EXACT SOLUTION AND A TRULY MULTIDIMENSIONAL GODUNOV SCHEME FOR THE ACOUSTIC EQUATIONS∗

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Abstract. The linearized Euler equations are a valuable system for which the behaviour of their multi-dimensional solutions can be studied explicitly. Here, the exact solution in three spatial dimensions is derived. It is then used to obtain a truly multidimensional Godunov finite volume scheme in two dimensions. Additionally the appearance of logarithmic singularities in the exact solution of the 4-quadrant Riemann Problem in two dimensions is discussed.

Key words. Godunov scheme, multidimensional Riemann Problem, acoustic equations, linearized Euler equations

AMS subject classifications. 35B08, 35B44, 35C05, 35E05, 35E15, 35F40, 35L03, 35Q35, 65M08

1. Introduction. Though the linearized Euler equations cannot grasp the full complexity of inviscid hydrodynamics, they are a paramount example of a hyperbolic system of equations in multiple spatial dimensions. Linearizing

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0$$
$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \text{div} (\rho \mathbf{v} \otimes \mathbf{v} + p) = 0$$

with $p(\rho) = K\rho^\gamma$ around the state $(\rho, \mathbf{v}) = (\bar{\rho}, 0)$ yields

$$\frac{\partial \rho}{\partial t} + \bar{\rho} \text{div} \mathbf{v} = 0 \quad (1)$$
$$\frac{\partial \mathbf{v}}{\partial t} + c^2 \frac{\text{grad} \rho}{\bar{\rho}} = 0 \quad (2)$$

where one defines $c = \sqrt{p'(\bar{\rho})}$. Linearization with respect to a fluid state moving at some constant speed $\mathbf{U}$ can be easily removed or added via a Galilei transform.

The same system can be obtained from the Euler equations endowed with an energy equation

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0$$
$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \text{div} (\rho \mathbf{v} \otimes \mathbf{v} + p) = 0$$
$$\frac{\partial e}{\partial t} + \text{div} (\mathbf{v}(e + p)) = 0$$

with $e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho |\mathbf{v}|^2$. Linearization around $(\rho, \mathbf{v}, p) = (\bar{\rho}, 0, \bar{p})$ yields

$$\frac{\partial \rho}{\partial t} + \bar{\rho} \text{div} \mathbf{v} = 0 \quad (3)$$
$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\text{grad} p}{\bar{\rho}} = 0 \quad (4)$$
$$\frac{\partial p}{\partial t} + \bar{p} c^2 \text{div} \mathbf{v} = 0 \quad (5)$$

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Equations (4)–(5) are (up to rescaling and renaming) the same as (1)–(2). Both can be linearly transformed to the symmetric version

\begin{align*}
\partial_t \mathbf{v} + c \text{grad} \ p &= 0 \\
\partial_t p + c \text{div} \ \mathbf{v} &= 0
\end{align*}

This system is the one to be studied in what follows. \( p \) will be called pressure and \( \mathbf{v} \) the velocity – just to have names. Due to the different linearizations and the symmetrization they are not exactly the physical pressure or velocity any more, but still closely related. These equations describe the time evolution of small perturbations to a constant state of the fluid. Therefore they will be called the equations of linear acoustics. Equations (6)–(7) have been studied among others by [14, 17, 1, 3].

It should be noted that only the equation for the scalar \( p \) is the usual scalar wave equation

\[ \partial_t^2 p - c^2 \Delta p = 0 \]

whereas \( \mathbf{v} \) fulfills

\[ \partial_t^2 \mathbf{v} - c^2 \text{grad} \text{div} \ \mathbf{v} = 0 \]

The identity \( \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v} \) links this operator to the vector Laplacian in 3-d. By (6)

\[ \partial_t (\nabla \times \mathbf{v}) = 0 \]

but \( \nabla \times \mathbf{v} \) needs not be zero initially.

The system (6)–(7) has a low Mach number limit just as the Euler equations (compare [3] and for the Euler case see e.g. [11, 12, 16]). One introduces a small parameter \( \epsilon \) and inserts the scaling \( \epsilon^{-2} \) in front of the pressure gradient in (4) such that the system and its symmetrized version read, respectively,

\begin{align*}
\partial_t \mathbf{v} + \frac{1}{\epsilon^2} \text{grad} \ p &= 0 \\
\partial_t p + c^2 \text{div} \ \mathbf{v} &= 0
\end{align*}

and

\begin{align*}
\partial_t \mathbf{v} + \frac{c}{\epsilon} \text{grad} \ p &= 0 \\
\partial_t p + \frac{c}{\epsilon} \text{div} \ \mathbf{v} &= 0
\end{align*}

The transformation which symmetrizes the Jacobian \( J = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix} \) is

\[ S = \begin{pmatrix} 1 \\ c\epsilon \end{pmatrix} \]

such that \( J = S \begin{pmatrix} 0 & \frac{\epsilon}{c} \\ \frac{1}{c} & 0 \end{pmatrix} S^{-1} \). In multiple spatial dimensions the upper left entry in \( S \) has to be replaced by an appropriate block-identity-matrix. In the following preference is given to the symmetric version if nothing else is stated. Regarding
the low Mach number limit the non-symmetrized version is more natural and will
be reintroduced for studying the low Mach number properties of the scheme. The
variables of the two systems are given the same names to simplify notation.

The three-dimensional solution to (6)–(7) is derived in this paper in Section 2 by
different methods, which are compared. In Section 3 it is applied to a two-dimensional
Riemann problem, and in Section 4 it is used to obtain a finite volume Godunov
method. Conclusions are drawn in Section 5.

2. Exact solution.

2.1. Review of the one-dimensional case. The 1-d system

\[ \partial_t p + c \partial_x v = 0 \] (16)
\[ \partial_t v + c \partial_x p = 0 \] (17)

is easily solved via characteristics

\[ \partial_t (p \pm v) \pm c \partial_x (p \pm v) = 0 \] (18)

The solution at a point \( x \) depends only on initial data at points \( y \) for which \( |y-x| = ct \).

This in general is different in multi-dimensional situations: the solution to the scalar
wave equation (8) at a point \( x \) in even spatial dimensions depends on initial data at
points \( y \) for which \( |x-y| \leq ct \) (cf. [4], Section 2.4.1e). In odd spatial dimensions the
solution depends on data in \( |x-y| = ct \).

The solution to the vector wave equation (9) will turn out to depend on initial
data at points \( y \) for which \( |x-y| \leq ct \) also in odd spatial dimensions.

With respect to the numerics of the 1-d system (16)–(17), one can write down the
exact Godunov scheme by solving the 1-d Riemann problem for this system which is
discussed in [8].

