

Viscosity-flux approximation method to inhomogeneous system of isentropic gas dynamics

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Abstract

In this paper, we apply the viscosity-flux approximation method coupled with the maximum principle to obtain the a priori L^∞ estimates $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(x - kt)$ and $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq B(x - kt)$ for the solutions $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon})$ of the Cauchy problem (2.1) and (2.2), where w and z are the Riemann invariants of (1.1) and $B(x - kt)$ is a nonnegative bounded function, and to obtain the global L^∞ entropy solutions for the Cauchy problem of inhomogeneous system of isentropic gas dynamics (1.1) with bounded initial data (1.2).

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1 Introduction

In this paper we studied the global entropy solutions for the Cauchy problem of the following system of isentropic gas dynamics with a moving source term

$$\begin{cases} \rho_t + (\rho u)_x = A(x - kt)\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = A(x - kt)\rho u^2 - \alpha(x, t)\rho u|u|, \end{cases} \quad (1.1)$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure and k is a constant ($k > 0$ denotes a moving speed [Liu1]) and $\alpha(x, t) \geq 0$ a friction function of the space variable x and the time variable t . For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, where $\gamma > 1$ is the adiabatic exponent.

Especially when the friction function α is a constant, the moving speed $k = 0$ and $A(x) = -\frac{a'(x)}{a(x)}$, then $a(x)$ denotes a slowly variable cross section area at x in the nozzle. For the case of the zero moving speed, the global entropy solutions for the Cauchy problem (1.1) with bounded initial data

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \quad (1.2)$$

was first studied in [CG, Ts1] for the usual gases $1 < \gamma \leq \frac{5}{3}$, and later, by the author in [Lu1] for any adiabatic exponent $\gamma > 1$, provided that the initial data are bounded and satisfy the strong restriction condition $z_0(\rho_0(x), u_0(x)) \leq 0$. The initial-boundary value problem of compressible Euler equations including friction and heating that model the transonic Fanno-Rayleigh flows through symmetric variable area nozzles is studied in [CHHQ].

It is well-known that after we have a method to obtain the global existence of solutions for the Cauchy problem of the following underlying nonlinear system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \end{cases} \quad (1.3)$$

with the bounded initial data (1.2), the unique difficulty to deal with the inhomogeneous system (1.1) is to obtain the a-priori L^∞ estimates of the approximation solutions of (1.1), for instance, the a-priori L^∞ estimates of the viscosity solutions for the Cauchy problem of the following parabolic system

$$\begin{cases} \rho_t + (\rho u)_x = A(x - kt)\rho u + \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = A(x - kt)\rho u^2 - \alpha(x, t)\rho u|u| + \varepsilon(\rho u)_{xx} \end{cases} \quad (1.4)$$

with the initial data (1.2).

When $A(x - kt) = 0$, (1.1) is the river flow equations, a shallow-water model describing the vertical depth ρ and mean velocity u , where $\alpha(x, t)\rho u|u|$ corresponds physically to a friction term and α is the friction coefficient. This kind of inhomogeneous systems is simple since the source terms have, in some senses, the symmetric behavior. We may introduce the Riemann invariants (w, z) of system (1.3) to rewrite (1.1) as the following symmetric, coupled system

$$\begin{cases} w_t + \lambda_2 w_x = \varepsilon w_{xx} + \frac{2\varepsilon}{\rho} \rho_x w_x - \frac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2 - \frac{1}{2} \alpha (w - z) |u| \\ z_t + \lambda_1 z_x = \varepsilon z_{xx} + \frac{2\varepsilon}{\rho} \rho_x z_x - \frac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2 - \frac{1}{2} \alpha (z - w) |u|, \end{cases} \quad (1.5)$$

where

$$\lambda_1 = \frac{m}{\rho} - \sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \sqrt{P'(\rho)}$$

are two eigenvalues of system (1.1);

$$z(\rho, u) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds - u, \quad w(\rho, u) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds + u \quad (1.6)$$

are the Riemann invariants, c is a constant and $m = \rho u$ denotes the momentum. We can apply the maximum principle directly to (1.5) to obtain the necessary a-priori L^∞ estimates $w(\rho^\varepsilon, u^\varepsilon) \leq M$ and $z(\rho^\varepsilon, u^\varepsilon) \leq N$, for two suitable constants M, N (see ([KL]) for the details).

When $\alpha(x, t) = 0$ and $k = 0$, i.e., the nozzle flow without friction, system (1.1) was well studied in (cf. [CG, Liu2, Liu3, Lu2, Ts2, Ts3, Ts4], and the references cited therein). Roughly speaking, the technique, introduced in these papers, is to control the super-linear source terms $A(x)\rho u$ and $A(x)\rho u^2$ by the flux functions ρu and $\rho u^2 + P(\rho)$ in (1.1) and to deduce a upper bound of w or z by a bounded nonnegative function $B(x)$, which depends on the function $A(x)$.

When $A(x) \not\equiv 0$ and $\alpha(x, t) \not\equiv 0$, both the above techniques do not work because the flux functions can not be used to control the super-linear friction source terms $\alpha \rho u |u|$, and the functions $A(x)\rho u$ destroy the symmetry of the Riemann invariants (w, z) (see [KPL] for the numerical analysis).

However, we may copy the following steps given in ([Lu2]) to overcome the above difficulty.

First, to avoid the singularity of the flux function ρu^2 near the vacuum $\rho = 0$, we still use the technique of the δ -flux-approximation given in [Lu3] and introduce the sequence of systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A(x - kt)(\rho - 2\delta)u \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x - kt)(\rho - 2\delta)u^2 - \alpha(x, t)\rho u|u| \end{cases} \quad (1.7)$$

to approximate system (1.1), where $\delta > 0$ denotes a regular perturbation constant and the perturbation pressure

