12\theta}}{m_1}}+\frac{m_1|v|^2}{4}\right).$$

By Lemma 4.1(3), a sufficiently small $\mathcal{E}(t)$ guarantees $T_{12\theta}\le 3/2$, so that

$$\begin{aligned}\langle A_{lm},g\rangle_{L^2_v}&\le C\left\|P(v)\exp\left(-\frac{2m_1|v-U_{12\theta}|^2}{3}+\frac{m_1|v|^2}{2}\right)\right\|_{L^2_v}\|g\|_{L^2_v}\\&\le C\left\|P(v)\exp\left(-\frac{m_1|v-4U_{12\theta}|^2}{6}+2m_1|U_{12\theta}|^2\right)\right\|_{L^2_v}\|g\|_{L^2_v}\\&\le C\|g\|_{L^2_v},\end{aligned}\tag{4.11}$$

where we used $e^{2m_1|U_{12\theta}|^2}\le C$ for sufficiently small $\mathcal{E}(t)$. Substituting (4.9) and (4.11) on (4.8) gives the desired result. □

4.3. Local existence. In this part, we prove the existence of a local-in-time classical solution of the mixture BGK model (1.1).

THEOREM 4.6. *Let $F_{10}=\mu_1+\sqrt{\mu_1}f_{10}\ge 0$ and $F_{20}=\mu_2+\sqrt{\mu_2}f_{20}\ge 0$. There exists $T_*>0$ and $M_0>0$ such that if $\mathcal{E}(0)\le\frac{M_0}{2}$, then there exists a unique local-in-time solution (F_1,F_2) of (1.1) such that*

1. *the distribution functions $F_1(x,v,t)$ and $F_2(x,v,t)$ are nonnegative;*

2. *the high-order energy* $\mathcal{E}(t)$ *is uniformly bounded:*

$$\sup_{0\le t\le T_*} \mathcal{E}(t) \le M_0,$$

3. *the high-order energy is continuous in* $t \in [0, T_*)$*;*
4. *the conservation laws* (2.17) *hold for all* $t \in [0, T_*)$.

Proof. We define an iteration of the mixture BGK model (1.1) as follows:

$$\begin{aligned}
\partial_t F_1^{n+1} + v\cdot\nabla_x F_1^{n+1} &= n_1(F_1^n)(\mathcal{M}_{11}(F_1^n) - F_1^{n+1}) \\
&\quad + n_2(F_2^n)(\mathcal{M}_{12}(F_1^n, F_2^n) - F_1^{n+1}), \\
\partial_t F_2^{n+1} + v\cdot\nabla_x F_2^{n+1} &= n_2(F_2^n)(\mathcal{M}_{22}(F_2^n) - F_2^{n+1}) \\
&\quad + n_1(F_1^n)(\mathcal{M}_{21}(F_1^n, F_2^n) - F_2^{n+1}),
\end{aligned} \tag{4.12}$$

and $F_1^{n+1}(x,v,0) = F_{10}(x,v)$ and $F_2^{n+1}(x,v,0) = F_{20}(x,v)$ for all $n \ge 0$. And $n_1(F_1^n)$ and $n_2(F_2^n)$ denote

$$n_1(F_1^n) = \int_{\mathbb{R}^3} F_1^n(x,v,t)dv, \qquad n_2(F_2^n) = \int_{\mathbb{R}^3} F_2^n(x,v,t)dv.$$

We start the iteration with $F_1^0(x,v,t) = F_{10}(x,v)$ and $F_2^0(x,v,t) = F_{20}(x,v)$.

We split $F_1^n = \mu_1 + \sqrt{\mu_1} f_1^n$ and $F_2^n = \mu_2 + \sqrt{\mu_2} f_2^n$ for all $n \in \mathbb{N}$ and use the linearization of the Maxwellian given in Proposition 2.3 and Lemma 2.2 to get

$$\begin{aligned}
\partial_t f_1^{n+1} + v\cdot\nabla_x f_1^{n+1} &= (n_{10}+n_{20})(P_1 f_1^n - f_1^{n+1}) + L_{12}^2(f_1^n, f_2^n) + \Gamma_{11}(f_1^n) + \Gamma_{12}(f_1^n, f_2^n), \\
\partial_t f_2^{n+1} + v\cdot\nabla_x f_2^{n+1} &= (n_{10}+n_{20})(P_2 f_2^n - f_2^{n+1}) + L_{21}^2(f_1^n, f_2^n) + \Gamma_{22}(f_2^n) + \Gamma_{21}(f_1^n, f_2^n).
\end{aligned}$$

Then the local existence can be constructed by the standard argument as in [39]. The key ingredient is the uniform control of the high-order energy norm in each iteration step. So we first prove the following auxiliary lemma below. ◻

LEMMA 4.7. *Let* $\mathcal{E}(0) < \frac{M_0}{2}$. *Then there exists* $T_* > 0$ *and* $M_0 > 0$ *such that* $\mathcal{E}(f^n(t)) < M_0$ *for all* $n \ge 0$ *and* $t \in [0, T_*]$.

Proof. We take ∂_β^α on each side of (2.13) and (2.15):

$$\begin{aligned}
\partial_\beta^\alpha \partial_t f_1^{n+1} + v\cdot\nabla_x \partial_\beta^\alpha f_1^{n+1} + \sum_{i=1}^3 \partial_{\beta-k_i}^{\alpha+\bar{k}_i} f_1^{n+1} &= (n_{10}+n_{20})(\partial_\beta P_1 \partial^\alpha f_1^n - \partial_\beta^\alpha f_1^{n+1}) \\
&\quad + \partial_\beta^\alpha L_{12}^2(f_1^n, f_2^n) + \partial_\beta^\alpha \Gamma_{11}(f_1^n) + \partial_\beta^\alpha \Gamma_{12}(f_1^n, f_2^n),
\end{aligned}$$

