Global Solutions for a Simplified Shallow Elastic Fluids Model

Yun-guang Lu, Christian Klingenberg, Leonardo Rendon, and De-Yin Zheng

1 Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China
2 Institute of Mathematics, University of Würzburg, Emil Fischer Straße 30, 97074 Würzburg, Germany
3 Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

Correspondence should be addressed to De-Yin Zheng; deyinzheng@yahoo.com.cn

Received 22 September 2013; Accepted 10 February 2014

1. Introduction

In [1], the authors proposed a new reduced model for gravity-driven free-surface flows of shallow elastic fluids. It is obtained by an asymptotic expansion of the upper-convected Maxwell model for elastic fluids. The viscosity is assumed small, but the relaxation time is kept finite. The simplified system of (5.6) in [1] is as follows:

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + \pi)_x &= 0, \\
\left( \rho \frac{\pi}{s^2} \right)_t + \left( \rho u \frac{\pi}{s^2} + u \right)_x &= 0, \\
s_t + us_x &= 0, \\
c_t + uc_x &= 0.
\end{align}

(1)

Since the variable \( c \) in the last equation is independent of the first four equations in (1), we remove it and let \( v = \pi/s^2, s = \text{const.} > 0 \) to obtain the following conservation system:

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + s^2 v)_x &= 0, \\
(\rho v)_t + (\rho u v + u)_x &= 0.
\end{align}

(2)

By simple calculations, three eigenvalues of system (2) are

\begin{align}
\lambda_1 &= u - \frac{s}{\rho}, \\
\lambda_2 &= u, \\
\lambda_3 &= u - \frac{s}{\rho}
\end{align}

(3)

with corresponding Riemann invariants

\begin{align}
R_1 &= s^2 v - su, \\
R_2 &= v + \frac{1}{\rho}, \\
R_3 &= s^2 v + su.
\end{align}

(4)

and all characteristic fields are of the Temple type, namely, the shock waves and rarefaction waves coincide.

In this paper, we will study the Cauchy problem of system (2) with bounded initial data

\begin{align}
(\rho(x,0), u(x,0), v(x,0)) &= (\rho_0(x), u_0(x), v_0(x)), \\
\rho_0(x) &\geq 0
\end{align}

(5)
and prove the existence of global weak solutions by using the vanishing viscosity method coupled with the compensated compactness argument.

The hyperbolic systems of two conservation laws whose shock and rarefaction curves coincide were first described by Temple in [2], and later the global existence of weak solutions for such systems, with any bounded variation initial condition, were proven by Serre in [3] (also [4] for a special system), where the key technique is that the total variation of the Riemann invariants is decreasing in time. Serre’s idea was extended by Heibig [5] to study any \( n \times n \) strictly hyperbolic system of Temple type, with smooth flux functions. For system (2), both the flux functions and the second Riemann invariant \( R_2 \) are singular near the zero layer depth \( \rho = 0 \). So, the technique proposed in [3, 5] cannot be used to study system (2) directly. Fortunately, we may accept the method in [3] to obtain that the total variation, of the first and third Riemann invariants, is decreasing in time, which implies the \( H^{1,1}_\text{loc} \) compactness of \( u_1 \) and \( v_1 \) for the viscosity solutions \((u^\varepsilon, v^\varepsilon)\) given by the Cauchy problem. Then we may apply for the div-curl lemma to the pairs of functions \((c, u^\varepsilon)\) and \((c, v^\varepsilon)\), where \( c \) is a constant, to give a very short proof of the existence of global weak solutions for the Cauchy problem (2) and (5).

**Theorem 1.** Suppose (I)

\[
\frac{1}{\rho_0(x)} + \varepsilon v_0(x) > c_5,
\]

where \( \varepsilon \) is a small positive constant and \( c_i, i = 1, 2, \ldots, 5 \) are suitable constants satisfying

\[
c_5 - \frac{c_4 - c_1}{2s} > 0;
\]

(II) the total variations of \( u_0(x) - s v_0(x) \) and \( u_0(x) + s v_0(x) \) are bounded, then the Cauchy problem (2) and (5) has a bounded measurable solution \((\rho, u, v), \rho \geq 0\) satisfying system (2) in the sense of distributions.