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t) dt. \quad (1.8)$$

Second, we add the viscosity terms to the right-hand side of (1.7) to obtain the following parabolic system

$$\begin{cases} \rho_t + ((\rho - 2\delta)u)_x = A(x - kt)(\rho - 2\delta)u + \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = A(x - kt)(\rho - 2\delta)u^2 - \alpha(x, t)\rho u|u| + \varepsilon(\rho u)_{xx} \end{cases} \quad (1.9)$$

with initial data

$$(\rho^{\delta, \varepsilon}(x, 0), u^{\delta, \varepsilon}(x, 0)) = (\rho_0(x) + 2\delta, u_0(x)), \quad (1.10)$$

where $(\rho_0(x), u_0(x))$ are given in (1.2). Now we multiply (1.9) by $(\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial m})$ and $(\frac{\partial z}{\partial \rho}, \frac{\partial z}{\partial m})$, respectively, where (w, z) are given in (1.6), to obtain

$$\begin{aligned} & w_t + \lambda_2^\delta w_x \\ &= \varepsilon w_{xx} + \frac{2\varepsilon}{\rho} \rho_x w_x - \frac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2 \\ &+ A(x - kt)(\rho - 2\delta)u \frac{\sqrt{P'(\rho)}}{\rho} - \alpha(x, t)u|u| \end{aligned} \quad (1.11)$$

and

$$\begin{aligned} & z_t + \lambda_1^\delta z_x \\ &= \varepsilon z_{xx} + \frac{2\varepsilon}{\rho} \rho_x z_x - \frac{\varepsilon}{2\rho^2 \sqrt{P'(\rho)}} (2P' + \rho P'') \rho_x^2 \\ &+ A(x - kt)(\rho - 2\delta)u \frac{\sqrt{P'(\rho)}}{\rho} + \alpha(x, t)u|u|, \end{aligned} \quad (1.12)$$

where

$$\lambda_1^\delta = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)}, \quad \lambda_2^\delta = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} \quad (1.13)$$

are two eigenvalues of the approximation system (1.7).

It is obvious that the terms $A(x - kt)(\rho - 2\delta)u \frac{\sqrt{P'(\rho)}}{\rho}$ in (1.11) and (1.12) are not symmetric with respect to the Riemann invariants w, z . However, with the strong restriction $z_0(\rho_0(x), u_0(x)) \leq 0$ on the initial data, in ([Ts1, Lu1]), we may obtain the uniformly upper bounds of z and w by using the maximum principle. In some senses, it is similar to obtain the estimate $u \leq 0$ for the following scalar equation

$$u_t + f(u)_x + S(u, x, t)u = \varepsilon u_{xx} \quad (1.14)$$

for a suitable function $S(u, x, t)$ when the initial data $u_0(x) \leq 0$.

Without the condition $z_0(\rho_0(x), u_0(x)) \leq 0$, we will meet some new technical difficulties. In this paper, we will unite the techniques given in [KL] and in [Ts1, Lu1] to obtain the estimates of $z \leq B(x - kt)$ and $w \leq B(x - kt)$, for a suitable uniformly bounded function $B(x - kt)$, and to prove the global existence of the entropy solutions for the Cauchy problem (1.1) and (1.2).

Before we introduce our main results, we first have the following

Definition 1 *One function $B(x)$ is called a member in the set $\in B_d^2(R)$ if (a): $B(x) \in C^1(R), 0 < B(x) \leq M, -MB(x) \leq B'(x) \leq 0$ and (b): $B''(x) \leq 0$ or $B''(x) = B_1(x) + B_2(x)$, where $B_1(x) \leq 0$ and $|B_2(x)| \leq M|B(x)B'(x)|$ for a suitable positive constant M ;*

Definition 2 *One function $C(x)$ is called a member in the set $\in C_d^2(R)$ if (a): $C(x) \in C^1(R), 0 < C(x) \leq M, 0 \leq C'(x) \leq MC(x)$ and (b): $C'''(x) \leq 0$ or $C'''(x) = C_1(x) + C_2(x)$, where $C_1(x) \leq 0$ and $|C_2(x)| \leq M|C(x)C'(x)|$ for a suitable positive constant M .*

Remark 1: For a given $B(x) \in B_d^2(R)$, if we let $C(x) = B(-x)$, then $C(x) \in C_d^2(R)$.

The main results of this paper are in the following Theorems 1-3.

Theorem 1 *Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma > 1, \alpha(x, t) \geq 0$ and $A(x - kt) \leq 0$.*

(I). Let $1 < \gamma \leq 3, k > 0$. If there exists a function $B(x) \in B_d^2(-\infty, \infty)$ such that $B(x) \leq M < k$ and

$$\begin{aligned} & \theta^2 A^2(x-kt) B^2(x-kt) + (1-\theta)^2 B_x^2 - 2\theta(1+\theta)A(x-kt)B(x-kt)B_x \\ & + 4\varepsilon_1 \theta A(x-kt)B(x-kt)B_x \leq 0 \end{aligned} \quad (1.15)$$

for a small $\varepsilon_1 > 0$, then we have

$$z(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{(\rho^{\delta,\varepsilon}(x,t))^\theta}{\theta} - u^{\delta,\varepsilon}(x,t) \leq B(x-kt) \quad (1.16)$$

and

$$w(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{(\rho^{\delta,\varepsilon}(x,t))^\theta}{\theta} + u^{\delta,\varepsilon}(x,t) \leq B(x-kt) \quad (1.17)$$

if the initial data $z_0(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq B(x)$ and $w_0(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq B(x)$, where $B_x = \frac{\partial B(x-kt)}{\partial x}$, the set B_d^2 is given in Definition 1 and $\theta = \frac{\gamma-1}{2}$.

(II). Let $\gamma > 3, k < 0$. If there exists a function $C(x) \in C_d^2(-\infty, \infty)$ such that $C(x) \leq M < -\frac{2}{\gamma+1}k$ and

$$\begin{aligned} & \theta^2 A^2(x-kt) C^2(x-kt) + (1-\theta)^2 C_x^2 + 2\theta(1+\theta)A(x-kt)C(x-kt)C_x \\ & - 4\varepsilon_1 \theta A(x-kt)C(x-kt)C_x \leq 0 \end{aligned} \quad (1.18)$$

for a small $\varepsilon_1 > 0$, then we have the estimates $z(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) \leq C(x-kt)$, $w(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) \leq C(x-kt)$, which are similar with the estimates (1.16) and (1.17).

Theorem 2 Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, \gamma > 1$, $\alpha(x,t) \geq 0$ and $A(x-kt) = A_-(x-kt) + A_+(x-kt)$, where $A_-(x-kt) \leq 0, A_+(x-kt) \geq 0$.