2.2. Multi-dimensional case. In the following, wherever appropriate, the de-
dpendent variables will be named \( q := (v,p) \) and the multi-dimensional system written
as

\[ \partial_t q + (J \cdot \nabla)q = 0 \] (19)

where \( J \) is the vector of the Jacobians into \( x, y \) and \( z \) directions, respectively. For the
system (7)–(6) in 3-d they are

\[ J = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ c & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \] (20)

The three different ways of deriving the solution lead to different, though of course
equivalent shapes of the solution formula. Certain properties of the solution are more
immediately seen in one form, and others in the other form. It thus seems appropriate
to discuss these different approaches.

2.2.1. Fourier transform and Green’s functions. The very standard pro-
dcedure of finding a solution to any linear equation such as (19) is to decompose the
initial data \( q_0(x) \) into Fourier modes

\[ q_0(x) = \frac{1}{(2\pi)^{d/2}} \int d^d k \tilde{q}_0(k) \exp(ik \cdot x) \] (21)
where $d$ is the dimensionality of the space. The coefficients $\tilde{q}_0(k)$ of this decomposition are called the Fourier transform of $q_0$ and $k$ here characterizes the mode. The time evolution of any single mode can be found via the ansatz

$$\exp(-i\omega(k)t + ik \cdot x)$$

where $\omega(k)$ is found from the equations by inserting the ansatz. Applying the procedure to (19) makes $\omega$ appear as the eigenvectors $\omega_n (n = 1, \ldots d + 1)$ of the matrix

$$J \cdot k$$

which for the acoustic system yields $\omega_n \in \{0, \pm c|k|\}$. Note that this matrix also shows up in the study of the Cauchy problem and bicharacteristics (see e.g. [2], VI, §3). One then decomposes every Fourier mode with respect to the (orthonormally chosen) eigenvectors $e_n (n = 1, \ldots, d + 1)$ and evolves every part with its own $\omega_n$. By linearity, the time evolution $q(t, x)$ of the initial data $q_0(x)$ is given by adding all the time evolutions of the individual modes:

$$q(t, x) = \frac{1}{(2\pi)^{d/2}} \int d^d k \sum_{n=1}^{d+1} e_n (e_n \cdot \tilde{q}_0(k)) \exp(-i\omega_n t - ik \cdot x)$$

Given the Fourier transform of any initial data therefore the solution can easily be constructed. Due to the fact that the diagonalization is very simple for the acoustic system, all the formulae can be worked out in full generality, and for the acoustic system one finds

$$p(t, x) = \frac{1}{(2\pi)^{d/2}} \int \left( \tilde{p}_0(k) + \frac{k \cdot \tilde{v}_0(k)}{|k|} \right) \exp(ik \cdot x - ic|k|t)$$

$$+ \frac{k \cdot \tilde{v}_0(k)}{|k|} \exp(ik \cdot x + ic|k|t) \right) \right) d^d k$$

$$v(t, x) = \frac{1}{(2\pi)^{d/2}} \int \left( \tilde{v}_0(k) + \frac{k}{|k|} \right) \exp(ik \cdot x - ic|k|t)$$

$$+ \frac{k}{|k|} \tilde{v}_0(k) \exp(ik \cdot x + ic|k|t) \right) \right) d^d k$$

$$+ \left\{ \tilde{v}_0(k) - \frac{k}{|k|} \cdot \frac{k}{|k|} \tilde{v}_0(k) \right\} \exp(ik \cdot x) \right) \right) d^d k$$

It is however quite inconvenient to first have to Fourier transform the initial data $q_0(x)$. A better decomposition is that of $\delta$-distributions, which is trivial:\n
$$q_0(x) = \int d^d x' \delta(x - x')q_0(x')$$

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The integration sign is meant formally here and denotes the application of the linear functional $\delta$ to the function $q_0$. 

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Again by linearity, the solution \( q(t, x) \) can thus be composed of time evolutions of the \( \delta \)-distributions. These functions deserve a name and are called Green’s functions \( G \); the time evolution of \( \delta(x - x') \) is \( G(t, x; x') \) and therefore the solution will then be written as the convolution

\[
q(t, x) = \int d^d x' G(t, x; x') q_0(x')
\]

of the initial data with Green’s function. The Fourier transformation now has to be performed only once for the \( \delta \)-distribution. Taking up the acoustic equations, after lengthy evaluations of integrals one finds (defining \( \xi := x - x' \) and \( \xi := |\xi| \))

\[
G_\rho(t, x; x') = -\left( \frac{\hat{\rho} + \hat{\nu} \cdot \xi}{ct} \right) \frac{\delta'(\xi - ct)}{\xi}
\]

\[
G_\nu(t, x; x') = \frac{2}{3} \frac{\hat{\nu} \delta(\xi)}{\xi} - \frac{1}{4\pi} \delta(\xi - ct)
\]

\[
-\frac{1}{4\pi} \left( \hat{\nu} - 3 \frac{\hat{v} \cdot \xi \xi}{\xi} \left( \frac{\hat{\theta}(\xi - ct)}{\xi^3} + \frac{\delta(\xi - ct)}{\xi} \right) + \frac{\hat{\nu} \cdot \xi \xi \delta'(\xi - ct)}{\xi} \right)
\]

Here \( \delta' \) denotes the (distributional) derivative of \( \delta \).

After Green’s function is determined, the solution is obtained as a convolution with the initial data. Evaluating the convolution in spherical polar coordinates \( (r, \vartheta, \varphi) \) makes the \( r \)-parts simplify, and one is left with only the integrations over \( \varphi \) and \( \vartheta \), which are abbreviated as

\[
\int d\Omega := \frac{1}{4\pi} \int_0^\pi d\vartheta \sin \vartheta \int_0^\pi d\varphi
\]

i.e. this is the average over the sphere. Such spherical means appear already in the study of the scalar wave equation ([10, 4]). After calculating the integrals one obtains

\[
p(t, x) = \partial_r \left( r \int d\Omega p_0 \right) - \frac{1}{r} \partial_r \left( r^2 \int d\Omega \mathbf{n} \cdot \mathbf{v}_0 \right)
\]

\[
\mathbf{v}(t, x) = \frac{2}{3} \mathbf{v}_0(x) - \frac{1}{r} \partial_r \left( r^2 \int d\Omega p_0 \mathbf{n} \right) + \partial_r \left( r \int d\Omega (\mathbf{v}_0 \cdot \mathbf{n}) \mathbf{n} \right)
\]

\[
- \int d\Omega [\mathbf{v}_0 - 3(\mathbf{v}_0 \cdot \mathbf{n}) \mathbf{n}] - \int_0^{ct} dr \frac{1}{r} \int d\Omega [\mathbf{v}_0 - 3(\mathbf{v}_0 \cdot \mathbf{n}) \mathbf{n}]
\]

