Global weak solutions for a nonlinear hyperbolic system of three equations

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Abstract

In this paper, we study the global existence of weak solutions for the Cauchy problem of the nonlinear hyperbolic system of three equations (1.1) with bounded initial data (1.2). When we fix the third variable $s$, the system about the variables $\rho$ and $u$ is the classical isentropic gas dynamics in Eulerian coordinates with the pressure function $P(\rho, s) = e^{s}e^{-\frac{s}{\rho}}$, which, in general, does not form a bounded invariant region. We introduce a variant of the viscosity argument, and construct the approximate solutions of (1.1) and (1.2) by adding the artificial viscosity to the Riemann invariants system ((2.1)). When the amplitude of the first two Riemann invariants $(w_1(x, 0), w_2(x, 0))$ of system (1.1) is small, $(w_1(x, 0), w_2(x, 0))$ are nondecreasing and the third Riemann invariant $s(x, 0)$ is of the bounded total variation, we obtained the necessary estimates and the pointwise convergence of the viscosity solutions by combining the compensated compactness theory with the approach given by DiPerna in [1].

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1 Introduction

In this paper, we study the global generalized solutions of the nonlinearly conservation system of three equations

\[
\begin{cases}
\rho_t + (\rho u)_x = 0 \\
(\rho u)_t + (\rho u^2 + P(\rho, s))_x = 0 \\
((\rho s)_t + (\rho u s)_x = 0
\end{cases}
\]  

(1.1)

with bounded initial data

\[
(\rho, u, s)|_{t=0} = (\rho_0(x), u_0(x), s_0(x)), \quad \rho_0(x) \geq 0, \quad s_0(x) \geq 0,
\]

(1.2)

where \(P(\rho, s)\) is fixed as \(e^s e^{-\frac{1}{\rho}}\).

Substituting the first equation in (1.1) into the second and the third, we have the following system about the variables \((\rho, u, s)\),

\[
\begin{cases}
\rho_t + u\rho_x + \rho u_x = 0 \\
u_t + \frac{1}{\rho} e^{s-\frac{1}{\rho}}\rho_x + uu_x + \frac{1}{\rho} e^{s-\frac{1}{\rho}}s_x = 0 \\
s_t + us_x = 0.
\end{cases}
\]

(1.3)

Let the matrix \(dF(U)\) be

\[
dF(U) = \begin{pmatrix}
    u & \rho & 0 \\
    \frac{1}{\rho} e^{s-\frac{1}{\rho}} & u & \frac{1}{\rho} e^{s-\frac{1}{\rho}} \\
    0 & 0 & u
\end{pmatrix}.
\]

(1.4)

Then three eigenvalues of (1.1) are

\[
\lambda_1 = u - \frac{1}{\rho} e^{\frac{s-1}{\rho}}, \quad \lambda_2 = u + \frac{1}{\rho} e^{\frac{s-1}{\rho}}, \quad \lambda_3 = u
\]

(1.5)

with corresponding three Riemann invariants

\[
w_1 = u - 2e^{\frac{s-1}{\rho}}, \quad w_2 = u + 2e^{\frac{s-1}{\rho}}, \quad w_3 = s.
\]

(1.6)

Based on the maximum principle on coupled parabolic system given in [2] and the following condition \((H)\), the existence of global smooth solution for the Cauchy problem (1.3) and (1.2) was studied in [3].
(H): \( \rho_0(x), u_0(x), s_0(x) \) are bounded in \( C^1(R) \), and there exists a positive constant \( M \) such that
\[
\begin{aligned}
0 &\leq \frac{d}{dx}(u_0(x) - 2e^{s_0(x)}e^{-\frac{1}{2}\rho_0(x)}) \leq M, \\
0 &\leq \frac{d}{dx}(u_0(x) + 2e^{s_0(x)}e^{-\frac{1}{2}\rho_0(x)}) \leq M, \\
\left|\frac{ds_0(x)}{dx}\right| &\leq M.
\end{aligned}
\] (1.7)

However, when we fix the third variable \( s \), the system about the variables \( \rho \) and \( u \) is the classical isentropic gas dynamics in Eulerian coordinates [4, 5, 6] with the pressure function \( P(\rho, s) = e^s e^{-\frac{1}{2}\rho} \), which, in general, does not form a bounded invariant region [7]. To overcome this difficulty, we have to assume that the amplitude of the first two Riemann invariants \( (w_1(x,0), w_2(x,0)) \) of system (1.1) is suitable small ((1.9)). Mainly we have the following result in this paper.