and

$$\begin{aligned}
\partial_\beta^\alpha \partial_t f_2^{n+1} + v\cdot\nabla_x \partial_\beta^\alpha f_2^{n+1} + \sum_{i=1}^3 \partial_{\beta-k_i}^{\alpha+\bar{k}_i} f_2^{n+1} &= (n_{10}+n_{20})(\partial_\beta P_2 \partial^\alpha f_2^n - \partial_\beta^\alpha f_2^{n+1}) \\
&\quad + \partial_\beta^\alpha L_{21}^2(f_1^n, f_2^n) + \partial_\beta^\alpha \Gamma_{22}(f_2^n) + \partial_\beta^\alpha \Gamma_{21}(f_1^n, f_2^n),
\end{aligned}$$

where $k_1 = (1,0,0)$, $k_2 = (0,1,0)$, $k_3 = (0,0,1)$, and $\bar{k}_1 = (0,1,0,0)$, $\bar{k}_2 = (0,0,1,0)$, $\bar{k}_3 = (0,0,0,1)$. We then take the inner product with $\partial_\beta^\alpha f_1^{n+1}$:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}} + (n_{10}+n_{20})\|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}}\\
&\quad = -\sum_{i=1}^3 \langle \partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_1^{n+1}, \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}}\\
&\qquad + \langle \partial_\beta P_1 \partial^\alpha f_1^n, \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}} + \langle \partial^\alpha_\beta L^2_{12}(f_1^n, f_2^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}}\\
&\qquad + \langle \partial^\alpha_\beta \Gamma_{11}(f_1^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}} + \langle \partial^\alpha_\beta \Gamma_{12}(f_1^n, f_2^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}}\\
&\quad = I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{4.13}
$$

Applying the Hölder inequality on I_1, we have

$$
\begin{aligned}
I_1 = \sum_{i=1}^3 \langle \partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_1^{n+1}, \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}} &\le \sum_{i=1}^3 \|\partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_1^{n+1}\|_{L^2_{x,v}} \|\partial^\alpha_\beta f_1^{n+1}\|_{L^2_{x,v}}\\
&\le \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}}.
\end{aligned}
$$

Since $\partial_\beta e_{1i}$ and $\partial_\beta e_{2i}$ have exponential decay,

$$
\|\partial_\beta P_1 \partial^\alpha f_1^n\|_{L^2_{x,v}} \le C_\beta \|\partial^\alpha f_1^n\|_{L^2_{x,v}}.
$$

Thus Young's inequality implies

$$
I_2 = \langle \partial_\beta P_1 \partial^\alpha f_1^n, \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}} \le C_\beta \|\partial^\alpha f_1^n\|^2_{L^2_{x,v}} + C\|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}}.
$$

To estimate I_3, we take ∂^α_β on L^2_{12},

$$
\begin{aligned}
\partial^\alpha_\beta L^2_{12}(f_1,f_2) = n_{20}\Bigg[(1-\delta)\sum_{2\le i\le 4}\left(\sqrt{\frac{n_{10}}{n_{20}}}\sqrt{\frac{m_1}{m_2}}\langle \partial^\alpha f_2, e_{2i}\rangle_{L^2_v} - \langle \partial^\alpha f_1, e_{1i}\rangle_{L^2_v}\right)\partial_\beta e_{1i}\\
+ (1-\omega)\left(\sqrt{\frac{n_{10}}{n_{20}}}\langle \partial^\alpha f_2, e_{25}\rangle_{L^2_v} - \langle \partial^\alpha f_1, e_{15}\rangle_{L^2_v}\right)\partial_\beta e_{15}\Bigg],
\end{aligned}
$$

and apply the Hölder inequality:

$$
\begin{aligned}
I_3 &= \langle \partial^\alpha_\beta L^2_{12}(f_1^n, f_2^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}}\\
&\le C\int_{\mathbb{T}^3}\left(\|\partial^\alpha f_2^n\|_{L^2_v} + C\|\partial^\alpha f_1^n\|_{L^2_v}\right)\|\partial^\alpha_\beta f_1^{n+1}\|_{L^2_v}\,dx\\
&\le C\|\partial^\alpha (f_1^n, f_2^n)\|_{L^2_{x,v}}\|\partial^\alpha_\beta f_1^{n+1}\|_{L^2_{x,v}}.
\end{aligned}
$$

Since I_4 and I_5 are similar, we only consider I_5. Applying Lemma 4.5, we have

$$
\begin{aligned}
I_5 = \langle \partial^\alpha_\beta \Gamma_{12}(f_1^n, f_2^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}} \le C \sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\int_{\mathbb{T}^3} \|\partial^{\alpha_1}(f_1^n, f_2^n)\|_{L^2_v}\\
\times \|\partial^{\alpha_2}(f_1^n, f_2^n)\|_{L^2_v}\|\partial^\alpha_\beta f_1^{n+1}\|_{L^2_v}\,dx.
\end{aligned}
$$

Without loss of generality, we assume that $|\alpha_1| \le |\alpha_2|$. Then the Sobolev embedding $H^2 \subset\subset L^\infty$ implies

$$
\begin{aligned}
I_5 &= \langle \partial^\alpha_\beta \Gamma_{12}(f_1^n, f_2^n), \partial^\alpha_\beta f_1^{n+1}\rangle_{L^2_{x,v}}\\
&\le C\Bigg(\sum_{|\alpha_1|\le|\alpha|}\|\partial^{\alpha_1}(f_1^n, f_2^n)\|_{L^2_{x,v}}\Bigg)^2 \|\partial^\alpha_\beta f_1^{n+1}\|_{L^2_{x,v}}.
\end{aligned}
$$

