

## MULTILEVEL MONTE CARLO FINITE VOLUME METHODS FOR RANDOM CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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**Abstract.** We consider a random scalar hyperbolic conservation law in one spatial dimension with bounded random flux functions which are discontinuous in the spatial variable. We show that there exists a unique random entropy solution to the conservation law for corresponding to the specific entropy condition used to solve the deterministic case. Using the empirical convergence rates of the underlying deterministic problem over a broad range of parameters, we present a convergence analysis of a multilevel Monte Carlo Finite Volume Method (MLMC-FVM). It is based on a pathwise application of the finite volume method for the deterministic conservation laws. We show that the work required to compute the MLMC-FVM solutions is an order lower than the work required to compute the Monte Carlo Finite Volume Method solutions with equal accuracy.

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### 1. INTRODUCTION

One dimensional scalar conservation laws with discontinuous flux in the space variables are often used to model different phenomena such as traffic flow [12, 22, 28], two phase flow in a porous media [?, 10, 11, 13–15] and sedimentation processes [5, 6, 9]. In one space dimension, a Cauchy problem for the model typically looks like

$$\frac{\partial u}{\partial t} + \frac{\partial f(k, u)}{\partial x} = 0 \quad x \in \mathbb{R}, t > 0 \quad (1a)$$

$$u(x, 0) = u_0(x) \quad (1b)$$

where  $k(x)$  is allowed to be discontinuous in  $x$ . Here  $u(x, t)$  is the unknown function while  $k(x)$  and the Cauchy data  $u_0(x)$  are assumed to be known.

Even for a differentiable  $k(x)$  and smooth initial conditions, the solutions are generally discontinuous. Hence, we consider the weak form of the equations. Weak solutions are non-unique, and we require admissibility conditions to select a unique solution. In the case of a differentiable  $k(x)$ , admissibility conditions guaranteeing uniqueness of solutions are well known [4, 8, 16].

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For a discontinuous flux, the equation (1) has been studied extensively by [3, 17, 29] and different admissibility conditions have been developed to select a unique solution. However, the different conditions, though yielding uniqueness, give different unique solutions. Finite volume methods for (1) have been developed in [17, 29]. The stability of solutions with respect to the Cauchy data was examined and shown in [18].

However, often Cauchy data is not known exactly [21] and is only given with via certain statistical quantities of interest like mean, variances and higher moments. The uncertainty in the Cauchy data is carried over to the uncertainty in the solution of the equation. Uncertainty in the Cauchy data and the corresponding solution is frequently modeled in a probabilistic manner. Here, we take the point of view that the Cauchy Data is a random variable described by a probability distribution. Furthermore, we adopt a simplification of the problem, wherein the flux function  $f$  is of the multiplicative form  $f(k, u) = k(x)f(u)$ . The equation then reads

$$\frac{\partial u}{\partial t} + \frac{\partial(k(\omega; x)f(u))}{\partial x} = 0 \quad x \in \mathbb{R}, t > 0 \quad (2a)$$

$$u(\omega; x, 0) = u_0(\omega; x) \quad (2b)$$

where  $k(\omega; \cdot)$  and  $u_0(\omega, \cdot)$  are some functions from a probability space  $\Omega$  into some relevant function space.

Development of efficient algorithms to quantify the uncertainty in the solutions of random conservation law is an active field of research. The challenge is to efficiently resolve the discontinuities that propagate from the physical space to the probability space in a robust manner. Also, the number of random sources driving the uncertainty may be very large, and possibly countably infinite as well. The numerical method should be able to deal with the corresponding possibly infinite dimensional spaces efficiently.

There are different methods to quantify the uncertainty in solutions of conservation laws, namely stochastic Galerkin method [1, 7, 23, 27, 31] based on generalized polynomial chaos, stochastic collocation [24, 32] and statistical sampling methods, especially Monte Carlo (MC) methods [25]. As was shown in [26], the MC methods converge to the mean at rate  $1/2$  as the number of samples  $M$  increase. This rate is due to the central limit theorem and hence, is optimal for these class of methods. This makes the MC methods computationally expensive.

In order to address this drawback, a multilevel Monte Carlo (MLMC) algorithm was proposed in [25] for a random initial condition  $u_0$  and flux of the form  $f = f(u)$  with differentiable  $f$ . Later, for the case of uncertainty in a smooth flux, a Multilevel Monte Carlo method was developed in [26]. A optimized combination of sampling sizes for different levels of spatial and temporal resolution were developed in the same paper to achieve the maximum accuracy in the statistical estimates of the first and higher order moments of the random solution. This analysis was vitally based on error analysis of the random conservation law.

Contrastingly, for the problems with discontinuous flux, Adimurthi et. al. [2] have demonstrated an example where the total variation of the solution is unbounded near the discontinuous interface. This precludes a determination of convergence rates which are uniformly valid. In this paper, we empirically determine a convergence rate and show that such an assumption is enough to derive an optimized combination of sample sizes to design an efficient MLMC method. We show that the resultant MLMC methods are indeed computationally more efficient than the Monte Carlo methods for the same problem. In specific, we show that the work required to compute an approximation with a given error using a MLMC is an order lower than that required by MC methods.

The remainder of this paper is organized as follows: We start by covering some preliminary background in Section 2. In Section 3, we prove that a unique solution exists to (2) under certain conditions and in Section 4, we analyze the Monte Carlo and the Multilevel Monte Carlo Methods. Finally, we test the method developed in this section on some numerical examples in the Section 5 and describe the conclusions of our analysis in Section 6.

## 2. PRELIMINARIES

We first introduce some preliminary concepts which are needed in the exposition. A large part of the exposition has been adapted from [20, 30]. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$  be a Banach space where  $\mathcal{B}(\mathbb{S})$  is the Borel  $\sigma$ -algebra over  $\mathbb{S}$ . A map  $G: \Omega \rightarrow \mathbb{S}$  is called a  $\mathbb{P}$ -simple function if it is of the form

$$G(\omega) = \sum_{j=1}^J g_j \mathbb{1}_{A_j}(\omega), \quad \text{where } \mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

$g_j \in \mathbb{S}$  and  $A_j \in \mathcal{F}$  for  $j = 1, 2, \dots, J$ . A map  $G: \Omega \rightarrow \mathbb{S}$  is *strongly  $\mathbb{P}$ -measurable* if there exists a sequence of simple functions  $G_n$  converging to  $G$  in the  $\mathbb{S}$ -norm  $\mathbb{P}$ -almost everywhere on  $\Omega$ . A strongly  $\mathbb{P}$ -measurable map  $G: \Omega \rightarrow \mathbb{S}$  is called an  *$\mathbb{S}$ -valued random variable*. We call two strongly  $\mathbb{P}$ -measurable functions,  $G_a, G_b: \Omega \rightarrow \mathbb{S}$ ,  $\mathbb{P}$ -versions of each other if they agree  $\mathbb{P}$ -almost everywhere on  $\Omega$ .

**Lemma 2.1.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\mathbb{S}_1$  and  $\mathbb{S}_2$  be two Banach spaces. Let  $f: \Omega \rightarrow \mathbb{S}_1$  be a strongly measurable function and  $g: \mathbb{S}_1 \rightarrow \mathbb{S}_2$  be a continuous function. Then the function  $g \circ f: \Omega \rightarrow \mathbb{S}_2$  is a strongly measurable function.*

**Definition 2.2** (Integration on Banach Spaces). The integral of a simple function  $G: \Omega \rightarrow \mathbb{S}$  is defined as

$$\int_{\Omega} G \, d\mathbb{P} := \sum_{j=1}^N g_j \mathbb{P}(A_j). \quad (4)$$

A strongly measurable map  $G: \Omega \rightarrow \mathbb{S}$  is said to be Bochner integrable if there exists a sequence of simple functions  $(G_n)_{n \geq 0}$ , converging to  $G$ ,  $\mathbb{P}$ -almost everywhere such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|G - G_n\|_{\mathbb{S}} \, d\mathbb{P} = 0. \quad (5)$$

The Bochner integral of a strongly measurable map  $G$  is then defined by

$$\int_{\Omega} G \, d\mathbb{P} := \lim_{n \rightarrow \infty} \int_{\Omega} G_n \, d\mathbb{P}. \quad (6)$$

**Theorem 2.3.** *A strongly measurable map  $G: \Omega \rightarrow \mathbb{S}$  is Bochner integrable if and only if*

$$\int_{\Omega} \|G\|_{\mathbb{S}} \, d\mathbb{P} < \infty, \quad (7)$$

in which case, we have

$$\left\| \int_{\Omega} G \, d\mathbb{P} \right\|_{\mathbb{S}} \leq \int_{\Omega} \|G\|_{\mathbb{S}} \, d\mathbb{P}. \quad (8)$$