**Theorem 1** Let \((\rho_0, u_0(x), s_0(x))\) be bounded, \( \rho_0 \geq 0, s_0 \geq 0 \). \( w_1(x,0), w_2(x,0) \) are nondecreasing, and there exist constants \( c_0, c_1, c_2, M \) such that
\[
\begin{aligned}
c_1 &\leq w_1(x,0) \leq c_0, \\
c_0 &\leq w_2(x,0) \leq c_2, \\
\int_R |s(x,0)| dx &\leq M,
\end{aligned}
\] (1.8)

where
\[
0 < c_2 - c_1 < 4 \tag{1.9}
\]

and
\[
w_1(x,0) = u_0(x) - 2e^{s_0(x)}e^{-\frac{1}{2}\rho_0(x)}, \quad w_2(x,0) = u_0(x) + 2e^{s_0(x)}e^{-\frac{1}{2}\rho_0(x)}. \tag{1.10}
\]

Then the Cauchy problem (1.1) and (1.2) has a generalized bounded solution \((\rho(x,t), u(x,t), s(x,t))\) satisfying that the first and the second Riemann invariants \( w_1(x,t), w_2(x,t) \) are nondecreasing and \( \int_R |s_{x}(x,t)| dx \leq M \).

It is worthwhile to point out that a similar result on nonlinear system of three equations was obtained in [8], where the authors studied the global weak solution for the following system
\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho, s))_x &= 0, \\
(\rho s)_t + (\rho u s)_x &= (\frac{1}{\rho} s_x)_x.
\end{aligned}
\] (1.11)
where $P(\rho, s) = e^{(\gamma-1)s} \rho^\gamma$.

This paper is organized as follows: In Section 2, we introduce a variant of the viscosity argument, and construct the approximated solutions of the Cauchy problem (1.1) and (1.2) by using the solutions of the parabolic system (2.1) with the initial data (2.2). Under the conditions in Theorem 1, we can easily obtain the necessary boundedness estimates (2.8), (2.9) and (2.10) on the approximated solutions $(w_{1, \delta}(x, t), w_{2, \delta}(x, t), w_{3, \delta}(x, t))$, where the bound $M(\delta, T)$ in (2.9) could tend to infinity as $\delta$ goes to zero or $T$ goes to infinity. In Section 3, based on the estimates (2.9) and (2.10), we obtain the pointwise convergence of the viscosity solutions $(\rho_{\delta}(x, t), u_{\delta}(x, t), s_{\delta}(x, t))$ by combining the compensated compactness theory [9, 10, 11] with the approach given by DiPerna in [1].

2 Viscosity Solutions

In this section we construct the approximated solutions of the Cauchy problem (1.1) and (1.2) by using the following parabolic systems

$$\begin{cases}
    w_{1t} + \lambda_1 w_{1x} = \varepsilon w_{1xx} \\
    w_{2t} + \lambda_2 w_{2x} = \varepsilon w_{2xx} \\
    w_{3t} + \lambda_3 w_{1x} = \varepsilon w_{3xx}
\end{cases} \quad (2.1)$$

with initial data

$$(w_1(x, 0), w_2(x, 0), w_3(x, 0)) = (w_{10}(x) * G\delta - \delta, w_{20}(x) * G\delta + \delta, s_0(x) * G\delta), \quad (2.2)$$

where $\varepsilon, \delta$ are small positive constants, $G\delta$ is a mollifier, and $(\rho_0, u_0(x), s_0(x))$ are given by (1.2). Thus $w_i(x, 0), i = 1, 2, 3$ are smooth functions, and satisfy

$$c_1 - \delta \leq w_1(x, 0) \leq c_0 - \delta, \quad c_0 + \delta \leq w_2(x, 0) \leq c_2 + \delta, \quad 0 \leq w_3(x, 0) \leq c_3, \quad (2.3)$$