Combining the estimate from I_1 to I_5, and taking $\sum_{|\alpha|+|\beta|\leq N}$ on (4.13), we have

$$\begin{aligned}\frac{1}{2}\sum_{|\alpha|+|\beta|\leq N}\frac{d}{dt}\|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}}+(n_{10}+n_{20})\sum_{|\alpha|+|\beta|\leq N}\|\partial^\alpha_\beta f_1^{n+1}\|^2_{L^2_{x,v}}\\ \leq C\mathcal{E}^n(t)+C\mathcal{E}^{n+1}(t)+C\sqrt{\mathcal{E}^n(t)}\sqrt{\mathcal{E}^{n+1}(t)}+C\mathcal{E}^n(t)\sqrt{\mathcal{E}^{n+1}(t)}.\end{aligned}\tag{4.14}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\sum_{|\alpha|+|\beta|\leq N}\frac{d}{dt}\|\partial^\alpha_\beta f_2^{n+1}\|^2_{L^2_{x,v}}+\sum_{|\alpha|+|\beta|\leq N}(n_{10}+n_{20})\|\partial^\alpha_\beta f_2^{n+1}\|^2_{L^2_{x,v}}\\ \leq C\mathcal{E}^n(t)+C\mathcal{E}^{n+1}(t)+C\sqrt{\mathcal{E}^n(t)}\sqrt{\mathcal{E}^{n+1}(t)}+C\mathcal{E}^n(t)\sqrt{\mathcal{E}^{n+1}(t)}.\end{aligned}\tag{4.15}$$

Combining (4.14) and (4.15) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{E}^{n+1}(t)+(n_{10}+n_{20})\mathcal{E}^{n+1}(t)&\leq C\mathcal{E}^n(t)+C\mathcal{E}^{n+1}(t)\\ &\quad+C\sqrt{\mathcal{E}^n(t)}\sqrt{\mathcal{E}^{n+1}(t)}+C\mathcal{E}^n(t)\sqrt{\mathcal{E}^{n+1}(t)}.\end{aligned}$$

We integrate in time to get

$$\begin{aligned}&\mathcal{E}^{n+1}(t)\leq\mathcal{E}^{n+1}(0)\\ &\quad+\int_0^t\Big(C\mathcal{E}^n(s)+C\mathcal{E}^{n+1}(s)+C\sqrt{\mathcal{E}^n(t)}\sqrt{\mathcal{E}^{n+1}(t)}+C\mathcal{E}^n(t)\sqrt{\mathcal{E}^{n+1}(t)}\Big)\,ds.\end{aligned}\tag{4.16}$$

We now apply an induction argument. We have $\mathcal{E}^0(0)<\frac{M_0}{2}$ from the assumption. Assume we have

$$\sup_{0\leq t\leq T_*}\mathcal{E}^n(t)\leq M_0,\quad \mathcal{E}^{n+1}(0)\leq M_0/2.$$

Then, from (4.16), we see that

$$\begin{aligned}\sup_{0\leq t\leq T_*}\mathcal{E}^{n+1}(t)&\leq\frac{M_0}{2}+CT_*M_0+CT_*\sup_{0\leq t\leq T_*}\mathcal{E}^{n+1}(t)\\ &\quad+CT_*\sqrt{M_0}\sqrt{\sup_{0\leq t\leq T_*}\mathcal{E}^{n+1}(t)}+CT_*M_0\sqrt{\sup_{0\leq t\leq T_*}\mathcal{E}^{n+1}(t)}.\end{aligned}$$

By using Young's inequality, we have

$$(1-3CT_*)\sup_{0\leq t\leq T_*}\mathcal{E}_1^{n+1}(t)\leq\frac{M_0}{2}+2CT_*M_0+CT_*M_0^2.$$

Therefore, for sufficiently small T_* and $M_0>0$, we can derive

$$\sup_{0\leq t\leq T_*}\mathcal{E}^{n+1}(t)\leq M_0.$$

This completes the proof. ∎

Now we can complete the proof of Theorem 4.6. First, Lemma 4.7 gives the strong compactness of $\{f^n\}$ with sufficient regularity. Therefore, taking the limit $n\to\infty$ yields the desired local-in-time solution.

For the nonnegativity of the solution, we see from (4.12) and the nonnegativity of the Maxwellian $\mathcal{M}_{ij}$ for $i,j=1,2$ that

$$\begin{aligned}&\partial_t F_1^{n+1}+v\cdot\nabla_x F_1^{n+1}+(n_1(F_1^n)+n_2(F_2^n))\,F_1^{n+1}\\&\quad=n_1(F_1^n)\mathcal{M}_{11}(F_1^n)+n_2(F_2^n)\mathcal{M}_{12}(F_1^n,F_2^n)\geq 0,\\&\partial_t F_2^{n+1}+v\cdot\nabla_x F_2^{n+1}+(n_1(F_1^n)+n_2(F_2^n))\,F_2^{n+1}\\&\quad=n_2(F_2^n)\mathcal{M}_{22}(F_2^n)+n_1(F_1^n)\mathcal{M}_{21}(F_1^n,F_2^n)\geq 0.\end{aligned}$$

Then the mild formulation of the solution F_k^{n+1} gives the desired nonnegativity for $(k=1,2)$:

$$F_k^{n+1}(x,v,t)\geq e^{-\int_0^t(n_1(F_1^n)+n_2(F_2^n))dt}F_k^{n+1}(x,v,0)\geq 0.$$

Therefore, an induction argument leads to the nonnegativity for F_k^n for all n, which gives the desired nonnegativity of the local-in-time solution F_k.

It remains to prove the continuity of the energy norm. For this, we sum (4.14) and (4.15), which also hold for F_k $(k=1,2)$, and integrate over time to get

$$|\mathcal{E}(t)-\mathcal{E}(s)|\leq C\sup_{s\leq\tau\leq t}\left(1+\sqrt{\mathcal{E}(\tau)}+\mathcal{E}(\tau)\right)\int_s^t\sum_{|\alpha|\leq N}\sum_{k=1,2}\|\partial^\alpha f_k(\cdot,\cdot,\tau)\|_{L^2_{x,v}}d\tau.$$

Since the energy norm is bounded, the R.H.S. converges to 0 when $s\to t$. Thus we have continuity of $\mathcal{E}(t)$.

Finally, substituting $F_k=\mu_k+\sqrt{\mu_k}f_k$ to (1.2) gives the conservation laws (2.17). The uniqueness and stability will be given at the end of section 6.