The last equation can be rewritten into

\[
\mathbf{v}(t, x) = \mathbf{v}_0(x) - \frac{1}{r} \partial_r \left( r^2 \int d\Omega p_0 \mathbf{n} \right)
\]

\[
+ \int_0^{ct} dr \frac{1}{r} \partial_r \left[ \frac{1}{r} \partial_r \left( r^3 \int d\Omega (\mathbf{v}_0 \cdot \mathbf{n}) \mathbf{n} \right) \right] - r \int d\Omega \mathbf{v}_0
\]

where everything (if not stated explicitly) is understood to be evaluated at \( x + r \mathbf{n} \), and wherever it remains, \( r = ct \) to be taken at the very end. \( \mathbf{n} \) is the unit outward
normal vector in spherical polar coordinates:
\[
\mathbf{n} = \begin{pmatrix}
\cos \varphi \sin \vartheta \\
\sin \varphi \sin \vartheta \\
\cos \vartheta
\end{pmatrix}
\] (41)

More details on the transition from (38) to (40) are given in Appendix A.

There is a number of striking differences to the 1-d case that appear in multiple spatial dimensions. The appearance of an integration over \( \frac{1}{r} \) in (38) or (40) rises the question of whether the integration actually gives finite results. It turns out that often the square bracket (in both integrals) is zero already, such that the integral is harmless. The example in Section 3 below however shows that this is not a general feature and that the presence of this term gives rise to singularities of the solution (as has been observed already in [1] using a different solution strategy).

2.2.2. Via the Helmholtz decomposition. Usage of the Helmholtz decomposition allows to write down a scalar wave equation for \( p \) and for the curl-free part of \( \mathbf{v} \), whereas the time evolution of the curl is given by \( \partial_t (\nabla \times \mathbf{v}) = 0 \). The solution to the scalar wave equation is well-known ([10]) and the solution can then be written down as (compare [5])

\[
p(t, \mathbf{x}) = p_0(\mathbf{x}) + \int_0^{ct} dt \int d\Omega (\text{div} \, \text{grad} \, p_0) - r \int d\Omega \, \text{div} \, \mathbf{v}_0
\] (42)

\[
\mathbf{v}(t, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \int_0^{ct} dt \int d\Omega (\text{grad} \, \text{div} \, \mathbf{v}_0) - r \int d\Omega (\text{grad} \, p_0)
\] (43)

Note that the Helmholtz decomposition does not need to be computed explicitly as the two time evolutions conveniently reassemble into (43). Again all the quantities are to be evaluated at \( \mathbf{x} + r \mathbf{n} \).

Equation (43) makes obvious that any change in time of \( \mathbf{v} \) is a gradient. Indeed, the curl must be stationary due to Eq. (10).

In order to transfer the \( r \)-derivatives in (36)–(40) into the derivative operators in (42)–(43) one uses the Gauss theorem for the sphere of radius \( r \). For example, differentiating

\[
\partial_r \left( r \int d\Omega p_0(\mathbf{x} + r \mathbf{n}) \right)
\] (44)

with respect to \( r \) yields

\[
\partial_r^2 \left( r \int d\Omega p_0(\mathbf{x} + r \mathbf{n}) \right) = \frac{1}{r} \partial_r \left( r^2 \partial_r \int d\Omega p_0(\mathbf{x} + r \mathbf{n}) \right)
\] (45)

by elementary manipulations. Differentiation with respect to \( r \) can be replaced by \( \mathbf{n} \cdot \nabla \) inside the spherical mean:

\[
\frac{1}{r} \partial_r \left( r^2 \partial_r \int d\Omega p_0(\mathbf{x} + r \mathbf{n}) \right) = \frac{1}{r} \partial_r \left( r^2 \int d\Omega \mathbf{n} \cdot \nabla p_0(\mathbf{x} + r \mathbf{n}) \right)
\] (46)

and by Gauss theorem

\[
\frac{1}{r} \partial_r \left( r^2 \int d\Omega \mathbf{n} \cdot \nabla p_0(\mathbf{x} + r' \mathbf{n}) \right) = \frac{1}{r} \int dv' \left( r'^2 \int d\Omega \nabla \cdot \nabla p_0(\mathbf{x} + r' \mathbf{n}) \right)
\] (47)

\[
= r \int d\Omega \nabla \cdot \nabla p_0(\mathbf{x} + r' \mathbf{n})
\] (48)
Integrating over $r$, and evaluating at $r = ct$ yields the sought identity

\begin{equation}
\partial_r \left( r \int d\Omega p_0(x + r n) \right) \bigg|_{r=ct} = p_0(x) + \int_0^{ct} dr \int d\Omega \nabla \cdot \nabla p_0(x + r n)
\end{equation}

In a similar but slightly more complicated way the equivalence of the other terms can be shown.

Formula (38) shows that the solution can be reformulated in a way employing only first spatial derivatives, whereas the other versions make the impression that second derivatives need to be computed. In one-dimensional problems only the values of the initial data appear in the solution formulae, not their derivatives. This is different in multiple dimensions and can already be observed for the scalar wave equation (as discussed e.g. in [4]).

Equation (49) shows that although the right-hand side makes you believe that the solution depends on initial data at $x$, and also on initial data inside all of the ball around $x$ with radius $ct$, the left-hand side makes obvious that this is not true. In particular the value of the initial data at $x$ has no influence on the solution and the explicit appearance of $p_0(x)$ on the other side of the equation was artificial, and is due to the fact that the integral actually contains $-p_0(x)$. The left-hand side shows that the solution depends only on the value and the first derivative of $p_0$ along the boundary of the aforementioned ball. Also it is not any derivative, but it is the normal derivative $\partial_r p_0 = n \cdot \nabla p_0$ which matters.

Similarly, comparison between Equations (43) and (38) shows that the expressions can be simplified. However, now the integral over $r$ is still present in both expressions.

One might wonder whether it is possible, by some manipulation, to rewrite Equation (38) such that it contains only terms involving initial data at $x$ and the boundary of the ball of radius $ct$ around $x$, rather than all of its interior. The following argument shows that this is not possible.