$$0 \leq w_{1x}(x, 0) \leq M(\delta), \quad 0 \leq w_{2x}(x, 0) \leq M(\delta), \quad |w_{3x}(x, 0)| \leq M(\delta) \quad (2.4)$$

and

$$\int_R |w_{3x}(x, 0)| dx \leq M, \quad (2.5)$$

where $M$ is a positive constant being independent of $\delta$ and $M(\delta)$ is a constant, which could tend to infinity as $\delta$ tends to zero.
First, following the standard theory of semilinear parabolic systems, the local existence result of the Cauchy problem (2.1), (2.2) can be easily obtained by applying the contraction mapping principle to an integral representation for a solution.

**Lemma 2** Let \( w_i(x, 0), i = 1, 2, 3 \) be bounded in \( C^1 \) space and satisfy (2.3) and (2.4). Then for any fixed \( \varepsilon > 0, \delta > 0 \), the Cauchy problem (2.1) and (2.2) always has a local smooth solution \( w_i^\varepsilon(\delta)(x, t) \in C^\infty(R \times (0, \tau)) \) for a small time \( \tau \), which depends only on the \( L^\infty \) norm of the initial data \( w_i(x, 0), i = 1, 2, 3 \), and satisfies

\[
0 \leq w_1^\varepsilon(\delta)(x, t) \leq c_0 - \frac{\delta}{2}, \quad c_0 + \frac{\delta}{2} \leq w_2^\varepsilon(\delta)(x, t) \leq c_2 + \frac{3\delta}{2}, \quad |w_3^\varepsilon(\delta)(x, t)| \leq 2c_3, \quad (2.6)
\]

and

\[
|w_1^\varepsilon(\delta)(x, t)| \leq 2M(\delta), \quad |w_2^\varepsilon(\delta)(x, t)| \leq 2M(\delta), \quad |w_3^\varepsilon(\delta)(x, t)| \leq 2M(\delta). \quad (2.7)
\]

Second, by using the maximum principle given in [2], we have the following a priori estimates on the solutions of the Cauchy problem (2.1) and (2.2)

**Lemma 3** Let \( w_i(x, 0), i = 1, 2, 3 \) be bounded in \( C^1 \) space and satisfy (2.3) and (2.4). Moreover, suppose that \( (w_1^\varepsilon(\delta)(x, t), w_2^\varepsilon(\delta)(x, t), w_3^\varepsilon(\delta)(x, t)) \) is a smooth solution of (2.1), (2.2) defined in a strip \( (-\infty, \infty) \times [0, T] \) with \( 0 < T < \infty \). Then

\[
c_1 - \frac{3\delta}{2} \leq w_1^\varepsilon(\delta)(x, t) \leq c_0 - \frac{\delta}{2}, \quad c_0 + \frac{\delta}{2} \leq w_2^\varepsilon(\delta)(x, t) \leq c_2 + \frac{3\delta}{2}, \quad 0 \leq w_3^\varepsilon(\delta)(x, t) \leq c_3, \quad (2.8)
\]

\[
0 \leq w_1^\varepsilon(\delta)(x, t) \leq M(\delta, T), \quad 0 \leq w_2^\varepsilon(\delta)(x, t) \leq M(\delta, T), \quad |w_3^\varepsilon(\delta)(x, t)| \leq M(\delta) \quad (2.9)
\]

and

\[
\int_R |w_3^\varepsilon(\delta)(x, t)|dx \leq M, \quad (2.10)
\]

where the bound \( M(\delta, T) \) could go to infinity as \( \delta \) goes to zero or \( T \) goes to infinity.

**Proof of Lemma 3.** The estimates in (2.8) can be obtained by using the maximum principle to (2.1), (2.2) and the condition (2.3) directly.