5. Coercivity estimate. We write the macroscopic part $P(f_1,f_2)$ of the distribution function (f_1,f_2) as

$$\begin{aligned}P(f_1,f_2)&=a_1(x,t)\,(\sqrt{\mu_1},0)+a_2(x,t)\,(0,\sqrt{\mu_2})+b(x,t)\cdot v\,(m_1\sqrt{\mu_1},m_2\sqrt{\mu_2})\\&\quad+c(x,t)|v|^2\,(m_1\sqrt{\mu_1},m_2\sqrt{\mu_2}),\end{aligned}$$

where

(5.1)
$$\begin{aligned}a_k(x,t)&=\frac{1}{n_{k0}}\int_{\mathbb{R}^3}f_k\sqrt{\mu_k}dv\\&\quad-\frac{1}{2n_{10}+2n_{20}}\left(\int_{\mathbb{R}^3}f_1(m_1|v|^2-3)\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right),\\b(x,t)&=\frac{1}{m_1n_{10}+m_2n_{20}}\left(\int_{\mathbb{R}^3}f_1m_1v\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f_2m_2v\sqrt{\mu_2}dv\right),\\c(x,t)&=\frac{1}{6n_{10}+6n_{20}}\left(\int_{\mathbb{R}^3}f_1(m_1|v|^2-3)\sqrt{\mu_1}dv+\int_{\mathbb{R}^3}f_2(m_2|v|^2-3)\sqrt{\mu_2}dv\right),\end{aligned}$$

for $k=1,2$. We substitute

$$(f_1,f_2)=(I-P)(f_1,f_2)+P(f_1,f_2),$$

into (3.1) to get

$$\begin{aligned}\{\partial_t+v\cdot\nabla_x\}P(f_1,f_2)&=-\{\partial_t+v\cdot\nabla_x-L\}(I-P)(f_1,f_2)\\&\quad+(\Gamma_{11}(f_1)+\Gamma_{12}(f_1,f_2),\Gamma_{22}(f_2)+\Gamma_{21}(f_1,f_2)).\end{aligned}\tag{5.2}$$

We write the L.H.S. of (5.2) in the form

$$\Big\{ (\partial_t a_1 + v\cdot\nabla_x a_1)\,(\sqrt{\mu_1},0) + (\partial_t a_2 + v\cdot\nabla_x a_2)\,(0,\sqrt{\mu_2}) \\ + v\cdot\partial_t b(m_1\sqrt{\mu_1}, m_2\sqrt{\mu_2}) + \sum_{1\le i<j\le 3} v_i v_j(\partial_{x_i} b_j + \partial_{x_j} b_i)(m_1\sqrt{\mu_1}, m_2\sqrt{\mu_2}) \\ + \sum_{1\le i\le 3}(\partial_{x_i} b_i + \partial_t c)v_i^2(m_1\sqrt{\mu_1}, m_2\sqrt{\mu_2}) + |v|^2 v\cdot\nabla_x c(m_1\sqrt{\mu_1}, m_2\sqrt{\mu_2})\Big\},$$

as a linear expansion with respect to the following 17 bases:

$$\{(\sqrt{\mu_1},0),(0,\sqrt{\mu_2}),v(\sqrt{\mu_1},0),v(0,\sqrt{\mu_2}), \\ v_iv_j(m_1\sqrt{\mu_1},m_2\sqrt{\mu_2}),v|v|^2(m_1\sqrt{\mu_1},m_2\sqrt{\mu_2})\}. \tag{5.3}$$

Therefore, comparing both sides of (5.2), we obtain the following system:

$$\begin{aligned} \partial_t a_1 &= l_{a1} + h_{a1},\\ \partial_t a_2 &= l_{a2} + h_{a2},\\ \partial_{x_i} a_1 + m_1\partial_t b_i &= l_{b1i} + h_{b1i},\\ \partial_{x_i} a_2 + m_2\partial_t b_i &= l_{b2i} + h_{b2i},\\ \partial_{x_i} b_j + \partial_{x_j} b_i &= l_{bbi} + h_{bbi} \quad (i\neq j),\\ \partial_{x_i} b_i + \partial_t c &= l_{bci} + h_{bci},\\ \partial_{x_i} c &= l_{ci} + h_{ci}, \end{aligned}$$

where $(l_{a1}, l_{a2}, l_{b1i}, l_{b2i}, l_{bbi}, l_{bci}, l_{ci})$ and $(h_{a1}, h_{a2}, h_{b1i}, h_{b2i}, h_{bbi}, h_{bci}, h_{ci})$ are the coefficients corresponding to the expansion of l and h,

$$\begin{aligned} l(f_1,f_2) &= -\{\partial_t + v\cdot\nabla_x - L\}(I-P)(f_1,f_2),\\ h(f_1,f_2) &= (\Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2), \Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2)), \end{aligned}$$

with respect to (5.3). For brevity, we denote

$$\begin{aligned} \tilde{l} &= l_{a1} + l_{a2} + \sum_{i=1}^{3}(l_{b1i} + l_{b2i} + l_{bbi} + l_{bci} + l_{ci}),\\ \tilde{h} &= h_{a1} + h_{a2} + \sum_{i=1}^{3}(h_{b1i} + h_{b2i} + h_{bbi} + h_{bci} + h_{ci}). \end{aligned}$$

LEMMA 5.1. *We have*

$$\int_{\mathbb{T}^3} a_1(x,t)dx = \int_{\mathbb{T}^3} a_2(x,t)dx = \int_{\mathbb{T}^3} b(x,t)dx = \int_{\mathbb{T}^3} c(x,t)dx = 0.$$

Proof. This follows from the conservation laws (2.17) and the definition of a_1, a_2, b, and c in (5.1). ∎

LEMMA 5.2 (see [39]). *Let $0\le|\alpha|\le N$ with $N\ge 3$; then we have*

$$\|\partial^\alpha a_1\|_{L^2_x} + \|\partial^\alpha a_2\|_{L^2_x} + \|\partial^\alpha b\|_{L^2_x} + \|\partial^\alpha c\|_{L^2_x} \le \sum_{|\alpha|\le N-1}\Big(\|\partial^\alpha \tilde{l}\|_{L^2_x} + \|\partial^\alpha \tilde{h}\|_{L^2_x}\Big).$$

Proof. The proof can be found in [39, Proof of Theorem 3, p. 620]. We omit it. ◻

LEMMA 5.3. *For sufficiently small energy norm $\mathcal{E}(t)$, we have*

$$(1) \sum_{|\alpha|\le N-1}\|\partial^\alpha \tilde{l}\|_{L^2_{x,v}} \le C\sum_{|\alpha|\le N}\|(I-P)\partial^\alpha(f_1,f_2)\|_{L^2_{x,v}},$$

$$(2) \sum_{|\alpha|\le N}\|\partial^\alpha \tilde{h}\|_{L^2_{x,v}} \le C\sqrt{M}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|_{L^2_{x,v}}.$$

Proof. (1) The proof can be found in [39, Lemma 7, p. 616]. We omit it.