(III). Let $1 < \gamma \leq 3, k > 0$. If there exists a function $B(x) \in B_d^2(-\infty, \infty)$ such that $B(x) \leq M < k$ and

$$\begin{aligned} & \theta^2 A_-^2(x-kt) B^2(x-kt) + (1-\theta)^2 B_x^2 - 2\theta(1+\theta)A_-(x-kt)B(x-kt)B_x \\ & + 4\varepsilon_1 \theta A_-(x-kt)B(x-kt)B_x \leq 0 \end{aligned} \quad (1.19)$$

and

$$4(-k + (1 + \varepsilon_1)B(x-kt))B_x(x-kt) - A_+(x-kt)B^2(x-kt) \geq 0 \quad (1.20)$$

for a small $\varepsilon_1 > 0$, then we have the same estimates like (1.16) and (1.17);

(IV). Let $\gamma > 3, k < 0$. If there exists a function $C(x) \in C_d^2(-\infty, \infty)$ such that $C(x) \leq M < -\frac{2}{\gamma+1}k$ and

$$\begin{aligned} & \theta^2 A_-^2(x-kt)C^2(x-kt) + (1-\theta)^2 C_x^2 + 2\theta(1+\theta)A_-(x-kt)C(x-kt)C_x \\ & - 4\varepsilon_1 \theta A_-(x-kt)C(x-kt)C_x \leq 0 \end{aligned} \quad (1.21)$$

and

$$4(-k + (1 + \varepsilon_1)C(x-kt))C_x(x-kt) - A_+(x-kt)C^2(x-kt) \geq 0 \quad (1.22)$$

for a small $\varepsilon_1 > 0$, then we have the estimates $z(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) \leq C(x-kt)$, $w(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) \leq C(x-kt)$.

Theorem 3 For such functions $A(x-kt), \alpha(x,t)$ and the initial data satisfying the conditions in Theorem 1 or Theorem 2, there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as δ, ε tend to zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2)

Definition 3 A pair of bounded functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.1)-(1.2) if

$$\left\{ \begin{array}{l} \int_0^\infty \int_{-\infty}^\infty \rho \phi_t + (\rho u) \phi_x + A(x-kt) \rho u \phi dx dt + \int_{-\infty}^\infty \rho_0(x) \phi(x,0) dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u \phi_t + (\rho u^2 + P(\rho)) \phi_x + (A(x-kt) \rho u^2 - \alpha(x,t) \rho u |u|) \phi dx dt \\ + \int_{-\infty}^\infty \rho_0(x) u_0(x) \phi(x,0) dx = 0 \end{array} \right. \quad (1.23)$$

holds for all test function $\phi \in C_0^1(R \times R^+)$ and

$$\left\{ \begin{array}{l} \int_0^\infty \int_{-\infty}^\infty \eta(\rho, m) \phi_t + q(\rho, m) \phi_x + A(x-kt) \rho u \eta(\rho, m)_\rho \\ + (A(x-kt) \rho u^2 - \alpha(x,t) \rho u |u|) \eta(\rho, m)_m \phi dx dt \geq 0 \end{array} \right. \quad (1.24)$$

holds for any non-negative test function $\phi \in C_0^\infty(R \times R^+ - \{t=0\})$, where $m = \rho u$ and (η, q) is a pair of convex entropy-entropy flux of system (1.3).

Example 1: For a given $A(x)$, there are many functions $B(x)$ in the set $B_d^2(R)$ satisfying (1.15) (or $C(x)$ in the set $C_d^2(R)$ satisfying (1.18)).

For instance, we consider the function $a(x) = x^2$, $A(x) = -\frac{a'(x)}{a(x)}$, $x > 1$ studied in the paper [Ts2] for the spherically, symmetric solutions and extend it to the whole space $x \in (-\infty, \infty)$:

$$A(x) = \begin{cases} -\frac{2}{x}, & \text{for } x > \varepsilon_0, \\ -\frac{2x}{\varepsilon_0^2}, & \text{for } 0 \leq x \leq \varepsilon_0, \\ 0, & \text{for } x < 0, \end{cases} \quad (1.25)$$

where $\varepsilon_0 > 0$ is a constant. Then it is easy to check that the following function

$$B(x) = \begin{cases} qx^\beta, & \text{for } x > \varepsilon_0, \\ q_1 e^{\frac{\beta x^2}{2\varepsilon_0^2}}, & \text{for } 0 \leq x \leq \varepsilon_0, \\ q_1 & \text{for } x < 0 \end{cases} \quad (1.26)$$

satisfies

$$B'(x) = \begin{cases} q\beta x^{\beta-1}, & \text{for } x > \varepsilon_0, \\ q_1 \frac{\beta x}{\varepsilon_0^2} e^{\frac{\beta x^2}{2\varepsilon_0^2}}, & \text{for } 0 \leq x \leq \varepsilon_0, \\ 0 & \text{for } x < 0 \end{cases} \quad (1.27)$$

and

$$B'(x) = -\frac{\beta}{2}A(x)B(x) \quad (1.28)$$

if

$$q\varepsilon_0^\beta = q_1 e^{\frac{\beta}{2}}, \quad (1.29)$$

where q, q_1 are two positive constants and β is a negative constant. Then we have $-MB(x) \leq B'(x) \leq 0$.

Moreover,

$$B''(x) = \begin{cases} q\beta(\beta-1)x^{\beta-2}, & \text{for } x > \varepsilon_0, \\ q_1 \frac{\beta}{\varepsilon_0^2} e^{\frac{\beta x^2}{2\varepsilon_0^2}} + q_1 \left(\frac{\beta x}{\varepsilon_0^2}\right)^2 e^{\frac{\beta x^2}{2\varepsilon_0^2}}, & \text{for } 0 \leq x \leq \varepsilon_0, \\ 0 & \text{for } x < 0 \end{cases} \quad (1.30)$$

and $B''(x) = B_1(x) + B_2(x)$, where $B_1(x), B_2(x)$ satisfy the conditions in Definition 1, so $B(x) \in B_d^2(R)$.

Let $-\frac{\beta}{2} = \frac{\theta}{1-\theta}$. Then it is easy to check that $B(x - kt)$, where $B(x)$ given by (1.28), satisfies (1.15).