We differentiate (2.1) with respect to \( x \) and let \( w_{ix} = \phi_i, i = 1, 2, 3; \) then we have the following parabolic system

\[
\phi_{it} + \lambda_i \phi_{ix} + \left( \sum_{j=1}^3 \lambda_{ij} \phi_j \right) \phi_i = \varepsilon \phi_{ixx}, \quad i = 1, 2, 3. \quad (2.11)
\]
The nonnegativity $w_{1x}^\epsilon \delta(x,t) \geq 0, w_{2x}^\epsilon \delta(x,t) \geq 0$ in (2.9) can be obtained by using the maximum principle to the first and second equations in (2.11) and the condition $w_{1x}(x,0) \geq 0, w_{2x}(x,0) \geq 0$ in (2.4).

Since $\lambda_3 = u, w_3 = s$, the third equation in (2.11) is
\[
\phi_{3t} + u \phi_{3x} + u_x \phi_3 = \epsilon \phi_{3xx}.
\]

(2.12)

Since $u_x = \phi_1 + \phi_2 \geq 0$, by using the maximum principle to (2.12) and the condition $|w_{3x}(x,0)| \leq M(\delta)$ in (2.4), we can easily prove the estimate $|w_{3x}^\epsilon \delta(x,t)| \leq M(\delta)$ in (2.9).

To prove the left estimates in (2.9), we first calculate $\lambda_{iwj}, i = 1, 2, j = 1, 2, 3$. By simple calculations, we have from the Riemann invariants given in (1.6)
\[
\begin{align*}
\lambda_{1w1} &= \lambda_{1u} w_{1w} + \lambda_{1\rho} w_{1w} + \lambda_{1s} w_{1w} = \frac{1}{4\rho}, \\
\lambda_{1w2} &= \frac{1}{4\rho}, \\
\lambda_{1w3} &= -\frac{1}{2\rho} \epsilon \phi_1 e^{-\frac{1}{4\rho}},
\end{align*}
\]

and so
\[
\begin{align*}
\lambda_{2w1} &= 1 - \frac{1}{4\rho}, \\
\lambda_{2w2} &= \frac{1}{4\rho}, \\
\lambda_{2w3} &= \frac{1}{2\rho} \epsilon \phi_1 e^{-\frac{1}{4\rho}}.
\end{align*}
\]

(2.14)

Then the first and second equations in (2.11) are
\[
\begin{align*}
\phi_{1t} + \lambda_1 \phi_{1x} + (\frac{1}{4\rho} \phi_1 + (1 - \frac{1}{4\rho}) \phi_2) \phi_1 - \frac{1}{2\rho} \epsilon \phi_1 e^{-\frac{1}{4\rho}} \phi_3 \phi_1 &= \epsilon \phi_{1xx}, \\
\phi_{2t} + \lambda_2 \phi_{2x} + ((1 - \frac{1}{4\rho}) \phi_1 + \frac{1}{4\rho} \phi_2) \phi_2 + \frac{1}{2\rho} \epsilon \phi_2 e^{-\frac{1}{4\rho}} \phi_3 \phi_2 &= \epsilon \phi_{1xx}.
\end{align*}
\]

(2.15)

By using the nonnegativity of $\phi_1, \phi_2$, we have from (2.15) that
\[
\begin{align*}
\phi_{1t} + \lambda_1 \phi_{1x} + (\frac{1}{4\rho} \phi_1 - \frac{1}{4\rho}) \phi_2) \phi_1 - \frac{1}{2\rho} \epsilon \phi_1 e^{-\frac{1}{4\rho}} \phi_3 \phi_1 &\leq \epsilon \phi_{1xx}, \\
\phi_{2t} + \lambda_2 \phi_{2x} + (-\frac{1}{4\rho} \phi_1 + \frac{1}{4\rho} \phi_2) \phi_2 + \frac{1}{2\rho} \epsilon \phi_2 e^{-\frac{1}{4\rho}} \phi_3 \phi_2 &\leq \epsilon \phi_{1xx}.
\end{align*}
\]

(2.16)

Let the bound of $|\frac{1}{2\rho} \epsilon \phi_2 e^{-\frac{1}{4\rho}} \phi_1|$ be $M_1(\delta)$, and
\[
\phi_1 = X e^{M_1(\delta)t}, \quad \phi_2 = Y e^{M_1(\delta)t}.
\]