Furthermore when \( F, G, \) and \( H \) are all convex, the entropy is convex.

We will prove Theorems 1 and 2 in Sections 2 and 3, respectively.

### 2. Proof of Theorem 1

We will prove Theorem 1 by several lemmas.

Consider the Cauchy problem for the related parabolic system

\[
\begin{align*}
\rho_t + (\rho u)_x &= \varepsilon \rho_{xx}, \\
(\rho u)_t + (\rho u^2 + s^2) &= \varepsilon (\rho u)_{xx}, \\
(\rho v)_t + (\rho uv + u) &= \varepsilon (\rho v)_{xx},
\end{align*}
\]

with initial data

\[
\begin{align*}
(\rho^\varepsilon(x, 0), u^\varepsilon(x, 0), v^\varepsilon(x, 0)) &= (\rho_0^\varepsilon(x), u_0^\varepsilon(x), v_0^\varepsilon(x)) \\
&= (\rho_0(x) + \varepsilon, u_0(x), v_0(x)) * G^\varepsilon,
\end{align*}
\]

where \( G^\varepsilon \) is a mollifier.

**Lemma 3.** If the initial data satisfy (6) and (7), then for any fixed \( \varepsilon > 0 \), the viscosity solution \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon)\) of the Cauchy problem (10) and (11) exists.

**Proof of Lemma 3.** We substitute the first equation into the second and third equations in (10), respectively, to obtain

\[
R_2 (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) + \lambda_1 R_{xx} (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) = \varepsilon R_{xx} (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) + \frac{2\varepsilon}{\rho} \rho_0 R_{xx} (\rho^\varepsilon, u^\varepsilon, v^\varepsilon),
\]

where \( R_t \) are Riemann invariants given in (4).

Then by applying the maximum principle to (12), we have

\[
\begin{align*}
c_1 &\leq R_1 (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) \leq c_2, \\
c_3 &\leq R_3 (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) \leq c_4,
\end{align*}
\]

due to the initial conditions given in (6), where \( R_i \) are Riemann invariants given in (4).

Using these estimates, we have the following bounded estimates on \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon)\):

\[
\frac{c_3 - c_1}{2s} \leq v^\varepsilon \leq \frac{c_4 - c_1}{2s}, \\
\frac{1}{2s} \leq \frac{c_3 + c_1}{2} \leq u^\varepsilon \leq \frac{c_2 + c_4}{2}, \\
\frac{1}{\rho^\varepsilon} \geq c_5 - \frac{c_4 - c_1}{2s} > 0
\]

due to the condition in (7). Since the boundedness of \( u^\varepsilon \), we have the positive, lower estimate

\[
\rho^\varepsilon(x, t) \geq c(t, \varepsilon) > 0,
\]
by using Theorem 1.0.2 in [6] and the first equation in (10), where \( c(t, \varepsilon) \) could tend to zero as the time \( t \) tends to infinity or \( \varepsilon \) tends to zero.

The standard theory of semilinear parabolic systems, namely, the local existence and the a priori bounded estimates in (14) and (15), gives us the global existence of smooth solution for the Cauchy problem (10) and (11). Lemma 3 is proven.

**Lemma 4.** If the total variations of \( u_0(x) - sv_0(x) \) and \( u_0(x) + sv_0(x) \) are bounded, then the viscosity solutions \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon)\) satisfy

\[
\int_{-\infty}^{\infty} \left| u_x^\varepsilon (x,t) - s v_x^\varepsilon (x,t) \right| dx \leq M, \\
\int_{-\infty}^{\infty} \left| u_x^\varepsilon (x,t) + s v_x^\varepsilon (x,t) \right| dx \leq M, \tag{16}
\]

where \( M \) is a positive constant, which is independent of \( \varepsilon \).