In fact, let $B_x = \frac{\partial B(x-kt)}{\partial x} = \frac{\theta}{1-\theta}A(x-kt)B(x-kt)$. Then $B_x \leq 0$ and (1.15) is equivalent to

$$\begin{aligned}
& \theta^2 A^2(x-kt)B^2(x-kt) + (1-\theta)^2 B_x^2 - 2\theta(1+\theta)A(x-kt)B(x-kt)B_x \\
& + 4\varepsilon_1 \theta A(x-kt)B(x-kt)B_x \\
& = 2\theta(1-\theta)A(x-kt)B(x-kt)B_x - 2\theta(1+\theta)A(x-kt)B(x-kt)B_x \\
& + 4\varepsilon_1 \theta A(x-kt)B(x-kt)B_x \\
& = (-4\theta + 4\varepsilon_1)\theta A(x-kt)B(x-kt)B_x \leq 0.
\end{aligned} \tag{1.31}$$

Example 2. We now construct a function $B(x) \in C^2(R) \cap B_d^2(R)$. Choose

$$a(x) = \begin{cases} c_1 + e^{c_2 x}, & \text{for } x \leq 0, \\ 1 + c_1 + c_2 x + \frac{c_2^2}{2} x^2, & \text{for } x > 0. \end{cases} \tag{1.32}$$

Then $a(x) \in C^2(R)$,

$$a'(x) = \begin{cases} c_2 e^{c_2 x}, & \text{for } x \leq 0, \\ c_2 + c_2^2 x, & \text{for } x > 0, \end{cases} \tag{1.33}$$

$$a''(x) = \begin{cases} c_2^2 e^{c_2 x}, & \text{for } x \leq 0, \\ c_2^2, & \text{for } x > 0. \end{cases} \tag{1.34}$$

and $0 < 1 + c_1 \leq a(x), 0 < -A(x) = \frac{a'(x)}{a(x)} \leq M, a''(x) \geq 0$. Let $B(x) = a(x)^{-c_0}$, where $c_0 \in (0, \frac{1}{2}]$ be a constant, then $B(x) \in C^2(R) \cap B_d^2(R)$.

In fact,

$$0 > B'(x) = c_0 B(x) A(x) \leq -c_0 M B(x), \tag{1.35}$$

and

$$B''(x) = -c_0 a(x)^{-c_0-1} a''(x) + (c_0 + 1) B'(x) A(x), \tag{1.36}$$

where

$$B_1(x) = -c_0 a(x)^{-c_0-1} a''(x) < 0 \tag{1.37}$$

and

$$|B_2(x)| = |(c_0 + 1)B'(x)A(x)| \leq MB(x)|B'(x)| \quad (1.38)$$

for any constant $c_0 \in (0, \frac{1}{2}]$.

Remark 2: We may choose suitable functions $A_-(x - kt)$ and $A_+(x - kt)$ to easily obtain the functions $B(x - kt), C(x - kt)$ satisfying (1.19) and (1.20), or (1.22) and (1.22).

For instance, if we let $A_-(x), B(x)$ satisfy (1.28) or

$$B_x(x - kt) = -\frac{\beta}{2}A_-(x - kt)B(x - kt), \quad (1.39)$$

then (1.19) is true. Clearly (1.20) is also true if

$$4(-k + (1 + \varepsilon_1)M)B_x(x - kt) - MA_+(x - kt)B(x - kt) \geq 0 \quad (1.40)$$

or

$$B_x(x - kt) \geq -l_0A_+(x - kt)B(x - kt) \quad (1.41)$$

for a suitable positive constant l_0 . If we choose $A_-(x - kt)$ and $A_+(x - kt)$ such that

$$-\frac{\beta}{2}A_-(x - kt) \geq -l_0A_+(x - kt), \quad (1.42)$$

then $B(x - kt)$ satisfies (1.20).

We shall prove Theorems 1-3 in the following several sections.

2 Proof of Theorem 1.

Letting $z = B(x, t) + v$, for a suitable function $B(x, t)$ in (1.12), we have

$$\begin{aligned} & v_t + B_t + (u - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)})(v_x + B_x) \\ &= \varepsilon v_{xx} + \varepsilon B_{xx} + \frac{2\varepsilon}{\rho}\rho_x v_x + \frac{2\varepsilon}{\rho}\rho_x B_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\ & - A(x - kt)\frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}(B(x, t) + v - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho) + \alpha(x, t)u|u| \end{aligned} \quad (2.1)$$

or

$$\begin{aligned}
& v_t + B_t + \left(u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}\right)v_x - B_x(B(x,t) + v - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho) - B_x \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} \\
&= \varepsilon v_{xx} - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 - \frac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x B_x + (\frac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}B_x)^2] \\
&+ \varepsilon B_{xx} + \frac{2\varepsilon}{\rho}\rho_x v_x + \frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - A(x-kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}B(x,t) \\
&- A(x-kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}v + A(x-kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho + \alpha(x,t)u|u|
\end{aligned} \tag{2.2}$$

or

$$\begin{aligned}
& v_t + B_t + a(x,t)v_x + b(x,t)v + \left[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x,t)B_x\right] \\
&+ \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho B_x - (1-\varepsilon_1)B(x,t)B_x - \alpha(x,t)u|u| \\
&+ [A(x-kt)B(x,t) - B_x](\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
&- A(x-kt)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \leq \varepsilon v_{xx},
\end{aligned} \tag{2.3}$$

where $\varepsilon_1 > 0$ is a suitable small constant, $a(x,t) = u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b(x,t) = -B_x + A(x-kt)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$.

Similarly, if letting $w = B(x,t) + s$ in (1.11), we have

$$\begin{aligned}
& s_t + B_t + \left(u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}\right)(s_x + B_x) \\
&= \varepsilon s_{xx} + \varepsilon B_{xx} + \frac{2\varepsilon}{\rho}\rho_x s_x + \frac{2\varepsilon}{\rho}\rho_x B_x - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')\rho_x^2 \\
&+ A(x-kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}u - \alpha(x,t)u|u|
\end{aligned} \tag{2.4}$$

or

$$\begin{aligned}
& s_t + B_t + \left(u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}\right)s_x + B_x(B(x,t) + s - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho) + B_x \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} \\
&= \varepsilon s_{xx} - \frac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P' + \rho P'')[\rho_x^2 + \frac{4\rho\sqrt{P'(\rho)}}{2P'+\rho P''}\rho_x B_x + (\frac{2\rho\sqrt{P'(\rho)}}{2P'+\rho P''}B_x)^2] \\
&+ \varepsilon B_{xx} + \frac{2\varepsilon}{\rho}\rho_x s_x + \frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 + A(x-kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}u - \alpha(x,t)u|u|
\end{aligned} \tag{2.5}$$

or

$$\begin{aligned}
& s_t + B_t + c(x, t)s_x + d(x, t)s + \left[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''} B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x, t)B_x\right] \\
& + B_x\left(\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho\right) + (1 + \varepsilon_1)B(x, t)B_x \\
& - A(x - kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}u + \alpha(x, t)u|u| \leq \varepsilon s_{xx},
\end{aligned} \tag{2.6}$$

where $c(x, t) = u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $d(x, t) = B_x$.