(2.17)
Then we have from (2.16) that
\[
\begin{cases}
X_t + \lambda_1 X_x + \frac{1}{4\rho} e^{M_1(\delta)t}(X - Y)X + (M_1(\delta) - \frac{1}{\delta} e^{\frac{t}{\delta}} e^{-\frac{1}{2\rho} \phi_3}) X \leq \varepsilon X_{xx}, \\
Y_t + \lambda_2 Y_x + \frac{1}{4\rho} e^{M_1(\delta)t}(Y - X)Y + (M_1(\delta) + \frac{1}{\delta} e^{\frac{t}{\delta}} e^{-\frac{1}{2\rho} \phi_3}) Y \leq \varepsilon Y_{xx}.
\end{cases}
\]
(2.18)

By using the nonnegativity of \(X, Y\) again, we have from (2.18) that
\[
\begin{cases}
X_t + \lambda_1 X_x + \frac{1}{4\rho} e^{M_1(\delta)t}(X - Y)X \leq \varepsilon X_{xx}, \\
Y_t + \lambda_2 Y_x + \frac{1}{4\rho} e^{M_1(\delta)t}(Y - X)Y \leq \varepsilon Y_{xx}.
\end{cases}
\]
(2.19)

From the conditions in (2.4), we have
\[
X(x,0) \leq M(\delta), \quad Y(x,0) \leq M(\delta).
\]
(2.20)

Repeating the proof of Lemma 2.4 in [2], where \(\lambda = 0\) in our case, we can obtain by using the maximum principle to (2.19), (2.20) that
\[
X(x,t) \leq M(\delta), \quad Y(x,t) \leq M(\delta) \quad \text{for} \ 0 < t \leq T,
\]
(2.21)

and so
\[
\phi_1 = X e^{M_1(\delta)t} \leq M(\delta) e^{M_1(\delta)t}, \quad \phi_2 = Y e^{M_1(\delta)t} \leq M(\delta) e^{M_1(\delta)t}.
\]
(2.22)

Finally using the same technique to estimate (2.8) in [12], we can prove (2.10) from (2.5), and so complete the proof of Lemma 3.

From the estimates (2.8) and (2.9), we have the following estimates about the functions \((\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, s^{\varepsilon,\delta})\). First, from (2.8), we have
\[
\frac{c_0 + c_3}{2} \leq u^{\varepsilon,\delta} = \frac{w_1^{\varepsilon,\delta} + w_2^{\varepsilon,\delta}}{2} \leq \frac{c_0 + c_2}{2}, \quad 0 \leq s^{\varepsilon,\delta} \leq c_3
\]
(2.23)

and
\[
4e^{\frac{\varepsilon}{\delta}} e^{-\frac{1}{2\rho^2\delta}} = \frac{w_2^{\varepsilon,\delta} - w_1^{\varepsilon,\delta}}{2} \geq 2\delta,
\]
which implies
\[
\rho^{\varepsilon,\delta} \geq c(\delta) > 0
\]
(2.24)

for a suitable constant \(c(\delta)\), which could go to zero as \(\delta\) goes to zero. Moreover,
\[
4e^{\frac{\varepsilon}{\delta}} e^{-\frac{1}{2\rho^2\delta}} \leq c_2 + \delta - (c_1 - \delta) = c_2 - c_1 + 2\delta
\]
or
\[ e^{-\frac{1}{2}\rho_{\epsilon,\delta}} \leq \frac{c_2 - c_1}{4} + \frac{\delta}{2} < c_4 < 1, \]
which implies
\[ \rho_{\epsilon,\delta} < -\frac{1}{2\ln c_4}. \]  
(2.25)

Second, from (2.9), we have
\[ 0 \leq u_{\epsilon,\delta} \leq M(\delta, T), \quad |\rho_{\epsilon,\delta}| \leq M(\delta, T), \quad |s_{\epsilon,\delta}| \leq M(\delta). \]  
(2.26)

Based on the a priori estimates given in (2.8) and (2.9) on the local solution, and the positive lower bound (2.24), we may extend the local time \( \tau \) in Lemma 2 step by step to arbitrary large time \( T \) and obtain the following global existence of solution for the Cauchy problem (2.1) and (2.2).