(2) Let us define $\{e_i^*\}_{i=1}^{17}$ to be the orthonormal basis corresponding to the basis (5.3). Then we can write

$$e_i^* = \sum_{j=1}^{17} C_{ij}e_j, \qquad h(f_1,f_2) = \sum_{i=1}^{17}\langle h, e_i^*\rangle_{L^2_v} e_i^*,$$

so that

$$\langle h, e_n^*\rangle_{L^2_v} = \sum_{1\le i,j\le 17} C_{ij}C_{ni}\langle h, e_i^*\rangle_{L^2_v},$$

for $n=1,\dots,17$. For the estimate of h, we compute

$$\begin{aligned}\left\|\int \partial^\alpha h(f_1,f_2)e_i^* dv\right\|_{L^2_x} \le &\left\|\int \partial^\alpha \Gamma_{11}(f_1)(|v|^k\sqrt{\mu_1})dv\right\|_{L^2_x}\\ &+\left\|\int \partial^\alpha \Gamma_{12}(f_1,f_2)(|v|^k\sqrt{\mu_1})dv\right\|_{L^2_x}\\ &+\left\|\int \partial^\alpha \Gamma_{22}(f_2)(|v|^k\sqrt{\mu_2})dv\right\|_{L^2_x}\\ &+\left\|\int \partial^\alpha \Gamma_{21}(f_1,f_2)(|v|^k\sqrt{\mu_2})dv\right\|_{L^2_x},\end{aligned}$$

for $k=0,1,2,3$. For sufficiently small $\mathcal{E}(t)$, by Lemma 4.4, we have

$$\left\|\int \partial^\alpha \Gamma_{mm}(f_m)|v|^k\sqrt{\mu_m}dv\right\|_{L^2_x} \le C\sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\left\|\|\partial^{\alpha_1}f_m\|_{L^2_v}\|\partial^{\alpha_2}f_m\|_{L^2_v}\right\|_{L^2_x}.$$

Similarly, we have from Lemma 4.5

$$\left\|\int \partial^\alpha \Gamma_{lm}(f_l,f_m)|v|^k\sqrt{\mu_l}dv\right\|_{L^2_x} \le C\sum_{|\alpha_1|+|\alpha_2|\le|\alpha|}\left\|\|\partial^{\alpha_1}(f_l,f_m)\|_{L^2_v}\|\partial^{\alpha_2}(f_l,f_m)\|_{L^2_v}\right\|_{L^2_x},$$

for $l\neq m$. Without loss of generality, we assume that $|\alpha_1|\le|\alpha_2|$ and apply the Sobolev embedding $H^2\subset\subset L^\infty$ to obtain

$$\begin{aligned}\sum_{|\alpha|\le N}\|\partial^\alpha\tilde{h}\|_{L^2_{x,v}} &\le C\sum_{|\alpha_1|\le|\alpha_2|}\sup_{x\in\mathbb{T}^3}\|\partial^{\alpha_1}(f_1,f_2)\|_{L^2_v}\sum_{|\alpha_2|\le N}\|\partial^{\alpha_2}(f_1,f_2)\|_{L^2_{x,v}}\\ &\le C\sqrt{\mathcal{E}(t)}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|_{L^2_{x,v}},\end{aligned}$$

which gives the desired result. ◻

We are now ready to derive the full coercivity estimate. By Lemma 5.2, we have

$$\begin{aligned}\sum_{|\alpha|\le N}\|\partial^\alpha P(f_1,f_2)\|^2_{L^2_{x,v}} &\le \sum_{|\alpha|\le N}\left(\|\partial^\alpha a_1\|^2_{L^2_x}+\|\partial^\alpha a_2\|^2_{L^2_x}+\|\partial^\alpha b\|^2_{L^2_x}+\|\partial^\alpha c\|^2_{L^2_x}\right)\\ &\le \sum_{|\alpha|\le N-1}\left(\|\partial^\alpha \tilde{l}\|^2_{L^2_x}+\|\partial^\alpha \tilde{h}\|^2_{L^2_x}\right).\end{aligned}$$

We then apply Lemma 5.3 to get

$$\begin{aligned}&\sum_{|\alpha|\le N}\|\partial^\alpha P(f_1,f_2)\|^2_{L^2_{x,v}}\\ &\quad\le C\sum_{|\alpha|\le N}\|(I-P)\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}+C\sqrt{M}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}.\end{aligned}$$

Adding $\sum_{|\alpha|\le N}\|(I-P)\partial^\alpha(f_1,f_2)\|^2_{L^2_x}$ on each side, we obtain

$$\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}\le\frac{C+1}{1-C\sqrt{M}}\sum_{|\alpha|\le N}\|(I-P)(\partial^\alpha(f_1,f_2))\|^2_{L^2_{x,v}}.$$

Combining it with the estimate in Proposition 3.1, we derive the full coercivity estimate

$$\langle L\partial^\alpha(f_1,f_2),\partial^\alpha(f_1,f_2)\rangle_{L^2_{x,v}}\le-\eta\min\{(1-\delta),(1-\omega)\}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}, \tag{5.4}$$

when $\mathcal{E}(t)$ is sufficiently small.