Using the first equation in system (1.9), we have the a priori estimate $\rho \geq 2\delta$. We can choose $\varepsilon = o(\delta)$ and suitable relation between ε and ε_1 such that the following three terms on the left-hand side of (2.3) and (2.6)

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''} B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x, t)B_x > 0. \tag{2.7}$$

Now we rewrite (2.3) and (2.6) as follows:

$$\begin{aligned}
& v_t + B_t + a(x, t)v_x + b(x, t)v \\
& + \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B_x - (1 - \varepsilon_1)B(x, t)B_x + \frac{1}{2}\alpha(x, t)(v - s)|u| \\
& + [A(x - kt)B(x, t) - B_x](\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
& - A(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \leq \varepsilon v_{xx},
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
& s_t + B_t + c(x, t)s_x + d(x, t)s \\
& + B_x\left(\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho\right) + (1 + \varepsilon_1)B(x, t)B_x \\
& + \frac{1}{2}(\alpha(x, t)|u| - A(x - kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(s - v) \leq \varepsilon s_{xx}.
\end{aligned} \tag{2.9}$$

If we may choose a suitable bounded function $B(x, t)$ such that the following inequalities hold

$$\begin{aligned}
& B_t + \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B_x - (1 - \varepsilon_1)B(x, t)B_x \\
& + [A(x - kt)B(x, t) - B_x](\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
& - A(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \geq 0
\end{aligned} \tag{2.10}$$

and

$$B_t + B_x \left(\frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right) + (1 + \varepsilon_1) B(x, t) B_x \geq 0, \quad (2.11)$$

then we have from (2.8) and (2.9) that

$$v_t + a(x, t)v_x + b(x, t)v + \frac{1}{2}\alpha(x, t)|u|(v - s) \leq \varepsilon v_{xx} \quad (2.12)$$

and

$$s_t + c(x, t)s_x + d(x, t)s + \frac{1}{2}(\alpha(x, t)|u| - A(x - kt) \frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)})(s - v) \leq \varepsilon s_{xx}. \quad (2.13)$$

Before we check the possibility of (2.10) and (2.11), we apply the inequalities (2.12) and (2.13) to prove the following Lemma 4 about the a priori estimates of v and s :

Lemma 4 *If at the time $t = 0$, $v(x, 0) \leq 0$ and $s(x, 0) \leq 0$, then the maximum principle is true to the functions $v(x, t)$ and $s(x, t)$, namely, $v(x, t) \leq 0$, $s(x, t) \leq 0$ for all $t > 0$.*

Proof of Lemma 4: Make a transformation

$$v = \left(\bar{v} + \frac{N(x^2 + qLe^t)}{L^2} \right) e^{\beta t}, \quad s = \left(\bar{s} + \frac{N(x^2 + qLe^t)}{L^2} \right) e^{\beta t}, \quad (2.14)$$

where L, q, β are suitable positive constants and N is the upper bound of v, s on $R \times [0, T]$ (N can be obtained by the local existence). The functions \bar{v}, \bar{s} , as are easily seen, satisfy the equations

$$\left\{ \begin{array}{l} \bar{v}_t + a(x, t)\bar{v}_x - \varepsilon\bar{v}_{xx} + (\beta + b(x, t) + \frac{1}{2}\alpha(x, t)|u|)\bar{v} - \frac{1}{2}\alpha(x, t)|u|\bar{s} \\ \leq -(qLe^t + 2xa(x, t) - 2\varepsilon)\frac{N}{L^2} - (\beta + b(x, t))\frac{N(x^2 + qLe^t)}{L^2}, \\ \bar{s}_t + c(x, t)\bar{s}_x - \varepsilon\bar{s}_{xx} + (\beta + d(x, t) + \frac{1}{2}\alpha(x, t)|u| - A(x - kt)\frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)})\bar{s} \\ - (\frac{1}{2}\alpha(x, t)|u| - A(x - kt)\frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)})\bar{v} \\ \leq -(qLe^t + 2xc(x, t) - 2\varepsilon)\frac{N}{L^2} - (\beta + d(x, t))\frac{N(x^2 + qLe^t)}{L^2}, \end{array} \right. \quad (2.15)$$

resulting from (2.12) and (2.13). Moreover

$$\bar{v}(x, 0) = v(x, 0) - \frac{N(x^2 + qL)}{L^2} < 0, \quad \bar{s}(x, 0) = s(x, 0) - \frac{N(x^2 + qL)}{L^2} < 0, \quad (2.16)$$

$$\bar{v}(+L, t) < 0, \quad \bar{v}(-L, t) < 0, \quad \bar{s}(+L, t) < 0, \quad \bar{s}(-L, t) < 0. \quad (2.17)$$

From (2.15),(2.16) and (2.17), we have

$$\bar{v}(x, t) < 0, \quad \bar{s}(x, t) < 0, \quad \text{on } (-L, L) \times (0, T). \quad (2.18)$$

If (2.18) is violated at a point $(x, t) \in (-L, L) \times (0, T)$, let \bar{t} be the least upper bound of values of t at which $\bar{v} < 0$ (or $\bar{s} < 0$); then by the continuity we see that $\bar{v} = 0, \bar{s} \leq 0$ at some points $(\bar{x}, \bar{t}) \in (-L, L) \times (0, T)$. So

$$\bar{v}_t \geq 0, \quad \bar{v}_x = 0, \quad -\varepsilon \bar{v}_{xx} \geq 0, \quad \text{at } (\bar{x}, \bar{t}). \quad (2.19)$$

If we choose sufficiently large constants q, β (which may depend on the bound of the local existence) such that

$$qL + 2xa(x, t) - 2\varepsilon > 0, \quad \beta + b(x, t) > 0 \quad \text{on } (-L, L) \times (0, T). \quad (2.20)$$

(2.19) and (2.20) give a conclusion contradicting the first inequality in (2.15). So (2.18) is proved. Therefore, for any point $(x_0, t_0) \in (-L, L) \times (0, T)$,

$$v(x_0, t_0) < \left(\frac{N(x_0^2 + qLe_0^t)}{L^2}\right)e^{\beta t_0}, \quad s(x_0, t_0) < \left(\frac{N(x_0^2 + qLe_0^t)}{L^2}\right)e^{\beta t_0}, \quad (2.21)$$

which gives the desired estimates $v \leq 0, s \leq 0$ if we let L go to infinity. So Lemma 4 is proved.