**Theorem 4** Let \( w_i(x, 0), i = 1, 2, 3 \) be bounded in \( C^1 \) space and satisfy (2.3)-(2.5), then the Cauchy problem (2.1) and (2.2) has a unique global smooth solution satisfying (2.8)-(2.10).

### 3 Proof of Theorem 1.

In this section, by using the method of the compensated compactness [9, 10, 1, 4], we prove Theorem 1.

First, based on the \( BV \) estimate (2.10) on the sequence of functions \( s_{\epsilon,\delta} \), we have the following lemmas

**Lemma 5** For any constant \( c \), \( c_t + s_{\epsilon,\delta} \) and \( s_{\epsilon,\delta} + c_x \) are compact in \( H_{loc}^{-1}(R \times R^+) \).

**Proof of Lemma 5.** Since \( c_t + s_{\epsilon,\delta} \) are bounded in \( L^1_{loc}(R \times R^+) \) from (2.10), hence compact in \( W_{loc}^{-1,\alpha} \) for \( \alpha \in (1, 2) \) by the Sobolev embedding theorem. Noticing that \( c_t + s_{\epsilon,\delta} \) are bounded in \( W^{-1,\infty} \), we get the proof by Murat’s theorem [10] that \( c_t + s_{\epsilon,\delta} \) are compact in \( H_{loc}^{-1}(R \times R^+) \).

From the third equation in (2.1), we have that
\[ s_{\epsilon,\delta} + c_x = -u_{\epsilon,\delta} s_{\epsilon,\delta} + \epsilon s_{\epsilon,\delta}, \]  
(3.1)
where \( u_{\epsilon,\delta} s_{\epsilon,\delta} \) are bounded in \( L^1_{loc}(R \times R^+) \) from (2.10) and (2.23). If we choose \( \epsilon \) to go zero more fast than \( \delta \), then \( \epsilon s_{\epsilon,\delta} \) are compact in \( H_{loc}^{-1}(R \times R^+) \) from the
last estimate in (2.26). Thus $c_t + s^{\varepsilon,\delta}_x$ are compact in $H_{\text{loc}}^{-1}(R \times R^+)$. The proof of Lemma 5 is completed.

Second, we prove the pointwise convergence of $s^{\varepsilon,\delta}$.

**Lemma 6** There exists a subsequence (still labelled) $s^{\varepsilon,\delta}$ such that when $\varepsilon$ goes to zero more fast than $\delta$,

$$s^{\varepsilon,\delta}(x,t) \to s(x,t), \text{ a.e. on } \Omega,$$

(3.2)

where $s(x,t)$ is a bounded function, and $\Omega \subset R \times R^+$ is any bounded open set.

**Proof of Lemma 6.** Using the results in Lemma 5, we may apply the div-curl lemma to the pairs of functions

$$(c, s^{\varepsilon,\delta}), \quad (s^{\varepsilon,\delta}, c)$$

(3.3)

to obtain

$$\overline{s^{\varepsilon,\delta}} \cdot s^{\varepsilon,\delta} = (s^{\varepsilon,\delta})^2,$$

(3.4)

where $\overline{s^{\varepsilon,\delta}}$ denotes the weak-star limit of $s^{\varepsilon,\delta}$, which gives us the pointwise convergence of $s^{\varepsilon,\delta}$. Thus Lemma 6 is proved.

We are going to complete the proof of Theorem 1 by proving the pointwise convergence of $\rho^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$.

Let the matrix $A$ be

$$A = \begin{pmatrix}
  w_{1\rho} & w_{1m} & w_{1s} \\
  w_{2\rho} & w_{2m} & w_{2s} \\
  w_{3\rho} & w_{3m} & w_{3s}
\end{pmatrix} = \begin{pmatrix}
  -\frac{m}{\rho^2} - \frac{1}{\rho} e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} & \frac{1}{\rho} & -e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} \\
  -\frac{m}{\rho^2} + \frac{1}{\rho} e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} & \frac{1}{\rho} & e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} \\
  0 & 0 & 1
\end{pmatrix}.\quad (3.5)
$$