Equation (38) contains an integration over $r$ and further terms. These depend on initial data at $x$ and on initial data and its normal derivatives at $r = ct$. Assume that it is possible to rewrite the integral over $r$ such that it also depends only on initial data and its normal derivatives at $x$ and at $r = ct$. Then it would be possible to vary any initial data in the region $0 < r < ct$ without effect on the solution. This however is not true, because the solution at $x = 0$ and time $ct = D$ of the initial data

\begin{equation}
p_0 = 0
\end{equation}

\begin{equation}
v_0 = e_z C x^3 y^2 z^3 (D^2 - x^2 - y^2 - z^2)^2
\end{equation}

is

\begin{equation}
v(D/c, 0) = \frac{CD^{12}}{46200} e_x
\end{equation}

and at the same time

\begin{equation}
v_0(rn) = e_z C \sin^2 \varphi \cos^3 \varphi \sin^5 \vartheta \cos^3 \vartheta \cdot r^8 (D^2 - r^2)^2
\end{equation}

such that

\begin{equation}
v_0(0) = 0
\end{equation}

\begin{equation}
v_0(rn)|_{r=ct} = 0
\end{equation}

\begin{equation}\partial_r v_0(rn)|_{r=ct} = 0
\end{equation}
The solution depends on the parameter $C$ which only modifies the initial data inside $0 < r < ct$. Therefore, contrary to the scalar wave equation, the acoustic system shows dependence on initial data inside the cone $x + r n$ (or in other words, the timelike past) even for three spatial dimensions.

2.2.3. Via exponential map. Using the exponential map, the solution of

\[
\partial_t \left( \begin{array}{c} v \\ p \end{array} \right) + \left( \begin{array}{c} \mathcal{G} p \\ \mathcal{D} v \end{array} \right) = 0
\]

with $\mathcal{G}$ and $\mathcal{D}$ (not necessarily commuting) operators is given by

\[
\left( \begin{array}{c} u \\ v \end{array} \right) = \exp(-t\mathcal{M}) \left( \begin{array}{c} u \\ v \end{array} \right)_0
\]

with

\[
\mathcal{M} = \left( \begin{array}{cc} 0 & \mathcal{G} \\ \mathcal{D} & 0 \end{array} \right)
\]

such that

\[
\left( \begin{array}{c} v \\ p \end{array} \right) = \left( \sum_{m=0}^{\infty} \frac{\mathcal{G}^m}{(2m)!} v_0 \right) - \left( \sum_{m=0}^{\infty} \frac{\mathcal{D}^m}{(2m)!} p_0 \right)
\]

In the case considered here, $\mathcal{G} = \text{grad}$ and $\mathcal{D} = \text{div}$. One can show the equivalence to the solution derived above by expanding all quantities around $r = 0$, e.g.

\[
p_0(x + r n) = \sum_{m=0}^{\infty} \frac{\partial^m}{m!} p_0(x)
\]

and using the identity

\[
\int d\Omega \, n_i n_j \cdots n_k n_\ell = \frac{1}{(m+1)!} \delta_{ij} \cdots \delta_{k\ell} \quad (m \text{ even})
\]

where $n_j$ denotes the $j$-th component of $n$, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$ the Kronecker symbol and the round brackets denote a symmetrization, i.e. a sum over all the permutations (without any prefactor included). Integration over an odd number of factors yields zero by considerations of symmetry. In case of an even number there is no other tensor available than the identity $\delta$, and the result has to remain unchanged with respect to the exchange of any two indices, which explains the symmetrization. The prefactor can be determined by evaluating the integral in some special case that is easy to check explicitly, e.g.

\[
\int d\Omega \, n_z n_z \cdots n_z n_z = \frac{1}{m+1}. \quad \text{The approach of writing the solution by employing the exponential map is used in [17] as a guideline for numerical solutions, but the authors do not derive the solution formulae.}
\]
As an example consider (40) with only the initial data in \( v_0 \). Here again (upper and lower) indices denote the different components

\[
(63) \\
v'(t, x) = v_0'(x) + \int_0^{ct} \frac{dr}{r} \partial_r \left[ \frac{1}{r} \partial_r \left( r^3 \int d\Omega(v_0^k n_k)n^i \right) - r \int d\Omega v_0^i \right]
\]

\[
(64) = v_0'(x) + \sum_{m=0}^{\infty} \int_0^{ct} \frac{dr}{r} \partial_r \left[ \frac{1}{r} \partial_r \left( r^{m+3} \int d\Omega \frac{(n^j \partial_j)^m}{m!} (v_0^k n_k)n^i \right) \right. \\
- \left. r^{m+1} \int d\Omega \frac{(n^j \partial_j)^m}{m!} v_0^i \right]
\]

\[
(65) = v_0'(x) + \sum_{m=0}^{\infty} \frac{m+1}{m!} \frac{r^m}{m} \left[ (m+3) \int d\Omega n_k n^i (n^j \partial_j)^m v_0^k - \int d\Omega (n^j \partial_j)^m v_0^i \right]
\]

\[
(66) = v_0'(x) + \sum_{m=1}^{\infty} \frac{r^{2m}}{(2m)!} \partial^i (\partial^a \partial_a)^{m-1} \partial_k v_0^k
\]

In the last equality the identity (62) and

\[
(68) (m+3) \int d\Omega n_k n^i (n^j \partial_j)^m v_0^k = \begin{cases} \\
\frac{1}{m+1} (\partial^a \partial_a)^{m/2} v_0^i + \frac{m}{m+1} \partial^i (\partial^a \partial_a)^{m-1} \partial_k v_0^k & m \geq 2 \\
v_0^i & m = 0
\end{cases}
\]

for \( m \) even was used. Equation (67) is the one from (60) which completes the example. Analogous computations confirm the other parts of the solution.

3. The two-dimensional Riemann problem. As an example, a particular feature of the exact solution (36)–(40), or (42)–(43) of a two-dimensional Riemann problem (in the \( x \)-\( z \)-plane for computational convenience) shall be discussed here. The initial velocity shall be \( v_0 = (0, 0, 1)^T \) in the first quadrant and vanish everywhere else (see Fig. 1). Also everywhere \( p_0 = 0 \).

\[\text{Fig. 1. Left: Setup of the 2-dimensional Riemann Problem. The only non-vanishing initial datum is the x-velocity in the first quadrant, indicated by the arrow. As the problem is linear its magnitude is of no importance and is chosen to be 1. Right: Sketch of the integration domain.}\]

Computing the \( x \)-velocity along \( x = 0, z > 0 \), say, results in the following expres-
\[ v_x(t, z) = \int_0^{ct} \int_0^r \frac{1}{r} \partial_t \left[ \frac{1}{r} \partial_r \left( r^3 \int d\Omega \, v_{0,z} n_z n_x \right) \right] \]  

\[ = \int_0^{ct} \frac{3}{r} \int d\Omega \, v_{0,z} n_z n_x + r \int d\Omega \partial_r v_{0,z} n_z n_x \bigg|_{r=ct} + 4 \int d\Omega v_{0,z} n_z n_x \bigg|_{r=0} \]

Here integration by parts has been used twice.