From $v \leq 0, s \leq 0$, we may immediately obtain the estimates $w \leq B(x, t)$ and $z \leq B(x, t)$ given in Part (I) of Theorem 1 if we may choose $B(x, t)$ such that (2.10) and (2.11) are true.

Lemma 5 *Let $P(\rho) = \frac{1}{\gamma}\rho^\gamma, 1 < \gamma \leq 3, c = 0$. For a given function $B(x) \in B_d^2(R)$, if $B(x) \leq M < k$ and satisfies the condition (1.15) in Theorem 1, then (2.10) and (2.11) are true if we choose $B(x, t) = B(x - kt)$.*

Proof of Lemma 5: We first prove (2.11). Let $B(x, t) = B(x - kt)$, then $B_t = -kB_x$. When $P(\rho) = \frac{1}{\gamma}\rho^\gamma, 1 < \gamma \leq 3$ and $c = 0$, we have

$$\frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho = (\rho - 2\delta)\rho^{\frac{\gamma-3}{2}} - \int_{2\delta}^\rho s^{\frac{\gamma-3}{2}} ds - \int_0^{2\delta} s^{\frac{\gamma-3}{2}} ds \leq 0, \quad (2.22)$$

and

$$B_t + (1 + \varepsilon_1)B(x, t)B_x = (-k + (1 + \varepsilon_1)B(x - kt))B_x \geq 0. \quad (2.23)$$

Thus (2.11) is proved.

To prove (2.10), we let the left side of (2.10) be L , then

$$\begin{aligned} L &= \left[\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - \frac{2\delta}{\rho} \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right] B_x \\ &+ \left(\frac{2\delta}{\rho} B_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - kB_x \right) - (1 - \varepsilon_1)B(x - kt)B_x \\ &+ [A(x - kt)B(x - kt) - B_x](\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \\ &- A(x - kt)(\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \\ &= -\frac{1}{\theta} A(x - kt)(\rho - 2\delta)^2 \rho^{2\theta-2} + (A(x - kt)B(x - kt) - B_x + \frac{B_x}{\theta})(\rho - 2\delta) \rho^{\theta-1} \\ &+ \left(\frac{2\delta}{\rho} B_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - kB_x \right) - (1 - \varepsilon_1)B(x - kt)B_x - \frac{2\delta}{\theta} A(x - kt)(\rho - 2\delta) \rho^{2\theta-2}. \end{aligned} \quad (2.24)$$

Since

$$-\frac{2\delta}{\theta} A(x)(\rho - 2\delta) \rho^{2\theta-2} \geq 0 \quad (2.25)$$

and

$$\left(\frac{2\delta}{\rho} B_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - kB_x \right) = \left(\frac{2\delta}{\theta} \rho^{\frac{\gamma-3}{2}} - k \right) B_x \geq \left(\frac{(2\delta)^\theta}{\theta} - k \right) B_x \geq 0, \quad (2.26)$$

we have

$$\begin{aligned} L &\geq -\frac{1}{\theta} A(x - kt)(\rho - 2\delta)^2 \rho^{2\theta-2} - (1 - \varepsilon_1)B(x - kt)B_x \\ &+ (A(x - kt)B(x - kt) - B_x + \frac{B_x}{\theta})(\rho - 2\delta) \rho^{\theta-1} \\ &= -\frac{1}{\theta} A(x - kt) \left[(\rho - 2\delta)^2 \rho^{2\theta-2} + \frac{\theta B_x - B_x - \theta A(x - kt)B(x - kt)}{A(x - kt)} (\rho - 2\delta) \rho^{\theta-1} \right] \\ &+ \left(\frac{\theta B_x - B_x - \theta A(x - kt)B(x - kt)}{2A(x - kt)} \right)^2 - (1 - \varepsilon_1)B(x - kt)B_x \\ &+ \frac{(\theta B_x - B_x - \theta A(x - kt)B(x - kt))^2}{4\theta A(x - kt)}. \end{aligned} \quad (2.27)$$

Since $A(x - kt) \leq 0$ and $B(x - kt) > 0$, then $L \geq 0$ if we let $B_x \leq 0$ satisfy

$$-(1 - \varepsilon_1)B(x - kt)B_x + \frac{(\theta B_x - B_x - \theta A(x - kt)B(x - kt))^2}{4\theta A(x - kt)} \geq 0, \quad (2.28)$$

which is equivalent to

$$\begin{aligned} & (\theta B_x - B_x - \theta A(x - kt)B(x - kt))^2 \\ & \leq 4\theta A(x - kt)B(x - kt)B_x - 4\theta\varepsilon_1 A(x - kt)B(x - kt)B_x \end{aligned} \quad (2.29)$$

or

$$\begin{aligned} & (\theta - 1)^2 B_x^2 - 2\theta(\theta + 1)A(x - kt)B(x - kt)B_x \\ & + 4\theta\varepsilon_1 A(x - kt)B(x - kt)B_x + \theta^2(A(x - kt)B(x - kt))^2 \leq 0. \end{aligned} \quad (2.30)$$

(2.30) is true for a suitably small $\varepsilon_1 > 0$ since the condition (1.15) given in Theorem 1. **Part (I) of Theorem 1 is proved.**

To prove Part (II), we let $z = C(x, t) + v, w = C(x, t) + s$ for a suitable function $C(x, t)$ in (1.12) and (1.11), we may repeat the process in the proof of (2.3) and (2.6) to obtain

$$\begin{aligned} & v_t + C_t + a(x, t)v_x + b(x, t)v + \left[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''}C_x^2 - \varepsilon C_{xx} + \varepsilon_1 C(x, t)C_x\right] \\ & + \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho C_x - (1 + \varepsilon_1)C(x, t)C_x - \alpha(x, t)u|u| \\ & + [A(x - kt)C(x, t) - C_x](\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\ & - A(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \leq \varepsilon v_{xx}, \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} & s_t + C_t + c(x, t)s_x + d(x, t)s + \left[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''}C_x^2 - \varepsilon C_{xx} + \varepsilon_1 C(x, t)C_x\right] \\ & + C_x\left(\frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho\right) + (1 - \varepsilon_1)C(x, t)C_x \\ & - A(x - kt)\frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}u + \alpha(x, t)u|u| \leq \varepsilon s_{xx}, \end{aligned} \quad (2.32)$$

where $\varepsilon_1 > 0$ is a suitable small constant.

