REGULARITY OF VISCOUS SOLUTIONS FOR A DEGENERATE NON-LINEAR CAUCHY PROBLEM

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Abstract. We consider the Cauchy problem for a class of nonlinear degenerate parabolic equation with forcing. By using the vanishing viscosity method we obtain generalized solutions. We prove some regularity results about this generalized solutions.

1. Introduction

We consider the Cauchy problem for the following nonlinear degenerate parabolic equation with forcing

\begin{align}
  u_t &= u\Delta u - \gamma|\nabla u|^2 + f(t,u), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,
  
  u(x,0) &= u_0(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),
\end{align}

where \( \gamma \) is a non-negative constant. Equation (1) arises in several applications of biology and physics, see [15], [12]. Equation (1) is of degenerate parabolic type; parabolicity it is loss at points where \( u = 0 \), see [15], [1] for a most detailed description. In [11] a weak solution for the homogeneous equation (1) is constructed by using the vanishing viscosity method, this method was introduced by Lions and Crandall [10], when they studied the existence of solutions to Hamilton-Jacobi equations

\[ u_t + H(x,t,u,Du) = 0 \]

and consists in view the equation (1) as the limit for \( \epsilon \rightarrow 0 \) of the equation

(3) \[ u_t = \epsilon \Delta u + u\Delta u - \gamma|\nabla u|^2 + f(t,u), \]

where \( \epsilon \) is a small positive number. The regularity of the weak solutions for the homogeneous Cauchy problem (1), (2) was studied by the author in [9]. In this paper we extend the above results for the inhomogeneous case, this extension is interesting, from physical viewpoint, since the equation (1) is related with non-equilibrium process in poros media due to external forces. We obtain the following main theorem,

**Theorem 1.1.** If \( \gamma \geq \sqrt{2N-1} \), \( |\nabla (u_0^{1+\frac{\gamma}{2}})| \leq M \), where \( M \) is a positive constant such as

\[ \alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0, \]

then the viscosity solutions of the Cauchy problem (1), (2) satisfies

(4) \[ |\nabla (u^{1+\frac{\gamma}{2}})| \leq M. \]

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2. Preliminaries

**Definition 2.1.** A function \( u \in L^\infty(\Omega) \cap L^2_{\text{loc}}([0, +\infty); H^1_{\text{loc}}(\mathbb{R}^N)) \), is called a weak solution of (1), (2) if it satisfies the following conditions:

(i) \( u(x, t) \geq 0 \), a.e in \( \Omega \).

(ii) \( u(x, t) \) satisfies the following relation

\[
\int_{\mathbb{R}^N} u_0 \psi(x, 0) \, dx + \int_0^t \int_{\Omega} (u \psi_t - u \nabla u \cdot \nabla \psi - (1 + \gamma) |\nabla u|^2 \psi - f(t, u) \psi) \, dx \, dt = 0,
\]

for any \( \psi \in C^{1,1}(\Omega) \) with compact support in \( \overline{\Omega} \).

For the construction of a weak solution to the Cauchy problem (1), (2), we use the viscosity method: we add the term \( \epsilon \Delta u \) in the equation (1) and we consider the following Cauchy problem

\[
u_t = u \Delta u - \gamma |\nabla u|^2 + f(t, u) + \epsilon \Delta u, \quad u \in \Omega,
\]

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N
\]

where \( \gamma \geq 0 \), the existence of solutions is guaranteed by the Maximum principle and then we investigate the convergence of the solutions when \( \epsilon \to 0 \), in fact, we will show that when \( \epsilon \to 0 \), \( u^\epsilon \) converges to the weak solution of (1), (2), but to cost of the loss of the uniqueness.

**Definition 2.2.** The weak solution for the Cauchy problem (1), (2) constructed by the vanishing viscosity method is called viscosity solution.

3. Estimates of Hölder

In this section we begin by collecting some a priori estimates for the function \( u \).

**Theorem 3.1.** If \( \gamma \geq \sqrt{2N} - 1 \), the initial data (1) satisfies \( |\nabla(u_0^{1+\gamma})| \leq M \), where \( M \) is a positive constant, \( \alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0 \) and \( f \in C^1(\mathbb{R}^n \times \mathbb{R}) \) satisfies, \( f \geq 0 \), \( f_0 \leq 0 \), then the viscosity solution \( u(x, t) \) of Cauchy problem (1), (2) satisfies

\[ |\nabla(u^{1+\gamma})| \leq M, \text{ in } \Omega. \]

**Proof.** Let

\[
w = \frac{1}{2} \sum_{i=1}^N u_{x_i}^2.
\]

