NOTE

The Vacuum Case in Diperna's Paper

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Lemma 4.1 in [Comm. Math. Phys. 91 (1983)] by R on Diperna pertaining to the vacuum case for an existence proof of polytropic gas dynamics using compensated compactness is incomplete as given. Here we give a quick fix of the lemma plus some generalization. © 1998 Academic Press

1. INTRODUCTION

In Diperna's famous article [1] he gave an elegant proof for the lower bound of the density for the viscously perturbed isentropic gas dynamic equation \( \rho^* \geq c(t, \epsilon) > 0 \), which is based on Lemma 4.1 in [1]. No proof of the lemma is given and as stated it seems not possible to prove it. We believe that actually it is a slip of the pen of Diperna when stating this lemma. We give a proof of his lemma under slightly changed hypotheses which snugly fits into the other results given in [1]. In addition, we will prove the validity of his lemma for more general pressure \( p(\rho) \) than in the polytropic gas dynamic case considered by Diperna.

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2. MAIN RESULTS

Consider the following viscous perturbation of the isentropic gas dynamics equations,

\[
\rho_t + (\rho u)_x = \epsilon \rho_{x x},
\]
\[
(\rho u)_t + (\rho u^2 + p(\rho))_x = \epsilon (\rho u)_{x x},
\]

with initial data,

\[(\rho, \rho u)|_{t=0} = (\rho_0(x), \rho_0(x)u_0(x)),\]

where \((\rho_0(x)u_0(x))\) are obtained by smoothing out a pair of bounded functions \((\rho_0(x), u_0(x))\) \((0 \leq \rho_0(x) \leq M, |u_0(x)| \leq M)\) with a mollifier \(G^*\),

\[(\rho_0^*(x), u_0^*(x)) = (\tilde{\rho}_0(x), \tilde{u}_0(x)) \ast G^*,\]

where

\[
\tilde{\rho}_0(x) = \begin{cases} 
\rho_0(x) + \epsilon, & |x| \leq L, \\
\bar{\rho}, & |x| > L,
\end{cases}
\]

\[
\tilde{u}_0(x) = \begin{cases} 
u_0(x), & |x| \leq L, \\
\bar{u}, & |x| > L.
\end{cases}
\]

Because the generalized solution of hyperbolic conservation laws is defined in a compact set of the plane \(R \times R^+\), we may take \(L\) to be large such that the compact set is contained in the region \(|x| < L\) and \(0 \leq t \leq T\) for some \(T\). \(\bar{\rho} > 0\) and \(\bar{u}\) in (4) and (5) are constants as needed in [1]. Therefore,

\[(\rho_0^*(x), u_0^*(x)) \in C^\infty \times C^\infty,\]

\(\epsilon \leq \rho_0^*(x) \leq M, \quad |u_0^*(x)| \leq M,\)

\[
\lim_{|x| \to \infty} (\rho_0^*(x), u_0^*(x)) = (\tilde{\rho}, \tilde{u}),
\]

\[(\rho_0^*(x) - \tilde{\rho}, u_0^*(x) - \tilde{u}) \in L^2(R) \times L^2(R)\].

By applying the general contraction mapping principle to an integral representation of (1), the following local existence of the Cauchy problems (1) and (2) is obtained:

**Lemma 1.** If \(p(\rho) \in C^1\) and the initial data satisfies the conditions (6) and (7), then for any fixed \(\epsilon\), there exists a smooth solution for the Cauchy problem (1) and (2) in some \(R_s = R \times [0, s]\), which satisfies

\[
\frac{\epsilon}{2} \leq \rho(x, t) \leq M, \quad |u(x, t)| \leq M,
\]
and
\[
\lim_{|x| \to \infty} (\rho(x,t), u(x,t)) = (\bar{\rho}, \bar{u}),
\]  \hspace{1cm} (10)

for any fixed \( t \in [0,s] \), where local time \( s \) depends on the bound of the initial data given in (6).