With the help of Lemma 4, we only need to choose a suitable function $C(x, t) \in C_d^2(R)$ and a constant c so that the following inequalities (which are similiar with

(2.10) and (2.11))

$$\begin{aligned}
& C_t + \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho C_x - (1 + \varepsilon_1)C(x, t)C_x \\
& + [A(x - kt)C(x, t) - C_x](\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \\
& - A(x - kt)(\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \geq 0
\end{aligned} \tag{2.33}$$

and

$$C_t + C_x \left(\frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right) + (1 - \varepsilon_1)C(x, t)C_x \geq 0 \tag{2.34}$$

are correct.

The proof of the inequality (2.34) is simple because when $\gamma > 3$ and $k < 0$ satisfies the condition in (II) of Theorem 1, we may choose $C(x, t) = C(x - kt)$, $c = 0$ so that

$$\begin{aligned}
& C_t + C_x \left(\frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right) + (1 - \varepsilon_1)C(x, t)C_x \\
& = (-k + (1 - \varepsilon_1)C(x - kt) + (\rho - 2\delta)\rho^{\frac{\gamma-3}{2}} - \int_{2\delta}^\rho s^{\frac{\gamma-3}{2}} ds - \int_0^{2\delta} s^{\frac{\gamma-3}{2}} ds)C_x \\
& \geq (-k + (1 - \varepsilon_1)C(x - kt) - \frac{1}{\theta}(2\delta)^\theta)C_x \geq 0.
\end{aligned} \tag{2.35}$$

To prove (2.33), we let the left side of (2.33) be L_1 , then

$$\begin{aligned}
L_1 & = \left[\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - \frac{2\delta}{\rho} \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right] C_x \\
& + \frac{2\delta}{\rho} C_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - kC_x - (1 + \varepsilon_1)C(x - kt)C_x \\
& + [A(x - kt)C(x - kt) - C_x](\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} - A(x - kt)(\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \\
& = -\frac{1}{\theta}A(x - kt)(\rho - 2\delta)^2 \rho^{2\theta-2} + (A(x - kt)C(x - kt) - C_x + \frac{C_x}{\theta})(\rho - 2\delta)\rho^{\theta-1} \\
& + \frac{2\delta}{\rho} C_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - kC_x - (1 + \varepsilon_1)C(x - kt)C_x - \frac{2\delta}{\theta}A(x - kt)(\rho - 2\delta)\rho^{2\theta-2}.
\end{aligned} \tag{2.36}$$

Since

$$\frac{2\delta}{\rho} C_x \int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \geq 0, \quad -\frac{2\delta}{\theta}A(x)(\rho - 2\delta)\rho^{2\theta-2} \geq 0, \tag{2.37}$$

we have

$$\begin{aligned}
L_1 &\geq -\frac{1}{\theta}A(x-kt)(\rho-2\delta)^2\rho^{2\theta-2} + (A(x-kt)C(x-kt) - C_x + \frac{C_x}{\theta})(\rho-2\delta)\rho^{\theta-1} \\
&\quad -kC_x - (1+\varepsilon_1)C(x-kt)C_x \\
&= -\frac{1}{\theta}A(x-kt)[(\rho-2\delta)^2\rho^{2\theta-2} + \frac{\theta C_x - C_x - \theta A(x-kt)C(x-kt)}{A(x-kt)}(\rho-2\delta)\rho^{\theta-1} \\
&\quad + (\frac{\theta C_x - C_x - \theta AB}{2A})^2] - kC_x - (1+\varepsilon_1)CC_x + \frac{(\theta C_x - C_x - \theta AB)^2}{4\theta A}.
\end{aligned} \tag{2.38}$$

Since $A(x-kt) \leq 0$ and $C(x-kt) > 0$, then $L_1 \geq 0$ if we let $C_x \geq 0$ satisfy

$$-kC_x - (1+\varepsilon_1)CC_x + \frac{(\theta C_x - C_x - \theta AC)^2}{4\theta A} \geq 0, \tag{2.39}$$

which is equivalent to

$$(\theta-1)^2C_x^2 + \theta^2(AC)^2 - 2\theta(\theta-1)ACC_x - (4\theta k + 4\theta(1+\varepsilon_1)C)AC_x \leq 0 \tag{2.40}$$

or

$$\begin{aligned}
&(\theta-1)^2C_x^2 + \theta^2(AC)^2 + 2\theta(\theta+1)ACC_x - 4\theta\varepsilon_1ACC_x \\
&-4\theta AC(k + (1+\theta)C) \leq 0.
\end{aligned} \tag{2.41}$$

Since $k + (1+\theta)C \leq k + \frac{\gamma+1}{2}M < 0$ and the condition (1.18) in (II) of Theorem 1, (2.41) is correct and so **Part (II) of Theorem 1 is proved.**

3 Proof of Theorem 2.

When $A(x-kt) = A_-(x-kt) + A_+(x-kt)$, where $A_-(x-kt) \leq 0$, $A_+(x-kt) \geq 0$, we may rewrite (2.3) as

$$\begin{aligned}
&v_t + B_t + a(x,t)v_x + b(x,t)v + [-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x,t)B_x] \\
&+ \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B_x - (1-\varepsilon_1)B(x,t)B_x - (\alpha(x,t)|u| + A_+(x-kt)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho})u \\
&+ [A_-(x-kt)B(x,t) - B_x](\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
&- A_-(x-kt)(\rho-2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \leq \varepsilon v_{xx}.
\end{aligned} \tag{3.1}$$

In a similar way to obtain (2.6), we may obtain the following inequality

$$\begin{aligned}
& s_t + B_t + c_1(x, t)s_x + d_1(x, t)s + \left[-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B_x^2 - \varepsilon B_{xx} - \varepsilon_1 B(x, t)B_x\right] \\
& + B_x\left(\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right) + (1 + \varepsilon_1)B(x, t)B_x \\
& - A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}B(x, t) + A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \\
& + (-A_-(x - kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} + \alpha(x, t)|u|)u \leq \varepsilon s_{xx},
\end{aligned} \tag{3.2}$$

where $c_1(x, t) = c(x, t) = u + \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $d_1(x, t) = B_x - A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$.