Deriving with respect to \( t \) in (3) and replacing in (1) we have

\[
w_t = \sum_{i=1}^N u_{x_i} \left[ u_{x_i} \Delta u + u \left( \sum_{j=1}^N u_{x_j x_i} \right) - 2\gamma w_{x_i} + f_{u} u_{x_i} \right].
\]

By other hand

\[
\Delta w = \frac{1}{2} \sum_{j=1}^N \left( \sum_{i=1}^N u_{x_i}^2 \right)_{x_j x_j}
\]

\[
= \frac{1}{2} \left[ \sum_{j=1}^N (2u_{x_1} u_{x_1 x_j})_{x_j} + \sum_{j=1}^N (2u_{x_2} u_{x_2 x_j})_{x_j} + \cdots + \sum_{j=1}^N (2u_{x_N} u_{x_N x_j})_{x_j} \right]
\]

\[
\Delta w = \sum_{i,j=1}^N u_{x_i x_j}^2 + \sum_{i,j=1}^N u_{x_i x_j} u_{x_i x_j},
\]

thereby,

\[
w_t = 2w \Delta u + u \Delta w + \sum_{i,j=1}^N u_{x_i x_j}^2 - 2\gamma \sum_{i=1}^N u_{x_i} w_{x_i} + 2f_{u} w.
\]
From equations (9), (12), (13) we have that,

\[ z = g(u)w. \]

Deriving two times with respect \( x_i \) in (11) we have

\[ w_{x_{i}} = (g^{-1})_{x_{i}}z + g^{-1}z_{x_{i}}, \]

\[ w_{x_{i}x_{i}} = (g^{-1})_{x_{i}x_{i}}z + 2(g^{-1})_{x_{i}x_{i}}z_{x_{i}} + g^{-1}z_{x_{i}x_{i}}. \]

From equations (11), (12), (13) we have that,

\[ \Delta w = \sum_{i=1}^{N} w_{x_{i}x_{i}} = \sum_{i=1}^{N} [(g^{-1})_{x_{i}x_{i}}z + 2(g^{-1})_{x_{i}x_{i}}z_{x_{i}} + g^{-1}z_{x_{i}x_{i}}], \]

Deriving two times with respect \( x_i \) in (11) we have

\[ (g^{-1}(u))_{x_{i}} = -g^{-2}g'_{u_{x_{i}}}, \]

\[ (g^{-1}(u))_{x_{i}x_{i}} = \left( \frac{2g'_{u} - gg''}{g^4} \right) g_{x_{i}x_{i}}^2 - \frac{g'}{g^2}u_{x_{i}x_{i}}, \]

then,

\[ \Delta w = \left( \frac{2g'_{u} - gg''}{g^4} \right) g \sum_{i=1}^{N} u_{x_{i}x_{i}}^2 - \frac{g'}{g^2} \sum_{i=1}^{N} u_{x_{i}x_{i}}^2 - 2g^{-2}g' \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + g^{-1} \sum_{i=1}^{N} z_{x_{i}x_{i}} \]

\[ = g^{-1} \sum_{i=1}^{N} z_{x_{i}x_{i}} - 2g^{-2}g' \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + 2 \left( \frac{2g'_{u} - gg''}{g^4} \right) \sum_{i=1}^{N} u_{x_{i}x_{i}} \]

\[ + \left( \frac{4ug'_{u}^2 - 2ug''}{g^3} + \frac{2ug'_{u}}{g^2} \right) \sum_{i=1}^{N} u_{x_{i}x_{i}}. \]

From (11), (12), (13), (14), we obtain

\[ z_t = u \Delta z - (2g^{-1}ug' + 2\gamma) \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + (2f_{u} + g' g^{-1} f(t, u)) z \]

\[ + \left( \frac{4ug'_{u}^2 - 2ug''}{g^3} + \frac{2ug'_{u}}{g^2} \right) \sum_{i=1}^{N} u_{x_{i}x_{i}}. \]

By choosing \( g(u) = u^\alpha \), and since

\[ \sum_{i,j=1}^{N} u_{x_{i}x_{j}}^2 \geq \frac{1}{N} (\Delta u)^2, \]

replacing \( g \) in (17), (18) we have

\[ z_t \leq u \Delta z - 2(\alpha + \gamma) \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + (2f_{u} + \alpha^{-1} f(t, u)) z \]

\[ + 2\alpha(\alpha + 1 + \gamma)u^{-\alpha-1}z^2 + 2z \Delta u - \frac{u^\alpha}{N} (\Delta u)^2. \]