**Definition 2.4** ( $L^p$  Spaces on Banach Spaces). For each  $1 \leq p < \infty$ , we define the space  $L^p(\Omega, \mathbb{S})$  to consist of all strongly measurable functions  $G$  for which  $\int_{\Omega} \|G\|_{\mathbb{S}}^p \, d\mathbb{P} < \infty$ . These spaces are Banach spaces under the norm

$$\|G\|_{L^p(\Omega, \mathbb{S})} := \left( \int_{\Omega} \|G\|_{\mathbb{S}}^p \, d\mathbb{P} \right)^{1/p}.$$

For  $p = \infty$ , we define the space  $L^\infty(\Omega, \mathbb{S})$  as the space of all strongly measurable functions  $G: \Omega \rightarrow \mathbb{S}$  for which there exists a  $r \geq 0$  such that  $\mathbb{P}(\|f\|_{\mathbb{S}} > r) = 0$ . This space is a Banach space under the norm

$$\|G\|_{L^\infty(\Omega, \mathbb{S})} := \inf \{r \geq 0 \mid \mathbb{P}(\|G\|_{\mathbb{S}} > r) = 0\}. \quad (9)$$

**Definition 2.5** (Banach Space of Type  $p$  [20, page 246]). Let  $Z_i, i \in \mathbb{N}$  be a sequence of independent Rademacher random variables. A Banach space  $\mathbb{S}$  is said to have the type  $1 \leq p \leq 2$  if there is a constant  $\kappa > 0$  (known as the type constant) such that for all finite sequences  $(x_i)_{i=1}^M \in \mathbb{S}$

$$\left\| \sum_{i=1}^M Z_i x_i \right\|_{\mathbb{S}} \leq \kappa \left( \sum_{i=1}^M \|x_i\|_{\mathbb{S}}^p \right)^{\frac{1}{p}}. \quad (10)$$

**Theorem 2.6** ([20, page 246]). Let  $1 \leq q < \infty$  and let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a measure space and let  $\mathbb{S}$  be a Banach space having the type  $p$ , then the space  $L^q(\Omega, \mathbb{S})$  has the type  $\min(q, p)$ . In particular, the Banach space  $L^q(\mathbb{R}^n)$  has the type  $\min\{q, 2\}$ .

**Theorem 2.7** ([20, Proposition 9.11]). Let  $\mathbb{S}$  be a Banach space having the type  $p$  with the type constant  $\kappa$ . Then, for every finite sequence  $(X_i)_{i=1}^M$  of zero mean independent random variables in  $L^p(\Omega, \mathbb{S})$ , we have

$$\mathbb{E} \left[ \left\| \sum_{i=1}^M X_i \right\|_{\mathbb{S}}^p \right] \leq (2\kappa)^p \sum_{i=1}^M \mathbb{E} [\|X_i\|_{\mathbb{S}}^p]. \quad (11)$$

Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a Banach space  $\mathbb{S}$ , let  $X: (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{S}$  be a random variable. Given  $M$  independent, identically distributed samples  $(\hat{X}_i)_{i=1}^M$  of  $X$ , the *Monte Carlo estimator*  $E_M[X]$  of  $\mathbb{E}[X]$  is defined as the sample average

$$E_M[X] := \frac{1}{M} \sum_{i=1}^M \hat{X}_i. \quad (12)$$

**Theorem 2.8** ([19, Corollary 2.5]). Let the Banach space  $\mathbb{S}$  have the type  $p$  with the type constant  $\kappa$ . Let  $X \in L^p(\Omega; \mathbb{S})$  be a zero mean random variable. Then for every finite sequence  $(X_i)_{i=1}^M$  of independent, identically distributed samples of  $X$ , we have

$$\mathbb{E} [\|E_M[X]\|_{\mathbb{S}}^p] \leq (2\kappa)^p M^{1-p} \mathbb{E} [\|X\|_{L^p(\Omega, \mathbb{S})}^p]. \quad (13)$$

**Theorem 2.9** ([19, Theorem 4.1]). Let  $X \in L^p(\Omega; L^q(\mathbb{R}))$ , then the Monte Carlo estimate  $E_M(X)$  converges in  $L^p(\Omega; L^q(\mathbb{R}))$  for  $p := \min\{2, q\}$  and we have the bound

$$\|\mathbb{E}[X] - E_M[X]\|_{L^p(\Omega; L^q(\mathbb{R}))} \leq 2\kappa M^{\frac{1-p}{p}} \|X\|_{L^p(\Omega; L^q(\mathbb{R}))}. \quad (14)$$

### 3. RANDOM CONSERVATION LAWS WITH DISCONTINUOUS FLUX

For a conservation law with a random flux and a random initial condition, we consider the Cauchy problem

$$\frac{\partial u(\omega; x, t)}{\partial t} + \frac{\partial(k(\omega; x)f(u(\omega; x, t)))}{\partial x} = 0, \quad (x, t) \in \mathbb{R} \times [0, \infty) \quad (15a)$$

$$u(\omega; x, 0) = u_0(\omega; x). \quad (15b)$$

We want to study the case where the flux coefficient,  $k(\omega; x)$ , is allowed to be discontinuous in the  $x$  variable. In this section, we review the results for the corresponding deterministic conservation laws, define the concept of a random entropy solution for (15) and prove the existence and uniqueness of the random entropy solution.

### 3.1. The Deterministic Problem: Scalar Conservation Law

For a fixed  $\omega$ , a deterministic realization of (15) is the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial(k(x)f(u))}{\partial x} = 0, \quad (16a)$$

$$u(x, 0) = u_0(x), \quad (16b)$$

where the flux coefficient  $k(x)$  is allowed to be discontinuous in  $x$  variable. Even for a differentiable flux coefficient, the solutions are known to be discontinuous. Hence, must we consider the framework of weak solutions.

**Definition 3.1** (Weak Solution). A weak solution to (16) is a bounded measurable function  $u: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $u \in L^\infty(\mathbb{R})$ , satisfying, for all  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+)$ ,

$$\int_{\mathbb{R} \times \mathbb{R}_+} \left[ u(x, t) \varphi_t(x, t) + k(x) f(u) \varphi_x(x, t) \right] dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0. \quad (17)$$

Assume there is an interval  $[a, b]$  such that

$$f(a) = f(b) = 0 \quad f \in C^2[a, b] \quad (18a)$$

There is a point  $u^* \in (a, b)$  such that

$$f'(u) > 0 \text{ for all } a < u < u^* \quad f'(u) < 0 \text{ for all } u^* < u < b \quad (18b)$$

$k(x)$  is a function such that

$$k(x) \in \text{BV}(\mathbb{R}) \cup L^1_{\text{loc}}(\mathbb{R}) \quad (18c)$$

$u_0(x)$  is a function such that

$$u_0(x) \in L^1_{\text{loc}}(\mathbb{R}) \quad u_0(x) \in L^\infty(\mathbb{R}) \quad (18d)$$

**Theorem 3.2** (Existence of Weak Solution [16]). *If the conditions (18) are satisfied, then there exists a weak solution  $u(x, t)$  to (16), and we have the weak solution  $u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and hence by interpolation for all  $1 \leq q \leq \infty$ ,  $u \in L^q(\mathbb{R})$  and  $\|u\|_{L^\infty(\mathbb{R})} \leq c = \max(|a|, |b|)$ .*

A weak solution is not unique, and in order to single out the relevant solution, we need to make use of a suitable entropy condition. The classical Kruzkov entropy condition is not valid for discontinuous flux and hence cannot be used in this situation. Instead, we use the modified Kruzkov entropy condition introduced in [29].