6. Global existence. In this section, we extend the local-in-time solution to the global one by establishing a uniform energy estimate. Let (f_1,f_2) be the classical local-in-time solution constructed in Theorem 4.6. We take ∂^α on (2.13) and take the inner product with $\partial^\alpha f_1$ in $L^2_{x,v}$ to have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f_1\|^2_{L^2_{x,v}}&=\langle\partial^\alpha L_{11}(f_1),\partial^\alpha f_1\rangle_{L^2_{x,v}}+\langle\partial^\alpha L_{12}(f_1,f_2),\partial^\alpha f_1\rangle_{L^2_{x,v}}\\ &\quad+\langle\partial^\alpha f_1,\partial^\alpha(\Gamma_{11}+\Gamma_{12})\rangle_{L^2_{x,v}}.\end{aligned}\tag{6.1}$$

Similarly, we get from (2.15) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f_2\|^2_{L^2_{x,v}}&=\langle\partial^\alpha L_{22}(f_2),\partial^\alpha f_2\rangle_{L^2_{x,v}}+\langle\partial^\alpha L_{21}(f_1,f_2),\partial^\alpha f_2\rangle_{L^2_{x,v}}\\ &\quad+\langle\partial^\alpha f_2,\partial^\alpha(\Gamma_{22}+\Gamma_{21})\rangle_{L^2_{x,v}}.\end{aligned}\tag{6.2}$$

Combining (6.1) and (6.2) yields

$$\begin{aligned}\sum_{k=1,2}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f_k\|^2_{L^2_{x,v}}&\le\langle L\partial^\alpha(f_1,f_2),\partial^\alpha(f_1,f_2)\rangle_{L^2_{x,v}}\\ &\quad+\langle\partial^\alpha f_1,\partial^\alpha(\Gamma_{11}+\Gamma_{12})\rangle_{L^2_{x,v}}+\langle\partial^\alpha f_2,\partial^\alpha(\Gamma_{22}+\Gamma_{21})\rangle_{L^2_{x,v}}.\end{aligned}$$

Then the first term of the R.H.S is controlled by the full coercivity estimate (5.4), and the nonlinear terms on the second line are estimated by Lemmas 4.4 and 4.5:

$$\begin{aligned}&\sum_{|\alpha|\le N}\sum_{k=1,2}\left(\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f_k\|^2_{L^2_{x,v}}+\eta\min\{(1-\delta),(1-\omega)\}\|\partial^\alpha f_k\|^2_{L^2_{x,v}}\right)\\ &\quad\le C_0\sqrt{\mathcal{E}_{N_1,0}(t)}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1,f_2)\|^2_{L^2_{x,v}}.\end{aligned}$$

For M_0 satisfying Theorem 4.6 and (5.4), we define

$$M = \left\{ \frac{M_0}{2}, \frac{\eta^2 \min\left\{(1-\delta)^2, (1-\omega)^2\right\}}{4C_0^2} \right\}, \qquad T = \sup_{t\in\mathbb{R}^+} \{t \mid \mathcal{E}_{N_1,0}(t) \le 2M\} > 0.$$

We restrict our initial data to satisfy the following energy bound:

$$\mathcal{E}_{N_1,0}(0) \le M \le 2M_0.$$

Once we define

$$y(t) = \sum_{|\alpha|\le N} \sum_{k=1,2} \|\partial^\alpha f_k\|^2_{L^2_{x,v}},$$

then $y(t)$ satisfies

$$\begin{aligned} y'(t) + 2\eta \min\{(1-\delta),(1-\omega)\} y(t) &\le 2C_0 \sqrt{\mathcal{E}_{N_1,0}(t)} y(t) \\ &\le \eta \min\{(1-\delta),(1-\omega)\} y(t). \end{aligned}$$

Thus we obtain

$$y(t) \le e^{-\eta \min\{(1-\delta),(1-\omega)\}t} y(0) \le y(0) \le M < 2M,$$

which is possible only when $T = \infty$. Note that this also gives

$$\sum_{|\alpha|\le N} \|\partial^\alpha (f_1(t), f_2(t))\|^2_{L^2_{x,v}} \le e^{-\eta \min\{(1-\delta),(1-\omega)\}t} \sum_{|\alpha|\le N} \|\partial^\alpha (f_1(0), f_2(0))\|^2_{L^2_{x,v}}.$$

Now we consider the general case of f having momentum derivatives. Taking ∂^α_β on (2.13) and (2.15) and applying an inner product with $\partial^\alpha_\beta f_1$ and $\partial^\alpha_\beta f_2$, respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f_1\|^2_{L^2_{x,v}} + (n_{10}+n_{20})\|\partial^\alpha_\beta f_1\|^2_{L^2_{x,v}} \\ &\quad = -\sum_{i=1}^3 \langle \partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_1, \partial^\alpha_\beta f_1\rangle_{L^2_{x,v}} \\ &\qquad + (n_{10}+n_{20})\langle \partial_\beta P_1 \partial^\alpha f_1, \partial^\alpha_\beta f_1\rangle_{L^2_{x,v}} + \langle \partial^\alpha_\beta L^2_{12}(f_1,f_2), \partial^\alpha_\beta f_1\rangle_{L^2_{x,v}} \\ &\qquad + \langle \partial^\alpha_\beta (\Gamma_{11}(f_1) + \Gamma_{12}(f_1,f_2)), \partial^\alpha_\beta f_1\rangle_{L^2_{x,v}}, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f_2\|^2_{L^2_{x,v}} + (n_{10}+n_{20})\|\partial^\alpha_\beta f_2\|^2_{L^2_{x,v}} \\ &\quad = -\sum_{i=1}^3 \langle \partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_2, \partial^\alpha_\beta f_2\rangle_{L^2_{x,v}} \\ &\qquad + (n_{10}+n_{20})\langle \partial_\beta P_2 \partial^\alpha f_2, \partial^\alpha_\beta f_2\rangle_{L^2_{x,v}} + \langle \partial^\alpha_\beta L^2_{21}(f_1,f_2), \partial^\alpha_\beta f_2\rangle_{L^2_{x,v}} \\ &\qquad + \langle \partial^\alpha_\beta (\Gamma_{22}(f_2) + \Gamma_{21}(f_1,f_2)), \partial^\alpha_\beta f_2\rangle_{L^2_{x,v}}. \end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4), and applying the Hölder inequality and Young's inequality, we can obtain