The initial data \( v_{0,z} \) at \( x + r \mathbf{n} \) have two regions that have to be treated separately (see Fig. 1). If \( \vartheta < \vartheta_0 \), then \( v_{0,z}(r, \vartheta) = 1 \) for all \( \varphi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). Otherwise, they are only non-zero if \( r \) is small enough, i.e. if \( z + r \cos \vartheta > 0 \):

\[ v_{0,z}(r, \vartheta) = 1 - \Theta \left( r + \frac{z}{\cos \vartheta} \right) = \begin{cases} 
0 & r > -\frac{z}{\cos \vartheta} \\
1 & \text{else}
\end{cases} \]

where

\[ \Theta(x - x_0) := \begin{cases} 
1 & x > x_0 \\
0 & \text{else}
\end{cases} \]

Mind for the inequalities that here \( \cos \vartheta < 0 \) because \( \vartheta_0 > \frac{\pi}{2} \). Additionally the initial data vanish again outside \( \varphi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), which will be taken care of by the integration bounds inside the spherical means.

The derivative exists only for \( \vartheta > \vartheta_0 \) and is

\[ \partial_r v_{0,z} = -\delta \left( r + \frac{z}{\cos \vartheta} \right) = -\frac{\delta(\vartheta - \vartheta_0)}{\cos^2 \vartheta} \]

with \( \vartheta_0 := -\frac{z}{r} \).

With this

\[ r \int d\Omega \partial_r v_{0,z} n_z n_x = -r \frac{1}{2\pi z} \cos^3 \vartheta_0 \sin \vartheta_0 = \frac{1}{2\pi} \left( \frac{z}{r} \right)^2 \sqrt{1 - \left( \frac{z}{r} \right)^2} \]

\[ \int d\Omega v_{0,z} n_z n_x = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{\vartheta_0} d\vartheta \sin^2 \vartheta \cos \vartheta \cos \varphi = \frac{1}{2\pi} \frac{1}{3} \left( 1 - \frac{z^2}{r^2} \right)^{3/2} \]

if \( r > z \) and zero otherwise.

\[ \int_0^{ct} \frac{3}{r} \int d\Omega v_{0,z} n_z n_x \]

\[ = \frac{1}{2\pi} \int_0^z \frac{3}{r} \int_0^{\vartheta_0} d\vartheta \sin^2 \vartheta \cos \vartheta + \frac{1}{2\pi} \int_0^{ct} \frac{3}{r} \int_0^{\vartheta_0} d\vartheta \sin^2 \vartheta \cos \vartheta \]

\[ = \frac{1}{2\pi} \int_z^{ct} d\vartheta \frac{1}{r} \left( 1 - \frac{z^2}{r^2} \right)^{3/2} \]

\[ = \frac{1}{2\pi} \left( \frac{1}{3} \frac{z^2}{(ct)^2} \right) \sqrt{1 - \left( \frac{z}{ct} \right)^2} - \frac{4}{3} \sqrt{1 - \left( \frac{z}{ct} \right)^2} + \ln \frac{1 + \sqrt{1 - \left( \frac{z}{ct} \right)^2}}{z} \]

In this paper an index is never meant to indicate a derivative; \( v_x \) is the \( x \)-component of the velocity \( \mathbf{v} \).
On total this gives

\[ v_x(t, z) = \frac{1}{2\pi} \ln \frac{1 + \sqrt{1 - \left(\frac{z}{ct}\right)^2}}{\frac{z}{ct}} = \frac{1}{2\pi} \mathcal{L} \left( \frac{z}{ct} \right) \]  

having defined

\[
\mathcal{L}(s) := \ln \frac{1 + \sqrt{1 - s^2}}{s} = -\ln \frac{s}{2} - \frac{s^2}{4} + O(s^4)
\]

One can verify that $e^{-\mathcal{L}(s)} = \tan \frac{\arcsin s}{2}$.

Of course the above formulae are valid for $z < ct$ only and $v_x(t, z)$ vanishes outside $|x| \leq ct$ by causality.

Therefore the $x$-component of the velocity has a logarithmic singularity at $z = 0$, the corner of the initial discontinuity of the $z$-component. Such a behaviour of the solution does not have analogs in one spatial dimension. This has already been mentioned in [1] in the context of self-similar solutions to Riemann Problems. Here it has been obtained by application of the general formula (36)–(40) which is not restricted to self-similar time evolution.

For Fig. 2–3 the integrals in (36)–(40) have been computed numerically at a finite number of points in the $x$-$y$-plane. The main difficulty is the appearance of derivatives of discontinuous functions, which has been solved by smoothing them out by a small amount. The ordinary derivative then approximates the $\delta$-distributions that appear in an exact calculation. This explains the smoothing of the jumps and the singularity in the plots.

A vector plot of the flow is shown in Fig. 3.

4. Godunov finite volume scheme.

4.1. Procedure. In this section we aim at deriving a two-dimensional finite volume scheme, which updates the numerical solution $q_{ij}^n$ in a Cartesian cell $C_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ at a time $t^n$ to a new solution $q_{ij}^{n+1}$ at time $t^{n+1} = t^n + \Delta t$ using $q_{ij}^n$ and information from the neighbours of $C_{ij}$. The grid locations are taken equidistant, such that $\Delta x := x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \text{const}$, and $\Delta y := y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = \text{const}$.

The knowledge of the exact solution makes it possible to derive a Godunov scheme.

This is similar in spirit to an idea by Gelfand mentioned in [8, 6] (a translated version...
Fig. 3. The direction of the velocity $v(t, x)$ is indicated by the arrows, color coded is the absolute value $|v|$.

is [7]), but the absence of a(n accessible) published work or details on the procedure make it hard to compare it to the approach taken here. From the scarce account it seems that an exact solution as presented here was not used and that the present approach is more general, in the sense that it might also be used to derive approximate solvers.

This work restricts itself to a derivation of a Godunov scheme via the procedure of reconstruction-evolution-averaging ([19, 13]). As it is formulated, the derivation encounters technical difficulties, because the solution is needed at all points inside the cell. This means an evaluation of the integrals (36)–(40), or (42)–(43), for basically all $x$ with no direct simplification. It afterwards undergoes the possibly nontrivial task of being integrated over the cell.

Employing the structure of the conservation law allows to rewrite the volume integral into a time integral over the boundary, which is a huge simplification. Now the solution is only needed along the boundary of the cell, and one of the components of $x$ is zero. Still however one needs to evaluate the solution formulae at a continuous set of $x$ values.

In the following it is shown that for linear systems a Godunov scheme can be written down using just one evaluation of the solution formula at a single point in space by suitably modifying the initial data.