**Proof of Lemma 4.** Let \( \phi = R_{1x} \). Choose \( i = 1 \) in (12). Differentiating (12) with respect to \( x \) and then multiplying the sequence of smooth functions \( h(\phi, \alpha) \) to the result, we have

\[
\phi_t + (\lambda_1 \phi)_x = \varepsilon \phi_{xx} + (2\rho^{-1} \rho_x \phi)_x, \\
h(\phi, \alpha)_x + (\lambda_1 h(\phi, \alpha))_x + h(\phi, \alpha) \phi - h(\phi, \alpha) \lambda_1 x \\
= \varepsilon h(\phi, \alpha)_xx - \varepsilon h''(\phi, \alpha) \phi_x^2 + 2\rho^{-1} \rho_x h(\phi, \alpha)_x \\
+ (2\rho^{-1} \rho_x)_x h(\phi, \alpha) - \varepsilon h(\phi, \alpha)_x. \tag{17}
\]

If we choose a convex \( h(\phi, \alpha), h''(\phi, \alpha) \geq 0 \) and \( h'(\phi, \alpha) \rightarrow \) sign \( \phi, h(\phi, \alpha) \rightarrow |\phi| \) as \( \alpha \rightarrow 0 \), then we let \( \alpha \rightarrow 0 \) in (17) and we have

\[
\phi_t + (\lambda_1 |\phi|)_x = \varepsilon \phi_{xx} + (2\rho^{-1} \rho_x |\phi|)_x \tag{18}
\]

in the sense of distributions. Integrating (18) in \( R \times [0, t] \), we have

\[
\int_{-\infty}^{\infty} \left| R_{1x} \right| (x,t) dx = \int_{-\infty}^{\infty} \left| \phi \right| (x,t) dx \\
\leq \int_{-\infty}^{\infty} \left| \phi \right| (x,0) dx \leq M 
\]

since \( TVR_{1\psi} (x) \) is bounded. So, we proved the first estimate in (16). Similarly we can prove the second estimate in (16). Lemma 4 is proven.

**Lemma 5.** \( f(u^\varepsilon (x,t), v^\varepsilon (x,t))_x \) are compact in \( H^{-1}_{loc}(R \times R^+) \),

\[
\tag{20}
\]

where \( f(u), g(v) \) are any smooth functions.

**Proof of Lemma 5.** Using (16) and (14), we know that \( f(u^\varepsilon (x,t), v^\varepsilon (x,t))_x \) are uniformly bounded both in \( L^1_{loc}(R \times R^+) \) and in \( W^{-1}_{loc}(R \times R^+) \) and so compact in \( H^{-1}_{loc}(R \times R^+) \) due to Murat’s lemma [7]. Lemma 5 is proven.

**Lemma 6.** \( \rho^\varepsilon u^\varepsilon + \rho^\varepsilon v^\varepsilon, (\rho^\varepsilon u^\varepsilon)_t + \rho^\varepsilon (u^\varepsilon)_x + s^2 v^\varepsilon, \rho^\varepsilon (u^\varepsilon)_x + s^2 v^\varepsilon \) are compact in \( H^{-1}_{loc}(R \times R^+) \).

**Proof of Lemma 6.** Multiplying \( l \rho^{-1} (l > 0) \) to the first equation in (10) for simplicity, we omit the superscript \( \varepsilon \), we have

\[
\rho_t^l + \left( \rho^l u^l \right)_x + (l - 1) \rho^l u_x = \varepsilon \rho^l_{xx} - \varepsilon l (l - 1) \rho^l - \rho^l \rho_x^2. \tag{22}
\]

Since \( u_x \) is bounded in \( L^1_{loc}(R \times R^+) \), multiplying a suitable nonnegative test function to (22), we have that

\[
\varepsilon \rho^l - \rho_x^2 \text{ is bounded in } L^1_{loc}(R \times R^+). \tag{23}
\]

So the right-hand side of the first equation in (10) is compact in \( H^{-1}_{loc}(R \times R^+) \), which implies the \( H^{-1}_{loc}(R \times R^+) \) compactness of \( \rho^\varepsilon u^\varepsilon + \rho^\varepsilon v^\varepsilon \).