**Definition 3.3** (Modified Kruzkov Entropy Condition [29]). Let  $(V, F)$  be a convex entropy pair for (16), and assume that  $V = C^2[0, 1]$ . Let  $\{\xi_1, \xi_2, \dots, \xi_M\}$  be a finite set of points in  $\mathbb{R}$ . A weak solution  $u$  for (16) is said to be an entropy solution if for every smooth test function  $\phi \geq 0$  with compact support in  $t > 0$ ,  $x \in \mathbb{R} \setminus \mathcal{D}$ ,  $\mathcal{D} = \{\xi_1, \xi_2, \dots, \xi_M\}$ , and every  $c \in \mathbb{R}$ ,  $u$  satisfies the inequality

$$\int_{\mathbb{R} \times \mathbb{R}_+} V(u) \phi_t + kF(u) \phi_x dx dt - \int_{\mathbb{R} \times \mathbb{R}_+} k'(x) \left( V'(u) f(u) - F(u) \right) \phi dx dt \geq 0 \quad (19)$$

**Theorem 3.4** (Uniqueness of Entropy Solution [16]). *In addition to (18), assume that*

$$k(x) \text{ has discontinuities separated by a finite distance } L \quad (20a)$$

$$u_0 \in L^1(\mathbb{R}) \quad k(x) \in \text{BV}(\mathbb{R}) \quad (20b)$$

then, there exists a unique weak entropy solution to (16) satisfying the inequality

$$\|u\|_{L^1(\mathbb{R})} \leq e^{C_d(k)t} \|u_0\|_{L^1(\mathbb{R})} \quad (21)$$

where

$$C_d(f, k) = \|k'\|_{L^\infty(\mathbb{R} \setminus \mathcal{D})} \|f\|_{L^\infty} \quad (22)$$

**Theorem 3.5** ( $L^1$  Stability Result [18]). *Assume that the flux function  $f \in \mathbb{F}$ , the flux coefficients  $k, l \in \mathbb{K}_s$  and the initial conditions  $u_0, v_0$  satisfy (18) and (20) and additionally the following conditions,*

$$k' \text{ is bounded, whenever defined, and has one sided limits at points of discontinuity} \quad (23a)$$

$$\text{There exists a constant } \alpha > 0 (< 0) \text{ such that } k(x) \geq \alpha (\leq \alpha) \text{ for all } x \quad (23b)$$

Then, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} + t \left( \|f\|_{L^\infty(\mathbb{R})} \text{TV}(k - l) + C_s(f, k, u_0) \|k - l\|_{L^\infty(\mathbb{R})} \right) \quad (24)$$

where

$$C_s(f, k, u_0) := \min \left( C_a(f, k, u_0), C_a(f, l, v_0) \right) \quad (25a)$$

$$C_a(f, k, u_0) := \frac{5 \max(\|k\|_{L^\infty}, 1) \|f\|_{L^\infty} (TV(\Psi(u_0, k)) + TV(k))}{\min(\alpha, \alpha^2)} \quad (25b)$$

$$\Psi(u, k)(x) := k(x) \operatorname{sgn}(u - u^*) \frac{f(u^*) - f(u)}{f(u^*)} \quad (25c)$$

### 3.2. Random Conservation Law

We are interested in solutions to (15) with random initial data  $u_0(\omega; x)$  and the random flux  $k(\omega; x)f(u)$ . However, we need to set up a couple of things before we can prove the existence and uniqueness of the random entropy solution.

**Definition 3.6** (Random Data). Define a norm

$$\|(u_0, k)\|_{\mathbb{D}} := \|u_0\|_{L^1(\mathbb{R})} + \|u_0\|_{L^\infty(\mathbb{R})} + \|k\|_{L^\infty(\mathbb{R})} + \|k\|_{\text{BV}} \quad (26)$$

where  $\|k\|_{\text{BV}} = \|k\|_{L^1} + \|k\|_{\text{TV}}$ . Let  $L > 0$  be a fixed constant, then we assume that  $\mathbb{D}$  is the space of functions  $(u_0(x), k(x))$  which satisfy the conditions in (18), (20) and (23) such that the discontinuities in  $k(x)$  are located only at  $x = nL$ ,  $n \in \mathbb{Z}$ . We note that  $\mathbb{D}$  is a Banach space under the given norm. Let  $\mathcal{B}(\mathbb{D})$  be the Borel  $\sigma$ -algebra on  $\mathbb{D}$  and let  $M < \infty$  be some fixed constant. Then, we assume that the random data is a strongly measurable map  $(u_0, k): (\Omega, \mathcal{F}) \rightarrow (\mathbb{D}, \mathcal{B}(\mathbb{D}))$  such that

$$\|(u_0, k)\|_{L^\infty(\Omega; \mathbb{D})} < M \quad (27)$$

From the results for the deterministic conservation law, we would expect the random solution to be a random variable taking values in  $C(\mathbb{R}, L^\infty(\mathbb{R}) \cap L^q(\mathbb{R}))$  for  $1 \leq q < \infty$ . Denote the space of solutions as  $\mathbb{S}$  and write the solution in terms of a mapping  $S: \mathbb{D} \rightarrow \mathbb{S}$ , whence we have,

$$\mathbb{S} = C(\mathbb{R}, L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, L^\infty(\mathbb{R})) \quad u(\cdot, t) = S(u_0, k)(t) \quad (28)$$

**Definition 3.7** (Random Entropy Solution). Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P}) \ni \omega$ , a random variable  $u: (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{S}$  is said to be a random entropy solution for (15) if the following conditions are satisfied:

(1) **Weak Solution** For  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $u(\omega; \cdot, \cdot)$  satisfies

$$\int_{I \times \mathbb{R}_+} \left[ u(\omega; x, t) \varphi_t(x, t) + k(\omega; x) f(u) \varphi_x(x, t) \right] dx dt + \int_I u_0(\omega; x) \phi(x, 0) dx = 0. \quad (29)$$

for all  $\phi \in C_c^\infty(I \times \mathbb{R}^+)$ .

(2) **Entropy Condition** For  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $u(\omega; \cdot, \cdot)$  satisfies the entropy condition as in Definition 3.3.

$$\int_{\mathbb{R} \times \mathbb{R}^+} V(u(\omega)) \phi_t + k(\omega) F(u(\omega)) \phi_x dx dt - \int_{\mathbb{R} \times \mathbb{R}^+} k'(\omega; x) \left( V'(u(\omega)) f(u(\omega)) - F(u(\omega)) \right) \phi dx dt \geq 0 \quad (30)$$

**Theorem 3.8** (Existence and  $\omega$ -wise Uniqueness of a Random Entropy Solution). *For each  $f$  satisfying (18), and  $(u_0, k)$  the random data as defined in Definition 3.6, then there exists a unique random entropy solution  $u: \Omega \rightarrow \mathbb{S}$  to the random conservation law (15).*

*Proof.* By (27) for almost all  $\omega \in \Omega$ , the random data  $(u_0, k)$  is such that there exists a corresponding unique entropy solution  $u(\omega; \cdot, \cdot) \in \mathbb{S}$ . By the assumptions of the theorem, the map  $(u_0, k): \Omega \rightarrow \mathbb{D}$  is a strongly measurable map. Further, by (24), the map  $u(x, t): \mathbb{D} \rightarrow \mathbb{S}$  is continuous. Then Lemma 2.1 shows that the map  $u(\omega; \cdot, \cdot): \Omega \rightarrow \mathbb{S}$  is strongly measurable. And hence, there exists a random entropy solution to (15).

Next, let the random variables  $(u_0, k) \in \mathbb{D}$  and  $(\tilde{u}_0, \tilde{k}) \in \mathbb{D}$  be  $\mathbb{P}$ -versions of each other. For all times  $t$ , let  $u(\cdot; \cdot, t)$  be the random entropy solution corresponding to  $(u_0, k)$  at time  $t$  and  $\tilde{u}(\cdot; \cdot, t)$  be the random entropy solution corresponding to  $(\tilde{u}_0, \tilde{k})$  at time  $t$ . Then by (24) and (26), for  $C_s = \max(C_a(\omega), C_a(\omega))$  and  $\omega$   $\mathbb{P}$ -almost everywhere we have

$$\begin{aligned} & \| u(\omega; \cdot, t) - \tilde{u}(\omega; \cdot, t) \|_{L^1(\mathbb{R})} \\ & \leq \| u_0 - \tilde{u}_0 \|_{L^1(\mathbb{R})} \\ & \quad + t \left( \| f \|_{L^\infty(\mathbb{R})} \mathbf{TV}(k - \tilde{k}) + C_s \| k - \tilde{k} \|_{L^\infty(\mathbb{R})} \right) \\ & = 0 \end{aligned}$$

This implies that for  $\omega$   $\mathbb{P}$ -almost everywhere we have  $u(\omega; x, t) = \tilde{u}(\omega; x, t)$  almost everywhere in  $\mathbb{R}$ . And hence, for all  $1 \leq q < \infty$ , the random entropy solution  $u$  is unique in  $L^\infty(\Omega; L^\infty(\mathbb{R}) \cap L^q(\mathbb{R}); d\mathbb{P})$  and therefore in  $\mathbb{S}$ .  $\square$

**Theorem 3.9.** *Let  $u(\cdot; \cdot, \cdot)$  be a random entropy solution to (15) as per Definition 3.7. For any  $1 \leq k < \infty$  and  $1 \leq q < \infty$ , we have  $u(\cdot; \cdot, \cdot) \in L^k(\Omega; C([0, T], L^q(\mathbb{R})))$  and*

$$\| u(\cdot; \cdot, \cdot) \|_{L^k(\Omega, C([0, T], L^q(\mathbb{R})))} \leq e^{CT} \| u_0 \|_{L^k(\Omega, L^q(\mathbb{R}))} \quad (31)$$

where  $C = \max_{\omega \in \Omega} C_d(f, k(\omega))$  is a constant dependent only on  $M$  and  $\|f\|_{L^\infty}$ . Given a bounded interval  $D$ , we also have the estimate

$$\|u(\cdot; \cdot, \cdot)\|_{L^k(\Omega, C([0, T], L^q(D)))} \leq c |D|^{1/q} \quad (32)$$

where  $c$  is as defined in Theorem 3.2.