By simple calculations, the inversion of $A$ is

$$A^{-1} = \begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
\end{pmatrix} = \begin{pmatrix}
  -\frac{\rho^2}{2} e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} & \frac{\rho^2}{2} e^{\frac{\varepsilon}{2}} e^{-\frac{1}{2\rho}} & -\rho^2 \\
  \frac{\rho}{2} (1 - me^{-\frac{\varepsilon}{2\rho}}) & \frac{\rho}{2} (1 + me^{-\frac{\varepsilon}{2\rho}}) & -\rho m \\
  0 & 0 & 1
\end{pmatrix}.\quad (3.6)
We multiply (2.1) by $A^{-1}$ to get
\[
\begin{pmatrix}
\rho_t \\
(\rho u)_t \\
s_t
\end{pmatrix} + \begin{pmatrix}
(\rho u)_x \\
(\rho u + P(\rho, s))_x \\
us_x
\end{pmatrix} = \varepsilon \begin{pmatrix}
\rho_{xx} \\
(\rho u)_{xx} \\
\s_{xx}
\end{pmatrix} - \varepsilon(A^{-1})_x \begin{pmatrix}
w_{1x} \\
w_{2x} \\
w_{3x}
\end{pmatrix}. \tag{3.7}
\]

Now we fix $s$ as a constant, and consider the following system about the variables $\rho, u$:
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho, s))_x &= 0.
\end{align*} \tag{3.8}
\]

A pair of smooth functions $(\eta(\rho, u, s), q(\rho, u, s))$ is called a pair of entropy-entropy flux of system (3.8) if $(\eta(\rho, u, s), q(\rho, u, s))$ satisfies the additional system
\[
(q_\rho, q_u) = (\eta_\rho, \eta_u) \cdot \begin{pmatrix}
u \\
P_s(\rho, s) \rho \\
u
\end{pmatrix} \tag{3.9}
\]
or equivalently
\[
q_\rho = u\eta_\rho + \frac{P_s(\rho, s)}{\rho} \eta_u, \quad q_u = \rho\eta_\rho + w_u. \tag{3.10}
\]

Eliminating the $q$ from (3.10), we have
\[
\eta_{pp} = \frac{P_s(\rho, s)}{\rho^2} \eta_{uu}. \tag{3.11}
\]

For any smooth pair of entropy-entropy flux $(\eta(\rho, u, s), q(\rho, u, s))$, we multiply $(\eta_\rho(\rho, u, s), \eta_m(\rho, u, s), \eta_s(\rho, u, s))$ to (3.7), where $m = pu$, to obtain
\[
\eta_\rho(\rho, u, s) + q_\rho(\rho, u, s) + (-q_s(\rho, u, s) + \eta_m P_s(\rho, s) + \eta_s u)s_x = R, \tag{3.12}
\]
where
\[
R = \varepsilon \begin{pmatrix}
\eta_\rho & \eta_m & \eta_s
\end{pmatrix} \begin{pmatrix}
\rho_{xx} \\
(\rho u)_{xx} \\
\s_{xx}
\end{pmatrix} - \varepsilon(A^{-1})_x \begin{pmatrix}
w_{1x} \\
w_{2x} \\
w_{3x}
\end{pmatrix}. \tag{3.13}
\]

By using the estimate (2.10), we have that $(-q_s(\rho, u, s) + \eta_m P_s(\rho, s) + \eta_s u)s_x$ is uniformly bounded in $L^1(R \times R^+)$, and hence compact in $W_{loc}^{-1,\alpha}$ for $\alpha \in (1, 2)$.
by the Sobolev embedding theorem. If we choose \( \varepsilon \) to go zero much fast than \( \delta \), by using the estimates in (2.26), we have that \( R \) is compact in \( H^{-1}_{loc}(R \times R^+) \). Therefore, we have the following Lemma:

**Lemma 7** For any fixed \( s \), let \((\eta(\rho,u,s),q(\rho,u,s))\) be an arbitrary pair of smooth entropy-entropy flux of system (3.8). Then

\[
\eta(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t),s^{\varepsilon,\delta}(x,t)) + q(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t),s^{\varepsilon,\delta}(x,t))_x
\]

are compact in \( H^{-1}_{loc}(R \times R^+) \), where \((\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t),s^{\varepsilon,\delta}(x,t))\) are determined by the Riemann invariants \((w_1,w_2,w_3)\), which are the solution of the Cauchy problem (2.1) and (2.2).