$$\begin{aligned} &\sum_{k=1,2} \left(\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}} + (n_{10}+n_{20}-2\epsilon)\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}} \right) \\ &\quad \le \frac{1}{2\epsilon} \sum_{k=1,2}\sum_{i=1}^3 \|\partial^{\alpha+\bar{k}_i}_{\beta-k_i} f_k\|^2_{L^2_{x,v}} + \frac{C}{2\epsilon}\sum_{k=1,2} \|\partial^\alpha f_k\|^2_{L^2_{x,v}} + C\mathcal{E}^{\frac{3}{2}}_{N_1,|\beta|}(t), \end{aligned}$$

for some positive constant ϵ satisfying $(n_{10}+n_{20})/2>\epsilon>0$. We sum this over $|\beta|=m+1$ and multiply both sides with $\epsilon\eta_m$:

$$\sum_{|\beta|=m+1}\left[\sum_{k=1,2}\left(\frac{\epsilon\eta_m}{2}\frac{d}{dt}\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}}+\epsilon\eta_m(n_{10}+n_{20}-2\epsilon)\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}}\right)\right]$$
$$\leq\sum_{|\beta|=m+1}\left[\frac{\eta_m}{2}\sum_{k=1,2}\sum_{i=1}^{3}\|\partial^{\alpha+\bar{k}_i}_{\beta-k_i}f_k\|^2_{L^2_{x,v}}+\frac{C\eta_m}{2}\sum_{k=1,2}\|\partial^\alpha f_k\|^2_{L^2_{x,v}}+C\mathcal{E}^{\frac{3}{2}}_{N_1,|\beta|}(t)\right].$$

Combining the previous cases $|\beta|\leq m$, the R.H.S of the inequality can be bounded by the energy $\mathcal{E}_{N_1,|\beta|}$ with $|\beta|\leq m$ and $\mathcal{E}_{N_1,0}$. Thus, we can conclude from induction that

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta|\leq m+1}}\sum_{k=1,2}\left(C_{m+1}\frac{d}{dt}\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}}+\eta_{m+1}\|\partial^\alpha_\beta f_k\|^2_{L^2_{x,v}}\right)\leq C^*_{m+1}\mathcal{E}^{\frac{3}{2}}_{N_1,|\beta|}(t).$$

Applying the same continuity argument as when $\beta=0$, we can construct the global-in-time classical solution. We mention that when $|\beta|=0$, the parameter η_0 depends on $1-\delta$ and $1-\omega$, and $C_0=1/2$. But when $|\beta|\geq 1$, both C_{m+1} and η_{m+1} depend on the parameter η_m. That is why we cannot extract a decay rate depending explicitly on the parameter δ and ω when the velocity derivatives are involved.

We now derive the uniqueness and the stability of the solutions. Let (f_1,f_2) and $(\bar f_1,\bar f_2)$ be the solutions to the system (2.16) with initial data (f_{10},f_{20}) and $(\bar f_{10},\bar f_{20})$, respectively. Note that L_{12} and L_{21} are bilinear,

$$L_{ij}(f_1,f_2)-L_{ij}(\bar f_1,\bar f_2)=L_{ij}(f_1-\bar f_1,f_2-\bar f_2)$$

so that substracting the two equations for (f_1,f_2) and $(\bar f_1,\bar f_2)$ gives

$$\begin{aligned}(\partial_t+v\cdot\nabla_x)(f_1-\bar f_1)&=L_{11}(f_1-\bar f_1)+L_{12}(f_1-\bar f_1,f_2-\bar f_2)\\&\quad+\Gamma_{11}(f_1)-\Gamma_{11}(\bar f_1)+\Gamma_{12}(f_1,f_2)-\Gamma_{12}(\bar f_1,\bar f_2),\\(\partial_t+v\cdot\nabla_x)(f_2-\bar f_2)&=L_{22}(f_2-\bar f_2)+L_{21}(f_1-\bar f_1,f_2-\bar f_2)\\&\quad+\Gamma_{22}(f_2)-\Gamma_{22}(\bar f_2)+\Gamma_{21}(f_1,f_2)-\Gamma_{21}(\bar f_1,\bar f_2).\end{aligned}$$

Following the same procedure as in (6.1) and (6.2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(f_1-\bar f_1)\|^2_{L^2_{x,v}}\\&\quad=\langle\partial^\alpha\left(L_{11}(f_1-\bar f_1)+L_{12}(f_1-\bar f_1,f_2-\bar f_2)\right),\partial^\alpha(f_1-\bar f_1)\rangle_{L^2_{x,v}}\\&\quad+\langle\partial^\alpha\left(\Gamma_{11}(f_1)-\Gamma_{11}(\bar f_1)\right),\partial^\alpha(f_1-\bar f_1)\rangle_{L^2_{x,v}}\\&\quad+\langle\partial^\alpha\left(\Gamma_{12}(f_1,f_2)-\Gamma_{12}(\bar f_1,\bar f_2)\right),\partial^\alpha(f_1-\bar f_1)\rangle_{L^2_{x,v}},\end{aligned}\tag{6.5}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(f_2-\bar f_2)\|^2_{L^2_{x,v}}\\&\quad=\langle\partial^\alpha\left(L_{22}(f_2-\bar f_2)+L_{21}(f_1-\bar f_1,f_2-\bar f_2)\right),\partial^\alpha(f_2-\bar f_2)\rangle_{L^2_{x,v}}\\&\quad+\langle\partial^\alpha\left(\Gamma_{22}(f_2)-\Gamma_{22}(\bar f_2)\right),\partial^\alpha(f_2-\bar f_2)\rangle_{L^2_{x,v}}\\&\quad+\langle\partial^\alpha\left(\Gamma_{21}(f_1,f_2)-\Gamma_{21}(\bar f_1,\bar f_2)\right),\partial^\alpha(f_2-\bar f_2)\rangle_{L^2_{x,v}}.\end{aligned}\tag{6.6}$$