Consider a piecewise constant reconstruction of the initial data $q_{ij}$ at some point in time, which then defines the piecewise constant function $q_0$ with $q_0(x) = q_{ij}$ if $x \in C_{ij}$ (Fig. 4). Define the evolution operator $T_t$ such that $(T_t q_0)(t, x)$ satisfies (19) with (at $t = 0$) initial data $q_0(x)$. Define also the sliding average operator

\begin{equation}
(Aq)(x) := \frac{1}{\Delta x \Delta y} \int_{-\Delta x/2}^{\Delta x/2} d\xi \int_{-\Delta y/2}^{\Delta y/2} d\eta \ q(x + s)
\end{equation}

where $s$ is the shift vector $s = (\xi, \eta)^T$. Then we have the following
Fig. 4. Left: Piecewise constant reconstruction. Right: Application of the sliding average to the same data amounts to a bilinear interpolation of the values \( q_{ij} \) interpreted as point values at \( x_{ij} \).

**Lemma 4.1.** The two operators commute:

\[
AT_{\Delta t} q_0 \bigg|_{x_{ij}} = T_{\Delta t} A q_0 \bigg|_{x_{ij}}
\]

**Proof.** By linearity of the equations the time evolution commutes with the sliding average (82).

It thus suffices to find the solution of the problem at time \( \Delta t \) at \( x_{ij} \) taking the sliding-averaged initial data \( A q_0 \). For a 2-d grid this amounts to a bilinear interpolation of the values \( q_{ij} \) taken at points \( x_{ij} \), see Fig. 4. In practice the four quadrants of the integration domain have still to be treated separately.

In short, for linear systems the last two steps of reconstruction-evolution-averaging can be turned around to be reconstruction-averaging-evolution which reduces the derivation effort but does not change the result.

**4.2. Notation.** In order to cope with the lengthy expressions for the numerical scheme, the following notation is used:

\[
[q]_{i+\frac{1}{2}} := q_{i+1} - q_i \quad \{q\}_{i+\frac{1}{2}} := q_{i+1} + q_i
\]

\[
[q]_{i\pm 1} := q_{i+1} - q_{i-1} \quad \{q\}_{i\pm 1} := \{q\}_{i+\frac{1}{2}} + \{q\}_{i-\frac{1}{2}}
\]

The only nontrivial identity is

\[
\{[q]\}_{i\pm \frac{1}{2}} = [q]_{i+\frac{1}{2}} + [q]_{i-\frac{1}{2}} = [q]_{i\pm 1}
\]

For multiple dimensions the notation is combined, e.g.

\[
[q]_{i\pm 1,j\pm 1} = q_{i+1,j+1} - q_{i-1,j+1} - q_{i+1,j-1} + q_{i-1,j-1}
\]

The brackets for different directions commute.

**4.3. Finite volume scheme.** It turns out that the solution (36)–(40) can be evaluated exactly. This has been performed with Mathematica [20] and one obtains
435 the numerical scheme
436 (89) \[ u^{n+1} = u^n_j - \frac{c\Delta t}{2\epsilon \Delta x} \left( [p]_{i\pm 1,j} - \left[ [u]\right]_{i\pm \frac{1}{2},j} \right) \]
437 \[ - \frac{1}{2} \frac{(c\Delta t)^2}{\epsilon^2 \Delta x \Delta y} \left( \frac{1}{4} \left[ [v]_{i,j\pm 1} \right]_{j\pm \frac{1}{2}} - \frac{1}{4} \left[ [p]_{i\pm 1,j} \right]_{j\pm \frac{1}{2}} \right) \]
438\]
439 (90) \[ v^{n+1} = v^n_j - \frac{c\Delta t}{2\epsilon \Delta y} \left( [p]_{i,j\pm 1} - \left[ [v]\right]_{i,j\pm \frac{1}{2}} \right) \]
440 \[ - \frac{1}{2} \frac{(c\Delta t)^2}{\epsilon^2 \Delta x \Delta y} \left( \frac{1}{4} \left[ [u]_{i\pm 1,j} \right]_{j\pm \frac{1}{2}} - \frac{1}{4} \left[ [p]_{i\pm \frac{1}{2},j} \right]_{j\pm 1} \right) \]
441\]
442 (91) \[ p^{n+1} = p^n_{ij} - \frac{c\Delta t}{2\epsilon \Delta x} \left( [u]_{i\pm 1,j} - \left[ [p]\right]_{i\pm \frac{1}{2},j} \right) \]
443 \[ - \frac{1}{2} \frac{(c\Delta t)^2}{\epsilon^2 \Delta x \Delta y} \left( \frac{1}{4} \left[ [u]_{i\pm 1,j} \right]_{j\pm \frac{1}{2}} + \frac{1}{4} \left[ [v]_{i\pm \frac{1}{2},j} \right]_{j\pm 1} - 2 \cdot \frac{1}{2\pi} \left[ \left[ [p]\right]_{i\pm \frac{1}{2}} \right]_{j\pm \frac{1}{2}} \right) \]
444\]
445 Of course this scheme is conservative and can be written as
446 (95) \[ q^{n+1} = q^n - \frac{\Delta t}{\Delta x} \left( \hat{f}(x)_{i+\frac{1}{2},j} - \hat{f}(x)_{i-\frac{1}{2},j} \right) - \frac{\Delta t}{\Delta y} \left( \hat{f}(y)_{i,j+\frac{1}{2}} - \hat{f}(y)_{i,j-\frac{1}{2}} \right) \]
447 because it is a Godunov scheme. One easily can identify the \( x \)-flux through the boundary located at \( x_{i+\frac{1}{2}} \):
448 \[ \hat{f}(x)_{i+\frac{1}{2}} = \frac{1}{2} \epsilon \left( \begin{array}{c} [p]_{i+\frac{1}{2},j} - [u]_{i+\frac{1}{2},j} \\ [u]_{i+\frac{1}{2},j} - [p]_{i+\frac{1}{2},j} \end{array} \right) \]
449 \[ + \frac{1}{2} \frac{c\Delta t}{\Delta y} \cdot \epsilon \left( \begin{array}{c} \frac{1}{4} \left[ [u]_{i+\frac{1}{2},j} \right]_{j\pm \frac{1}{2}} - \frac{1}{4} \left[ [v]_{i+\frac{1}{2},j} \right]_{j\pm 1} + \frac{1}{4} \left[ [p]_{i+\frac{1}{2},j} \right]_{j\pm \frac{1}{2}} \\ 0 \\ \frac{1}{2} \left[ [v]_{i+\frac{1}{2},j} \right]_{j\pm 1} - \frac{1}{2\pi} \left[ [p]_{i+\frac{1}{2}} \right]_{j\pm \frac{1}{2}} \end{array} \right) \]
450\]
451 The corresponding perpendicular flux is its symmetric analogon. The first bracket is easily identified as the flux obtained in a 1-d or dimensionally split situation.
452 The appearance of prefactors which contain \( \pi \) in schemes derived with the exact multi-dimensional evolution operators has already been noticed in [15], but none of the schemes mentioned therein matches the one presented here.
453 For better analysis of the low Mach number limit the scheme is given in Appendix B in the usual variables, i.e. by applying the transformation (15) or, which is equivalent, by replacing \( p \mapsto \frac{p}{\epsilon c} \).
454\]
455 4.4. Numerical results. The scheme (90)–(94) is applied to two test cases. The first one is the Riemann Problem considered analytically in Section 3. The second is a test of the low Mach number abilities of the scheme.
456 4.4.1. Riemann Problem. The initial setup is that of Section 3 (Fig. 1); it is solved on a square grid of 101 \( \times \) 101 cells on a domain that is large enough such that \( \frac{1}{10} \) the disturbance produced by the corner has not reached the boundaries for \( t = 0.25 \).
457 Here, \( c = \epsilon = 1 \). The particular feature of this simulation is that it has been performed using a CFL number of 1. This is not possible with dimensionally split schemes, as the CFL condition there is ([8], Eq. 8.15, p. 63)
458 \[ c\Delta t < \frac{1}{\Delta x} + \frac{1}{\Delta y} \]
459 \]
460 This manuscript is for review purposes only.
which for square grids gives a maximum CFL number of 0.5. As the present scheme is an exact multidimensional Godunov scheme it is stable up to the physical CFL number. The results are shown in Fig. 5.