*Proof.* By Theorem 3.2, for any  $1 \leq k < \infty$ , we have for  $\mathbb{P}$ -almost everywhere  $\omega \in \Omega$ ,

$$\begin{aligned} \|u(\cdot; \cdot, \cdot)\|_{L^k(\Omega, C([0, T], L^q(\mathbb{R})))} &\leq \left[ \int_{\Omega} \sup_{t \in [0, T]} \|u(\omega; \cdot, t)\|_{L^q(\mathbb{R})}^k d\mathbb{P} \right]^{1/k} \\ &\leq \left[ \int_{\Omega} e^{C_d(f, k(\omega)T)} \|u(\omega; \cdot, 0)\|_{L^q(\mathbb{R})}^k d\mathbb{P} \right]^{1/k} \\ &\leq e^{CT} \left[ \int_{\Omega} \|u(\omega; \cdot, 0)\|_{L^q(\mathbb{R})}^k d\mathbb{P} \right]^{1/k} \\ &\leq e^{CT} \|u_0\|_{L^k(\Omega, L^q(\mathbb{R}))} \end{aligned}$$

And hence, we have, for all  $1 \leq k < \infty$ ,  $u \in L^k(\Omega; C([0, T], L^q(\mathbb{R})))$ . Also, by the fact that  $\|u\|_{L^\infty(\mathbb{R})} \leq c$ , we have

$$\begin{aligned} \|u(\cdot; \cdot, \cdot)\|_{L^k(\Omega, C([0, T], L^q(D)))} &\leq \left[ \int_{\Omega} \sup_{t \in [0, T]} \|u(\omega; \cdot, t)\|_{L^q(D)}^k d\mathbb{P} \right]^{1/k} \\ &\leq c |D|^{1/q} \end{aligned}$$

□

#### 4. MULTILEVEL MONTE CARLO FINITE VOLUME METHOD

Monte Carlo methods are a class of methods where repeated random sampling is used to obtain the mean value and the subsequent moments of the random variable. In our case, the samples are the entropy solution to the deterministic Cauchy problem for the corresponding samples of Cauchy data. The exact solutions to those deterministic problems are however unavailable and instead, we must use a numerical approximation. Here, we use a finite volume method to compute the numerical approximation to the deterministic problem.

The error introduced by the Monte Carlo methods depends on the number of samples used, while the error introduced by the finite volume methods depends on the resolution of the grid. Different combinations of grids can be used to compute the finite volume approximations for different samples. In this section, we will describe and analyze two such combinations, denoted by the Monte Carlo Finite Volume Method (MC-FVM) and the Multilevel Monte Carlo Finite Volume Method (MLMC-FVM).

We solve our random conservation law on a compact interval  $D = [x_L, x_R]$  and in the time interval  $[0, T]$ , wherein, we can rewrite (15) as

$$\frac{\partial u(\omega; x, t)}{\partial t} + \frac{\partial (k(\omega; x)f(u(\omega; x, t)))}{\partial x} = 0, \quad (x, t) \in D \times [0, T] \quad (33)$$

$$u(\omega; x, 0) = u_0(\omega; x). \quad (34)$$

**Definition 4.1** (Cauchy Problem Sample, Solution Sample and FVM Solution Sample). In context of Monte Carlo methods, given a sample  $(u_0, k)(\omega_0)$  of the random variable  $(u_0, k)$ , the corresponding deterministic Cauchy problem will be referred to as the Cauchy problem sample for  $\omega_0$ . Similarly, the unique entropy solution and the finite volume solution for the Cauchy problem sample will be referred to as the solution sample  $u(\omega_0; \cdot, \cdot)$  for  $\omega_0$  and the FVM solution sample  $U(\omega; \cdot, \cdot)$  for  $\omega_0$  respectively.

**Definition 4.2** ( $N$ -discretization). Let the domain  $D = [x_L, x_R]$ . Divide the domain  $D$  into  $N$  uniform cells  $D_j$ ,  $j = 0, 1, \dots, N-1$  with  $D_j = [x_{j-1/2}, x_{j+1/2}]$  where  $x_{1/2} = x_L$  and  $x_{N+1/2} = x_R$ . The  $N$ -discretization is defined as  $\cup_{j=0}^{N-1} D_j$ . Additionally, the cell center  $x_j$  and the cell length  $h_j$  are given as

$$x_j = \frac{x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}}}{2}, \quad h = \frac{|D|}{N} \quad (35)$$

#### 4.1. Finite Volume Method For Conservation Law with Discontinuous Flux

Given a Cauchy problem sample and a  $N$ -discretization  $\Sigma$  of  $D$  containing  $N$  cells, we now describe a method to compute the FVM solution  $U(x, t)$  to the solution of the given Cauchy problem. The initial conditions of the Cauchy problem dictate that

$$U_j^0 = \frac{1}{h_j} \int_{D_j} u_0(x) dx \quad (36)$$

We denote the cell averages of the solution  $U(x, t)$  for the  $j$ -th cell at  $n$ -th time step as  $U_{n,j}$ ,

$$U_j^n = \frac{1}{|D_j|} \int_{D_j} U(x, t_n) dx \quad j = 0, 1, \dots, N-1 \quad (37)$$

The function  $k(x)$  is computed at the face interfaces by considering the cell averages on the staggered grid as shown below

$$k_{j+\frac{1}{2}} = \int_{x_j}^{x_{j+1}} k(x) dx \quad (38)$$

The finite volume scheme is then defined as

$$U_j^{n+1} = U_j^n - \frac{\Delta t_n}{h_j} \left[ H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}} \right] \quad j = 0, 1, \dots, N-1 \quad (39a)$$

where  $H$  is given by

$$H_{j+\frac{1}{2}} = k_{j+\frac{1}{2}} F(U_{j+1}^n, U_j^n) \quad j = 0, 1, \dots, N-1 \quad (39b)$$

where  $F$  is an monotone numerical flux. There are a few different numerical fluxes that are appropriate in this problem. We use the following numerical fluxes due to [29].

(1) Godunov Flux

$$F(U_l, U_r) = \begin{cases} f(u^*) & \text{if } U_l \leq u^* \leq U_r \\ \min_{U_l \leq q \leq U_r} f(u) & U_l \leq U_r \leq u^* \text{ or } u^* \leq U_l \leq U_r \\ \min_{U_r \leq q \leq U_l} f(u) & U_r \leq U_l \leq u^* \text{ or } u^* \leq U_r \leq U_l \end{cases} \quad (40)$$

(2) Engquist-Osher Flux

$$F(U_l, U_r) = \frac{f(U_l) + f(U_r)}{2} - \frac{1}{2} \int_{U_l}^{U_r} |f'(\theta)| d\theta \quad (41)$$

**Theorem 4.3** ([29, Theorem 3.2]). *Let  $u_0, k, f$  satisfy the conditions in (18), (20) and (23). Additionally, assume*

$$f''(u) > 0 \quad \text{for all } u \in [a, b] \quad (42)$$

*Then, the finite volume scheme (39) converges to the entropy solution of the Cauchy problem (16) on  $D$  provided that the time step  $\Delta t$  follows the CFL condition*

$$\Delta t \leq \frac{\Delta x}{\|k\|_\infty \|f'\|_\infty} \quad (43)$$

**Remark 4.4.** The convergence rates for the numerical methods are yet unknown. However, we do need the convergence rates to determine the optimal sample numbers for the analysis of MLMC method. Hence, we assume that the convergence rate for the  $\ell^q$ -error between the numerical solution and the exact solution is  $s_q$ . Then, the  $\ell^q$ -error can be written as

$$\|U(x, t) - u(x, t)\|_{\ell^q} < C_b(q) \Delta x^{s_q} \quad (44)$$

## 4.2. Monte Carlo Finite Volume Methods

We now describe and analyze the MC-FVM to compute the numerical approximation to the random entropy solution  $u$  for conservation law (33). The underlying idea of MC-FVM is to use identical uniform discretization to solve each of the Cauchy problem sample.