For \( \gamma \geq \sqrt{2N} - 1 \), if \( \alpha \) satisfies

\[ \alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0, \]

where \( \alpha^2 + (\gamma + 1)\alpha \leq \frac{N}{2} \), then,

\[ 2\alpha(\alpha + 1 + \gamma)u^{-\alpha-1}z^2 + 2z \Delta u - \frac{u^\alpha}{N} (\Delta u)^2 \leq 0. \]
Therefore from (19) and (21) we have
\begin{equation}
(22) \quad z_t \leq u \Delta z - 2(\alpha + \gamma) \sum_{i=1}^{N} u_{x_i} z_{x_i} + (2f_u + \alpha u^{-1} f(t,u))z.
\end{equation}

By an application of the maximum principle in (22) we have
\begin{equation}
|z|_{\infty} \leq |z_0|_{\infty}.
\end{equation}

Now, from (8), (11), with \( g(u) = u^\alpha \), since the initial data (2) satisfies
\begin{equation}
|\nabla (u_0^{1+\frac{\alpha}{2}})| \leq M,
\end{equation}
with \( M \) a positive constant and \( \alpha \) satisfies (20), we have
\begin{align*}
|\nabla (u^{1+\frac{\alpha}{2}})|^2 &= \left| \sum_{i=1}^{N} (u^{1+\frac{\alpha}{2}})_{x_i} e_i \right|^2 \\
&= \sum_{i=1}^{N} \left( (u^{1+\frac{\alpha}{2}})_{x_i} \right)^2 \\
&= \sum_{i=1}^{N} \left( (1 + \frac{\alpha}{2}) u^\frac{\alpha}{\alpha} u_{x_i} \right)^2 \\
&= \left( 1 + \frac{\alpha}{2} \right)^2 \alpha u^\alpha \sum_{i=1}^{N} u_{x_i}^2 \\
&= 2 \left( 1 + \frac{\alpha}{2} \right)^2 \alpha u^\alpha w \\
&= 2 \left( 1 + \frac{\alpha}{2} \right)^2 z,
\end{align*}

therefore
\begin{equation}
|\nabla (u^{1+\frac{\alpha}{2}})| \leq M.
\end{equation}
\hfill \Box

4. Hölder Continuity of \( u(x,t) \)

Now, using Theorem 3.1, we have the following corollary about the regularity of the viscosity solution \( u(x,t) \) to the Cauchy problem (1), (2).

**Corollary 4.1.** Let \( f \) be a continuous function such that
\begin{equation} |f(t,w)| \leq k|w|^m, \end{equation}
where \( w \) is a real valued function and \( m, k \) non-negative constants. Under conditions of the Theorem 3.1 the viscosity solution \( u(x,t) \) of the Cauchy problem (1), (2) is Lipschitz continuous with respect to \( x \) and locally Hölder continuous with exponent \( \frac{1}{2} \) with respect to \( t \) in \( \Omega \).

**Proof.** From Theorem 3.1 there exists \( \alpha \in \mathbb{R} \) with \( \alpha^2 + (\gamma + 1)\alpha + \frac{\gamma}{2} \leq 0 \), with \( \alpha < 0 \), or,
\begin{equation}
-\frac{\sqrt{(\gamma + 1)^2 - 2N}}{2} - \frac{\gamma + 1}{2} \leq \alpha \leq -\frac{\gamma + 1}{2} + \frac{\sqrt{(\gamma + 1)^2 - 2N}}{2} < 0.
\end{equation}

Since \( \alpha < 0 \), taking \( \alpha = -2 \), we have the estimate,
\begin{align*}
|\nabla (u^{1+\frac{\alpha}{2}})| &= \left| (1 + \frac{\alpha}{2}) u^\frac{\alpha}{\alpha} \nabla u \right| \\
&= \left| 1 + \frac{\alpha}{2} \right| u^\frac{\alpha}{\alpha} \nabla u \leq M.
\end{align*}
Now, as $u \geq 0$, we have that
\begin{equation}
|\nabla u| \leq \left| 1 + \frac{\alpha}{2} \right|^{-1} u^{1-\frac{\alpha}{2}} M \leq M_1 \quad \text{in } \Omega,
\end{equation}
since $u$ is bounded.

Using the value mean theorem we have
\begin{equation}
u(x_1, t) - u(x_2, t) = \nabla u(x_1 + \theta(x_2 - x_1), t) \cdot (x_1 - x_2),
\end{equation}
for any $\theta \in (0, 1)$. From (23), (24) we have,
\begin{align*}
|u(x_1, t) - u(x_2, t)| & \leq |\nabla u(x_1 + \theta(x_2 - x_1), t)||x_1 - x_2| \\
& \leq M_1|x_1 - x_2|, \quad \forall (x_1, t), (x_2, t) \in \Omega.
\end{align*}
Therefore $u(x, t)$ is a Lipschitz continuous with respect to the spatial variable.