Fig. 5. Solution of Riemann problem at time $ct = 0.25$ using scheme (90)–(94). Left: Pressure. Center: $x$-velocity. Right: $y$-velocity. Compare the images to Fig. 2. Although the present figure has been obtained with a finite volume scheme, and Fig. 2 by a sampled evaluation of the exact solution, the latter, despite larger computational effort, displays more noise. This is due to the particular choice $CFL = 1$ (see text).

It is well-known that a CFL number of 1 makes the upwind scheme solve Riemann Problems for one-dimensional advection without error. A similar feature is observed in this simulation: for $y > 0.25$ the Riemann Problem in $x$-direction is yet uninfluenced by the corner and the solution is that of the usual one-dimensional Riemann Problem, which decays into two waves. These waves don’t show diffusion. However, just as in the case of the upwind scheme of linear advection this is an academic artefact. Any misalignment of the discontinuity with the grid will destroy this feature and smear out shocks just as any other first order scheme. Therefore the described test is not meant to stress any particularly good resolution of discontinuities, but it is rather meant to show the ability of the scheme to be integrated in a stable way with a CFL number which is double the CFL number possible with its dimensionally split counterpart.

In Fig. 6 the $y$-component of the velocity obtained with the numerical scheme is compared to the analytic solution (80) found in Section 3.

4.4.2. Low Mach number vortex. The second test shows the properties of the scheme in the limit $\epsilon \to 0$. The setup is that of a stationary, divergencefree velocity field and constant pressure:

\begin{align}
  p_0(x) &= 1 \\
  v(x) &= e^{\phi} \begin{cases} 
  \frac{r}{d} & r < d \\
  2 - \frac{r}{d} & d \leq r < 2d \\
  0 & \text{else}
  \end{cases}
\end{align}

The velocity thus has a compact support, which is entirely contained in the computational domain, discretized by $51 \times 51$ square cells. Here $c = 1$ and $d = 0.2$. Zero-gradient boundaries are enforced.

Fig. 7 shows the error norm at time $t = 1$ for different CFL numbers; results for the scheme (90)–(94) are shown as solid lines. As has been stated earlier, it is stable until a CFL number of 1, which is confirmed by a rapid increase of error beyond this value. Additionally, the error drops significantly when the CFL number approaches 1 from below. This drop is more and more abrupt the lower $\epsilon$ is.
The dimensionally split solver is known to display artefacts in the limit $\epsilon \to 0$ (see e.g. [9]). Results obtained with this scheme are shown in the same Figure by dashed lines. For small CFL numbers, the flux (97) approaches the dimensionally split case, which is confirmed experimentally. For the dimensionally split scheme the error does not depend on the CFL number. Also the small stability region $\text{CFL} < 0.5$, as well as the growth of the error for decreasing $\epsilon$ are prominent in the Figure.

This growth is present also for the case $\text{CFL} = 1$ for the scheme presented here. Therefore strictly speaking it is not suitable for the low Mach number regime. At the same time, in comparison to the dimensionally split case, the choice $\text{CFL} = 1$ gives a considerable improvement, as is shown in Fig. 5 for the case $\epsilon = 10^{-2}$. The reason for this behaviour can be understood by considering the modified equation for the scheme (119)–(123) (here it is of advantage to use the scheme in its non-symmetrized form).
512 shape. The modified equation reads

513 \begin{align}
514 \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{\epsilon^2} \partial_x p &= \Delta x \frac{c}{2\epsilon} \partial_x (u + \text{CFL} \, \partial_y v) + O(\Delta x^2) \\
515 \frac{v^{n+1} - v^n}{\Delta t} + \frac{1}{\epsilon^2} \partial_y p &= \Delta x \frac{c}{2\epsilon} \partial_y (\text{CFL} \, \partial_x u + \partial_y v) + O(\Delta x^2) \\
516 \frac{p^{n+1} - p^n}{\Delta t} + c^2 (\partial_x u + \partial_y v) &= \Delta x \frac{c}{2\epsilon} (\partial_x^2 p + \partial_y^2 p) + O(\Delta x^2)
517 \end{align}

518 Here, for simplicity, \( \Delta x = \Delta y \) has been used. One observes that choosing a CFL number of 1 makes the velocity diffusion become a gradient of the divergence. In the limit \( \epsilon \to 0 \) the divergence becomes \( O(\epsilon) \), and thus the highest order terms in the modified equation become \( O(1) \). The low Mach number artefacts are then entirely due to contributions from the \( O(\Delta x^2) \) terms.

5. Conclusions and outlook. This paper presented an exact solution of the linearized Euler equations in three spatial dimensions. It is a paramount example of a solution very different to that of advection, with characteristic cones and spherical means replacing simple advection along a one-dimensional characteristic. On the other hand it also shows differences to the well-understood scalar wave equation, thus emphasizing the additional difficulties when dealing with systems of equations. Three different ways of derivation have been shown. The advantage of doing so is that they make different properties of the solution obvious. Some features are the dependence on first derivatives and function values, rather than function values in the initial data alone, the dependence on initial data that cannot be reached by traveling with one of the characteristic speeds and the appearance of divergent integrals in certain situations.