**Definition 4.5** (MC-FVM Approximation). Given  $M \in \mathbb{N}$ , generate  $M$  independent, identically distributed samples  $((\hat{u}_{0,i}, \hat{k}_i))_{i=1}^M$ . Let  $\Sigma$  be a  $N$ -discretization of the spatial domain  $D$ . Let  $\hat{U}_i$  denoted the FVM solution sample corresponding to  $(\hat{u}_{0,i}, \hat{k}_i)$  at time  $T$ . Then, the  $M$ -sample MC-FVM  $E_{\text{MC}}(u)$  to  $\mathbb{E}[u]$  is defined as

$$E_{\text{MC}}(u) := E_M[U] := \frac{1}{M} \sum_{i=1}^M \hat{U}_i \quad (45)$$

The variance is defined as

$$\text{Var}_{\text{MC}}(u) := E_M[U^2] - E_M[U]^2 \quad (46)$$

**Theorem 4.6** (MC-FVM Error Bound). *Assume that  $f \in \mathbb{F}_n$  and  $(u_0, k)$  is the random data as defined in Definition 3.6, then for the random conservation law (33), for  $p = \min(2, q)$  the MC-FVM approximation converges to  $\mathbb{E}[u]$  in  $L^p(\Omega; L^q(D))$  as  $M \rightarrow \infty$  and  $\Delta x \rightarrow 0$ . Furthermore, we have the bounds*

$$\mathcal{E}_{MC}(u) := \|\mathbb{E}[u] - E_{MC}(u)\|_{L^p(\Omega; L^q(D))} \leq |D|^{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} 2\kappa c M^{-\frac{1}{2}} + C_b(q) \Delta x^{s_q} \quad (47)$$

where  $s_q$  is the rate of convergence determined empirically from the numerical experiments.

*Proof.* For the first inequality, using the triangle inequality, we can write

$$\|\mathbb{E}[u] - E_{MC}(u)\|_{L^p(\Omega; L^q(D))} \leq \|\mathbb{E}[u] - E_M(u)\|_{L^p(\Omega; L^q(D))} + \|E_M[u] - E_M(U)\|_{L^p(\Omega; L^q(D))}$$

Using the fact that  $p \leq 2$  and applying Hölder inequality, Theorem 2.9 and (31) to the first term, we have

$$\begin{aligned} \|\mathbb{E}[u] - E_M(u)\|_{L^p(\Omega; L^q(D))} &\leq |D|^{\frac{1}{p} - \frac{1}{2}} \|\mathbb{E}[u] - E_M(u)\|_{L^2(\Omega; L^q(D))} \\ &\leq |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa M^{-\frac{1}{2}} \|u\|_{L^2(\Omega; L^q(D))} \\ &\leq |D|^{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} 2\kappa c M^{-\frac{1}{2}} \end{aligned}$$

For the second term, by (44), we have

$$\begin{aligned} \|E_M[u] - E_{MC}(u)\|_{L^p(\Omega; L^q(D))} &\leq \sup_i \left\| \left( \hat{u}_i - \hat{U}_i \right) \right\|_{L^q(D)} \\ &\leq C_b(q) \Delta x^{s_q} \end{aligned}$$

□

### 4.3. Multilevel Monte Carlo Finite Volume Methods

The underlying idea of the MLMC-FVM is to use a hierarchy of nested set of discretization and solve several Cauchy problem samples on each of them.

**Definition 4.7** (MLMC-FVM Approximation). Let  $(\Sigma_l)_{l=0}^L$  be a sequence of nested  $2^l N$  discretization of the spatial domain  $D$ . For each level  $l$ , define by  $U_l$  the random Finite Volume approximation on  $\Sigma_l$  with  $U_{-1} = 0$ . Then, given  $(M_l)_{l=0}^L \in \mathbb{N}$ , the MLMC-FVM approximation  $E_{MLMC}(u)$  to  $\mathbb{E}[u]$  is then defined as

$$E_{MLMC}(u) = \sum_{l=0}^L E_{M_l}(U_l - U_{l-1}) \quad (48a)$$

$$\text{Var}_{MLMC}(u) = \sum_{l=0}^L \Delta V_l \quad (48b)$$

$$\Delta V_l = \text{Var}_{MC}(U_l - U_{l-1}) \quad (48c)$$

$$= E_{M_l} [(u_l - u_{l-1} - E_{M_l}[u_l - u_{l-1}])^2] \quad (48d)$$

**Theorem 4.8** (MLMC-FVM Error Bound). *Assume that  $f \in \mathbb{F}_n$  and  $(u_0, k)$  is the random data as defined in Definition 3.6, then for the random conservation law (33), for  $p = \min\{2, q\}$ , the MLMC-FVM approximation converges to  $\mathbb{E}[u]$  as  $M_l \rightarrow \infty$  for  $l = 0, 1, \dots, L$  and  $\Delta x \rightarrow 0$ . Further, we have the bound*

$$\mathcal{E}_{MLMC}(u) := \|\mathbb{E}[u] - E_{MLMC}(u)\|_{L^p(\Omega; L^q(D))} \leq C_b(q) (\Delta x)^{s_q} \left( 2^{-L s_q} + C_m(p, q) \sum_{l=0}^L M_l^{-\frac{1}{2}} 2^{-l s_q} \right) \quad (49)$$

where  $C_m(p, q) = |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa(1 + 2^{s_q})$ .

*Proof.* We first derive the result for error in  $\mathbb{E}$ . Using the triangle inequality, we can write

$$\|\mathbb{E}[u] - E_{\text{MLMC}}(u)\|_{L^p(\Omega, L^q(D))} \leq \|\mathbb{E}[u] - \mathbb{E}[U_L]\|_{L^p(\Omega, L^q(D))} + \|\mathbb{E}[U_L] - E_{\text{MLMC}}(u)\|_{L^p(\Omega, L^q(D))}$$

For the first term, we have

$$\begin{aligned} \|\mathbb{E}[u] - \mathbb{E}[U_L]\|_{L^p(\Omega, L^q(D))}^p &\leq \|u - U_L\|_{L^1(\Omega, L^q(D))}^p \\ &\leq \|u - U_L\|_{L^\infty(\Omega, L^q(D))} \\ &\leq C_b(q) 2^{-L s_q} (\Delta x)^{s_q} \end{aligned}$$

For the second term, using the fact that  $p \leq 2$ , and by Hölder inequality, Theorem 2.9 and Theorem 4.3

$$\begin{aligned} \|\mathbb{E}[U_L] - E_{\text{MLMC}}(u)\|_{L^p(\Omega, L^q(D))} &= \left\| \sum_{l=0}^L \mathbb{E}[U_l - U_{l-1}] - \sum_{l=0}^L E_{M_l}(U_l - U_{l-1}) \right\|_{L^p(\Omega, L^q(D))} \\ &\leq \sum_{l=0}^L \|\mathbb{E}[U_l - U_{l-1}] - E_{M_l}(U_l - U_{l-1})\|_{L^p(\Omega, L^q(D))} \\ &\leq \sum_{l=0}^L |D|^{\frac{1}{p} - \frac{1}{2}} \|\mathbb{E}[U_l - U_{l-1}] - E_{M_l}(U_l - U_{l-1})\|_{L^2(\Omega, L^q(D))} \\ &\leq \sum_{l=0}^L |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa M_l^{-\frac{1}{2}} \|U_l - U_{l-1}\|_{L^q(D)} \\ &\leq \sum_{l=0}^L |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa M_l^{-\frac{1}{2}} \left( \|U_l - u\|_{L^q(D)} + \|U_{l-1} - u\|_{L^q(D)} \right) \\ &\leq \sum_{l=0}^L |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa M_l^{-\frac{1}{2}} \left( C_b(q) (2^{-l} \Delta x)^{s_q} + C_b(q) (2^{-l+1} \Delta x)^{s_q} \right) \\ &\leq |D|^{\frac{1}{p} - \frac{1}{2}} 2\kappa C_b(q) (\Delta x)^{s_q} (1 + 2^{s_q}) \sum_{l=0}^L M_l^{-\frac{1}{2}} 2^{-l s_q} \\ &\leq C_b(q) C_m(p, q) (\Delta x)^{s_q} \sum_{l=0}^L M_l^{-\frac{1}{2}} 2^{-l s_q} \end{aligned}$$

□

#### 4.4. Work Estimates and Sample Number Optimization

The work required to compute a MCFVM or a MLMC-FVM approximation and the corresponding error depends on multiple factors, namely the grid resolution  $\Delta x$ , number of levels  $L$  and the sample numbers at each level  $M_l$ ,  $l = 0, 1, \dots, L$ . In order to get optimal error rates, we need to select these parameters to either minimize the work required to be done for a specified error or to minimize the error given the work needed.