**Lemma 2.** Let (6) hold and \( p(\rho) \in C^2([0,\infty), p'(\rho) > 0, 2p'(\rho) + \rho p''(\rho) > 0 \) for \( \rho > 0 \) and \( \int_0^\infty \sqrt{p'(\rho)} / \rho \, d\rho = \infty \) for any positive constant \( c \). Suppose that \( (\rho(x,t), \rho(x,t)u(x,t)) \) is a smooth solution of (1) and (2) defined in a strip \( T = R \times [0,t] \), which satisfies
\[
0 < \rho(x,t) < M(\epsilon,t), \quad |u(x,t)| \leq M(\epsilon,t).
\]

Then
\[
0 < \rho(x,t) \leq M, \quad |u(x,t)| \leq M, \quad (11)
\]

if \( \int_0^\infty \sqrt{p'(\rho)} / \rho \, d\rho \leq M \) is finite;

\[
0 < \rho(x,t) \leq M, \quad |\rho u| \leq M, \quad (12)
\]

if \( \int_0^\infty \sqrt{p'(\rho)} / \rho \, d\rho = \infty \) but \( \int_0^\infty \sqrt{p'(\rho)} / \rho \, d\rho \) is finite.

Lemma 2 comes from [3].

Before giving the lower bound of \( \tau \), we first prove Diperna's Lemma 4.1 in [1].

**Lemma 3 (Diperna).** If \( \phi(t) \) is a nonnegative continuous function in \([0,T]\) satisfying
\[
\phi(0) > 0, \quad (13)
\]
\[
\phi(t) - \phi(s) \geq -c_4(t-s)^{1/2}, \quad \text{if } t > s, \quad (14)
\]
\[
\int_0^T \phi^{-\alpha}(t) \, dt \geq c_2, \quad \text{for } \alpha \geq 2, \quad (15)
\]

then \( \phi(t) \geq c_3 \) on the interval \([0,T]\), where \( c_i (i = 1, 2, 3) \) are all positive constants and \( c_3 \) depends on \( c_1, c_2, T, \) and \( \alpha \).

**Proof.** If \( \phi(t) = 0 \) at some points \( t \in (0,T) \), let \( t_1 \leq T \) be the least point such that \( \phi(t) > 0 \) for \( t \in [0,t_1] \) and \( \phi(t_1) = 0 \). Then
\[
\phi(t_1) - \phi(s) \geq -c_4(t_1-s)^{1/2}, \quad (16)
\]
and so
\[ \phi(s) \leq c_1(t_1 - s)^{3/2}, \quad \text{for } 0 \leq s < t_1. \]  
(17)

Thus,
\[ \int_0^{t_1} \phi^{-a}(s) \, ds \geq \int_0^{t_1} c_1^{-a}(t_1 - s)^{-\alpha/2} \, ds = \infty, \]  
(18)

which contradicts (15). The lemma is proved.

We are going to give the lower bound of \( \rho \) following Diperna’s method except correcting a few of what we consider misprints and we are going to extend the result from \( \gamma \)-law gas to general \( p(\rho) \).

Using the normalized convex entropy \( \rho'p(\rho) > 0 \),
\[ \eta = \frac{1}{2} \rho(u - \bar{u})^2 + Q\sigma(\rho, \bar{\rho}), \]
where \( \sigma = \rho p(s)/s^2 \, ds, \; Q\sigma = \sigma(\rho) - \sigma(\bar{\rho}) - \sigma'(\bar{\rho})(\sigma - \bar{\sigma}). \)

We have from (8) and (10),
\[ \int_{-\infty}^{\infty} \frac{1}{2} \rho(u - \bar{\rho})^2 + Q\sigma(\rho, \bar{\rho}) \, dx \leq c. \]  
(19)

We construct a function \( h(\rho) \) in the class of strictly convex, nonnegative \( C^2 \) functions with the following properties,
\[ h(\bar{\rho}) = h'(\bar{\rho}) = 0, \]
\[ h(\rho) = \rho^{-\alpha}, \quad \text{on } \left( 0, \frac{\bar{\rho}}{2} \right) \text{ for some } \alpha \geq 2, \]
and
\[ h \leq c(\rho - \bar{\rho}) \rho \text{ near } \bar{\rho}, \]
\[ \rho^2 h'' \leq c \rho, \quad \text{for } \frac{\bar{\rho}}{2} \leq \rho \leq M^{1/2}, \]
\[ \rho^2 h'' \leq c h(\rho), \quad \text{for } 0 < \rho \leq \frac{\bar{\rho}}{2}. \]