534 The multi-dimensional Riemann Problem for the acoustic equations is a prominent example for the appearance of such a singularity, which is an intrinsically multi-dimensional feature. This paper presented the exact shape of the solution and showed that the singularity is logarithmic.

538 Furthermore, using the exact solution formulae, a multi-dimensional Godunov scheme has been obtained. It has been found that, as expected, its stability region extends right up to the maximum allowed physical CFL number, which is twice what is possible with a dimensionally split scheme. Additionally, the accuracy increases significantly in the vicinity of a CFL number of 1. The method has been shown to perform well when applied to Riemann Problems, but not to allow calculations in the
Concentrating on initial data in the velocity only the solution (38) reads

\[ v(t, x) = \frac{2}{3} v_0(x) + \partial_r \left( r \int d\Omega (v_0 \cdot n)n \right) - \int d\Omega \left[ v_0 - 3(v_0 \cdot n)n \right] \]

\[ - \int_0^{ct} dr \frac{1}{r} \int d\Omega \left[ v_0 - 3(v_0 \cdot n)n \right] \]

In order to make the case of stationary solutions \( v(t, x) = v_0(x) \) appear explicitly in the formula one might wonder where the remaining \( \frac{1}{3} v_0(x) \) are hiding. Recall that all the functions inside the integrals are to be evaluated at \( x + r n \). One rewrites (using \( \int d\Omega n \otimes n = \frac{1}{3} \text{id} \) from (62) or by explicit calculation)

\[ \int d\Omega (v_0 \cdot n)n \bigg|_{r=ct} - \frac{1}{3} v_0(x) = \int d\Omega (v_0 \cdot n)n \bigg|_{r=ct} - \int d\Omega (v_0 \cdot n)n \bigg|_{r=0} \]

\[ = \int_0^{ct} dr \partial_r \int d\Omega (v_0 \cdot n)n \]

Now one can regroup the terms. By product rule at the location indicated by an arrow

\[ \int_0^{ct} dr \frac{1}{r} \partial_r \left( r \int d\Omega (v_0 \cdot n)n \right) \]

\[ = \int_0^{ct} dr \partial_r \left( r \partial_r \int d\Omega (v_0 \cdot n)n \right) + \int_0^{ct} dr \partial_r \int d\Omega (v_0 \cdot n)n \]

\[ = r \partial_r \int d\Omega (v_0 \cdot n)n \bigg|_{r=ct} + \int_0^{ct} dr \partial_r \int d\Omega (v_0 \cdot n)n \]

which are both terms appearing in the expression for \( v(t, x) \). Here

\[ \lim_{r \to 0} r \partial_r \int d\Omega (v_0 \cdot n)n = 0 \]

was used. Similarly,

\[ \int_0^{ct} dr \frac{1}{r} \partial_r \left( r \int d\Omega (v_0 - 3(v_0 \cdot n)n) \right) \]

\[ = \int_0^{ct} dr \partial_r \int d\Omega (v_0 - 3(v_0 \cdot n)n) + \int_0^{ct} dr \frac{1}{r} \int d\Omega (v_0 - 3(v_0 \cdot n)n) \]

By

\[ \lim_{r \to 0} \int d\Omega (v_0 - 3(v_0 \cdot n)n) = 0 \]
one obtains

\begin{equation}
\mathbf{v}(t, \mathbf{x}) = \frac{2}{3} \mathbf{v}_0(\mathbf{x}) + \frac{1}{3} \mathbf{v}_0(\mathbf{x})
\end{equation}

\begin{equation}
\frac{d}{dt} \int_0^r \frac{1}{r} \partial_r \int d\Omega (\mathbf{v}_0 \cdot \mathbf{n}) d^n - \int_0^r \frac{d}{dt} \frac{1}{r} \partial_r \left( r \int d\Omega [\mathbf{v}_0 - 3(\mathbf{v}_0 \cdot \mathbf{n})] \right)
\end{equation}

\begin{equation}
\mathbf{v}(\mathbf{x}) + \int_0^t \frac{d}{dt} \frac{1}{r} \partial_r \left[ \frac{1}{r} \partial_r \left( r^3 \int d\Omega (\mathbf{v}_0 \cdot \mathbf{n}) \right) - r \int d\tilde{\Omega} \mathbf{v}_0 \right]
\end{equation}

as claimed in (40).

**Appendix B.**

For better comparison to other schemes, here the scheme (90)–(94) is given in the variables prior to symmetrization, i.e. such that it is a numerical approximation to (11)–(12).

\begin{align}
\mathbf{u}^{n+1} &= \mathbf{u}_i^n - \frac{\Delta t}{2\Delta x} \left( \frac{1}{\epsilon^2} [p]_{i,\pm 1,j} - \frac{c}{\epsilon} [\mathbf{v}]_{i,\pm 1,j} \right) \\
&\quad - \frac{\Delta t}{2\Delta y} \left( \frac{1}{\epsilon^2} [\mathbf{v}]_{i,j,\pm 1} - \frac{c}{\epsilon} [\mathbf{v}]_{i,j,\pm 1} \right) \\
\mathbf{v}^{n+1} &= \mathbf{v}_i^n - \frac{\Delta t}{2\Delta y} \left( \frac{1}{\epsilon^2} [p]_{i,j,\pm 1} - \frac{c}{\epsilon} [\mathbf{v}]_{i,j,\pm 1} \right) \\
&\quad - \frac{\Delta t}{2\Delta x} \left( \frac{1}{\epsilon^2} [\mathbf{v}]_{i,\pm 1,j} - \frac{c}{\epsilon} [\mathbf{v}]_{i,\pm 1,j} \right) \\
p^{n+1} &= p_j^n - \frac{\Delta t}{2\Delta x} \left( c^2 [\mathbf{u}]_{i,\pm 1,j} - \frac{c}{\epsilon} [\mathbf{v}]_{i,\pm 1,j} \right) - \frac{\Delta t}{2\Delta y} \left( c^2 [\mathbf{v}]_{i,j,\pm 1} - \frac{c}{\epsilon} [\mathbf{v}]_{i,j,\pm 1} \right) \\
&\quad - \frac{1}{\epsilon^2} [p]_{i,j,\pm 1} + \frac{1}{\epsilon^2} [\mathbf{v}]_{i,j,\pm 1} - \frac{1}{\epsilon^2} [\mathbf{v}]_{i,j,\pm 1} - \frac{1}{\epsilon^2} [\mathbf{v}]_{i,j,\pm 1}
\end{align}

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