A low-Mach Roe-type solver for the Euler equations allowing for gravity source terms

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A time-continuous finite volume scheme for a system of conservation laws \( \partial_t q + \nabla \cdot f(q) = 0 \) with \( q : \mathbb{R}^d \times \mathbb{R}_+^n \to \mathbb{R}^n \)
\( f : \mathbb{R}^n \to \mathbb{R}^n \)
reads (e.g. in \( d = 2 \)):
\[
\partial_t q_{i,j} + \frac{f^{(i)}_j - f^{(s)}_j}{\Delta x} + \frac{f^{(i)}_j - f^{(s)}_j}{\Delta y} = 0
\]
Roe-type schemes:
\[
f_{j+1/2}^{(i)} = \frac{1}{2}(f(q_{i+1}) + f(q_i)) - \frac{1}{2}D_{j+1/2}(q_{i+1} - q_i)
\]
E.g. Roe scheme: \( D_{j+1/2} = \frac{f_j}{f_{j+1} - f_j} \) evaluated at \((q_{i+1}, q_i)\)

Modification of the diffusion matrix

Roe matrix:
\[
D_{\text{Roe}} \in \left( \begin{array}{c|c} 0 & (\frac{1}{2}) \\ \hline (\frac{1}{2}) & 0 \end{array} \right)
\]

We suggest (Miczek+ 15, Barsukow+ in prep.) a \( D \) such that
\[
\begin{array}{c|c} O(1) & O(1) \\ \hline O(1) & O(1) \end{array}
\]

Incompressible limit

For the Euler equations in \( d \) spatial dimensions
\[
n = d + 2 \quad q = (\rho, \rho v, e)^T \quad f = \left( \rho v, \rho v \otimes v + \frac{p}{e}, v(e + p) \right)^T
\]

with \( \epsilon = \frac{p}{\gamma - 1} + \frac{1}{2} \rho |v|^2 \) and the local Mach number \( M = \frac{\sqrt{\rho |v|^2}}{\sqrt{\rho g}} \sim \epsilon \).

Expand quantities as power series in \( \epsilon \), e.g.
\[
p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \ldots
\]

Limit in the continuous case: \textbf{incompressible hydrodynamics} with \( p^{(2)} \) as the dynamic pressure, as well as \( \nabla p^{(2)} = \nabla p^{(3)} = 0 \)

Kinetic energy

The equation for the kinetic energy \( e_{\text{kin}} = \frac{1}{2} \rho v^2 \) can be written as
\[
\partial_t e_{\text{kin}} + \nabla \cdot \left( v \left( e_{\text{kin}} + \frac{p}{\epsilon} \right) \right) = \frac{p}{\epsilon} v \cdot \nabla \epsilon \not\in O(\epsilon).
\]
and is equivalent to
\[
\partial_t e_{\text{kin}} + \nabla \cdot \left( v \left( e_{\text{kin}} + p^{(2)} \right) \right) \in O(\epsilon) \quad (\rho^{(2)} \nabla \cdot v \in O(\epsilon)).
\]
Kinetic energy is a conserved quantity in the limit \( \epsilon \to 0 \).

INFLUENCE OF GRAavity SOURCE TERMS

With \( M_{\text{A}}/M_{\text{E}} = 1, M_{\text{A}} = \epsilon \to 0 \)
\[
\partial_t \left( \frac{\rho v}{\epsilon} \right) + \nabla \cdot \left( \frac{\rho v}{\epsilon} \otimes v + \frac{p}{\epsilon} \right) = \frac{\epsilon}{\rho g} \nabla \cdot \frac{\rho g}{\epsilon} v
\]

Energy as time-continuous scheme for Weisskoph Smith / Turkel 99 (bottom row of matrix \( D_{\text{Roe}} \)):
\[
\partial_t \epsilon + \frac{1}{\Delta x} (\text{central fluxes}) + \frac{1}{\Delta x} (\text{diffusive part}) = \frac{\epsilon g}{\epsilon} \nabla \cdot \frac{\rho g}{\epsilon} v
\]

The highest order equation (formally) would still impose \( \nabla p^{(2)} = 0 \) – the method is thus not asymptotic preserving.

The new modification overcomes this problem:

References:
Barsukow, W., Edelmann, P. V. F., Klingenberg, C. Röpke, F. K. in prep.