Thus for a solution with velocity bounded by \( M \) we have
\[ h''(\rho) \rho^2 (u - \bar{u})^2 \leq c\left( \rho(u - \bar{u})^2 + h(\rho) \right). \]  
(20)
Multiplying the mass equation (the first equation in (1)) by \( h'(\rho) \) and integrating over \( R \times (0, t) \), we get

\[
h_i + (h \rho u)_x = h'' \rho u_x = \epsilon h' - \epsilon h'' \rho_x^2,
\]

\[
\int_{-\infty}^{\infty} h(\rho) - h(\rho_0) \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h'' \rho_x^2 \, dx \, dt
\]

\[
= \int_0^t \int_{-\infty}^{\infty} h''(\rho) \rho u_x \, dx \, dt
\]

\[
= \int_0^t \int_{-\infty}^{\infty} h''(\rho) \rho u_x (u - \bar{u}) \, dx \, dt + \int_0^t \int_{-\infty}^{\infty} \left( \int_0^\rho h''(\rho) \, d\rho \right) \bar{u} \, dx \, dt
\]

\[
\leq \int_0^t \int_{-\infty}^{\infty} \frac{\epsilon}{2} h'' \rho_x^2 + \frac{2}{\epsilon} h'' \rho_x^2 (u - \bar{u})^2 \, dx \, dt.
\]

Then from (8), (19), and (20),

\[
\int_{-\infty}^{\infty} h(\rho) \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h'' \rho_x^2 \, dx \, dt \leq c + \frac{ct}{\epsilon} + \frac{c}{\epsilon} \int_0^t \int_{-\infty}^{\infty} h(\rho) \, dx \, dt,
\]

(21)

for a suitable positive constant \( c \). Thus,

\[
\int_{-\infty}^{\infty} h(\rho) \, dx + \int_0^t \int_{-\infty}^{\infty} h'' \rho_x^2 \, dx \, dt \leq M(T, \epsilon).
\]

The rest of the proof is the same as DiPerna's proof. Thus we have

**Lemma 4.** Let (6), (7), and (8) hold and \( p(\rho) \in C^2(0, \infty) \), \( p'(\rho) > 0 \), \( 2p'' + pp'' > 0 \) for \( \rho > 0 \), \( \int_0^\rho p'(\rho) / \rho \, d\rho = \infty \), \( \int_0^\rho p''(\rho) / \rho \, d\rho \) is finite for any positive constant \( c \). Then

\[
\rho \geq c(\epsilon, t) > 0,
\]

(22)

for an appropriate function \( c(\epsilon, t) \).

Lemmas 1, 2, and 4 give the following global existence of the Cauchy problem (1) and (2).

**Theorem 1.** Let the conditions in Lemma 4 be satisfied. Then for any fixed \( \epsilon > 0 \), there exists a smooth solution for the Cauchy problem (1) and (2) in \( R_T = R \times [0, T] \) (for arbitrary \( T \)), which satisfies

\[
0 < c(\epsilon, t) \leq \rho(x, t) \leq M, \quad |u(x, t)| \leq M,
\]

\[
\lim_{|x| \to \infty} (\rho(x, t), u(x, t)) = (\bar{\rho}, \bar{u}),
\]
for any fixed $t \in [0, T]$, and
\[(\rho(\cdot, t) - \bar{\rho}, u(\cdot, t) - \bar{u}) \in L^2(R) \times L^2(R).\]

Remark. Conditions (7) and (8) are technical. In [2] they are replaced by a periodicity condition on the initial data for $\gamma$-law gas. In [3], conditions (7) and (8) are omitted, but a stronger condition on $p(\rho)$ is introduced: $p'(\rho) - \rho p''(\rho) > 0$, $p''(\rho) < 0$, for $\rho > 0$.

REFERENCES