Thus for fixed \( s \), for smooth entropy-entropy flux pairs \((\eta_i(\rho,u,s),q_i(\rho,u,s))\), \(i=1,2\), of system (3.8), the following measure equations or the communicate relations are satisfied

\[
<\nu(x,t),\eta_1(\cdot,s)q_2(\cdot,s) - \eta_2(\cdot,s)q_1(\cdot,s)>
\]

\[
= <\nu(x,t),\eta_1(\cdot,s)><\nu(x,t),\eta_2(\cdot,s)><\nu(x,t),q_1(\cdot,s)>,
\]

where \(\nu(x,t)\) is the family of positive probability measures with respect to the viscosity solutions \((\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},s^{\varepsilon,\delta})\) of the Cauchy problem (2.1) and (2.2).

To prove the pointwise convergence of \(\rho^{\varepsilon,\delta}\) and \(u^{\varepsilon,\delta}\) and to finish the proof of Theorem 1, it is enough to prove that Young measures given in (3.15) are Dirac measures.

To this end, to use the approach in [1], for fixed \( s \), we construct four families of entropy-entropy flux pairs of system (3.8) of Lax’s type in the following special form:

\[
\eta^1_k = e^{kw_2}(a_1(\rho,s) + \frac{b_1(\rho,s,k)}{k}),
\eta^2_k = \eta^1_k(\lambda_2 + \frac{c_1(\rho,s,k)}{k} + \frac{d_1(\rho,s,k)}{k^2});
\]

\[
\eta^2_{-k} = e^{-kw_2}(a_2(\rho,s) + \frac{b_2(\rho,s,k)}{k}),
\eta^1_{-k} = \eta^2_{-k}(\lambda_2 + \frac{c_2(\rho,s,k)}{k} + \frac{d_2(\rho,s,k)}{k^2});
\]

\[
\eta^1_k = e^{kw_1}(a_3(\rho,s) + \frac{b_3(\rho,s,k)}{k}),
\eta^2_k = \eta^1_k(\lambda_1 + \frac{c_3(\rho,s,k)}{k} + \frac{d_3(\rho,s,k)}{k^2});
\]

\[
\eta^1_{-k} = e^{-kw_1}(a_4(\rho,s) + \frac{b_4(\rho,s,k)}{k}),
\eta^1_{-k} = \eta^1_{-k}(\lambda_1 + \frac{c_4(\rho,s,k)}{k} + \frac{d_4(\rho,s,k)}{k^2}),
\]

where \(w_1, w_2\) are the first two Riemann invariants of system (1.1) given by (1.6). Notice that all the unknown functions \(a_i,b_i(i=1,2,3,4)\) are only of a single
variable $\rho$ (for fixed $s$). This special simple construction yields an ordinary differential equation of second order with a singular coefficient $1/k$ before the term of the second order derivative. Then the following necessary estimates for functions $a_i(\rho, s), b_i(\rho, s, k)$ are obtained by the use of the singular perturbation theory of ordinary differential equations:

$$0 < a_i(\rho, s) \leq M, \quad |b_i(\rho, s, k)| \leq M, \quad (3.20)$$

$$0 < c_i(\rho, s) \leq M, \quad \text{or} -M \leq c_i(\rho, s) < 0, \quad |d_i(\rho, s, k)| \leq M \quad (3.21)$$

uniformly for $0 < \rho \leq M_1$, where $i = 1, 2, 3, 4$ and $M$ is a positive constant independent of $k$ (the details can be found in Lemma 10.2.1 in [11]). By using the entropy-entropy flux pairs in (3.16)-(3.19) in combination with the compensated compactness theory and the approach in [1], we obtain the the pointwise convergence of $(\rho^\epsilon, u^\epsilon)$ and hence the proof of Theorem 1.

References