For Hölder continuity of $u(x, t)$ with respect to the temporary variable, we are going to use the ideas developed in [5]. Let $u_\epsilon(x, t) \in C^{2,1}(\Omega) \cap C(\overline{\Omega}) \cap L^\infty(\Omega)$ the classical solution to the Cauchy problem $\Omega$, (4), namely,
\begin{align*}
\begin{cases}
\frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon - \gamma |\nabla u_\epsilon|^2 + f(t, u_\epsilon) = 0 & \text{in } \Omega \\
u(x, 0) = u_0(x) + \epsilon_\epsilon & \text{on } \mathbb{R}^N.
\end{cases}
\end{align*}
We have that
\begin{align*}
|\nabla (u_0 + \epsilon)^{1+\frac{\alpha}{2}}| &= \left| \left(1 + \frac{\alpha}{2}\right)(u_0 + \epsilon)^{\frac{\alpha}{2}} \nabla u_0 \right| \\
& \leq \left| 1 + \frac{\alpha}{2}(u_0)^{\frac{\alpha}{2}} |\nabla u_0| \right| \\
& \leq M.
\end{align*}
Then, the conditions of Theorem 3.1 holds. Thereby
\begin{equation}
|\nabla (u_0 + \epsilon)^{1+\frac{\alpha}{2}}| \leq M.
\end{equation}
Since $u_\epsilon$ is a classical solution, $u$ is also a weak solution of the Cauchy problem (3). Hence, using the same arguments in the proof of Theorem 3.1, we have that $u_\epsilon$ is a Lipschitz continuous with respect to the spatial variable, with constant $M$, namely
\begin{equation}
|u_\epsilon(x_1, t) - u_\epsilon(x_2, t)| \leq M|x_1 - x_2| \quad \forall (x_1, t), (x_2, t) \in \Omega.
\end{equation}
Now, let $z = u_\epsilon$ be, then we have,
\begin{equation}
z_t = u_\epsilon \Delta u_\epsilon - \gamma |\nabla u_\epsilon|^2 + f(t, u_\epsilon)
\end{equation}
or,
\begin{equation}
u_\epsilon \Delta z - z_t = \gamma |\nabla u_\epsilon|^2 - f(t, u_\epsilon) \quad \text{in } \Omega.
\end{equation}
Using (26) we have that for all $T > 0, R > 0$, $z$ satisfies the equation
\begin{equation}
u_\epsilon \Delta z - z_t = \gamma |\nabla u_\epsilon|^2 - f(t, u_\epsilon) \quad \text{in } B_{2R}(0) \times (0, T],
\end{equation}
where \( B_{2R}(0) \) is the open ball centered in 0, with radius \( 2R \) in \( \mathbb{R}^N \). Noticing that \( u_\epsilon \in C^{2,1}(B_{2R}(0)) \times (0,T] \).

Now, since \( u_\epsilon \) and \( \nabla u_\epsilon \) are bounded in \( \overline{B_{2R}(0)} \times (0,T] \), there exists a constant \( \mu > 0 \), such that
\[
\sum_{i=1}^{N} u_\epsilon(x,t) = Nu_\epsilon(x,t) \leq \mu, \\
\gamma|\nabla u_\epsilon(x,t)| \leq \mu, \quad \forall (x,t) \in B_{2R}(0) \times (0,T],
\]
and
\[
f(t, u_\epsilon) \leq \mu.
\]

From (25), we have also
\[
|z(x_1,t) - z(x_2,t)| \leq M|x_1 - x_2| \quad \forall (x,t) \in \overline{B_{2R}(0)} \times (0,T].
\]
In acording with [5], there exists a positive constant \( \delta \) (which depends only of \( \mu \) and \( R \)) and a positive constant \( K \), which depends only of \( \mu, R \) and \( M \), such that
\[
|z(x,t) - z(x,t_0)| \leq K|t-t_0|^\frac{p}{p-1},
\]
for all \((x,t), (x,t_0) \in B_R(0) \times (0,T] \) with \(|t-t_0| < \delta\).

That is,
\[
|u_\epsilon(x,t) - u_\epsilon(x,t_0)| \leq K|t-t_0|^\frac{p}{p-1},
\]
for all \((x,t), (x,t_0) \in B_R(0) \times (0,T] \) with \(|t-t_0| < \delta\).

Whenever \( K \) is independent of \( \epsilon \), taken \( \epsilon \searrow 0 \), we obtain
\[
|u(x,t) - u(x,t_0)| \leq K|t-t_0|^\frac{p}{p-1},
\]
for all \((x,t), (x,t_0) \in B_R(0) \times (0,T] \) with \(|t-t_0| < \delta\). \( \square \)

References