In this section, we calculate the work required for MC and MLMC methods as a function of sample numbers and then, we use those functions along with the error expressions previously derived to select the optimal parameters. We assume that the time required to compute the finite volume method solution in a single cell for a single time step is a constant, and we denote it as a unit work. For the purpose of optimization, we will

assume that all parameters except for the sample numbers are fixed. In particular, we mandate that the number of levels is not a parameter for optimization, and instead, is fixed.

#### 4.4.1. Optimal Sample Numbers for MC-FVM

The error of an MC-FVM approximation for a grid resolution of  $\Delta x$  and  $M$  samples is given by (47). We use the fact that for a fixed domain  $N = |D|/\Delta x$  and incorporate terms independent of  $M$  and  $N$  into constants  $A$  and  $B$ , when we rewrite the error term as below.

$$\epsilon \leq AM^{-\frac{1}{2}} + BN^{-s_q} \quad A = |D|^{\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} 2\kappa b, \quad B = |D|^{s_q} C_b(q) \quad (50)$$

The work required to compute the solution is given by

$$W = MN^2 \quad (51)$$

Assume that we have the fixed the error  $\epsilon = \epsilon_0$ . Then, using (50), we can write

$$N = \left( \frac{\epsilon_0 - AM^{-\frac{1}{2}}}{B} \right)^{-\frac{1}{s_q}} \quad (52)$$

The work as a function of sample numbers can then be written as

$$W = M \left( \frac{\epsilon_0 - AM^{-\frac{1}{2}}}{B} \right)^{-\frac{2}{s_q}} \quad (53)$$

Quick analysis shows that for  $s_q \leq 1$  and  $M \geq 1$ , the attains a minimum at a single point at which the derivative w.r.t  $M$  is 0. Performing the analysis gives us

$$M = \left( \frac{A}{Bs_q} \right)^2 N^{2s_q} \quad (54)$$

#### 4.4.2. Optimal Sample Numbers for MLMC-FVM

Consider a  $L$ -level MLMC-FVM with  $N$ -cells at level 0. By (49), the error for the method can be written in terms of the number of samples  $M_0, M_1, \dots, M_L$  as

$$\epsilon = A \left( 1 + B \sum_{l=0}^L 2^{-ls_q} M_l^{-\frac{1}{2}} \right) \quad A = C_b(q) N^{-s_q} |D|^{s_q} 2^{-Ls_q}, \quad B = C_m(p, q) 2^{Ls_q} \quad (55)$$

The work done to calculate the MLMC-FVM approximation can be written as

$$W = \sum_{l=0}^L M_l N^2 2^{2l} \quad (56)$$

We use the method of Lagrange multipliers in order to optimize the quantities of interest over multiple variables. We will need the following expressions during optimization,

$$\frac{\partial W}{\partial M_l} = N^2 2^{2l} \quad (57a)$$

$$\frac{\partial \epsilon}{\partial M_l} = -\frac{1}{2} AB 2^{-ls_q} M_l^{-\frac{3}{2}} \quad (57b)$$

We fixed the error at  $\epsilon_0$ . The Lagrangian  $\mathcal{L}$  is then given by

$$\mathcal{L} = W + \lambda(\epsilon - \epsilon_0) \quad (58)$$

$W$  will be minimum when  $M_l, l = 0, 1, \dots, L$  are such that the following condition is satisfied

$$\left. \frac{\partial \mathcal{L}}{\partial M_l} \right|_{M_l} = 0 \quad l = 0, 1, \dots, L \quad (59)$$

which gives us

$$M_l = \left( \frac{AB}{2} \right)^{\frac{2}{3}} \lambda^{\frac{2}{3}} N^{-\frac{4}{3}} 2^{-\frac{4l}{3}} 2^{-\frac{2ls_q}{3}} \quad (60)$$

Putting the value in the value for error, we can get the value of  $\lambda$  as

$$\lambda^{\frac{2}{3}} = \left( \frac{1}{B} \right)^{-2} \left( \frac{\epsilon_0}{A} - 1 \right)^{-2} \left( \frac{AB}{2} \right)^{-\frac{2}{3}} N^{\frac{4}{3}} C^2 \quad (61)$$

and the expression for  $M_l$  independent of  $\lambda$  is given by

$$M_l = 2^{-\frac{4l}{3}} 2^{-\frac{2ls_q}{3}} B^2 \left( \frac{\epsilon_0}{A} - 1 \right)^{-2} C^2 \quad (62)$$

## 5. NUMERICAL EXPERIMENTS

We now present some numerical experiments which validate the method developed over the previous sections<sup>1</sup>. The numerical examples are motivated by the traffic flow problems. In particular, we consider the inhomogeneous LWR model which can be written as show below, where the multiplying factor of  $k$  can represent space-dependent factors in the equation like a change in the number of lanes and the speed limit.

$$\frac{\partial u}{\partial t} + \frac{\partial k(x)f(u)}{\partial x} = 0 \quad f(u) = 4u(1 - u) \quad (63)$$

The purpose of the numerical experiments is to verify that the multilevel Monte Carlo methods are computationally more efficient than simple Monte Carlo method. To that end, we have designed our setup in such a way, that the errors produced by both the methods, the Monte Carlo and the Multi-Level Monte Carlo methods are almost equal for a given grid. We then measure the computational effort required to compute the two solution and we compare them. The computational effort is measured by calculating the CPU time required to run the two different programs.

As noted in Remark 4.4, the convergence rates for the deterministic case are not theoretically known, and instead, we have to use an empirical convergence rate for determination of optimal sample numbers. Experience tells us that we should expect a convergence rate of at least  $1/2$ . We verify that for the above problem, that the convergence rates are at least  $1/2$ .

We now describe the relation between the grid used for the multilevel Monte Carlo method and the grid used for the Monte Carlo method. The base grid is with  $N$  cells, and then, for multilevel Monte Carlo method, we use a total for 4 levels, resulting in  $2^3 N$  cells at the finest grid. For comparison with monte carlo method, we have observed that the grid containing  $2N$  cells produces an error that is of the same order as the one produced using MLMC method of 4 levels. Hence, we can then compare the two methods purely on the basis of computational work taken to calculate the two solutions.

<sup>1</sup>The code can be found at [https://www.jayeshbadwaik.in/uq-conlaw/mlmc.sconlaw\\_1d](https://www.jayeshbadwaik.in/uq-conlaw/mlmc.sconlaw_1d)

<b>Cells</b>	128	256	512	1024	2048	4096
<b>Error</b>	1.071e-2	6.090e-3	2.807e-3	1.376e-3	7.302e-4	3.513e-4
<b>Rate</b>	1.123	0.814	1.112	1.029	0.914	1.056

TABLE 1. Pulse Coefficient : Error Rates for Deterministic Case

Next, we consider the Monte Carlo and the multilevel Monte Carlo methods, wherein, we now calculate the sample numbers as described before using the experimental convergence rates of 0.5 and calculate the optimal sample numbers. We then verify that the deterministic convergence rates of the problem are above 1/2 as required in our analysis and the results are shown in Table 5.2. We use a MCFVM solution calculated on 16384 cells as a reference solution to calculate the error rates. Next, we compute the MC-FVM and the MLMC-FVM solutions. We make sure that the errors from both the computations are of the same order and then compare the work done for the both the cases.

Let  $U_e$  be the reference solution as mentioned above and let  $U_l$  be the computed MC or MLMC solution for a grid with  $2^l$  points, then the error is calculated as

$$\text{err}_l = \sum_{i=0}^{2^l-1} \frac{|U_l(i) - U_e(i)|}{|U_e(i)|} \quad (64)$$

where  $U_e(i)$  is calculated using linear interpolation. The error rates are computed as

$$r_l = \frac{\log(\text{err}_{l-1}/\text{err}_l)}{\log(2)} \quad (65)$$

### 5.1. Traffic Flow Problem with Pulse Coefficient

For the first case, we consider the traffic flow problem, where there are two discontinuities in the function  $k$ . The discontinuities form a constant width pulse whose position is uncertain. We use periodic boundary conditions for the problem.

$$\frac{\partial u}{\partial t} + \frac{\partial k(\omega; x)f(u)}{\partial x} = 0 \quad (x, t) \in [-2, 2] \times \left[0, \frac{1}{10}\right] \quad (66a)$$

$$u(x, 0) = \frac{1}{4} \sin\left(\frac{\pi x}{2}\right) + \frac{1}{2} \quad f(u) = 4u(1 - u) \quad (66b)$$

where

$$k(\omega, x) = \begin{cases} 1 & \text{if } x < -0.5 + \omega \\ 2 & \text{if } -0.5 + \omega \leq x < 0.5 + \omega \\ 1 & \text{if } 0.5 + \omega \leq x \end{cases} \quad \omega \in \mathcal{U}(-1, 1) \quad (66c)$$