We substitute the first equation into the second in (10) to obtain

\[
u_t + u u_x + s^2 v_x = \varepsilon u_{xx} + \frac{2\varepsilon}{\rho} \rho_x u_x. \tag{24}
\]

Then we multiply (24) by \( \rho h'(u) \), the first equation in (10) by \( h(u) \), then add the result to obtain

\[
\varepsilon h(u)_x + (\rho h(u) + s^2 h'(u)) v_x = (\rho h(u))_{xx} - \varepsilon h''(u) \rho u_x^2. \tag{25}
\]

Choosing a strictly convex function \( h(u) \) and multiplying a suitable nonnegative test function to (25), we have that

\[
\varepsilon \rho u_x^2 \text{ is bounded in } L^1_{loc}(R \times R^+). \tag{26}
\]

Similarly we can prove that

\[
\varepsilon \rho v_x^2 \text{ is bounded in } L^1_{loc}(R \times R^+). \tag{27}
\]

Using (23), (26), (27), and (20), we may prove the \( H^{-1}_{loc}(R \times R^+) \) compactness of the last two terms in (21). Lemma 6 is proven.

Now we give the proof of Theorem 1. Let \( (\rho, u, v, h(\rho, u^\varepsilon, v^\varepsilon)) \) be the weak star limits of

\[
(\rho^\varepsilon, u^\varepsilon, v^\varepsilon, h(\rho^\varepsilon, u^\varepsilon, v^\varepsilon)).
\]

Using the div-curl lemma [6, 8] to the following pairs of functions, respectively:

\[
(c, u^\varepsilon), (\rho^\varepsilon, \rho^\varepsilon u^\varepsilon); \]

\[
(c, u^\varepsilon), (\rho^\varepsilon u^\varepsilon, \rho^\varepsilon (u^\varepsilon)_x + s^2 v^\varepsilon); \]

\[
(c, v^\varepsilon), (\rho^\varepsilon, \rho^\varepsilon u^\varepsilon); \]

\[
(c, v^\varepsilon), (\rho^\varepsilon u^\varepsilon, \rho^\varepsilon (u^\varepsilon)_x + s^2 v^\varepsilon); \tag{28}
\]
where \( c \) is a constant, we have
\[
\overbrace{\rho^2 u^2} = \rho u, \quad \overbrace{\rho^4 (u^2)} = \rho^2 u^2 \cdot u^2 = \rho u^2, \quad \overbrace{\rho^4 v^2} = \rho v, \quad \overbrace{\rho^2 v^2} = \rho^2 v^2, \quad \overbrace{\rho^2 u^2} = \rho u^2 \cdot \rho v = \rho uv.
\] (29)

Therefore we may prove that the limit \((\rho, u, v)\) satisfies (2) in the sense of distributions by letting \( \varepsilon \) go to zero in (10). Theorem 1 is proven.

### 3. Proof of Theorem 2

In this section, we will prove Theorem 2. It is easy to prove that any pair of entropy-entropy flux \((\eta, q, q, u, v)\), of system (2), must satisfy the following system:
\[
q_p = \frac{\partial \eta}{\partial \rho},
q_u = \frac{\partial \eta}{\partial \rho} \rho u + \frac{1}{\rho} \eta_v,
q_v = \frac{s^2}{\rho} \eta_v + \frac{1}{\rho} \eta_v.
\] (30)

Then we have
\[
q_p = \eta_p + u \eta_p + \rho \eta_{pp} + \rho \eta_{pv},
q_u = \frac{1}{\rho} \eta_p + \frac{1}{\rho} \eta_{pp},
q_v = \frac{s^2}{\rho} \eta_v + \frac{1}{\rho} \eta_v.
\] (31)

Resolving the second equation in (32), we have
\[
\frac{\eta_v}{\rho} = \frac{-\rho \eta_p + \eta + h_2 (u, v)}{\rho} \quad \text{or} \quad \eta_v = \rho \left( \eta - \rho \eta_p \right) + \rho h_2 (u, v),
\] (36)

where \( h_2 \) is an arbitrary function of \((u, v)\).