Therefore, if we may choose a suitable bounded function $B(x, t)$ such that the following inequalities hold

$$\begin{aligned}
& B_t + \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho B_x - (1 - \varepsilon_1)B(x, t)B_x \\
& + [A_-(x - kt)B(x, t) - B_x](\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\
& - A_-(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \geq 0
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
& B_t + B_x\left(\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right) + (1 + \varepsilon_1)B(x, t)B_x \\
& - A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}B(x, t) \\
& + A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \geq 0,
\end{aligned} \tag{3.4}$$

then we have from (3.1) and (3.2) that

$$\begin{aligned}
& v_t + a(x, t)v_x + b(x, t)v \\
& + \frac{1}{2}(\alpha(x, t)|u| + A_+(x - kt)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho})(v - s) \leq \varepsilon v_{xx}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
& s_t + c_1(x, t)s_x + d_1(x, t)s \\
& + \frac{1}{2}(\alpha(x, t)|u| - A_-(x - kt)\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(s - v) \leq \varepsilon s_{xx}.
\end{aligned} \tag{3.6}$$

From the condition (1.19), if we let $B(x, t) = B(x - kt)$, we may prove (3.3) in a similar way like the proof of (2.10).

Under the conditions in (III) of Theorem 2, (3.4) is true because

$$B_x \left(\frac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} - \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \right) \geq 0 \quad (3.7)$$

and

$$\begin{aligned} & B_t + (1 + \varepsilon_1)B(x, t)B_x \\ & - A_+(x - kt)(\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} B(x, t) \\ & + A_+(x - kt)(\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho} \int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \geq (-k + (1 + \varepsilon_1)B(x - kt))B_x \\ & - A_+(x - kt)B(x - kt)f(\rho) + A_+(x - kt)f^2(\rho) \quad (3.8) \\ & = (-k + (1 + \varepsilon_1)B(x - kt))B_x - \frac{1}{4}A_+(x - kt)B^2(x - kt) \\ & + A_+(f(\rho) - \frac{1}{2}B(x - kt))^2 \\ & \geq (-k + (1 + \varepsilon_1)B(x - kt))B_x - \frac{1}{4}A_+(x - kt)B^2(x - kt) \geq 0 \end{aligned}$$

due to the condition (1.20), where $f(\rho) = (\rho - 2\delta) \frac{\sqrt{P'(\rho)}}{\rho}$. Thus **Part (III) of Theorem 2 is proved**. Similarly we may prove Part (IV) of Theorem 2 and complete the proof of Theorem 2.

4 Proof of Theorem 3.

After we have the upper estimates on the Riemann invariants obtained in Theorems 1-2, we have the following estimates on $(\rho^{\delta, \varepsilon}, u^{\delta, \varepsilon})$,

$$2\delta \leq \rho^{\delta, \varepsilon}(x, t) \leq N(x, t), \quad |u^{\delta, \varepsilon}(x, t)| \leq N(x, t), \quad (4.1)$$

where $N(x, t)$ is a positive, bounded function, which depending on the bound of the initial data, but independent of ε, δ .

The local existence result of the Cauchy problem (1.9)-(1.10) can be easily obtained by applying the contraction mapping principle to an integral representation of a solution, following the standard theory of semilinear parabolic systems.

Whenever we have an a priori L^∞ estimate (4.1) of the local solution, it is clear that the local time can be extended to an arbitrary time T step by step since the step time depends only on the L^∞ norm.

Since the original system (1.1) and the approximated system (1.7) have the same entropy equation or the same entropies ([Lu3]), also for any weak entropy-entropy flux pair $(\eta(\rho, u), q(\rho, u))$ of system (1.3), it was proved in [Lu3] that

$$\eta_t(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t)) + q_x(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t)) \quad (4.2)$$

are compact in $H_{loc}^{-1}(R \times R^+)$, then there exists a subsequence of $(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t))$, which converges pointwisely to a pair of bounded functions $(\rho(x, t), u(x, t))$ as δ, ε tend to zero by using the compactness framework given in [D, LPS] for $1 < \gamma < 3$ and in [LPT] for $\gamma \geq 3$. It is easy to prove that the limit $(\rho(x, t), u(x, t))$ satisfies (1.23). Moreover, for any weak convex entropy-entropy flux pair $(\eta(\rho, u), q(\rho, u))$ of system (1.3), we multiply (1.9) by (η_ρ, η_m) to obtain that

$$\begin{aligned} & \eta_t(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t)) + q_x(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t)) + \delta q_{1x}(\rho^{\delta,\varepsilon}(x, t), u^{\delta,\varepsilon}(x, t)) \\ &= \varepsilon \eta(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})_{xx} - \varepsilon (\rho_x^{\delta,\varepsilon}, m_x^{\delta,\varepsilon}) \cdot \nabla^2 \eta(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}) \cdot (\rho_x^{\delta,\varepsilon}, m_x^{\delta,\varepsilon})^T \\ &+ A(x - kt) u^{\delta,\varepsilon} m^{\delta,\varepsilon} \eta_\rho(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}) \\ &+ (A(x - kt) u^{\delta,\varepsilon} m^{\delta,\varepsilon} - \alpha(x, t) m^{\delta,\varepsilon} |u^{\delta,\varepsilon}|) \eta_m(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}) \\ &\leq \varepsilon \eta(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})_{xx} + A(x - kt) u^{\delta,\varepsilon} m^{\delta,\varepsilon} \eta_\rho(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}) \\ &+ (A(x - kt) u^{\delta,\varepsilon} m^{\delta,\varepsilon} - \alpha(x, t) m^{\delta,\varepsilon} |u^{\delta,\varepsilon}|) \eta_m(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}), \end{aligned} \quad (4.3)$$

where $q + \delta q_1$ is the entropy flux of system (1.7) corresponding to the entropy η . Thus the entropy inequality (1.24) is proved if we multiply a test function to (4.3) and let ε, δ go to zero. Thus **Theorem 3 is proved.**

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