Cells	64	128	256	512	1024
MC	6.2149e-2	3.60286e-1	2.4019e-1	1.7014e-1	1.2009e-1
Rate	-	0.784	0.584	0.497	0.502
MLMC	7.1394e-1	3.7205e-1	2.4133e-1	1.5083e-1	1.106e-1
Rate	-	0.940	0.624	0.678	0.44

TABLE 2. Pulse Coefficient : MC and MLMC Errors

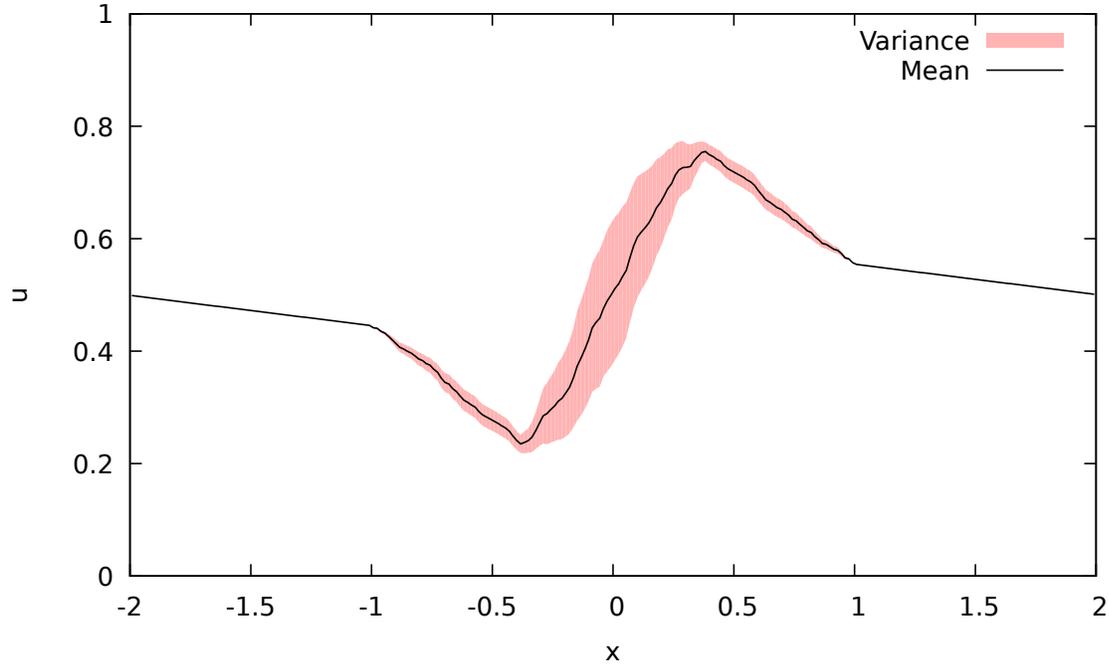


FIGURE 1. Traffic Flow Problem with Pulse Function Coefficient

Cells	64	128	256	512	1024
MC	4	18.52	135	1092	8490
Rate	-	2.211	2.865	3.015	2.958
MLMC	5.5	15.02	40	132	489
Rate	-	1.449	1.413	1.722	1.889

TABLE 3. Pulse Coefficient : MC and MLMC Work (in compute seconds)

## 5.2. Traffic Flow Problem with a Brownian Bridge with Jumps Coefficient

For the second example, we consider the case where the coefficient  $k$  is given by a Brownian Bridge with Jumps. We construct such a bridge from a Brownian bridge pinned at both ends by adding discontinuities at random points in the interval. Let  $b(x)$  be a Brownian bridge pinned at 2 at  $x = -2, 2$ . Next, we start introducing jumps of random magnitude  $h_j$  in the Brownian bridge  $b(x)$  at random points  $x_j$ ,  $j = 1, 2, \dots, n$ . We ensure that the number of jumps are finite and that  $b(x) > 1$  to fulfill the condition  $k(x) > 1$ . The resultant function  $B(x)$  can be written as

$$B(x) = b(x) + \sum_{k=0}^j h_k \quad \text{for } x_j < x < x_{j+1}, x_0 = -2, x_{n+1} = 2 \quad (67)$$

<b>Cells</b>	128	256	512	1024	2048	4096
<b>Error</b>	2.665e-2	1.423e-2	8.438e-3	4.855e-3	2.678e-3	1.325e-3
<b>Rate</b>	0.662	0.900	0.758	0.797	0.858	1.015

TABLE 4. Brownian Bridge with Jumps : Error Rates for Deterministic Case

$$\frac{\partial u}{\partial t} + \frac{\partial k(\omega; x)f(u)}{\partial x} = 0 \quad (x, t) \in [-2, 2] \times \left[0, \frac{1}{10}\right] \quad (68a)$$

$$u(x, 0) = \frac{1}{4} \sin\left(\frac{\pi x}{2}\right) + \frac{1}{2} \quad f(u) = 4u(1 - u) \quad (68b)$$

Cells	64	128	256	512	1024
MC-FVM	2.132e-1	1.3872e-1	1.003e-1	7.6139e-2	5.7780e-2
Rate	-	0.620	0.467	0.397	0.398
MLMC-FVM	2.3452e-1	1.3925e-1	9.8519e-2	6.6764e-2	5.299e-1
Rate	-	0.752	0.499	0.561	0.335

TABLE 5. Brownian Bridge with Jumps : MC and MLMC Errors

Cells	64	128	256	512	1024
MC-FVM	6.7	31.58	254	2012	16380
Rate	-	2.236	3.001	2.986	3.025
MLMC-FVM	10.82	30.85	64	252	922
Rate	-	1.511	1.052	1.977	1.871

TABLE 6. Brownian Bridge with Jumps : MC and MLMC Work (in compute second)

## 6. CONCLUSION

In this paper, we have considered a scalar conservation law with discontinuous flux in space in one dimension. We have defined a random entropy solution for the conservation law and have proved its existence and uniqueness. Further, we have adapted the Multilevel Monte Carlo Finite Volume Method for the problem and have compared its performance with the Monte Carlo Finite Volume Method, wherein, we have shown that the Multilevel Monte Carlo Method behaves as expected in the theoretical analysis. In particular, we show that the Multilevel Monte Carlo method is a more efficient alternative to Monte Carlo methods.

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## REFERENCES

- [1] Remi Abgrall. A simple, flexible and generic deterministic approach to uncertainty quantifications in non linear problems: application to fluid flow problems. 2008.
- [2] Adimurthi, Rajib Dutta, Shyam Sundar Ghoshal, and G. D. Veerappa Gowda. Existence and nonexistence of TV bounds for scalar conservation laws with discontinuous flux. *Communications on Pure and Applied Mathematics*, 64(1):84–115, 2011.