Differentiating (36) with respect to \( v \) and using the first equation in (32), we have
\[
\frac{\eta_{vv}}{\rho} = \rho \left( \eta_v - \rho \eta_{vp} \right) + \rho h_{2v} (u, v) = \rho \left( \eta_p + \rho h_{2v} \right) (u, v).
\] (37)

Differentiating (34) with respect to \( v \) and differentiating (36) with respect to \( u \) and using the second equation in (32), we have
\[
h_{2u} (u, v) = \frac{s^2}{\rho} h_{1u} (u, v).
\] (38)

Moreover, differentiating (34) with respect to \( v \) and differentiating (36) with respect to \( u \) and using the second equation in (32), we have
\[
h_{2u} (u, v) = h_1 (u, v).
\] (39)

We resolve (38) and (39) to obtain
\[
h_1 (u, v) = f (u + sv) + g (u - sv),

h_2 (u, v) = sf (u + sv) - sg (u - sv),
\] (40)

where \( f, g \) are two arbitrary functions.

Integrating (34) with respect to \( u \), we have
\[
\eta (\rho, u, v) = \rho \left( F (u + sv) + G (u - sv) + h \right) (\rho, v),
\] (41)

where \( F' = f, G' = g \), and \( h \) is a suitable function of \((\rho, v)\) such that \( \eta \) given in (41) must satisfy (36). Thus \( h \) satisfies
\[
h_v (\rho, v) = \rho h (\rho, v) - \rho^2 h_p (\rho, v) \quad \text{or} \quad \left( \frac{h}{\rho} \right)_v + \rho \left( \frac{h}{\rho} \right)_v
\] (42)

and so \( h/\rho = H (v + 1/\rho) \) or \( h (\rho, v) = \rho H (v + 1/\rho) \). Thus (8) is proven.

To prove (9), we remove all the terms on the right-hand side of (12) to obtain
\[
(u - sv)_t + \left( u - s \right) (u - sv)_x = 0,

\left( \frac{1}{\rho} + v \right)_t + u \left( \frac{1}{\rho} + v \right)_x = 0,
\] (43)

\[
(u + sv)_t + \left( u + s \right) (u + sv)_x = 0.
\]

Multiplying the first equation in (43) by \( \rho G' (u - sv) \) and the first equation in (2) by \( G (u - sv) \), then adding the result, we have
\[
(\rho G)_t + ((\rho u - s) G)_x = 0.
\] (44)

Thus the entropy flux corresponding to the entropy \( \rho G (u - sv) \) is \( q_1 = (\rho u - s) G (u - sv) \).
Similarly, we may prove that the entropy flux corresponding to the entropy \( \rho F(u + sv) \) is \( q_2 = (\rho u + s)F'(u + sv) \), the flux corresponding to the entropy \( \rho H(v + 1/\rho) \) is \( q_3 = \rho u H(v + 1/\rho) \). Thus (9) is proven.

Let \( \eta = \rho F(u + sv) \), \( m = \rho u \) and \( n = \rho v \). By simple calculations

\[
\eta_{pp} = \frac{1}{\rho} \left( \frac{m + s n}{\rho} \right)^2 F'', \quad \eta_{mm} = \frac{1}{\rho} F'', \\
\eta_{nn} = \frac{s^2}{\rho} F'', \quad \eta_{pm} = -\frac{m + s n}{\rho^2} F'', \\
\eta_{pn} = -\frac{s m + s n}{\rho^2} F'', \quad \eta_{mn} = \frac{s}{\rho} F''
\]

(45)

and for any vector \((a, b, c)\)

\[
\eta_{pp} a^2 + \eta_{mm} b^2 + \eta_{nn} c^2 + 2\eta_{pm} ab + 2\eta_{pn} ac + 2\eta_{mn} bc \\
= \frac{1}{\rho} F'' \left( \frac{m + sn}{\rho} a - b - sc \right)^2 \geq 0, \quad \text{as} \quad F'' \geq 0.
\]

(46)

Thus \( \eta = \rho F(u + sv) \) is convex when \( F \) is convex. Similarly we can prove both \( \rho G(u - sv) \) and \( \rho H(v + 1/\rho) \) are convex when \( G, H \) are convex. Theorem 2 is proven.

**Conflict of Interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This work was partially supported by the Natural Science Foundation of Zhejiang Province of China (Grant no. LY12A01030 and No. LZ13A010002) and the National Natural Science Foundation of China (Grant no. 11271105).

**References**