Combining (6.5) and (6.6) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}\\
&\quad\le -\eta_{\delta,\omega}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}\\
&\qquad+\sum_{k=1}\langle\partial^\alpha\left(\Gamma_{kk}(f_k)-\Gamma_{kk}(\bar{f}_k)\right),\partial^\alpha(f_k-\bar{f}_k)\rangle_{L^2_{x,v}}\\
&\qquad+\langle\partial^\alpha\left(\Gamma_{12}(f_1,f_2)-\Gamma_{12}(\bar{f}_1,\bar{f}_2)\right),\partial^\alpha(f_1-\bar{f}_1)\rangle_{L^2_{x,v}}\\
&\qquad+\langle\partial^\alpha\left(\Gamma_{21}(f_1,f_2)-\Gamma_{21}(\bar{f}_1,\bar{f}_2)\right),\partial^\alpha(f_2-\bar{f}_2)\rangle_{L^2_{x,v}},
\end{aligned}
\tag{6.7}
$$

where $\eta_{\delta,\omega}=\eta\min\{(1-\delta),(1-\omega)\}$, and we used the full coercivity estimate (5.4) for $(f_1-\bar{f}_1,f_2-\bar{f}_2)$.

We now turn to the estimates of nonlinear terms. We only consider the estimate of the second line of (6.7), because the remaining part can be treated in an almost identitcal manner. For this, we consider

$$
\begin{aligned}
\Gamma_{kk}(f_k,g_k,h_k)&=(n_k^f-n_{k0}^f)(P_kg-g)\\
&\quad+(H_k^f-H_{k0}^f)^T\int_0^1\frac{n_k^g}{\sqrt{\mu_k}}\left\{\nabla^2_{H_{k\theta}}\mathcal{M}^g_{kk}(\theta)\right\}(1-\theta)d\theta(H_k^h-H_{k0}^h),
\end{aligned}
$$

where the definition of H can be found in (4.3). We split the nonlinear term as

$$
\begin{aligned}
\Gamma_{kk}(f_k,f_k,f_k)-\Gamma_{kk}(\bar{f}_k,\bar{f}_k,\bar{f}_k)&=\Gamma_{kk}(f_k,f_k,f_k)-\Gamma_{kk}(\bar{f}_k,f_k,f_k)\\
&\quad+\Gamma_{kk}(\bar{f}_k,f_k,f_k)-\Gamma_{kk}(\bar{f}_k,\bar{f}_k,f_k)\\
&\quad+\Gamma_{kk}(\bar{f}_k,\bar{f}_k,f_k)-\Gamma_{kk}(\bar{f}_k,\bar{f}_k,\bar{f}_k).
\end{aligned}
$$

Following the procedure of the proof of Lemma 4.5, the quantity

$$
\left\langle\int_0^1\frac{n_k^f}{\sqrt{\mu_k}}\left\{\nabla^2_{H_{k\theta}}\mathcal{M}^f_{kk}(\theta)\right\}(1-\theta)d\theta-\int_0^1\frac{n_k^{\bar{f}}}{\sqrt{\mu_k}}\left\{\nabla^2_{H_{k\theta}}\mathcal{M}^{\bar{f}}_{kk}(\theta)\right\}(1-\theta)d\theta,f_k-\bar{f}_k\right\rangle_{L^2_{x,v}}
$$

can be bounded by $C\|f_k-\bar{f}_k\|^2_{L^2_{x,v}}$ for sufficiently small energy norm. Thus, applying Lemma 4.4, we have

$$
\begin{aligned}
&\sum_{|\alpha|\le N}\langle\partial^\alpha\left(\Gamma_{kk}(f_k)-\Gamma_{kk}(\bar{f}_k)\right),\partial^\alpha(f_k-\bar{f}_k)\rangle_{L^2_{x,v}}\\
&\quad\le C\sum_{|\alpha|\le N}\Bigg(\|\partial^\alpha f_k\|_{L^2_{x,v}}+\|\partial^\alpha\bar{f}_k\|_{L^2_{x,v}}\\
&\qquad+\|\partial^\alpha f_k\|^2_{L^2_{x,v}}+\|\partial^\alpha f_k\|_{L^2_{x,v}}\|\partial^\alpha\bar{f}_k\|_{L^2_{x,v}}+\|\partial^\alpha\bar{f}_k\|^2_{L^2_{x,v}}\Bigg)\times\|\partial^\alpha(f_k-\bar{f}_k)\|^2_{L^2_{x,v}}\\
&\quad\le\sqrt{\mathcal{E}(t)}\sum_{|\alpha|\le N}\|\partial^\alpha(f_k-\bar{f}_k)\|^2_{L^2_{x,v}}.
\end{aligned}
$$

Similarly, the third line and the fourth line of (6.7) are estimated as

$$
\begin{aligned}
&\sum_{|\alpha|\le N}\langle\partial^\alpha\left(\Gamma_{ij}(f_1,f_2)-\Gamma_{ij}(\bar{f}_1,\bar{f}_2)\right),\partial^\alpha(f_i-\bar{f}_i)\rangle_{L^2_{x,v}}\\
&\quad\le\sqrt{\mathcal{E}(t)}\sum_{|\alpha|\le N}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}.
\end{aligned}
$$

Combining all the estimates, we have

$$\begin{aligned}\frac{1}{2}\sum_{|\alpha|\leq N}\frac{d}{dt}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}\\ \leq -\left(\eta_{\delta,\omega}-\sqrt{\mathcal{E}(t)}\right)\sum_{|\alpha|\leq N}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}.\end{aligned}$$

Therefore, when $\mathcal{E}(t)$ is sufficiently small, the Grönwall inequality yields the desired result,

$$\sum_{|\alpha|\leq N}\|\partial^\alpha(f_1-\bar{f}_1),\partial^\alpha(f_2-\bar{f}_2)\|^2_{L^2_{x,v}}\leq e^{-\eta_{\delta,\omega}t}\sum_{|\alpha|\leq N}\|\partial^\alpha(f_{10}-\bar{f}_{10}),\partial^\alpha(f_{20}-\bar{f}_{20})\|^2_{L^2_{x,v}},$$

which also gives the uniqueness of the solution.

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