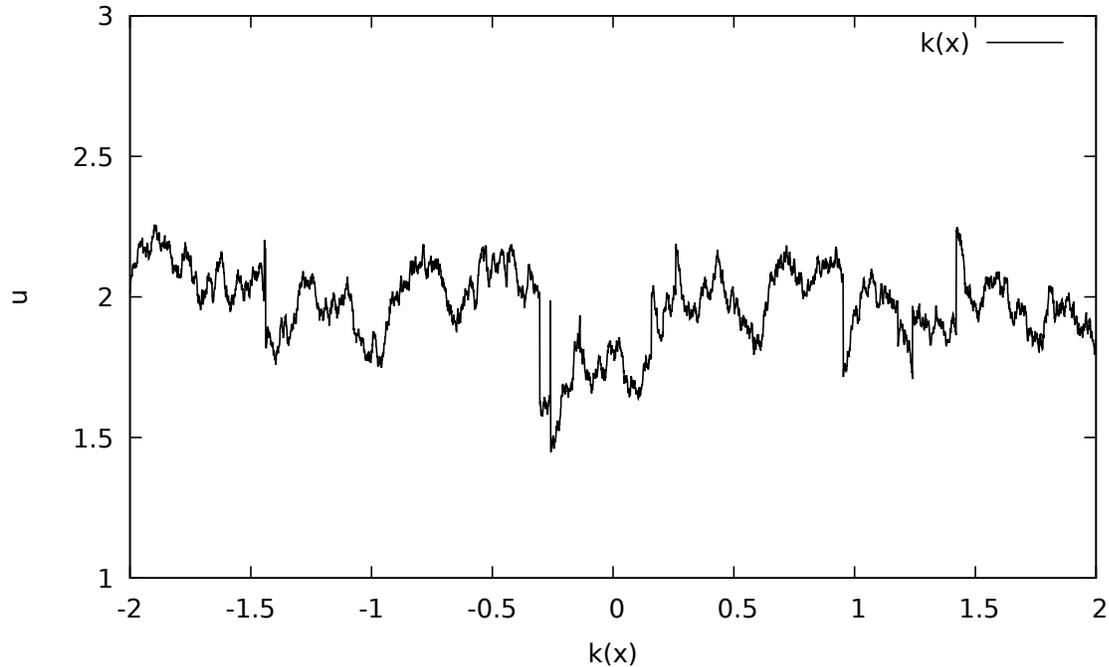


FIGURE 2. One Instance of Brownian Bridge with Jumps

- [3] Boris Andreianov, Kenneth Hvistendahl Karlsen, and Nils Henrik Risebro. A theory of  $l_1$ -dissipative solvers for scalar conservation laws with discontinuous flux. *Archive for rational mechanics and analysis*, 201(1):27–86, 2011.
- [4] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.
- [5] R. Bürger, K. H. Karlsen, C. Klingenberg, and N. H. Risebro. A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units. *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, 4(3):457–481, 2003.
- [6] Raimund Bürger and Wolfgang L Wendland. Sedimentation and suspension flows: Historical perspective and some recent developments. *Journal of Engineering Mathematics*, 41(2):101–116, 2001.
- [7] Qian-Yong Chen, David Gottlieb, and Jan S Hesthaven. Uncertainty analysis for the steady-state flows in a dual throat nozzle. *Journal of Computational Physics*, 204(1):378–398, 2005.
- [8] Constantine M Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *grundlehren der mathematischen wissenschaften [fundamental principles of mathematical sciences]*, 2010.
- [9] Stefan Diehl. A conservation law with point source and discontinuous flux function modelling continuous sedimentation. *SIAM Journal on Applied Mathematics*, 56(2):388–419, 1996.
- [10] Tore Gimse and Nils Henrik Risebro. Solution of the cauchy problem for a conservation law with a discontinuous flux function. *SIAM Journal on Mathematical Analysis*, 23(3):635–648, 1992.
- [11] William G Gray. General conservation equations for multi-phase systems: 4. constitutive theory including phase change. *Advances in Water Resources*, 6(3):130–140, 1983.
- [12] BD Greenshields, Ws Channing, Hh Miller, et al. A study of traffic capacity. In *Highway research board proceedings*, volume 1935. National Research Council (USA), Highway Research Board, 1935.
- [13] M Hassanizadeh and WG Gray. General conservation equations for multi-phase system: 2. mass, momenta, energy and entropy equations. *Advances in Water Resources*, 2:191–203, 1979.
- [14] Majid Hassanizadeh and William G Gray. General conservation equations for multi-phase systems: 1. averaging procedure. *Advances in water resources*, 2:131–144, 1979.
- [15] Majid Hassanizadeh and William G Gray. General conservation equations for multi-phase systems: 3. constitutive theory for porous media flow. *Advances in Water Resources*, 3(1):25–40, 1980.
- [16] Helge Holden and Nils Henrik Risebro. *Front tracking for hyperbolic conservation laws*, volume 152. Springer, 2015.

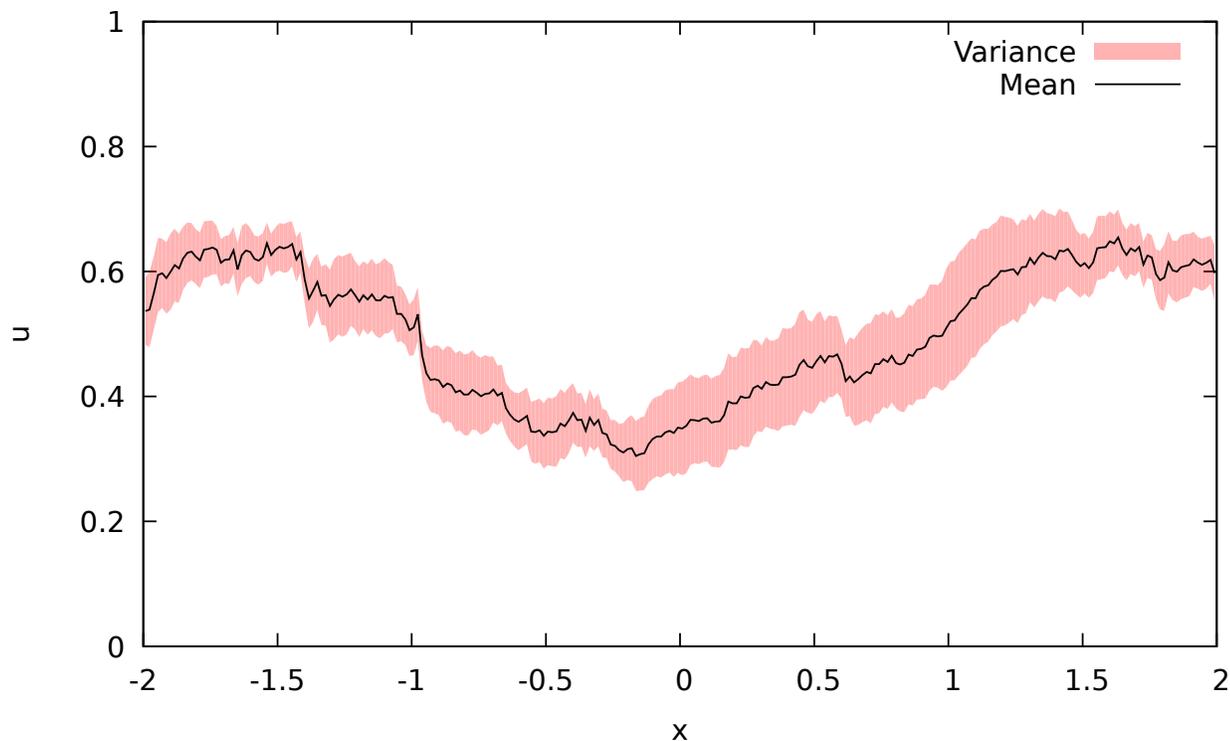


FIGURE 3. Traffic Flow Problem with Brownian Bridge with Jumps

- [17] Kenneth H Karlsen and John D Towers. Convergence of the lax-friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux. *Chinese Annals of Mathematics*, 25(03):287–318, 2004.
- [18] Runhild Aae Klausen and Nils Henrik Risebro. Stability of conservation laws with discontinuous coefficients. *Journal of differential equations*, 157(1):41–60, 1999.
- [19] U. Koley, N. H. Risebro, Ch. Schwab, and F. Weber. Multilevel monte carlo for random degenerate scalar convection diffusion equation.
- [20] Michel Ledoux and Michel Talagrand. *Probability in Banach Spaces: isoperimetry and processes*. Springer Science & Business Media, 2013.
- [21] Randall J LeVeque, David L George, and Marsha J Berger. Tsunami modelling with adaptively refined finite volume methods. *Acta Numerica*, 20:211–289, 2011.
- [22] Michael J Lighthill and Gerald Beresford Whitham. On kinematic waves. ii. a theory of traffic flow on long crowded roads. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 229, pages 317–345. The Royal Society, 1955.
- [23] G Lin, CH Su, and GE Karniadakis. The stochastic piston problem. *Proceedings of the National Academy of Sciences of the United States of America*, 101(45):15840–15845, 2004.
- [24] Xiang Ma and Nicholas Zabaras. An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations. *Journal of Computational Physics*, 228(8):3084–3113, 2009.
- [25] Siddhartha Mishra, Nils Henrik Risebro, Christoph Schwab, and Svetlana Tokareva. Numerical solution of scalar conservation laws with random flux functions. *SIAM/ASA Journal on Uncertainty Quantification*, 4(1):552–591, 2016.
- [26] Siddhartha Mishra and Ch Schwab. Sparse tensor multi-level monte carlo finite volume methods for hyperbolic conservation laws with random initial data. *Mathematics of Computation*, 81(280):1979–2018, 2012.
- [27] Gaël Poëtte, Bruno Després, and Didier Lucor. Uncertainty quantification for systems of conservation laws. *Journal of Computational Physics*, 228(7):2443–2467, 2009.
- [28] Paul I Richards. Shock waves on the highway. *Operations research*, 4(1):42–51, 1956.
- [29] John D Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM journal on numerical analysis*, 38(2):681–698, 2000.

- [30] JMAM Van Neerven. Stochastic evolution equations. *ISEM lecture notes*, 2008.
- [31] Xiaoliang Wan and George Em Karniadakis. Long-term behavior of polynomial chaos in stochastic flow simulations. *Computer methods in applied mechanics and engineering*, 195(41):5582–5596, 2006.
- [32] Jeroen AS Witteveen, Alex Loeven, and Hester Bijl. An adaptive stochastic finite elements approach based on newton-cotes quadrature in simplex elements. *Computers & Fluids*, 38(6):1270–1288, 2009.